Some Applications of Decomposition Techniques to Systems of Coupled Variational Inequalities

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Abstract: We consider a set of problems which consists of systems of coupled variational inequalities. To solve these problems we use a decomposition–coordination method which allows us to solve the coupled problem through the solution of simple independent problems – in general, they are linear problems or simple obstacle problems. In this approach, the original problem is stated in terms of some appropriately defined auxiliary variables. These variables are modified (by an iterative algorithm in the coordination phase of the procedure) until the desired global solution is obtained..

Key-words: junction problems, variational inequalities, decomposition methods, convex function, unilateral condition, bilateral condition, numerical solution, iterative algorithm

(Résumé : tsvp)

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Applications de techniques de décomposition aux systèmes d'inéquations variationnelles couplées.

Résumé : On considère ici un ensemble de problèmes définis par des systèmes d'inéquations variationelles couplées. Pour résoudre le problème originel on utilise une méthode de décomposition-coordination. Cette méthode permet de résoudre le problème couplé à travers la solution des simples problèmes indépendants – en général, ce sont des problèmes linéaires ou des problémes d'obstacle. Dans cette méthode le problème originel est redéfini en termes des variables auxilliaires. Ces variables sont modifiées (par un algorithme itératif dans la phase de coordination du procédé) jusqu' à ce que la solution globale soit obtenue.

Mots-clé : problèmes de jonctions, inégalités variationelles, méthodes de décomposition, fonctions convexes, condition unilatérale, condition bilatérale, solution numérique, algorithmes itératifs.

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1 Introduction.

This paper deals with some problems of coupled variational inequalities, where the coupling may take different forms. Sometimes the solution is defined as a n-uple of functions with different domains, which have some overlapping regions (known as the interface); in that case, the coupling restrictions are defined in terms of the common values of the functions at the interface, but in other cases the coupling involves some global functionals.

In this paper we restrict our attention to the solution of some problems of coupled variational inequalities which were presented in [8].

Starting from the V.I. formulation, we obtain the solution via a decomposition–coordination method (the theoretical justification of the method is given in [5]; the method itself stems from the general methodology presented in [9]).

We use this procedure because it allows us to solve the coupled problem through the solution of simple independent problems – in general, they are linear problems or simple obstacle problems. These problems depend on some auxiliary variables which are modified (by the coordination phase of the procedure) until the desired global solution is obtained.

This rapport is organized in the following way: In this section we give a general introduction to the problem of coupled variational inequalities, Section 2 presents the decomposition method and Section 3 contains the detailed solution of five selected problems

1.1 Description of the original problem

1.1.1 A brief presentation of junction problems

This work is originated by what is known as *junction problems* – we can see [1], [2], [4], [5] and the bibliography therein, for a more detailed description of these problems and the analysis of some related topics. Specifically, our work deals with some issues that appear when the variational inequality approach is used to analyze these problems (see [8]). In order to fix the ideas, we consider the geometrical

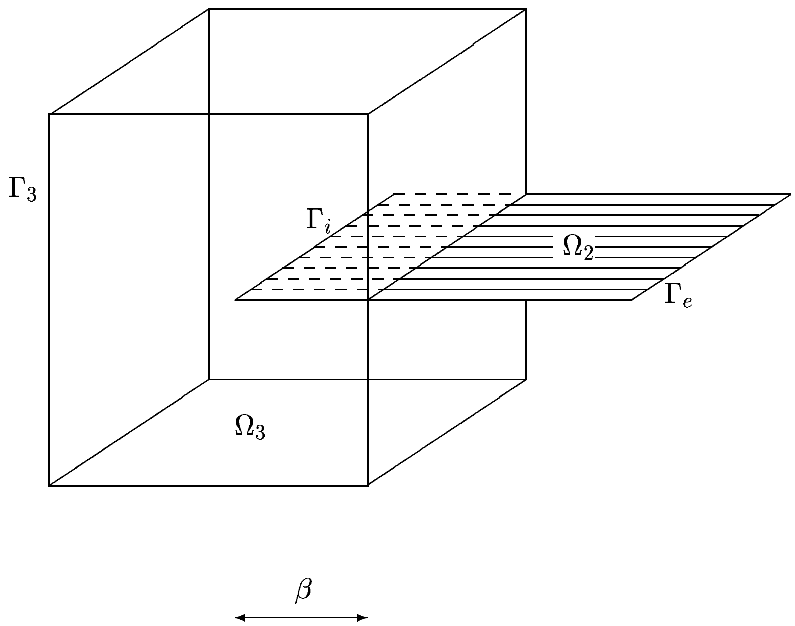

Figure 1: A 2-3 dimensional coupled problem

situation shown in Fig. 1, where

$$\Omega_2 = \{x \in \Re^3 \,:\, -\beta < x_1 < 1,\ -1 < x_2 < 1,\ x_3 = 0\} \,,$$

$$\Phi = \{x \in \Re^3 \,:\, -1 < x_1 < 0,\ -1 < x_2 < 1,\ -1 < x_3 < 1\} \,,$$

$$\Omega_3 = \Phi \setminus \Omega_2 \,.$$

In $\Re^3$ we have two open domains Ω_2 and Ω_3, of dimensions 2 and 3 respectively; Ω_2 is in the plane $x_3 = 0$. The boundary $\partial\Omega_2$ of Ω_2 consists of Γ_{2i} (where i stands for *interface or junctions*) and Γ_{2e} (where e stands for exterior). The domain Ω_3 is an open domain in $\Re^3$; we denote by $\Gamma_3 = \partial\Omega_3$ its boundary.

Note 1.1 *Other geometrical situations, and also regions consisting of several* pieces *of dimensions 1,2,3, can be considered. We want to study boundary value problems in domains of type* $\Omega_2 \cup \Omega_3$.

1.1.2 System state

The state of the system (which may represent some variables of interest: temperature, deplacement, etc) is given by a real function $(u_2, u_3) : \Omega_2 \cup \Omega_3 \to \Re$. We set

$$\left| \begin{array}{l} X_2 = H^1(\Omega_2)\,, \qquad X_3 = H^1(\Omega_3)\,, \quad X = X_2 \oplus X_3\,, \\ \\ \partial\Omega_2 = \Gamma_{2i} \cup \Gamma_{2e}\,, \quad \partial\Omega_3 = \Gamma_3\,. \end{array} \right. \tag{1}$$

1.1.3 Connections: Operators M_i

Around the interface Γ_{2i}, both in Ω_2 and in Ω_3, *connections* are established between u_2 and u_3. These connections may be local or not; they are defined in terms of linear continuous operators $M_i \in \mathcal{L}(X_i\,, \mathcal{H}), \quad i = 2, 3$, where $\mathcal{H}$ is a given Hilbert space. Specifically, we consider a closed convex subset $\mathcal{K}$ of $\mathcal{H}$ and we define the *connection* in the following form:

$$M_2\, u_2 - M_3\, u_3 \in \mathcal{K} \subset \mathcal{H}\,. \tag{2}$$

We define the set of *admissible states* as the set of states that verify the connection (2), i.e.

$$K = \{(v_2, v_3) \in X_2 \oplus X_3 \;:\; M_2\, v_2 - M_3\, v_3 \in \mathcal{K}\}\;. \tag{3}$$

Obviously, as M_i are linear continuous operators, the set K is a closed convex subset of X.

1.1.4 The energy functional: The bilinear forms

As it is usual in physical problems, we find the state of the system looking for the admissible state that minimizes a functional of energy, which in our problem will be described in terms of a couple of bilinear forms a_2, a_3. We will suppose that these bilinear forms are symmetric and we define the functional $J : X_2 \oplus X_3 \to \Re$ in the following way:

$$J(u_2, u_3) = \frac{1}{2}\, a_2(u_2, u_2) - (f_2, u_2) + \frac{1}{2}\, a_3(u_3, u_3) - (f_3, u_3)\,, \tag{4}$$

where the bilinear forms have the following expressions ($\alpha > 0$, $\beta > 0$)

$$a_2(u_2, v_2) = \int_{\Omega_2} (\nabla\, u_2 \nabla v_2 + \alpha\; u_2\, v_2)\, dx\,, \tag{5}$$

$$a_3(u_3, v_3) = \int_{\Omega_3} (\nabla\, u_3 \nabla v_3 + \beta\; u_3\, v_3)\, dx\,. \tag{6}$$

We associate to (5), (6) the differential operators A_2 and A_3 :

$$A_2 = -\triangle + \alpha\,, \qquad A_3 = -\triangle + \beta\,.$$

In the examples analyzed in this paper, we will restrict the study to bilinear forms associated to these simple second order differential operators, although extensions to more general operators are straightforward and without difficulties.
We also define the functional operators $\mathcal{A}_2$ and $\mathcal{A}_3$ such that

$$a_2(u_2, v_2) = \langle \mathcal{A}_2 u_2, v_2 \rangle \text{ and } a_3(u_3, v_3) = \langle \mathcal{A}_3 u_3, v_3 \rangle\,, \tag{7}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $H^1(\Omega_2)$ or $H^1(\Omega_3)$.

1.2 Coupled variational inequalities

We must minimize the functional (4) in the set of admissible states K or, in a equivalent way, we will consider the variational inequality: Find $u = (u_2, u_3) \in K$ such that

$$a_2(u_2, v_2 - u_2) + a_3(u_3, v_3 - u_3) \geq (f_2, v_2 - u_2) + (f_3, v_3 - u_3), \quad \forall\ (v_2, v_3) \in K\,, \tag{8}$$

where $f_2 \in L^2(\Omega_2)$, $f_3 \in L^2(\Omega_3)$ and (v, w) denotes the inner product in $L^2(\Omega_2)$ or $L^2(\Omega_3)$. By virtue of (5), (6), this variational inequality has a unique solution (see [10]).

Remark 1 *By assumption, the bilinear forms a_2 and a_3 are symmetric and in consequence, the inequality* (8) *is the necessary condition that must hold at the point that realizes the minimum of the functional J on the set K.*

Using the general framework above presented, we introduce de following example which will be completely defined once we have specified $\mathcal{H}$, $\mathcal{K}$, M_2 and M_3. This problem will be solved, among others, by decomposition techniques in Section 3.

Example 1 *We take $\mathcal{H} = H^{1/2}(\Gamma_3)$, $\mathcal{K} = 0$, and we define the operator M_3 to be*

$$M_3\, v_3 = \gamma_3\, v_3\,,$$

where γ_3 is the trace of v_3 on Γ_3.

We consider now the trace operator $T \in H^1(\Gamma_3) \longrightarrow H^{1/2}(\Gamma_{2i})$ *such that* $T\phi = \phi|_{\Gamma_{2i}}$. *This operator has inverse, an operator* R *such that*

$$R \in \mathcal{L}(H^{1/2}(\Gamma_{2i}); H^1(\Gamma_3)), \qquad T\,R\,\phi = \phi \qquad \forall \phi \in H^{1/2}(\Gamma_{2i}). \tag{9}$$

We introduce now the operator $E \in \mathcal{L}(H^{1/2}(\Gamma_{2i}); H^{1/2}(\Gamma_3))$ *; we take, for example,* $E = R$. *Let* $\gamma_{2i}\, v_2$ *be the trace of* v_2 *on* Γ_{2i}, *and so we have* $\gamma_{2i}\, v_2 \in H^{1/2}(\Gamma_{2i})$.
We define $M_2\, v_2 = E\,\gamma_{2i}\, v_2$ *and we consider the convex set of admissible states*

$$K = \{(v_2, v_3) \in X_2 \oplus X_3 \; : \; M_2\, v_2 - M_3\, v_3 = 0\} \; . \tag{10}$$

In this case, the connection condition

$$M_2\, v_2 = M_3\; v_3 \,, \tag{11}$$

means that the extension to Γ_3 *of the trace* $\gamma_{2i}\, v_2$ *must be equal to the trace of* v_3.

2 A decomposition method

The idea of solving (8) by hierarchical optimization and decomposition stems from the fact that the solution $u = (u_2, u_3)$ depends on the values of a function defined on some intermediate space and if we choose the correct (and unique) intermediate value, then the original problem (8) is solved.

The procedure starts choosing a value u_I and then – through the solution of some ad-hoc defined separated problems – that value is corrected until condition (8) is verified.

Decomposition of the convex K

We introduce a linear space X_I and a convex set K_I. Also, $\forall\, u_I \in K_I$, the associated convex closed sets $K_2(u_I)$ and $K_3(u_I)$ with the property

$$K = \bigcup_{u_I \in K_I} (K_2(u_I) \oplus K_3(u_I)) \,. \tag{12}$$

Definition 1 *We introduce the notation*

$$\varphi_2(u_I) = \min_{u_2 \in K_2(u_I)} J(u_2, 0)\,, \tag{13}$$

$$\varphi_3(u_I) = \min_{u_3 \in K_3(u_I)} J(0, u_3)\,. \tag{14}$$

Definition 2 The separated problems

To compute the functions φ_2 *and* φ_3 *we define the problems* $\mathcal{P}_2(u_I)$ *and* $\mathcal{P}_3(u_I)$*:*

Problem $\mathcal{P}_2(u_I)$

$$\textit{Find } \bar{u}_2(u_I) \in K_2(u_I) \;\; \textit{such that} \;\; J(\bar{u}_2\,,\,0) = \varphi_2(u_I)\,. \tag{15}$$

Problem $\mathcal{P}_3(u_I)$

$$\textit{Find } \bar{u}_3(u_I) \in K_3(u_I) \;\; \textit{such that} \;\; J(0, \bar{u}_3) = \varphi_3(u_I)\,. \tag{16}$$

Remark 2 *By* (5) *and* (6)*, there exists unique solution of* (15) *and* (16)*.*

Definition 3 The hierarchical optimization problem
Let us define the auxiliary function φ

$$\varphi(u_I) = \varphi_2(u_I) + \varphi_3(u_I)\,. \tag{17}$$

By virtue of (12) *we have*

$$\min_{(u_2,u_3)\in K} J(u_2\,,u_3) = \min_{u_I \in K_I} \left(\min_{K_2(u_I)\oplus K_3(u_I)} J(u_2\,,u_3) \right). \tag{18}$$

But from (4) *we have*

$$\begin{aligned} \min_{K_2(u_I)\oplus K_3(u_I)} J(u_2\,,u_3) \; &= \left(\min_{u_2\in K_2(u_I)} J(u_2\,,0) \right) + \left(\min_{u_3 \in K_3(u_I)} J(0, u_3) \right) \\ &= \varphi_2(u_I) + \varphi_3(u_I) = \varphi(u_I)\,. \end{aligned} \tag{19}$$

Then

$$\min_{(u_2,u_3)\in K} J(u_2\,,u_3) = \min_{u_I\in K_I} \varphi(u_I)\,, \tag{20}$$

and we conclude that problem (8) *is equivalent to the following:*

Problem $\mathcal{P}_I$

$$Find\ \bar{u}_I \in K_I \ \ such\ that \ \ \varphi(\bar{u}_I) = \min_{u_I\in K_I} \varphi(u_I)\,. \tag{21}$$

Remark 3 *From* (18)-(21) *it follows that the new problem $\mathcal{P}_I$ is a* <u>*decomposable hierarchical optimization problem*</u> [9].

The keystone of the decomposition method is the fact that the function φ appearing in (21) *verify* (17) *and that* $(\varphi_2,\ \varphi_3)$ *can be computed by solving two separated optimization problems.*

Remark 4 $(\bar{u}_2(\bar{u}_I), \bar{u}_3(\bar{u}_I))$ *is the solution of the variational inequality* (8) *when* $\bar{u}_I$ *realizes the minimum of* φ.

Remark 5 *In general, the problem $\mathcal{P}_I$ is easier than* (8)*, because*

1. *The dimension of Ω_I (where X_I is defined) is smaller than the dimensions of Ω_2, Ω_3.*
2. *X_I has a simpler structure.*
3. *Problem $\mathcal{P}_I$ is the minimum of a convex function on the set K_I which involves sometimes simpler restrictions.*
4. *By solving the problems $\mathcal{P}_2$ and $\mathcal{P}_3$ it is possible to compute a descent direction for φ.*

In the following we will suppose that the hypotheses of Theorem 1 hold and in consequence, φ_2 and φ_3 are convex functions. Finally, from (17), φ is also convex.

Theorem 1 *Let X_I, X be vector spaces, K_I a closed convex subset of X_I, $\phi(\cdot)$ a strictly convex continuous function defined on X and (for each $v \in K_I$) a closed convex subset $K^v \subset X$ such that we can define on K_I the function*

$$\psi(v) = \inf_{x\in K^v} \phi(x) \tag{22}$$

and such that there exists a unique $\bar{u}(v) \in K^v$ *with the property*

$$\phi(\bar{u}(v)) = \psi(v). \tag{23}$$

Assume the multivalued mapping $v \to K^v$ *is a convex mapping in the sense that*

$$\lambda\, K^v + (1-\lambda)\, K^{\tilde{v}} \subset K^{\lambda v + (1-\lambda)\tilde{v}}\,, \tag{24}$$

then ψ *is a convex function.*

Proof: Let v and $\tilde{v}$ be elements of K_I and u, $\tilde{u}$ such that

$$u = \bar{u}(v)\,, \qquad \tilde{u} = \bar{u}(\tilde{v}), \tag{25}$$

then, ψ is convex because

$$\begin{aligned} \psi(\lambda\, v + (1-\lambda)\,\tilde{v}) &\leq \phi(\lambda\, u + (1-\lambda)\,\tilde{u}) && \text{by (22) and (24)} \\ &\leq \lambda\, \phi(u) + (1-\lambda)\,\phi(\tilde{u}) && \text{by the convexity of } \phi \\ &= \lambda\, \psi(v) + (1-\lambda)\,\psi(\tilde{v}) && \text{by (23) and (25).} \end{aligned}$$

□

Definition 4 The auxiliary function g
Let us suppose that φ_2, φ_3 *are differentiable and let us denote* g_2, g_3 *the corresponding derivatives. For each* $v_I \in X_I$, *we define*

$$g(v_I) = g_2(v_I) + g_3(v_I)\,. \tag{26}$$

Definition 5 The equilibrium condition
The optimality condition (that we will refer as the equilibrium condition) for $\bar{u}_I$ *to be the solution of Problem* $\mathcal{P}_I$ *is:*

$$(g(\bar{u}_I), v_I - \bar{u}_I) \geq 0\,, \qquad \forall v_I \in K_I\,. \tag{27}$$

Remark 6 *As* φ *is convex this condition is also a sufficient condition of optimality. In addition, if* φ *is strictly convex, the point* $\bar{u}_I$ *that verifies this optimality condition is unique.*

Definition 6 An iterative algorithm
The above mentioned unique value $\bar{u}_I$ can be found iteratively using the algorithm described below, which employs the information given by the function g. The algorithm generates a sequence $(\bar{u}_2(v_I^\nu),\, \bar{u}_3(v_I^\nu))$, which starts at an initial pair $\bar{u}_2(v_I^0),\, \bar{u}_3(v_I^0)$ and that converges to the solution $(u_2\,, u_3)$ of (8).

Let $\{\gamma_\nu : \nu = 0, \ldots\}$ be a sequence of positive numbers such that

$$\lim_{\nu \to +\infty} \gamma_\nu = 0 \;\; \textit{and} \;\; \sum_{\nu=0}^{\infty} \gamma_\nu = +\infty\,.$$

Let us denote by $P_{K_I}(q)$ the projection of q on K_I.

ALGORITHM

Step 1 : $\nu = 0,\; v_I^0 \in K_I\,.$

Step 2 : Solve problems $\mathcal{P}_2(v_I^\nu)$ and $\mathcal{P}_3(v_I^\nu)$. Set $\gamma = \gamma_\nu\,.$

Step 3 : $\hat{v}_I = P_{K_I}(v_I^\nu - \gamma\, g(v_I^\nu))\,.$

Step 4 : If $\varphi(\hat{v}_I) < \varphi(v_I^\nu) + \frac{1}{2}(g(v_I^\nu), \hat{v}_I - v_I^\nu)\,,$ set $v_I^{\nu+1} = \hat{v}_I\,,\; \nu = \nu + 1$ and go to Step 2;

else, $\gamma = \frac{\gamma}{2}$, and go to Step 3.

3 Applications

We give in this section some examples where we apply the methodology described in the previous section. In the first four cases, we obtain the solution in an explicit form and in the last example, the solution is reduced to the application of a simple iterative method to find the zero of a monotone real function.

3.1 Example 1

3.1.1 The original problem

The description of this problem was given in the section 1.2; so here we have

$$
\left|
\begin{array}{ll}
\mathcal{H} = H^{1/2}(\Gamma_3)\,, & \mathcal{K} = 0\,, \\[2ex]
(M_2\, v_2)(x) = (E\,\gamma_{2i}\, v_2)(x)\,, & x \in \Gamma_3\,, \\[2ex]
(M_3\, v_3)(x) = (\gamma_3\, v_3)(x)\,, & x \in \Gamma_3\,.
\end{array}
\right.
\tag{28}
$$

In this case we have

$$
K = \{(v_2, v_3) \in X_2 \oplus X_3 \;:\; E\,\gamma_{2i}\, v_2 = \gamma_3\, v_3\}\,. \tag{29}
$$

3.1.2 Solution by decomposition

Elements of the decomposition

We define:

$$
\left|
\begin{array}{l}
X_I = H^{1/2}(\Gamma_{2i})\,, \\[2ex]
K_2(v_I) = \{v_2 \in X_2 \;:\; \gamma_{2i}\; v_2(x) = v_I(x),\; a.e.\; x \in \Gamma_{2i}\}\,, \\[2ex]
K_3(v_I) = \{v_3 \in X_3 \;:\; \gamma_3\; v_3(x) = (E v_I)\,(x),\; a.e.\; x \in \Gamma_3\}\,.
\end{array}
\right.
\tag{30}
$$

The separated optimization problems

With these data, we consider the problem $\mathcal{P}_2(v_I)$

$$
\boxed{\mathcal{P}_2(v_I)}\; : \qquad \varphi_2(v_I) = \min_{u_2 \in K_2(v_I)} \tfrac{1}{2}\,\langle \mathcal{A}_2 u_2, u_2\rangle - \langle \hat{f}_2, u_2\rangle\,, \tag{31}
$$

which is equivalent to find the solution $\bar{u}_2(v_I)$ of the Dirichlet-Neumann problem:

$$\begin{cases} A_2\bar{u}_2 = f_2\,, \\ \bar{u}_2|_{\Gamma_{2i}} = v_I\,, \\ \frac{\partial \bar{u}_2}{\partial n}|_{\Gamma_{2e}} = 0\,. \end{cases} \tag{32}$$

To make explicit the dependence on v_I of the solution $\bar{u}_2(v_I)$, we introduce the following definitions: Let w_2 be the solution of

$$\begin{cases} A_2 w_2 = f_2\,, \\ w_2|_{\Gamma_{2i}} = 0\,, \\ \frac{\partial w_2}{\partial n}|_{\Gamma_{2e}} = 0 \end{cases} \tag{33}$$

and let G_2 be the operator

$$\begin{gathered} G_2 : H^{1/2}(\Gamma_{2i}) \longrightarrow H^1(\Omega_2)\,, \\ \phi \longrightarrow G_2\phi\,, \end{gathered} \tag{34}$$

such that

$$\begin{cases} A_2(G_2\phi) = 0\,, \\ (G_2\phi)|_{\Gamma_{2i}} = \phi\,, \\ \frac{\partial (G_2\phi)}{\partial n}|_{\Gamma_{2e}} = 0\,. \end{cases} \tag{35}$$

By virtue of (32), (33) and (35), we have

$$\bar{u}_2(v_I) = G_2 v_I + w_2\,. \tag{36}$$

In consequence, for the auxiliary function $\varphi_2(v_I)$ we have

$$\varphi_2(v_I) = \frac{1}{2}\langle \mathcal{A}_2(G_2 v_I + w_2), G_2 v_I + w_2\rangle - \left\langle \hat{f}_2, G_2 v_I + w_2 \right\rangle \tag{37}$$

and then, for the derivative (in the Frechet sense) we obtain: $\forall \zeta \in X_I$

$$\langle D\varphi_2(v_I), \zeta\rangle = \langle G_2^* \mathcal{A}_2 G_2 v_I + G_2^*(\mathcal{A}_2 w_2 - \hat{f}_2)\,, \zeta\rangle. \tag{38}$$

Now, we consider the problem $\mathcal{P}_3(v_I)$

$$\boxed{\mathcal{P}_3(v_I)} : \qquad \varphi_3(v_I) = \min_{u_3 \in K_3(v_I)} \tfrac{1}{2}\left\langle \mathcal{A}_3 u_3, u_3\right\rangle - \langle \hat{f}_3, u_3\rangle\,. \tag{39}$$

In this case, the solution $\bar{u}_3(v_I)$ is given by the solution of the Dirichlet problem:

$$\begin{cases} A_3 \bar{u}_3 = f_3\,, \\ \\ \bar{u}_3|_{\Gamma_3} = E v_I\,. \end{cases} \tag{40}$$

Let be w_3 the solution of

$$\begin{cases} A_3 w_3 = f_3\,, \\ \\ w_3|_{\Gamma_3} = 0 \end{cases} \tag{41}$$

and let G_3 be the operator

$$\begin{aligned} G_3 : H^{1/2}(\Gamma_3) &\longrightarrow H^1(\Omega_3)\,, \\ \\ \phi &\longrightarrow G_3\phi\,, \end{aligned} \tag{42}$$

such that

$$\begin{cases} A_3(G_3\phi) = 0\,, \\ \\ (G_3\phi)|_{\Gamma_3} = \phi\,. \end{cases} \tag{43}$$

By virtue of (40), (41) and (43), we have

$$\bar{u}_3(v_I) = G_3 E v_I + w_3\,. \tag{44}$$

Then, we obtain

$$\varphi_3(v_I) = \frac{1}{2}\langle \mathcal{A}_3(G_3 E v_I + w_3)\,,\ G_3 E v_I + w_3\rangle - \left\langle \hat{f}_3\,,\, G_3 E v_I + w_3 \right\rangle\,. \tag{45}$$

Therefore, for the derivative (in the Frechet sense) we have: $\forall \zeta \in X_I$

$$\langle D\varphi_3(v_I), \zeta \rangle = \langle (E^* G_3^* \mathcal{A}_3 G_3 E) v_I + E^* G_3^* (\mathcal{A}_3 w_3 - \hat{f}_3) \,,\, \zeta \rangle \,. \tag{46}$$

Coordination phase

The solution $\bar{u}_I$ is given by solving the equation $g(v_I) = 0$, where $g(v_I) = D\varphi_2(v_I) + D\varphi_3(v_I)$. By virtue of (38) and (46), we have

$$G_2^* \mathcal{A}_2 G_2 \bar{u}_I + G_2^* (\mathcal{A}_2 w_2 - \hat{f}_2) + (E^* G_3^* \mathcal{A}_3 G_3 E) \bar{u}_I + E^* G_3^* (\mathcal{A}_3 w_3 - \hat{f}_3) = 0 \,. \tag{47}$$

In consequence we get

$$\bar{u}_I = Q \left(G_2^* (\mathcal{A}_2 w_2 - \hat{f}_2) + E^* G_3^* (\mathcal{A}_3 w_3 - \hat{f}_3) \right) \,, \tag{48}$$

where

$$Q = -(G_2^* \mathcal{A}_2 G_2 + E^* G_3^* \mathcal{A}_3 G_3 E)^{-1} \,. \tag{49}$$

So, the solution is given by

$$\boxed{\begin{cases} u_2(\bar{u}_I) = G_2 \, Q \left(G_2^* (\mathcal{A}_2 w_2 - \hat{f}_2) + E^* G_3^* (\mathcal{A}_3 w_3 - \hat{f}_3) \right) + w_2 \,, \\[2ex] u_3(\bar{u}_I) = G_3 E \, Q \left(G_2^* (\mathcal{A}_2 w_2 - \hat{f}_2) + E^* G_3^* (\mathcal{A}_3 w_3 - \hat{f}_3) \right) + w_3 \,. \end{cases}} \tag{50}$$

3.2 Example 2

3.2.1 The original problem

We take $X_2 = H^1(\Omega_2)$, $X_3 = H^1(\Omega_3)$, $X = X_2 \oplus X_3$, $\mathcal{H} = \Re^p$, $\mathcal{K} = 0$.

We define the operators $W_2 : X_2 \mapsto \Re^p$ such that $(W_2 \, u_2)_j = (w_{2j}, u_2)$ and $W_3 : X_3 \mapsto \Re^p$ such that $(W_3 \, u_3)_j = (w_{3j}, u_3)$, with $w_{2j} \in L^2(\Omega_2)$ and $w_{3j} \in L^2(\Omega_3)$, $\forall j = 1, \cdots, p$.

We define the operators M_i by

$$M_2\, v_2 = W_2\, v_2\,, \qquad M_3\, v_3 = W_3\, v_3\,.$$

In this case we have

$$K = \{(v_2, v_3) \in X_2 \oplus X_3\ :\ M_2\, v_2 - M_3\, v_3 = 0\}\,, \tag{51}$$

or

$$K = \{(v_2, v_3) \in X_2 \oplus X_3\ :\ (v_2, w_{2j}) = (v_3, w_{3j})\,,\ 1 \le j \le p\}\,. \tag{52}$$

Note 3.1 *In this example we will suppose that the operators W_2, W_3 defined above verify:* rank $(W_i) = p$.

3.2.2 Solution by decomposition. Case of dimension 1

$$\left|\begin{array}{l} X_I = R\,, \\ \\ K_2(\xi) = \{u_2 \in X_2 : (w_2, u_2) = \xi\}\,, \\ \\ K_3(\xi) = \{u_3 \in X_3 : (w_3, u_3) = \xi\}\,. \end{array}\right. \tag{53}$$

We compute φ_2 and φ_3 by solving the decoupled optimization problems $\mathcal{P}_2(\xi)$ and $\mathcal{P}_3(\xi)$, with solutions $u_2(\xi)$ and $u_3(\xi)$ respectively.

The optimization problem $\mathcal{P}_2(\xi)$

$$\boxed{\mathcal{P}_2(\xi)}\ : \qquad \varphi_2(\xi) = \min_{u_2 \in K_2(\xi)} \tfrac{1}{2} a_2(u_2, u_2) - (f_2, u_2)\,. \tag{54}$$

The necessary condition of optimality in the convex set $K_2(\xi)$ is:

$$\nabla \varphi_2^0(u_2)(v_2 - u_2) \ge 0 \qquad \forall v_2 \in K_2(\xi)\,. \tag{55}$$

Let $\mathcal{A}_2$ be the operator associated to a_2, i.e. $a_2(u, v) = \langle \mathcal{A}_2 u, v\rangle$ and let $\hat{f}$ be the element given by the Riesz representation theorem, i.e. $\langle \hat{f}_2\,, v_2\rangle = (f_2\,, v_2)\,,\ \forall v_2 \in X_2$. Then (55) is equivalent to

$$\langle \mathcal{A}_2 u_2 - \hat{f}_2, v_2 - u_2\rangle \ge 0 \qquad \forall v_2\ /\ \langle \hat{w}_2, v_2\rangle = \xi\,. \tag{56}$$

So, we have

$$\langle \mathcal{A}_2 u_2 - \hat{f}_2,\ z_2 \rangle \geq 0 \qquad \forall z_2 \ / \ \langle \hat{w}_2, z_2 \rangle = 0\,, \tag{57}$$

that is

$$\langle \mathcal{A}_2 u_2 - \hat{f}_2,\ z_2 \rangle = 0 \qquad \forall z_2 \ / \ \langle \hat{w}_2, z_2 \rangle = 0\,, \tag{58}$$

therefore $\exists \lambda_2 \in \Re$ such that

$$\begin{cases} \mathcal{A}_2 u_2 - \hat{f}_2 = \lambda_2 \hat{w}_2\,, \\ \langle \hat{w}_2, u_2 \rangle = \xi\,. \end{cases} \tag{59}$$

We obtain $u_2(\xi)$ in the following form

$$\mathcal{A}_2 u_2 = \hat{f}_2 + \lambda_2 \hat{w}_2\,. \tag{60}$$

To compute λ_2, we consider that

$$\begin{aligned} \xi \ &= (w_2, u_2) = \langle \hat{w}_2, \mathcal{A}_2^{-1} \mathcal{A}_2 u_2 \rangle = \langle \hat{w}_2, \mathcal{A}_2^{-1} (\,\hat{f}_2 + \lambda_2 \hat{w}_2) \rangle \\ &= \langle \hat{w}_2, \mathcal{A}_2^{-1}\, \hat{f}_2 \rangle + \lambda_2 \, \langle \hat{w}_2, \mathcal{A}_2^{-1} \hat{w}_2 \rangle \end{aligned} \tag{61}$$

and then, we have

$$\begin{cases} \lambda_2 = \left\langle \hat{w}_2, \mathcal{A}_2^{-1} \hat{w}_2 \right\rangle^{-1} \left(\xi - \langle \hat{w}_2, \mathcal{A}_2^{-1}\, \hat{f}_2 \rangle \right), \\ u_2(\xi) = \mathcal{A}_2^{-1} \hat{f}_2 + \left\langle \hat{w}_2, \mathcal{A}_2^{-1} \hat{w}_2 \right\rangle^{-1} \left(\xi - \langle \hat{w}_2, \mathcal{A}_2^{-1}\, \hat{f}_2 \rangle \right) \, \mathcal{A}_2^{-1}\, \hat{w}_2\,. \end{cases} \tag{62}$$

Finally, from (62) we get

$$\varphi_2(\xi) = \frac{1}{2} \langle \mathcal{A}_2\, u_2, u_2 \rangle - \langle \hat{f}_2, u_2 \rangle\,. \tag{63}$$

The optimization problem $\mathcal{P}_3(\xi)$

$$\boxed{\mathcal{P}_3(\xi)} \ : \qquad \varphi_3(\xi) = \min_{u_3 \in K_3(\xi)} \tfrac{1}{2} a_3(u_3, u_3) - (f_3, u_3)\,. \tag{64}$$

With a similar procedure to that one used above, we obtain λ_3 and $u_3(\xi)$ in the following form

$$\left\{\begin{array}{l} \lambda_3 = \left\langle \hat{w}_3, \mathcal{A}_3^{-1} \hat{w}_3 \right\rangle^{-1} \left(\xi - \langle \hat{w}_3, \mathcal{A}_3^{-1} \hat{f}_3 \rangle \right) , \\ \\ u_3(\xi) = \mathcal{A}_3^{-1} \hat{f}_3 + \left\langle \hat{w}_3, \mathcal{A}_3^{-1} \hat{w}_3 \right\rangle^{-1} \left(\xi - \langle \hat{w}_3, \mathcal{A}_3^{-1} \hat{f}_3 \rangle \right) \mathcal{A}_3^{-1} \hat{w}_3 \, . \end{array}\right. \tag{65}$$

In consequence, we get the following explicit expression of φ_3

$$\varphi_3(\xi) = \frac{1}{2} \langle \mathcal{A}_3 \, u_3, u_3 \rangle - \langle \hat{f}_3, u_3 \rangle \, . \tag{66}$$

Computation of φ_2' and φ_3'

We denote

$$\boxed{\begin{array}{ll} \vartheta_2 = \left\langle \hat{w}_2, \mathcal{A}_2^{-1} \hat{w}_2 \right\rangle^{-1} & \vartheta_3 = \left\langle \hat{w}_3, \mathcal{A}_3^{-1} \hat{w}_3 \right\rangle^{-1} \\ \\ \kappa_2 = \langle \hat{w}_2, \mathcal{A}_2^{-1} \hat{f}_2 \rangle & \kappa_3 = \langle \hat{w}_3, \mathcal{A}_3^{-1} \hat{f}_3 \rangle \end{array}} \tag{67}$$

then, from (62) and (63), we have

$$u_2(\xi) = \mathcal{A}_2^{-1} \hat{f}_2 - \vartheta_2 \, \kappa_2 \, \mathcal{A}_2^{-1} \, \hat{w}_2 \, + \vartheta_2 \, \mathcal{A}_2^{-1} \, \hat{w}_2 \, \xi \tag{68}$$

and so,

$$\begin{aligned}
\varphi_2(\xi) &= \tfrac{1}{2}\left\langle \mathcal{A}_2\left(\mathcal{A}_2^{-1}\hat{f}_2 - \vartheta_2\kappa_2\mathcal{A}_2^{-1}\,\hat{w}_2\ + \vartheta_2\mathcal{A}_2^{-1}\,\hat{w}_2\xi\right),\, \mathcal{A}_2^{-1}\hat{f}_2 - \vartheta_2\kappa_2\mathcal{A}_2^{-1}\hat{w}_2 + \vartheta_2\mathcal{A}_2^{-1}\,\hat{w}_2\xi\right\rangle \\
&\quad - \left\langle \hat{f}_2,\, \mathcal{A}_2^{-1}\hat{f}_2 - \vartheta_2\kappa_2\mathcal{A}_2^{-1}\,\hat{w}_2 + \vartheta_2\mathcal{A}_2^{-1}\,\hat{w}_2\xi\right\rangle \\
&= \tfrac{1}{2}\left\langle \hat{f}_2 - \vartheta_2\kappa_2\,\hat{w}_2\ + \vartheta_2\,\hat{w}_2\xi,\, \mathcal{A}_2^{-1}\hat{f}_2 - \vartheta_2\kappa_2\mathcal{A}_2^{-1}\,\hat{w}_2\ + \vartheta_2\mathcal{A}_2^{-1}\,\hat{w}_2\xi\right\rangle \\
&\quad - \left\langle \hat{f}_2,\, \mathcal{A}_2^{-1}\hat{f}_2\right\rangle + \left\langle \hat{f}_2,\, \vartheta_2\kappa_2\mathcal{A}_2^{-1}\,\hat{w}_2\ \right\rangle - \left\langle \hat{f}_2,\, \vartheta_2\mathcal{A}_2^{-1}\,\hat{w}_2\xi\right\rangle \\
&= \tfrac{1}{2}\left\langle \hat{f}_2,\, \mathcal{A}_2^{-1}\hat{f}_2\right\rangle - \tfrac{1}{2}\left\langle \hat{f}_2,\, \vartheta_2\kappa_2\mathcal{A}_2^{-1}\,\hat{w}_2\ \right\rangle + \tfrac{1}{2}\left\langle \hat{f}_2,\, \vartheta_2\mathcal{A}_2^{-1}\,\hat{w}_2\xi\right\rangle \\
&\quad -\tfrac{1}{2}\left\langle \vartheta_2\kappa_2\,\hat{w}_2,\, \mathcal{A}_2^{-1}\hat{f}_2\right\rangle + \tfrac{1}{2}\left\langle \vartheta_2\kappa_2\,\hat{w}_2,\, \vartheta_2\kappa_2\mathcal{A}_2^{-1}\,\hat{w}_2\ \right\rangle - \tfrac{1}{2}\left\langle \vartheta_2\kappa_2\,\hat{w}_2,\, \vartheta_2\mathcal{A}_2^{-1}\,\hat{w}_2\xi\right\rangle \\
&\quad +\tfrac{1}{2}\left\langle \vartheta_2\,\hat{w}_2\xi,\, \mathcal{A}_2^{-1}\hat{f}_2\right\rangle - \tfrac{1}{2}\left\langle \vartheta_2\,\hat{w}_2\xi,\, \vartheta_2\kappa_2\mathcal{A}_2^{-1}\,\hat{w}_2\ \right\rangle + \tfrac{1}{2}\left\langle \vartheta_2\,\hat{w}_2\xi,\, \vartheta_2\mathcal{A}_2^{-1}\,\hat{w}_2\xi\right\rangle \\
&\quad - \left\langle \hat{f}_2,\, \mathcal{A}_2^{-1}\hat{f}_2\right\rangle + \left\langle \hat{f}_2,\, \vartheta_2\kappa_2\mathcal{A}_2^{-1}\,\hat{w}_2\ \right\rangle - \left\langle \hat{f}_2,\, \vartheta_2\mathcal{A}_2^{-1}\,\hat{w}_2\xi\right\rangle .
\end{aligned}$$

After some simplifications, we obtain

$$\varphi_2(\xi) = \tfrac{1}{2}\vartheta_2^2\ \left\langle \hat{w}_2\xi,\, \mathcal{A}_2^{-1}\,\hat{w}_2\xi\right\rangle - \vartheta_2\,\kappa_2\,\xi - \tfrac{1}{2}\left\langle \hat{f}_2,\, \mathcal{A}_2^{-1}\hat{f}_2\right\rangle + \tfrac{1}{2}\vartheta_2\,\kappa_2^2\,, \tag{69}$$

in consequence, we get for φ_2' the expression

$$\varphi_2'(\xi) \quad = \vartheta_2\,\xi - \vartheta_2\,\kappa_2 = \left\langle \hat{w}_2, \mathcal{A}_2^{-1}\hat{w}_2\right\rangle^{-1}\left(\xi - \langle \hat{w}_2, \mathcal{A}_2^{-1}\,\hat{f}_2\rangle\right) . \tag{70}$$

In a similar way, we get

$$\varphi_3(\xi) = \tfrac{1}{2}\vartheta_3^2\ \left\langle \hat{w}_3\xi,\, \mathcal{A}_3^{-1}\,\hat{w}_3\xi\right\rangle - \vartheta_3\,\kappa_3\,\xi - \tfrac{1}{2}\left\langle \hat{f}_3, \mathcal{A}_3^{-1}\hat{f}_3\right\rangle + \tfrac{1}{2}\vartheta_3\,\kappa_3^2\,, \tag{71}$$

$$\varphi_3'(\xi) \quad = \vartheta_3\,\xi - \vartheta_3\,\kappa_3 = \left\langle \hat{w}_3, \mathcal{A}_3^{-1}\hat{w}_3\right\rangle^{-1}\left(\xi - \langle \hat{w}_3, \mathcal{A}_3^{-1}\hat{f}_3\rangle\right) . \tag{72}$$

So, by virtue of (26), we have

$$\begin{cases} g_2(\xi) = \left\langle \hat{w}_2, \mathcal{A}_2^{-1}\hat{w}_2 \right\rangle^{-1} \left(\xi - \langle \hat{w}_2, \mathcal{A}_2^{-1}\hat{f}_2\rangle\right) , \\ \\ g_3(\xi) = \left\langle \hat{w}_3, \mathcal{A}_3^{-1}\hat{w}_3 \right\rangle^{-1} \left(\xi - \langle \hat{w}_3, \mathcal{A}_3^{-1}\hat{f}_3\rangle\right) . \end{cases} \tag{73}$$

Coordination phase

We obtain the solution by solving the equation $g(\xi) = 0$. So, we have

$$\left\langle \hat{w}_2, \mathcal{A}_2^{-1}\hat{w}_2 \right\rangle^{-1} \left(\xi - \langle \hat{w}_2, \mathcal{A}_2^{-1}\hat{f}_2\rangle\right) + \left\langle \hat{w}_3, \mathcal{A}_3^{-1}\hat{w}_3 \right\rangle^{-1} \left(\xi - \langle \hat{w}_3, \mathcal{A}_3^{-1}\hat{f}_3\rangle\right) = 0 \tag{74}$$

and then

$$\xi = \frac{\langle \hat{w}_2, \mathcal{A}_2^{-1}\hat{f}_2\rangle \left\langle \hat{w}_3, \mathcal{A}_3^{-1}\hat{w}_3 \right\rangle + \langle \hat{w}_3, \mathcal{A}_3^{-1}\hat{f}_3\rangle \left\langle \hat{w}_2, \mathcal{A}_2^{-1}\hat{w}_2 \right\rangle}{\left\langle \hat{w}_3, \mathcal{A}_3^{-1}\hat{w}_3 \right\rangle + \left\langle \hat{w}_2, \mathcal{A}_2^{-1}\hat{w}_2 \right\rangle} . \tag{75}$$

Therefore, the solution is given by

$$\boxed{\begin{cases} u_2(\xi) = \mathcal{A}_2^{-1}\hat{f}_2 + \frac{\langle \hat{w}_3, \mathcal{A}_3^{-1}\hat{f}_3\rangle - \langle \hat{w}_2, \mathcal{A}_2^{-1}\hat{f}_2\rangle}{\langle \hat{w}_2, \mathcal{A}_2^{-1}\hat{w}_2\rangle + \langle \hat{w}_3, \mathcal{A}_3^{-1}\hat{w}_3\rangle} \mathcal{A}_2^{-1}\hat{w}_2 , \\ \\ u_3(\xi) = \mathcal{A}_3^{-1}\hat{f}_3 + \frac{\langle \hat{w}_2, \mathcal{A}_2^{-1}\hat{f}_2\rangle - \langle \hat{w}_3, \mathcal{A}_3^{-1}\hat{f}_3\rangle}{\langle \hat{w}_2, \mathcal{A}_2^{-1}\hat{w}_2\rangle + \langle \hat{w}_3, \mathcal{A}_3^{-1}\hat{w}_3\rangle} \mathcal{A}_3^{-1}\hat{w}_3 . \end{cases}} \tag{76}$$

3.2.3 Solution by decomposition. Case of dimension p

We take

$$\left| \begin{array}{l} X_I = R^p , \quad \mathcal{K} = 0 , \\ \\ K_2(\xi) = \{v_2 \in X_2 : (W_2\, v_2)_j = \xi_j, \ \forall j = 1, \cdots, p\} , \\ \\ K_3(\xi) = \{v_3 \in X_3 : (W_3\, v_3)_j = \xi_j, \ \forall j = 1, \cdots, p\} . \end{array} \right. \tag{77}$$

We compute φ_2 and φ_3 by solving the decoupled optimization problems $\mathcal{P}_2(\xi)$ and $\mathcal{P}_3(\xi)$, with solutions $u_2(\xi)$ and $u_3(\xi)$ respectively.

The optimization problem $\mathcal{P}_2(\xi)$

$$\boxed{\mathcal{P}_2(\xi)} : \qquad \min_{u_2 \in K_2(\xi)} \varphi_2^0(\xi) = \min_{u_2 \in K_2(\xi)} \tfrac{1}{2} a_2(u_2, u_2) - (f_2, u_2). \tag{78}$$

Computation of $\nabla\varphi_2$

We have

$$\varphi_2(\xi) = \min_{u_2 \in K_2(\xi)} \left\{ \tfrac{1}{2}\langle \mathcal{A}_2 u_2, u_2\rangle - \langle \hat{f}_2, u_2\rangle \right\}. \tag{79}$$

With the same argument used for the case of dimension 1, $\exists \lambda_2 \in \Re^p$ such that

$$\left\{ \begin{array}{l} \mathcal{A}_2 u_2 - \hat{f}_2 + W_2^\star \lambda_2 = 0\,, \\ \\ W_2 u_2 = \xi\,. \end{array} \right. \tag{80}$$

Then

$$\left\{ \begin{array}{rl} u_2 & = \mathcal{A}_2^{-1}(\hat{f}_2 - W_2^\star \lambda_2)\,, \\ \\ W_2 u_2 & = W_2 \mathcal{A}_2^{-1} \hat{f}_2 - (W_2 \mathcal{A}_2^{-1} W_2^\star)\lambda_2 = \xi\,. \end{array} \right. \tag{81}$$

We denote $P_2 \in \Re^{p\times p}$ as

$$P_2 = W_2 \mathcal{A}_2^{-1} W_2^\star\,, \tag{82}$$

which is a positive definite matrix. Then, we obtain

$$\lambda_2 = P_2^{-1}(W_2 \mathcal{A}_2^{-1} \hat{f}_2 - \xi) \tag{83}$$

and

$$u_2(\xi) = S_2 \xi + (I - S_2 W_2)\, \mathcal{A}_2^{-1} \hat{f}_2, \tag{84}$$

where

$$S_2 = \mathcal{A}_2^{-1} W_2^\star P_2^{-1}. \tag{85}$$

In consequence, we get

$$\begin{aligned}
\varphi_2(\xi) &= \frac{1}{2}\left\langle \mathcal{A}_2 S_2\xi + \mathcal{A}_2(I - S_2W_2)\mathcal{A}_2^{-1}\hat{f}_2,\, S_2\xi + (I - S_2W_2)\mathcal{A}_2^{-1}\hat{f}_2\right\rangle \\
&\quad - \left\langle \hat{f}_2,\, S_2\xi + (I - S_2W_2)\mathcal{A}_2^{-1}\hat{f}_2\right\rangle \\
&= \frac{1}{2}\left\langle \mathcal{A}_2 S_2\xi,\, S_2\xi\right\rangle + \frac{1}{2}\left\langle \mathcal{A}_2 S_2\xi,\, (I - S_2W_2)\mathcal{A}_2^{-1}\hat{f}_2\right\rangle \\
&\quad + \frac{1}{2}\left\langle \mathcal{A}_2(I - S_2W_2)\mathcal{A}_2^{-1}\hat{f}_2,\, S_2\xi\right\rangle - \left\langle \hat{f}_2,\, S_2\xi\right\rangle \qquad (86)\\
&\quad + \left\langle \frac{1}{2}\mathcal{A}_2(I - S_2W_2)\mathcal{A}_2^{-1}\hat{f}_2 - \hat{f}_2,\, (I - S_2W_2)\mathcal{A}_2^{-1}\hat{f}_2\right\rangle \\
&= \frac{1}{2}\left\langle S_2^\star\mathcal{A}_2 S_2\xi,\, \xi\right\rangle - \left\langle \xi,\, S_2^\star\mathcal{A}_2 S_2 W_2\mathcal{A}_2^{-1}\hat{f}_2\right\rangle \\
&\quad + \left\langle \frac{1}{2}\mathcal{A}_2(I - S_2W_2)\mathcal{A}_2^{-1}\hat{f}_2 - \hat{f}_2,\, (I - S_2W_2)\mathcal{A}_2^{-1}\hat{f}_2\right\rangle,
\end{aligned}$$

therefore

$$\begin{aligned}
\nabla\varphi_2 &= S_2^\star\mathcal{A}_2 S_2\xi - S_2^\star\mathcal{A}_2 S_2 W_2\,\mathcal{A}_2^{-1}\hat{f}_2 \\
&= P_2^{-1}W_2\mathcal{A}_2^{-1}W_2^\star P_2^{-1}\xi - P_2^{-1}W_2\mathcal{A}_2^{-1}W_2^\star P_2^{-1}W_2\,\mathcal{A}_2^{-1}\hat{f}_2\,.
\end{aligned} \qquad (87)$$

By virtue of (82) we obtain

$$\nabla\varphi_2 = P_2^{-1}\xi - S_2^\star\,\hat{f}_2\,. \qquad (88)$$

The optimization problem $\mathcal{P}_3(\xi)$

Now, we study the following problem

$$\boxed{\mathcal{P}_3(\xi)}\ : \qquad \min_{u_3\in K_3(\xi)} \varphi_3^0(\xi) = \min_{u_3\in K_3(\xi)} \left\{\tfrac{1}{2}a_3(u_3,u_3) - (f_3,u_3)\right\}. \qquad (89)$$

Computation of $\nabla\varphi_3$

In a similar way to that one used above, we have

$$\varphi_3(\xi) = \min_{u_3 \in K_3(\xi)} \left\{ \tfrac{1}{2}\langle \mathcal{A}_3 u_3, u_3\rangle - \langle \hat{f}_3, u_3\rangle \right\}. \tag{90}$$

The solution u_3 is given by:

$$u_3(\xi) = S_3\xi + (I - S_3 W_3)\,\mathcal{A}_3^{-1}\hat{f}_3\,, \tag{91}$$

where we define $P_3 \in \Re^{p\times p}$ as

$$P_3 = W_3\mathcal{A}_3^{-1}W_3^\star \tag{92}$$

and

$$S_3 = \mathcal{A}_3^{-1}W_3^\star P_3^{-1}. \tag{93}$$

Therefore

$$\nabla\varphi_3 = P_3^{-1}\xi - S_3^\star\,\hat{f}_3\,. \tag{94}$$

Coordination phase

We define $g(\xi) = \nabla\varphi_2 + \nabla\varphi_3$ and then, the solution is given by solving the equation $g(\xi) = 0\,.$
So, we must find ξ such that

$$P_2^{-1}\xi - S_2^\star\,\hat{f}_2 + P_3^{-1}\xi - S_3^\star\,\hat{f}_3 = 0\,, \tag{95}$$

then

$$\xi = \hat{P}\,\left(S_2^\star\,\hat{f}_2 + S_3^\star\,\hat{f}_3\right)\,, \tag{96}$$

where

$$\hat{P} = (P_2^{-1} + P_3^{-1})^{-1}\,. \tag{97}$$

By virtue of (84) and (91), we have

$$
\boxed{\begin{cases} u_2(\xi) = \mathcal{A}_2^{-1} \hat{f}_2 + S_2 \hat{P} \left(S_2^\star \hat{f}_2 + S_3^\star \hat{f}_3 \right) - S_2 W_2 \, \mathcal{A}_2^{-1} \hat{f}_2 \,, \\ \\ u_3(\xi) = \mathcal{A}_3^{-1} \hat{f}_3 + S_3 \hat{P} \left(S_2^\star \hat{f}_2 + S_3^\star \hat{f}_3 \right) - S_3 W_3 \, \mathcal{A}_3^{-1} \hat{f}_3 \,. \end{cases}} \tag{98}
$$

3.3 Example 3

3.3.1 The original problem

We take $\mathcal{H} = L^2(\Omega_3)$, $\mathcal{K} = 0$, and we define

$$
(M_3 \, v_3)(x) = m_3(x) \, v_3(x) \,, \tag{99}
$$

where $m_3 \in L^\infty(\Omega_3)$.

We consider $\gamma_{2i} \, v_2$ (trace of v_2 on Γ_{2i}) and we extend $\gamma_{2i} \, v_2$ to Ω_3. This can be done in the following ways. Let P be any extension operator:

$$
\begin{aligned} & P \in \mathcal{L}(H^{1/2}(\Gamma_{2i}), H^{3/2}(\Omega_3)) \,, \\ & P \, \varphi|_{\Gamma_{2i}} = \varphi \,, \quad \forall \varphi \in H^{1/2}(\Gamma_{2i}) \end{aligned} \tag{100}
$$

and we define

$$
(M_2 \, v_2)(x) = (P \, \gamma_{2i} \, v_2)(x) \,, \; x \in \Gamma_{2i} \,.
$$

In this way, (2) becomes

$$
m_3 \, u_3 = P \, \gamma_{2i} \, u_2 \,, \qquad in \; \Omega_3 \,. \tag{101}
$$

So, in this case we have

$$
K = \{ (v_2, v_3) \in X_2 \oplus X_3 \, : \, m_3(x) \, v_3(x) = \left(P \gamma_{2i} \; v_2 \right)(x), \; a.e. \; x \in \Omega_3 \} \,. \tag{102}
$$

3.3.2 Solution by decomposition

Elements of the decomposition

$$\left| \begin{array}{l} X_I = H^{1/2}(\Gamma_{2i})\,, \quad K_I = X_I\,, \\[2ex] K_2(v_I) = \{ v_2 \in X_2 \,:\, \gamma_{2i}\; v_2(x) = v_I(x),\; a.e.\; x \in \Gamma_{2i} \}\;, \\[2ex] K_3(v_I) = \{ v_3 \in X_3 \,:\, m_3(x)\; v_3(x) = P(v_I)(x),\; a.e.\; x \in \Omega_3 \}\;. \end{array} \right. \tag{103}$$

Analysis of problems $\mathcal{P}_2$, $\mathcal{P}_3$

We consider the problem $\mathcal{P}_2(v_I)$ defined by

$$\boxed{\mathcal{P}_2(v_I)} \;: \qquad \varphi_2(v_I) = \min_{u_2 \in K_2(v_I)} \tfrac{1}{2} \left\langle \mathcal{A}_2 u_2, u_2 \right\rangle - \langle \hat{f}_2, u_2 \rangle\,. \tag{104}$$

To solve this problem $\mathcal{P}_2$ we work in a similar way as that one appearing in section 3.1. So, the solution has the following form

$$\bar{u}_2(v_I) = G_2 v_I + w_2\,. \tag{105}$$

In consequence, for the auxiliary function $\varphi_2(v_I)$ we have

$$\varphi_2(v_I) = \frac{1}{2}\langle \mathcal{A}_2(G_2 v_I + w_2), G_2 v_I + w_2 \rangle - \langle \hat{f}_2, G_2 v_I + w_2 \rangle\,, \tag{106}$$

then, for the derivative (in the Frechet sense) we obtain: $\forall \zeta \in X_I$

$$\langle D\varphi_2(v_I), \zeta \rangle = \langle G_2^* \mathcal{A}_2 G_2 v_I + G_2^*(\mathcal{A}_2 w_2 - \hat{f}_2)\,, \zeta \rangle. \tag{107}$$

Now, we consider the problem $\mathcal{P}_3(v_I)$ defined by

$$\boxed{\mathcal{P}_3(v_I)} \;: \qquad \varphi_3(v_I) = \min_{u_3 \in K_3(v_I)} \tfrac{1}{2} \left\langle \mathcal{A}_3 u_3, u_3 \right\rangle - \langle \hat{f}_3, u_3 \rangle\,. \tag{108}$$

Let D and I be subsets of Ω_3 such that

$$D = \{ x_3 : m_3(x_3) \neq 0 \}\;, \qquad\qquad I = \{ x_3 : m_3(x_3) = 0 \}\,. \tag{109}$$

Then

$$u_3(x_3) = \frac{1}{m_3}\, P(v_I)(x_3)\,, \quad \forall x_3 \in D\,. \tag{110}$$

We define the operator G_3 such that

$$(G_3\, v_I)(x_3) = \begin{cases} \frac{1}{m_3}\, P(v_I)(x_3) & \forall x_3 \in D\,, \\ \\ 0 & \forall x_3 \in I\,, \end{cases} \tag{111}$$

and w_3 the solution of

$$\begin{cases} A_3\, w_3 = f_3\,, \\ \\ w_3\,|_D = 0\,. \end{cases} \tag{112}$$

Then we can write the solution $\bar{u}_3$ in the following form

$$\bar{u}_3 \ = G_3 v_I + w_3\,, \tag{113}$$

and so,

$$\begin{aligned} \varphi_3(v_I) &= \frac{1}{2}\, \langle \mathcal{A}_3\, u_3, u_3 \rangle - \langle \hat{f}_3, u_3 \rangle \\ &= \frac{1}{2}\, \langle \mathcal{A}_3(G_3 v_I + w_3)\,,\, (G_3 v_I + w_3) \rangle - \left\langle \hat{f}_3, (G_3 v_I + w_3) \right\rangle \\ &= \frac{1}{2}\, \langle \mathcal{A}_3 w_3\,, w_3 \rangle - \left\langle \hat{f}_3, w_3 \right\rangle + \frac{1}{2}\, \langle \mathcal{A}_3 G_3 v_I\,, G_3 v_I \rangle - \left\langle \hat{f}_3, G_3 v_I \right\rangle \\ &\quad + \frac{1}{2}\, \langle \mathcal{A}_3 G_3 v_I\,, w_3 \rangle + \frac{1}{2}\, \langle \mathcal{A}_3 w_3\,, G_3 v_I \rangle\ . \end{aligned} \tag{114}$$

The support of the functions $G_3 v_I$ and w_3 are disjoint, then we have

$$\langle \mathcal{A}_3 G_3 v_I\,, w_3 \rangle = \langle \mathcal{A}_3 w_3\,, G_3 v_I \rangle = (A_3 w_3, G_3 v_I) = 0\,, \tag{115}$$

therefore, we obtain

$$\varphi_3(v_I) = \overline{\varphi}_3 + \tilde{\varphi}_3(v_I)\,, \tag{116}$$

where $\overline{\varphi}_3$ is independent of v_I, that is

$$\overline{\varphi}_3 = \frac{1}{2}\, \langle \mathcal{A}_3 w_3\,, w_3 \rangle - \left\langle \hat{f}_3, w_3 \right\rangle \tag{117}$$

and

$$\tilde{\varphi}_3(v_I) = \frac{1}{2}\langle \mathcal{A}_3 G_3 v_I \,, G_3 v_I\rangle - \left\langle \hat{f}_3, G_3 v_I \right\rangle = \frac{1}{2}\langle\, G_3^\star \mathcal{A}_3 G_3\, v_I \;\;, v_I\rangle - \left\langle\, G_3^\star\, \hat{f}_3 \,, v_I \right\rangle . \tag{118}$$

Then, for the derivative (in the Frechet sense) we obtain: $\forall \zeta \in X_I$

$$\langle D\varphi_3(v_I), \zeta\rangle = \langle G_3^* \mathcal{A}_3 G_3 v_I - G_3^* \hat{f}_3 \,, \zeta\rangle. \tag{119}$$

Coordination phase

The solution $\bar{u}_I$ is given by solving the equation $g(v_I) = 0$, where $g(v_I) = D\varphi_2(v_I) + D\varphi_3(v_I)$. By virtue of (107) and (119), we have

$$G_2^* \mathcal{A}_2 G_2 \bar{u}_I + G_2^*(\mathcal{A}_2 w_2 - \hat{f}_2) + G_3^* \mathcal{A}_3 G_3 \bar{u}_I - G_3^* \hat{f}_3 = 0\,. \tag{120}$$

In consequence we get

$$\bar{u}_I = R\left(-G_2^*\,(\mathcal{A}_2 w_2 - \hat{f}_2) + G_3^*\,\hat{f}_3\right), \tag{121}$$

where

$$R = (G_2^* \mathcal{A}_2 G_2 + G_3^* \mathcal{A}_3 G_3)^{-1}\,. \tag{122}$$

So, the solution is given by

$$\boxed{\begin{cases} u_2(\bar{u}_I) = G_2\, R\left(-G_2^*(\mathcal{A}_2 w_2 - \hat{f}_2) + G_3^* \hat{f}_3\right) + w_2\,, \\ \\ u_3(\bar{u}_I) = G_3 R\left(-G_2^*(\mathcal{A}_2 w_2 - \hat{f}_2) + G_3^* \hat{f}_3\right) + w_3\,. \end{cases}} \tag{123}$$

3.4 Example 4

3.4.1 The original problem

We consider here $\mathcal{H} = \Re$, $\mathcal{K} = \Re^+$, and we define

$$M_2\, v_2 = \int_{\Omega_2} v_2\, dx\,, \qquad M_3\, v_3 = \int_{\Omega_3} v_3\, dx\,. \tag{124}$$

Note 3.2 *We denote*

$$\overline{v_i} = (v_i, e_i) = \int_{\Omega_i} v_i \, dx\,, \tag{125}$$

where e_i is the constant function $\mathbf{1}$ in Ω_i.

So, in this case we have

$$K = \{(v_2, v_3) \in X_2 \oplus X_3 \,:\, \overline{v_2} - \overline{v_3} \geq 0\}\ . \tag{126}$$

3.4.2 Solution by decomposition

$$\left| \begin{array}{l} X_I = \Re^2\,, \\[2ex] K_I = \{(x_2, x_3) \in \Re^2 \,:\, x_2 - x_3 \geq 0\}\,, \\[2ex] K_2(x_2, x_3) = \{v_2 \in X_2 \,:\, \overline{v_2} = x_2\}\,, \\[2ex] K_3(x_2, x_3) = \{v_3 \in X_3 \,:\, \overline{v_3} = x_3\}\,. \end{array} \right. \tag{127}$$

We have the problem $\mathcal{P}_2(x_2)$

$$\boxed{\mathcal{P}_2(x_2)} \;: \qquad \varphi_2(x_2) = \min_{u_2 \in K_2} \tfrac{1}{2} \langle \mathcal{A}_2 u_2, u_2 \rangle - \langle \hat{f}_2, u_2 \rangle \tag{128}$$

and the problem $\mathcal{P}_3(x_3)$

$$\boxed{\mathcal{P}_3(x_3)} \;: \qquad \varphi_3(x_3) = \min_{u_3 \in K_3} \tfrac{1}{2} \langle \mathcal{A}_3 u_3, u_3 \rangle - \langle \hat{f}_3, u_3 \rangle\,. \tag{129}$$

From (127)-(129) it is easy to check that there exist $c_i \in \Re$ such that

$$\left\{ \begin{array}{l} \mathcal{A}_i\, u_i - \hat{f}_i = c_i \hat{e}_i\,, \qquad i = 2, 3\,, \\[2ex] \langle u_i, \hat{e}_i \rangle = x_i\,, \end{array} \right. \tag{130}$$

then

$$u_i = \mathcal{A}_i^{-1} (\hat{f}_i + c_i \hat{e}_i)\,. \tag{131}$$

To get the value c_i we take into account that

$$x_i = \langle \hat{e}_i, u_i \rangle = \left\langle \hat{e}_i, \mathcal{A}_i^{-1} (\hat{f}_i + c_i\, \hat{e}_i) \right\rangle = \overline{\mathcal{A}_i^{-1}\, \hat{f}_i} + c_i\, \overline{\mathcal{A}_i^{-1} \hat{e}_i}\,, \tag{132}$$

where
$$\overline{\mathcal{A}_i^{-1}\hat{f}_i} = \left\langle \hat{e}_i, \mathcal{A}_i^{-1}\hat{f}_i \right\rangle \quad \text{and} \quad \overline{\mathcal{A}_i^{-1}\hat{e}_i} = \left\langle \hat{e}_i, \mathcal{A}_i^{-1}\hat{e}_i \right\rangle . \tag{133}$$

Therefore
$$c_i = \frac{1}{\overline{\mathcal{A}_i^{-1}\hat{e}_i}} \left(x_i - \overline{\mathcal{A}_i^{-1}\hat{f}_i}\right) \tag{134}$$

and in consequence
$$u_i = \mathcal{A}_i^{-1}\hat{f}_i + \frac{1}{\overline{\mathcal{A}_i^{-1}\hat{e}_i}} \left(x_i - \overline{\mathcal{A}_i^{-1}\hat{f}_i}\right) \mathcal{A}_i^{-1}\hat{e}_i . \tag{135}$$

From these relations we get
$$\varphi_i(x_i) = \frac{1}{2}\frac{1}{\overline{A_i^{-1}\hat{e}_i}} \left(x_i - \overline{A_i^{-1}\hat{f}_i}\right)^2 - \frac{1}{2}\left\langle \hat{f}_i, A_i^{-1}\hat{f}_i \right\rangle \tag{136}$$

and
$$\frac{\partial \varphi_i}{\partial x_i} = \frac{\left(x_i - \overline{\mathcal{A}_i^{-1}\hat{f}_i}\right)}{\overline{\mathcal{A}_i^{-1}\hat{e}_i}} . \tag{137}$$

We have two alternatives:

1. $x_2 - x_3 > 0$

   In this case the necessary condition that must be satisfied at this interior point is: $\frac{\partial \varphi_i}{\partial x_i} = 0$. This condition implies
   $$\left(x_i - \overline{\mathcal{A}_i^{-1}\hat{f}_i}\right) = 0 .$$

   Then, by virtue of (135), we have the solution that we may call *"free solution"*:
   $$\boxed{u_i = \mathcal{A}_i^{-1}\hat{f}_i \qquad i = 2, 3 .} \tag{138}$$

   So, if
   $$\overline{\mathcal{A}_2^{-1}\hat{f}_2} - \overline{\mathcal{A}_3^{-1}\hat{f}_3} > 0 , \tag{139}$$
   the solution is given by the *free solution.* Otherwise, we will have the following case:

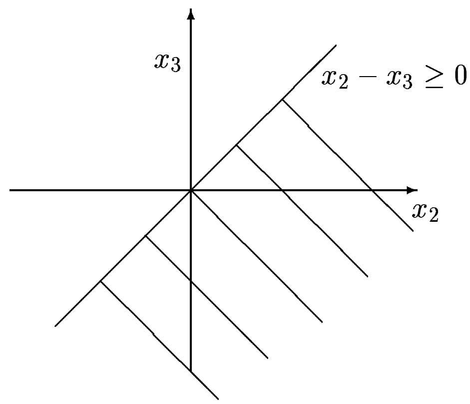

Figure 2: $K_I = \{(x_2, x_3) \in \Re^2 \ : \ x_2 - x_3 \geq 0\}$

2. $x_2 - x_3 = 0$

   In this case the necessary condition that must be satisfied at this boundary point is: $\exists \lambda \neq 0$ such that

$$\begin{pmatrix} \frac{\partial \varphi_2}{\partial x_2} \\ \\ \frac{\partial \varphi_3}{\partial x_3} \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ \\ -1 \end{pmatrix} = 0 \qquad (140)$$

   This condition implies that

$$\frac{\partial \varphi_2}{\partial x_2} = -\frac{\partial \varphi_3}{\partial x_3} \, . \qquad (141)$$

   By virtue of (137) we get

$$\frac{\left(x_2 - \overline{\mathcal{A}_2^{-1} \hat{f}_2}\right)}{\overline{\mathcal{A}_2^{-1} \hat{e}_2}} + \frac{\left(x_3 - \overline{\mathcal{A}_3^{-1} \hat{f}_3}\right)}{\overline{\mathcal{A}_3^{-1} \hat{e}_3}} = 0 \, , \qquad (142)$$

then, as $x_2 = x_3$ we have

$$\frac{\left(x_2 - \overline{\mathcal{A}_2^{-1}\hat{f}_2}\right)}{\overline{\mathcal{A}_2^{-1}\hat{e}_2}} + \frac{\left(x_2 - \overline{\mathcal{A}_3^{-1}\hat{f}_3}\right)}{\overline{\mathcal{A}_3^{-1}\hat{e}_3}} = 0 \tag{143}$$

and therefore

$$x_2 = \frac{\overline{\mathcal{A}_3^{-1}\hat{e}_3}\;\overline{\mathcal{A}_2^{-1}\hat{f}_2} + \overline{\mathcal{A}_2^{-1}\hat{e}_2}\;\overline{\mathcal{A}_3^{-1}\hat{f}_3}}{\left(\overline{\mathcal{A}_2^{-1}\hat{e}_2} + \overline{\mathcal{A}_3^{-1}\hat{e}_3}\right)}\,. \tag{144}$$

Finally, the solution is given by

$$\boxed{\begin{cases} u_2 &= \mathcal{A}_2^{-1}\left(\hat{f}_2 - \frac{\overline{\mathcal{A}_2^{-1}\hat{f}_2}}{\overline{\mathcal{A}_2^{-1}\hat{e}_2}}\;\hat{e}_2\right) + \frac{1}{\overline{\mathcal{A}_2^{-1}\hat{e}_2}}\frac{\overline{\mathcal{A}_3^{-1}\hat{e}_3}\;\overline{\mathcal{A}_2^{-1}\hat{f}_2}+\overline{\mathcal{A}_2^{-1}\hat{e}_2}\;\overline{\mathcal{A}_3^{-1}\hat{f}_3}}{\left(\overline{\mathcal{A}_2^{-1}\hat{e}_2}+\overline{\mathcal{A}_3^{-1}\hat{e}_3}\right)}\mathcal{A}_2^{-1}\,\hat{e}_2\,, \\ \\ u_3 &= \mathcal{A}_3^{-1}\left(\hat{f}_3 - \frac{\overline{\mathcal{A}_3^{-1}\hat{f}_3}}{\overline{\mathcal{A}_3^{-1}\hat{e}_3}}\;\hat{e}_3\right) + \frac{1}{\overline{\mathcal{A}_3^{-1}\hat{e}_3}}\frac{\overline{\mathcal{A}_3^{-1}\hat{e}_3}\;\overline{\mathcal{A}_2^{-1}\hat{f}_2}+\overline{\mathcal{A}_2^{-1}\hat{e}_2}\;\overline{\mathcal{A}_3^{-1}\hat{f}_3}}{\left(\overline{\mathcal{A}_2^{-1}\hat{e}_2}+\overline{\mathcal{A}_3^{-1}\hat{e}_3}\right)}\mathcal{A}_3^{-1}\,\hat{e}_3\,. \end{cases}} \tag{145}$$

3.5 Example 5

3.5.1 The original problem

We will deal now with an example in a simpler geometry. We consider (cf. Fig. 3) a body which consists of domains Ω_1 and Ω_2 which are respectively of dimensions 1 and 2. In the present situation

$$\partial\Omega_1 = \Gamma_{1i} \cup \Gamma_{1e}\,, \quad \Gamma_{1i} = 0\,, \quad \Gamma_{1e} = 1\,.$$

We consider $\mathcal{H} = L^2(\Omega_2)\,,\ \mathcal{K} = (L^2(\Omega_2))^+$, and we define the operators M_i by

$$(M_1\,v_1)(x) = v_1(0)\,,\ \forall x \in \Omega_2 \quad \text{and} \quad (M_2\,v_2)(x) = v_2(x)\,,\ \forall x \in \Omega_2\,; \tag{146}$$

then

$$K = \{(v_1, v_2) \in X_1 \oplus X_2\ :\ v_2(x) \le v_1(0)\,,\ a.e.\ in\ \Omega_2\}\,. \tag{147}$$

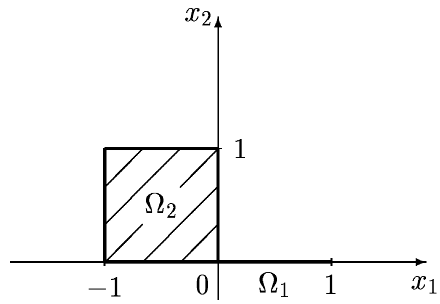

Figure 3: A 1-2 dimensional coupled problem

3.5.2 Properties of the solution

As K is a cone with vertex 0, the V.I. (8) is equivalent to (148) and (149)

$$a_1(u_1, u_1) + a_2(u_2, u_2) = (f_1, u_1) + (f_2, u_2), \qquad (u_1, u_2) \in K, \tag{148}$$

and

$$a_1(u_1, v_1) + a_2(u_2, v_2) \geq (f_1, v_1) + (f_2, v_2), \qquad \forall (v_1, v_2) \in K. \tag{149}$$

From these relations, the following characterization of the solution is given in [8]:

$$A_1 u_1 = f_1, \qquad in\ \Omega_1, \tag{150}$$

$$\frac{\partial u_1}{\partial n}(1) = 0, \qquad \frac{\partial u_1}{\partial n}(0) \geq 0, \tag{151}$$

$$A_2 u_2 - f_2 \leq 0, \qquad u_2 - u_1(0) \leq 0, \qquad in\ \Omega_2, \tag{152}$$

$$(A_2 u_2 - f_2) \times (u_2 - u_1(0)) = 0, \qquad in\ \Omega_2, \tag{153}$$

$$\frac{\partial u_2}{\partial n} = 0, \qquad in\ \partial\Omega_2. \tag{154}$$

It is also given the following relation, which characterizes the equilibrium between both systems:

$$\frac{\partial u_1}{\partial n}(0) + (A_2 u_2 - f_2, 1) = 0\,. \tag{155}$$

We will prove that

$$\frac{\partial u_1}{\partial n}(0) = \int_S (f_2(x) - \beta\, u_2(x))\, dx\,, \tag{156}$$

expression which gives the equilibrium condition (155) in terms of the data at the *interaction zone* S, where

$$S := \{x \,:\, u_2(x) = u_1(0)\}\,. \tag{157}$$

By virtue of (154), we have

$$\int_{\Omega_2} \triangle u_2(x)\, dx = 0 \tag{158}$$

and from (157), it is verified

$$\triangle u_2(x) = 0\,, \quad a.e.\ x \in S\,. \tag{159}$$

So

$$\int_{CS} \triangle u_2(x)\, dx = \int_{\Omega_2} \triangle u_2(x)\, dx = 0\,. \tag{160}$$

By virtue of (153) we have

$$\int_{CS} (-\triangle u_2 + \beta\, u_2 - f_2)(x)\, dx = 0 \tag{161}$$

and from relations (160) and (161), we obtain

$$(\beta\, u_2 - f_2, 1) = \int_S (\beta\, u_2 - f_2)(x)\, dx\,. \tag{162}$$

Finally, from (155) and (162), we obtain (156).

3.5.3 Solution by decomposition techniques

We can solve the V.I. (8) by decomposition techniques. Let us introduce a real parameter ξ. We take

$$\left| \begin{array}{l} X_I = \Re\,, \qquad K_I = \Re^+\,, \\ \\ K_1(\xi) = \{v_1 \in X_1 \,:\, v_1(0) = \xi\}\,, \\ \\ K_2(\xi) = \{v_2 \in X_2 \,:\, v_2(x) \le \xi,\ a.e.\ x \in \Omega_2\}\,. \end{array} \right. \tag{163}$$

For each ξ we define u_1^ξ as the unique solution of the following V.I.

$$\left| \begin{array}{l} a_1(u_1, v_1 - u_1) \ge (f_1, v_1 - u_1)\,, \\ \\ \forall v_1 \in K_1(\xi)\,,\, u_1 \in K_1(\xi)\,, \end{array} \right. \tag{164}$$

and u_2^ξ as the unique solution of the following V.I.

$$\left| \begin{array}{l} a_2(u_2, v_2 - u_2) \ge (f_2, v_2 - u_2)\,, \\ \\ \forall v_2 \in K_2(\xi)\,,\, u_2 \in K_2(\xi)\,, \end{array} \right. \tag{165}$$

In this case, from (164) and (165) we get for the derivative of the auxiliary function (defined in (17)) $\varphi(\xi) = \varphi_1(\xi) + \varphi_2(\xi)$

$$\frac{\partial \varphi}{\partial \xi} := \frac{\partial u_1^\xi}{\partial n}(0) - \int_S (f_2(x) - \beta\, u_2(x))\, dx\,. \tag{166}$$

From (166), it can be easily seen that $\frac{\partial \varphi}{\partial \xi}$ is an increasing continuous function with the property

$$\lim_{\xi \to \pm\infty} \frac{\partial \varphi}{\partial \xi} = \pm\infty\,. \tag{167}$$

Then, solving the original V.I. (8) is equivalent to solve the equation

$$\frac{\partial \varphi}{\partial \xi} = 0\,, \tag{168}$$

which can be solved by any iterative method. In particular, using that one described in section 2.

Conclusions

In this paper we have analyzed some issues concerning the solution of coupled variational inequalities. Specifically, we have studied the solution of some junction problems presented in [8]. Starting from a V.I. formulation, we obtain the solution via a decomposition–coordination method. We have used this procedure because it allows us to solve the coupled problems through the solution of simple independent problems – in general, they are linear problems or simple obstacle problems. These problems depend on some auxiliary variables which are modified by an iterative algorithm. The convergence of this algorithm can be improved using techniques of Newton's type. The study of this methodology will be contained in [11].

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References

[1] Aufranc M., *Étude limite de l'insertion d'une plaque mince dans un massif trimensionnel*, Rapport de Recherche INRIA N°868, Rocquencourt, France, 1988.

[2] Bernadou M., Fayolle S., Lene F., *Numerical analysis of junctions between plates*, Rapport de Recherche INRIA N°865, Rocquencourt, France, 1988.

[3] Ciarlet P.G., *Plates and junctions in elastic multi-structures: an asymptotic analysis*, Masson, Paris, 1990.

[4] D'Hennezel F., *Domain decomposition method and elastic multistructures: the stiffened plate problem*, Rapports de Recherche INRIA N°1800, Rocquencourt, France, 1992.

[5] González R.L.V., Rofman E., *On some junction problems*, Rapport de Recherche INRIA N°2937, Rocquencourt, France, 1996.

[6] Le Dret H., *Problèmes variationnels dans les multi-domaines. Modélisation des jonctions et applications*, Masson, Paris, 1991.

[7] Le Dret H., Raoult A., *The nonlinear membrane model as variational limit of nonlinear three-dimensional elasticity*, J. Math. Pure Appl., **74**, 6, pp. 549-578, 1995.

[8] Lions J.L., *Some more remarks on boundary value problems and junctions*, Colloque Lisbonne 1993.

[9] Lions J.L., Marchouk G.I., *Sur les méthodes numériques en sciences physiques et économiques*, Dunod, Paris, 1974.

[10] Lions J.L., Stampacchia G., *Variational inequalities*, Comm. Pure Appl. Math., **20**, pp. 493-519, 1967.

[11] Lotito P.A., Reyero G.F., González R.L.V., *On the numerical solution of some junction problems*, Working paper.

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