Multi-Item Single Machine Scheduling Optimization. The Case with Piecewise Deterministic Demands

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Abstract: In this work we study the optimization of a production system comprising a multi-item single machine with piecewise deterministic demands. Demands can only take a finite number of values and the demand changes are described by Poisson processes. We present the theoretical characterization of the solution and a numerical procedure to solve it. We establish the rate of convergence of the discrete solution toward the original continuous solution.

Key-words: scheduling problems, piecewise deterministic demands, quasi-variational inequalities, Hamilton-Jacobi-Bellman equation, numerical solution

(Résumé : tsvp)

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Optimisation d'un système des machines multiproduits. Le cas de demandes aléatoires déterministes par morceaux

Résumé : Dans ce papier on étudie l'optimisation d'un programme de production d'une machine multiproduit, où les demandes accumulées de chaque produit sont des processus aléatoires déterministes par morceaux. Les demandes instantanées prennent un nombre fini de valeurs et les instants de changement de demandes sont décrits par un processus de Poisson. On présente la caractérisation théorique de la solution du problème et un procédé d'approximation numérique. On établie aussi la vitesse de convergence des solutions discrètes vers la solution du problème originel.

Mots-clé : problèmes d'ordonnancement, demandes aléatoires déterministes par morceaux, inégalités quasi-variationelles, équation de Hamilton-Jacobi-Bellman, solution numérique

Contents

1 Introduction

We consider the optimization of a production system comprising a multi-item single machine with piecewise deterministic demands. The demand changes are described by Poisson processes and the demand values are taken among a finite number of values.

These piecewise deterministic processes were introduced by Davis in [9] and some results about their control can be seen in [17]-[19], analytical related results can be seen in [11].

This type of problems belongs to the class of optimal control problems of jump processes with state space constraints. The space constraint means here that the trajectories of the controlled process have to remain in a given bounded closed subset (a polyhedron in our case) of $\mathbb{R}^m$.

The deterministic counterpart of this problem was studied in [1], [2], [12]-[15]. In that work, taking into account the switching costs, the optimal cost function was characterized as the viscosity solution of the associated Hamilton-Jacobi-Bellman (HJB) equation, see [3], [8], [10], [22].

Here, we generalize the results mentioned above to the case where demands vary randomly according to a piecewise deterministic process, previous drafts about these work can be seen in [20] and [21].

The paper is organized as follows:

- In §2 we describe the problem. We prove some properties of boundedness of the cost function. As this cost function is unbounded in a special part of the boundary, we introduce an approximate "bounded" problem defined in terms of a penalization parameter (denoted by "A" in the following). The penalized problem is defined in such a way that when the parameter "A" of penalization tends to infinity the solution of the approximate problem converges to the solution of the original problem.

- In §3 we study the existence, uniqueness and continuity of the optimal cost function V^A associated to the approximate problem and the HJB equation corresponding to this problem.

- In §4 we present a discretization procedure and we prove (in addition to the existence and uniqueness of the discrete solution) the convergence of the discrete solution to the continuous solution. We also give an estimate of the rate of convergence.

2 Description of the problem

2.1 Production system

At any time the machine is either idle or producing any of m different items. We denote with $D = \{0, 1, ..., m\}$, and we assign the following values to the machine setting
$d = 0$, the idle state of the machine;
$d = 1, \cdots, m$, when it is producing the item d.
For each item $d = 1, ..., m$, we define the problem data as follows:

- p_d the production quantity by unit time when the machine state is d.
- n_d the quantity of possible demands for item d.
- $\mathcal{J}$ the set of possible demands,

$$\mathcal{J} = \prod_{d=1}^{m} \{1, ..., n_d\}, \text{ and } J = |\mathcal{J}| = \prod_{d=1}^{m} n_d \, .$$

  For each $j \in \mathcal{J}$, $r_j = (r_{1\,j}, \cdots, r_{m\,j})$ is the demand vector by unit time.

- λ_{ij} the commutation rate between the states of demand i and j.
- M_d the inventory capacity constraint for item d.
- $f(x, d)$ the instantaneous inventory holding/production cost.
- $q(d, \widetilde{d})$ the switching cost for the machine from state d to $\widetilde{d}$.
- α the discount rate.

We assume that the following conditions hold. The commutation costs satisfy:

$$\begin{cases} q\,(d, d) = 0, & \forall d \in D, \\ q(d, \widetilde{d}) \geq q_0 > 0, & \forall \widetilde{d} \neq d, \\ q(d, \widehat{d}) < q(d, \widetilde{d}) + q(\widetilde{d}, \widehat{d}), & \forall d \neq \widetilde{d} \neq \widehat{d}. \end{cases} \tag{1}$$

We also suppose instantaneous commutations and that the following compatibility condition between demands and productions holds:

$$\sum_{d=1}^{m} \frac{r_{d\,j}}{p_d} < 1 \qquad \forall j \in \mathcal{J}. \tag{2}$$

2.2 Admissible states

Let $y_d(t)$ be the inventory level of item d at time t, starting at $y_d(0) = x_d$. Therefore, for the system global state y, we have

$$y\,(t) \equiv (y_1(t), \cdots, y_m(t)), \qquad (y_1(0), \cdots, y_m(0)) = (x_{1,\cdots,}x_m).$$

As neither backlogging nor production over capacity constraints are allowed for the inventory state y_d, the following restriction for y_d holds:

$$0 \leq y_d \leq M_d, \quad \forall d = 1, \cdots, m.$$

In order to describe the admissible state space, we will need the following notation. Let

$$\Gamma(a_{1,\cdots,}a_m) = \left\{ x \in \prod_{d=1}^{m} [0, M_d] : a(x) = (a_{1,\cdots,}a_m) \right\},$$

where

$$a_i(x) = \begin{cases} 0 & \text{if } x_i = 0, \\ 1 & \text{if } x_i \in (0, M_i)\,, \\ 2 & \text{if } x_i = M_i. \end{cases}$$

If the stock levels of at least two items reach zero simultaneously, the shortage of at least one item is inevitable with any admissible control policy. Therefore, the admissible state space Q comprises only the set of points with at most one zero component, i.e.,

$$Q = \bigcup_a \left\{\Gamma\left(a_1, ..., a_i, ..., a_m\right) : \text{ at most one component } a_i = 0\right\}.$$

We denote ∂Q^+ the points of Q that are not admissible, i.e.,

$$\partial Q^+ = \bigcup_a \left\{\Gamma\left(a_1, ..., a_i, ..., a_m\right) : \text{ at least two components } a_i = 0\right\}.$$

If we denote with Ω the interior points of Q, we have

$$\Omega = \{x : 0 < x_i < M_i, \; i = 1, ..., m\} = \Gamma(1, ..., 1).$$

We also define

$$\gamma_d^- = \bigcup_a \left\{\Gamma(a_1, ..., a_d, ..., a_m) : \; a_d = 0\right\} \cap Q\,, \tag{3}$$

$$\gamma_d^+ = \bigcup_a \left\{\Gamma(a_1, ..., a_d, ..., a_m) : \; a_d = 2\right\} \cap Q\,. \tag{4}$$

2.3 System evolution

The state y follows an evolution given by

$$\frac{dy}{dt} = g(\beta(t),\, j) \quad \text{where} \quad g_d(\beta(t),\, j) = \begin{cases} -r_{d\,j} & \text{if } \beta \neq d, \\ \\ p_d - r_{d\,j} & \text{if } \beta = d. \end{cases} \tag{5}$$

The evolution of the system depends on the current state of demands. The state of demands r is given by a continuous time Markov chain with transition rates $\lambda_{i\,j}$ between states of demand i and j.

Remark 2.1 *Throughout the paper we suppose that the following inequalities hold*

$$\left| \begin{array}{lll} |f(x,d)| \leq M_f & \forall x \in \overline{Q}, & \forall d \in D, \\ \\ q(d,\tilde{d}) \leq M_q & \forall d, \tilde{d} \in D, & \\ \\ \|g(d,j)\| \leq M_g & \forall j \in \mathcal{J}, & \forall d \in D, \\ \\ |f(x,d) - f(\overline{x},d)| \leq L_f \, \|x - \overline{x}\| & \forall x, \overline{x} \in \overline{Q}, & \forall d \in D. \end{array} \right. \tag{6}$$

2.4 Admissible controls

An admissible control is given by a sequence of pairs (θ_i, d_i) where θ_i are the commutation times of the control (state of production) and d_i are the state of production (θ_i are stopping times adapted to the process and d_i are random step functions also adapted to the process (see [4], [17])). The control must also verify the additional constraint condition: $y(t) \in Q$, $\forall t \in [0,\infty)$. We denote with $B_x^{d\,j}$ the set of these admissible controls.

Remark 2.2 *It can be proved that for the optimization purpose it is only necessary to consider the subset of Markovian admissible controls. Given a function $\mathcal{D} : Q \times D \times \mathcal{J} \longrightarrow D$, such that*

$$\mathcal{D}(x, \mathcal{D}(x,d,j),\, j) = \mathcal{D}(x,d,j), \;\; \forall x \in Q, \; \forall j \in \mathcal{J}, \; \forall d \in D, \tag{7}$$

the feedback control $d(\cdot)$ (see [10]) is defined as

$$d(t) = \mathcal{D}(y(t), d(t-),\, j(t))\,.$$

Under suitable additional conditions, the state production $d(t)$ results a piecewise constant function with a finite number of switchings in any finite interval, i.e. $d = d_i$ in the interval $(\theta_i, \theta_{i+1}]$, with

$$0 = \theta_0 \leq \theta_1 < ... < \theta_i < \theta_{i+1} < ...;$$

$$d_i \in \{0, 1, ..., m\}\ ;\ d_i \neq d_{i+1};\ i = 0, 1, ...$$

Remark 2.3 *Condition (7) is a technical restriction aimed to avoid the existence of instantaneous closed loops of control switchings.*

2.5 Optimal cost function

Our purpose is to find an optimal feedback control policy that minimizes the criterion $J_x^{dj}(\beta)$ which takes into account the production cost and the commutation cost of control policies. In other words, we look for a policy $\bar{\beta}$ such that:

$$V_{dj}(x) = J_x^{dj}(\overline{\beta}(\cdot)) = \inf\left\{J_x^{dj}(\beta(\cdot)) : \beta(\cdot) \in B_x^{dj}\right\},$$

where:

$$J_x^{dj}(\beta) = E\left\{\sum_{i=1}^{\infty}\left(\int_{\theta_{i-1}}^{\theta_i} f(y(s), d_{i-1})e^{-\alpha s}ds \ + q(d_{i-1}, d_i)\, e^{-\alpha\,\theta_i}\right)\right\}.$$

Here, j represents the initial state of demands. A special feature of our problem is that neither the shortage of products nor the production over the maximum stocks are allowed. This additional requirement imposes a constraint on the admissible trajectories and on the admissible policies.

2.6 Properties of the optimal cost function

The value function is rather regular at interior points, but it becomes singular at points $x \in \partial Q^+$. Specifically, we have the following properties:

- i) $\exists\ C_1 > 0$ (a constant independent of the discount rate α) such that

$$|V_{dj}(x)| \leq C_1\left(1 + \frac{1}{\alpha} - \log(d(x, \partial Q^+))\right),$$

- ii) $\lim\limits_{x \to \partial Q^+} V_{dj}(x) = +\infty.$

In the next theorems we prove these properties. To simplify the proof we only consider the case Q$\subset \mathbb{R}^2$.

Theorem 2.1 *Let* $0 < b < \min(\frac{M_1}{p_1}, \frac{M_2}{p_2})$, $E = Q \cap \left\{ x \in \mathbb{R}^2 : \frac{x_1}{p_1} + \frac{x_2}{p_2} \leq b \right\}$. *Let* $x \in Q \setminus E$, *then there exists* $C_1 > 0$ *such that:*

$$|V_{d,j}(x)| \leq C_1 \left(\frac{1}{\alpha} + 1 \right). \tag{8}$$

Proof. Without losing of generality we consider $p_1 = p_2 = 1$ (by a change of variables, we can always reduce the original problem to this case).

Let $\max(M_1, M_2) < \eta_1 < \eta_2 < M_1 + M_2$ and

$$\left| \begin{array}{l} B = \{ x \in Q : \eta_2 < x_1 + x_2 \}, \\ \\ \Xi = \{ x \in Q : \eta_1 < x_1 + x_2 \leq \eta_2 \}, \\ \\ F = \{ x \in Q : b \leq x_1 + x_2 \leq \eta_1 \}. \end{array} \right. \tag{9}$$

We define the following feedback control policy ($\forall j \in \mathcal{J}$)

$$\begin{array}{llll} \mathcal{D}(x,0,j) = 1, & x \in F \setminus (\gamma_2^- \cup \gamma_1^+) & \mathcal{D}(x,0,j) = 0, & x \in \Xi \\ \mathcal{D}(x,0,j) = 2, & x \in F \cap (\gamma_2^- \cup \gamma_1^+) & \mathcal{D}(x,0,j) = 0, & x \in B \\ \mathcal{D}(x,1,j) = 1, & x \in (\Xi \cup F) \setminus (\gamma_2^- \cup \gamma_1^+) & \mathcal{D}(x,1,j) = 1, & x \in B \setminus \gamma_1^+ \\ \mathcal{D}(x,1,j) = 2, & x \in (\Xi \cup F) \cap (\gamma_2^- \cup \gamma_1^+) & \mathcal{D}(x,2,j) = 2, & x \in B \setminus \gamma_2^+ \\ \mathcal{D}(x,2,j) = 2, & x \in (\Xi \cup F) \setminus (\gamma_1^- \cup \gamma_2^+) & \mathcal{D}(x,1,j) = 0, & x \in B \cap \gamma_1^+ \\ \mathcal{D}(x,2,j) = 1, & x \in (\Xi \cup F) \cap (\gamma_1^- \cup \gamma_2^+) & \mathcal{D}(x,2,j) = 0, & x \in B \cap \gamma_2^+. \end{array} \tag{10}$$

It can be proved (after some lengthy calculations) that, if we use this feedback control policy $\mathcal{D}$ starting at a point $x \in Q \setminus E$, we obtain a control policy $\beta(.)$ whose associated switching times verify: $\forall \nu \geq 1$

$$0 < \epsilon \leq \theta_{\nu+1} - \theta_\nu, \tag{11}$$

where ϵ is a positive constant defined in terms of η_1, η_2 and the data of the problem (but independent of the discount factor α).

In consequence, for the associated functional cost $J_x^{d\,j}(\beta)$ we have:

$$\begin{aligned}
J_x^{d\,j}(\beta) &= E\left\{\sum_{\nu=1}^{\infty}\int_{\theta_{\nu-1}}^{\theta_\nu} f(y(s),d_{\nu-1})e^{-\alpha s}ds + q(d_{\nu-1},d_\nu)e^{-\alpha\theta_\nu}\right\} \\
&\leq \frac{M_f}{\alpha} + E\{\sum_{\nu=1}^{\infty} M_q e^{-\alpha\theta_\nu}\} \leq \frac{M_f}{\alpha} + E\{\sum_{\nu=1}^{\infty} M_q e^{-\alpha\nu\epsilon}\} \\
&= \frac{M_f}{\alpha} + \frac{M_q}{1-e^{\alpha\epsilon}} \leq \frac{M_f}{\alpha} + M_q(1+\frac{1}{\alpha\epsilon}) \\
&\leq C_1(\frac{1}{\alpha}+1),
\end{aligned}$$

as we wanted to prove.

□

Theorem 2.2 *Let $x \in E$ then there exists $C_1 > 0$ (which depends only on problem data but not on the discount factor α) such that:*

$$V_{d\,j}(x) \leq C_1\left(\frac{1}{\alpha} + 1 + (log(d(x,\partial Q^+)))^-\right).$$

Proof. In order to simplify the proof we also consider here the case $p_1 = p_2 = 1$.

Let $x \in E$, and suppose $x_1 = 0$, $x_2 < b$. We apply the control policy defined in (10). So, at the switching time θ_ν we have

$$\begin{array}{lcl}
y(\theta_1) & = & (x_1^1, 0) \\
y(\theta_2) & = & (0, x_2^2) \\
y(\theta_3) & = & (x_1^3, 0) \\
\ldots\ldots\ldots\ldots & & \ldots\ldots\ldots\ldots\ldots \\
y(\theta_{2\nu}) & = & (0, l^\nu) \\
y(\theta_{2\nu+1}) & = & (l^{\nu+1}, 0).
\end{array}$$

It is clear that

$$\theta_{\nu+1} - \theta_1 \geq l^\nu \min_j \min_d (\frac{1}{r_{d\,j}}) \qquad (12)$$

and

$$l^{\nu n+1} \geq \gamma\, l^\nu, \qquad (13)$$

where $l^0 = \|x\|$ and

$$\gamma = \min(\min_j (1 - r_{1\,j}) \min_i \frac{1}{r_{2\,i}},\ \min_j (1 - r_{2\,j}) \min_i \frac{1}{r_{1\,i}}) \,. \tag{14}$$

¿From (2) it results $\gamma > 1$. In consequence, for a time $\overline{t} \leq \theta_{\overline{\nu}}$, where

$$\overline{\nu} = \left[\frac{\log b - \log \|x\|}{\log \gamma} \right] + 1,$$

the system reaches $Q \setminus E$.

If after $\overline{t}$ we apply the feedback policy defined in (10), we have for the corresponding cost

$$J_x^{d\,j}(\beta) \leq \left(1 + \frac{\log b - \log \|x\|}{\log \gamma} \right) M_q + \frac{M_f}{\alpha} + C_1 \left(\frac{1}{\alpha} + 1 \right).$$

And so for a suitable constant C we have

$$J_x^{d\,j}(\beta) \leq C \left(1 + \frac{1}{\alpha} + (\log \|x\|)^- \right).$$

For the general case $x \in E$ it is possible to get a similar result and then we have the general relation

$$V_{d\,j}(x) \leq C \left(1 + \frac{1}{\alpha} + (\log(\|x\|)^- \right).$$

□

Theorem 2.3 *For each $x \in Q$ it holds that*

$$V_{d\,j}(x) \geq -C_1 \log \|x\| - C_2. \tag{15}$$

Proof. Let $\beta(.) \in B_x^{d\,j}$. We want to obtain a bound of the first switching time θ_1 as a linear function of $\|x\|$.
Always at least one of the stocks $y_d(\cdot)$ is a decreasing function in $[0, \theta_1]$ and the shortage of any item is forbidden, we have

$$\theta_1 \leq \max_d x_d \ \max_j \left(\tfrac{1}{r_{d\,j}} \right) \leq \|x\| \left\| \tfrac{1}{r} \right\|_\infty ,$$

where

$$\left\| \frac{1}{r} \right\|_\infty = \max_j \max \left(\frac{1}{r_{1\,j}}, \frac{1}{r_{2\,j}} \right).$$

For a generic $\theta_{\nu+1}$ we have

$$\theta_{\nu+1} - \theta_\nu \leq \|y(\theta_\nu)\| \left\|\tfrac{1}{r}\right\|_\infty . \tag{16}$$

By virtue of (6) we have

$$\|y(\theta_{\nu+1})\| \leq \|y(\theta_\nu)\| + M_g(\theta_{\nu+1} - \theta_\nu) \leq \chi \|y(\theta_\nu)\| , \tag{17}$$

where

$$\chi = 1 + M_g \left\|\frac{1}{r}\right\|_\infty .$$

¿From (16) and (17) we get

$$\theta_\nu \leq \left\|\frac{1}{r}\right\|_\infty \|x\|_\infty \frac{1-\chi^\nu}{1-\chi}. \tag{18}$$

Let us define $\overline{\nu} = \min\{\nu : \theta_\nu \geq 1\}$. Then we have

$$\overline{\nu} \geq \frac{\log\left(1 + \frac{-1+\chi}{\left\|\frac{1}{r}\right\|_\infty \|x\|_\infty}\right)}{\log\chi} \tag{19}$$

and in consequence, for the corresponding functional value $J_x^{d\,j}(\beta)$ we have

$$J_x^{d\,j}(\beta) \geq -\frac{M_f}{\alpha} + (\overline{\nu} - 1) q_0\, e^{-\alpha}.$$

So, by virtue of (19) we get

$$V(x) \geq -\frac{M_f}{\alpha} + e^{-\alpha} q_0 \left(\frac{\log\left(\frac{M_g}{\|x\|}\right)}{\log\chi} - 1\right)$$

then

$$V_{d\,j}(x) \geq -C_1 \log\|x\| - C_2 \,, \tag{20}$$

where

$$\begin{aligned} C_1 &= e^{-\alpha} \frac{q_0}{\log\chi} \,, \\ C_2 &= \frac{M_f}{\alpha} + e^{-\alpha} q_0 \left(1 - \frac{\log M_g}{\log\chi}\right). \end{aligned}$$

$\square$

2.7 Solution by perturbations

The singular behavior in ∂Q^+ brings technical difficulties both in the analytical sense and in the numerical approximation. For this reason we introduce a perturbed problem where external purchases are allowed. In this case a penalty cost A must be paid and the system jumps instantaneously to the point of joint maximum stock. We call $V_{dj}^A(x)$ the optimal cost corresponding to this problem. The relation between the two problems is the following: $\forall x \in Q, \forall d \in D, \forall j \in \mathcal{J}$,

- $V_{dj}^A(x) \leq V_{dj}(x)$,
- $V_{dj}^{\widetilde{A}}(x) \leq V_{dj}^A(x)$, $\forall A \geq \widetilde{A}$,
- $\lim_{A \to +\infty} V_{dj}^A(x) = V_{dj}(x)$,
- $\forall\, x \in Q$, $\exists\, \widetilde{A}(x)$ / $V_{dj}^A(x) = V_{dj}(x)\ \forall\, A \geq \widetilde{A}(x)$.

Remark 2.4
The singular behavior of $V(x)$ when $x \to \partial Q^+$ is of type $-\log(d(x, \partial Q^+))$. From a practical point of view if we want to compute V in a compact K, the constant A can be chosen as

$$A = C(-\log(d(K, \partial Q^+))).$$

Remark 2.5 *We consider in this work only the perturbed problem where the constant A verifies*

$$A \geq \max(q(d, \widetilde{d}) : d, \widetilde{d} \in D). \tag{21}$$

For the sake of simplicity we will denote in the following with V the function that we have called V^A in the present paragraph.

3 The optimal cost function V

3.1 The function and its properties

In the following sections we present the dynamic programming principle (DPP) verified by the optimal cost function of our problem. To prove the existence, uniqueness and continuity of the solution of the set of integral equations and boundary conditions associated to the DPP, we introduce an operator P and we will obtain that V is the only fixed point of P.

Construction of an optimal feedback policy

When the optimal cost function V is known, optimal control policies can be found in the following way: given a state (x, d, j), an optimal policy is to continue the production of item d till the time of commutation

$$t_{x,d,j} = \inf\{t \geq 0 \,:\, V_{d\,j}(y(t)) = (SV)_{d\,j(t)}(y(t))\}, \tag{22}$$

which is a well defined non anticipative commutation time (see [6], [10]). In (22) we have used the operator S defined as

$$(SV)_{d\,j}\,(x) = \min(\min_{\widetilde{d}\neq d}(V_{\widetilde{d}\,j}(x) + q(d,\widetilde{d})), V_{d\,j}(e) + A),$$

being

$$e = (M_1, M_2, ..., M_m).$$

Remark 3.1 *By virtue of (1), the technical condition (7) is implicitly verified.*

3.2 Dynamic programming principle

The *Dynamic Programming Principle* allows us to establish the Hamilton-Jacobi-Bellman (HJB) equations associated to this problem and its associated boundary conditions (see [3], [4], [10], [22]).

3.2.1 The integral form of the Hamilton-Jacobi-Bellman equation

Definition 3.1 *We use the following notation and auxiliary variables*

$$\widetilde{\alpha}_j = \alpha + \sum_{i\neq j} \lambda_{j\,i}\,,$$

$$\tau(x, d, j) = \sup\{t : x + t\,g(d, j) \in \Omega\}. \tag{23}$$

Remark 3.2 $\tau(x, d, j)$ *(the time at which the trajectory reaches the boundary from the initial position x if there have not been demand changes) is a Lipschitz function of the initial position because this function is given by:*

$$\tau(x, d, j) = \min\left(\frac{M_d - x_d}{p_d - r_{d\,j}},\; \min_{\widetilde{d}\neq d}\left(\frac{x_{\widetilde{d}}}{r_{\widetilde{d}}}\right)\right).$$

This function is a minimum of Lipschitz functions and therefore it is a Lipschitz function, in fact

$$|\tau(x,d,j) - \tau(\overline{x},d,j)| \leq L_\tau \|x - \overline{x}\|,$$

where

$$L_\tau = \max_{d=1,m} \max_{j \in \mathcal{J}} \max \left(\frac{1}{p_d - r_{d\,j}}, \frac{1}{r_{\widetilde{d}}} \right).$$

The following form of the DPP can be proved using classical arguments and the Ito's formula corresponding to the process appearing in this production problem.

Theorem 3.1 *Let V be continuous, then $\forall x \in \Omega$, $\forall d \in D$, $\forall j \in \mathcal{J}$ the following conditions are verified*

1. $V_{d\,j}(x) \leq (SV)_{d\,j}(x)$.

2. $$V_{d\,j}(x) \leq \int_0^t e^{-\widetilde{\alpha}_j\, s}(f(y(s),d) + \sum_{i \neq j} \lambda_{j,i} V_{d,i}(y(s)))ds + e^{-\widetilde{\alpha}_j\, t}\, V_{d,j}(x + t\, g(d,j)),$$

   $$\forall\, t \leq \tau(x,d,j), \text{ being } y(s) = x + s\, g(d,j).$$

3. If furthermore, for some point $x \in \Omega$, a strict inequality holds in 1, then there exists $t_{x,d,j} > 0$ such that

$$V_{d\,j}(x) = \int_0^t e^{-\widetilde{\alpha}_j\, s}(f(y(s),d) + \sum_{i \neq j} \lambda_{j,i} V_{d\,i}(y(s)))ds + e^{-\widetilde{\alpha}_j\, t}\, V_{d\,j}(x + t\, g(d,j)), \quad (24)$$

$$\forall\, 0 \leq t \leq t_{x,d,j} \leq \tau(x,d,j).$$

Definition 3.2 *It will be useful to define $t_{x,d,j}$ in the following way*

$$t_{x,d,j} = \min\left\{t : V_{d\,j}(x + tg(d,j)) = (SV)_{d\,j}(x + t\, g(d,j))\right\}. \quad (25)$$

¿From the properties 1 and 3 it follows that $t_{x,d,j}$ defined in (25) verifies (24).

3.2.2 Boundary conditions for the HJB equation.

As the evolution of the system is restricted to the set $\overline{Q}$ there are some conditions involving the values of the function V at the boundary ∂Q.

Theorem 3.2 *In ∂Q the following boundary conditions are verified for all $d \neq 0$:*

$$\begin{aligned} V_{\widetilde{d}\,j}(x) &= (SV)_{\widetilde{d}\,j}(x) \qquad \forall x \in \gamma_d^-, \ \forall \widetilde{d} \neq d, \\ V_{d\,j}(x) &= (SV)_{d\,j}(x) \qquad \forall x \in \gamma_d^+, \end{aligned} \tag{26}$$

where γ_d^- and γ_d^+ are defined in (3) and (4) respectively.

3.3 A fixed point problem associated to the DPP

3.3.1 An auxiliary operator and its properties

Definition 3.3 *Let P be the operator $P : B(\overline{Q})^{(m+1)\times J} \to B(\overline{Q})^{(m+1)\times J}$, where $B(\bar{\Omega})^{m+1)\times J}$ is the set of Borel-measurable and bounded functions, given by* ($\forall d \in D, \forall j \in \mathcal{J}, \ \forall x \in Q$)

$$(Pw)_{d\,j}(x) =$$

$$\inf_{\tau \leq \tau(x,d,j)} \Big(\int_0^\tau e^{-\widetilde{\alpha}_j\, s} (f(y(s),d) + \sum_{i \neq j} \lambda_{j\,i}\, w_{d\,i}(y(s)))ds + e^{-\widetilde{\alpha}_j\, \tau} (Sw)_{d\,j}(y(\tau))\Big), \tag{27}$$

where $y(s) = x + s\, g(d,j)$.

Definition 3.4 *Let $\beta_{x\,d\,j}$ be the function given by*

$$\beta_{x\,d\,j}(\tau, w) = \int_0^\tau e^{-\widetilde{\alpha}_j\, s}(f(y(s),d) + \sum_{i \neq j} \lambda_{j\,i}\, w_{d\,i}(y(s)))ds + (Sw)_{d\,j}(x + \tau\, g(d,j))e^{-\widetilde{\alpha}_j\, \tau},$$

then we can write the operator P as

$$(Pw)_{d\,j}(x) = \min_{\tau \leq \tau(x,d,j)} \beta_{x\,d\,j}(\tau, w).$$

Let us denote by

$$\overline{\tau}(x,d,j) = \inf\{\tau : (Pw)_{d\,j}(x) = \beta_{x\,d\,j}(\tau, w)\}. \tag{28}$$

Lemma 3.1 *P is an operator from $\left(W^{1,\infty}(Q)\right)^{(m+1)\times J}$ to $\left(W^{1,\infty}(Q)\right)^{(m+1)\times J}$ and it verifies*

$$\|\nabla Pw\|_\infty \leq (1+\eta)(\|\nabla w\|_\infty + \rho),$$

where

$$\left|\begin{array}{ll} \eta = & 1 + L_\tau M_g + \dfrac{1}{\tilde{\alpha}_j}\displaystyle\sum_{i\neq j}\lambda_{j\,i}, \\ \\ \rho = & \dfrac{L_f}{\tilde{\alpha}_j} + L_\tau(M_f + A\displaystyle\sum_{i\neq j}\lambda_{j\,i}) . \end{array}\right. \tag{29}$$

In other words if w is a Lipschitz function (with associated Lipschitz constant L_w), then Pw is a Lipschitz function too, whose associated Lipschitz constant verifies

$$L_{Pw} = (1+\eta)(L_w + \rho).$$

Proof.

We want to prove that

$$\begin{aligned} \|(Pw)_{d\,j}(x) - (Pw)_{d\,j}(\overline{x})\| &= \left\| \min_{\tau\,\leq\tau(x,d,j)} \beta_{x\,d\,j}(\tau,w) - \min_{\tau\,\leq\tau(\overline{x},d,j)} \beta_{\overline{x}\,d\,j}(\tau,w) \right\| \\ &= \|\beta_{x\,d\,j}(\tau_x,w) - \beta_{\overline{x}\,d\,j}(\tau_{\overline{x}},w)\| \leq L_{Pw}\,\|x-\overline{x}\| . \end{aligned}$$

where we denote with τ_x and $\tau_{\overline{x}}$ the times which realize the minimum of β associated to the initial conditions x and $\overline{x}$ respectively.

We will suppose that $\tau(x,d,j) \leq \tau(\overline{x},d,j)$

• Case 1: $\tau_{\overline{x}} \leq \tau(x,d,j)$

$$\begin{aligned} (Pw)_{d\,j}(x) &= \beta_{x\,d\,j}(\tau_x,w), \\ (Pw)_{d\,j}(\overline{x}) &= \beta_{\overline{x}\,d\,j}(\tau_{\overline{x}},w) \quad \leq \beta_{\overline{x}\,d\,j}(\tau_x,w) \end{aligned}$$

then

$$
\begin{aligned}
&(Pw)_{d\,j}(\overline{x}) - (Pw)_{d\,j}(x) \leq \\
&\quad \leq \quad \beta_{\overline{x}\,d\,j}(\tau_x) - \beta_{x\,d\,j}(\tau_x) \\
&\quad \leq \quad \int_0^{\tau_x} e^{-\widetilde{\alpha}_j\,s} \left(|f(\overline{y}(s),d) - f(y(s),d)| + \sum_{i\neq j} \lambda_{j\,i}\, |w_{d\,i}(\overline{y}(s)) - w_{d\,i}(y(s))| \right) ds \\
&\qquad\quad + e^{-\widetilde{\alpha}_j\,\tau_x} \, |(Sw)_{d\,j}(\overline{x} + \tau_x\, g(d,j)) - (Sw)_{d\,j}(x + \tau_x\, g(d,j))| \\
&\quad \leq \quad \int_0^{\tau_x} e^{-\widetilde{\alpha}_j\,s} \, \|x - \overline{x}\| \left(L_f + L_w \sum_{i\neq j} \lambda_{j\,i} \right) ds + e^{-\widetilde{\alpha}_j\,\tau} L_w \,(1 + L_\tau M_g)\, \|x - \overline{x}\| \\
&\quad \leq \quad \frac{1}{\widetilde{\alpha}_j} \left(L_f + L_w \sum_{i\neq j} \lambda_{j\,i} \right) \|x - \overline{x}\| + L_w\,(1 + L_\tau M_g)\, \|x - \overline{x}\| \\
&\quad \leq \quad \left(L_w \left(1 + L_\tau M_g + \frac{1}{\widetilde{\alpha}_j} \sum_{i\neq j} \lambda_{j\,i} \right) + \frac{L_f}{\widetilde{\alpha}_j} \right) \|x - \overline{x}\| \,.
\end{aligned}
$$

We can also consider the following inequalities

$$
\begin{aligned}
(Pw)_{d\,j}(x) &= \beta_{x\,d\,j}(\tau_x, w) \quad \leq \beta_{x\,d\,j}(\tau_{\overline{x}}, w) \\
(Pw)_{d\,j}(\overline{x}) &= \beta_{\overline{x}\,d\,j}(\tau_{\overline{x}}, w)
\end{aligned}
$$

and by similar calculations we have

$$
(Pw)_{d\,j}(x) - (Pw)_{d\,j}(\overline{x}) \leq \left(L_w \left(1 + L_\tau M_g + \frac{1}{\widetilde{\alpha}_j} \sum_{i\neq j} \lambda_{j\,i} \right) + \frac{L_f}{\widetilde{\alpha}_j} \right) \|x - \overline{x}\| \,,
$$

therefore

$$
\|(Pw)_{d\,j}(x) - (Pw)_{d\,j}(\overline{x})\| \leq \left(L_w \left(1 + L_\tau M_g + \frac{1}{\widetilde{\alpha}_j} \sum_{i\neq j} \lambda_{j\,i} \right) + \frac{L_f}{\widetilde{\alpha}_j} \right) \|x - \overline{x}\| \,.
$$

- Case 2: $\tau(x,d,j) \leq \tau_{\overline{x}} \leq \tau(\overline{x},d,j)$

$$(Pw)_{d\,j}(x) \quad = \beta_{x\,d\,j}(\tau_x) \quad \leq \beta_{x\,d\,j}(\tau(x,d,j)),$$

$$(Pw)_{d\,j}(\overline{x}) \quad = \beta_{\overline{x}\,d\,j}(\tau_{\overline{x}}).$$

¿From these relations we have

$$\begin{aligned}
&(Pw)_{d\,j}(x) - (Pw)_{d\,j}(\overline{x}) \leq \\
&\leq \quad \beta_{x\,d\,j}(\tau(x,d,j)) - \beta_{\overline{x}\,d\,j}(\tau_{\overline{x}}) \\
&\leq \quad \int_0^{\tau(x,d,j)} e^{-\widetilde{\alpha}_j\,s} \left(|f(y(s),d) - f(\overline{y}(s),d)| + \sum_{i\neq j} \lambda_{j\,i}\, |w_{d\,i}(y(s)) - w_{d\,i}(\overline{y}(s))| \right) ds \\
&\quad + \int_{\tau(x,d,j)}^{\tau_{\overline{x}}} e^{-\widetilde{\alpha}_j\,s} \left(f(\overline{y}(s),d) + \sum_{i\neq j} \lambda_{ji}\, w_{di}(\overline{y}(s)) \right) ds \\
&\quad e^{-\widetilde{\alpha}_j \tau(x,d,j)} (Sw)_{d\,j}(x + \tau(x,d,j)\; g(d,j)) - e^{-\widetilde{\alpha}_j\,\tau_{\overline{x}}} (Sw)_{d\,j}(\overline{x} + \tau_{\overline{x}}\; g(d,j)) \\
&\leq \quad \frac{1}{\widetilde{\alpha}_j} \left(L_f + L_w \sum_{i\neq j} \lambda_{j\,i} \right) \|x - \overline{x}\| + \frac{1}{\widetilde{\alpha}_j} (M_f + A \sum_{i\neq j} \lambda_{j\,i}) \left(e^{-\widetilde{\alpha}_j \tau(x,d,j)} - e^{-\widetilde{\alpha}_j\,\tau_{\overline{x}}} \right) \\
&\quad + e^{-\widetilde{\alpha}_j \tau(x,d,j)} \left((Sw)_{d\,j}(x + \tau(x,d,j) g(d,j)) - (Sw)_{d\,j}(\overline{x} + \tau_{\overline{x}} g(d,j)) \right) \\
&\leq \quad \frac{1}{\widetilde{\alpha}_j} \left(L_f + L_w \sum_{i\neq j} \lambda_{j\,i} \right) \|x - \overline{x}\| + (M_f + A \sum_{i\neq j} \lambda_{j\,i}) \|\tau_{\overline{x}} - \tau(x,d,j)\| \\
&\quad + L_w (1 + L_\tau M_g) \|x - \overline{x}\| \\
&\leq \quad \left(\frac{1}{\widetilde{\alpha}_j} \left(L_f + L_w \sum_{i\neq j} \lambda_{j\,i} \right) + L_\tau (M_f + A \sum_{i\neq j} \lambda_{ji}) + L_w (1 + L_\tau M_g) \right) \|x - \overline{x}\| \\
&\leq \quad \left(L_w \left(1 + L_\tau M_g + \frac{1}{\widetilde{\alpha}_j} \sum_{i\neq j} \lambda_{j\,i} \right) + \frac{L_f}{\widetilde{\alpha}_j} + L_\tau (M_f + A \sum_{i\neq j} \lambda_{ji}) \right) \|x - \overline{x}\|\,.
\end{aligned}$$

We can also consider the following inequalities

$(Pw)_{dj}(x) \quad = \beta_{x\,dj}(\tau_x),$

$(Pw)_{dj}(\overline{x}) \quad = \beta_{\overline{x}\,dj}(\tau_{\overline{x}}) \quad \leq \beta_{\overline{x}\,dj}(\tau_x),$

obtaining in this case

$$(Pw)_{dj}(\overline{x}) - (Pw)_{dj}(x) \leq \left(L_w \left(1 + L_\tau M_g + \frac{1}{\widetilde{\alpha}_j} \sum_{i \neq j} \lambda_{j\,i} \right) + \frac{L_f}{\widetilde{\alpha}_j} \right) \|x - \overline{x}\|,$$

therefore

$$\|(Pw)_{dj}(x) - (Pw)_{dj}(\overline{x})\|$$
$$\leq \left(L_w \left(1 + L_\tau M_g + \frac{1}{\widetilde{\alpha}_j} \sum_{i \neq j} \lambda_{j\,i} \right) + \frac{L_f}{\widetilde{\alpha}_j} + L_\tau (M_f + A \sum_{i \neq j} \lambda_{j\,i}) \right) \|x - \overline{x}\|.$$

Hence by symmetry between x and $\overline{x}$ this inequality holds for any x, $\overline{x} \in \overline{Q}$. Then by virtue of (29) it results $L_{Pw} \leq \eta L_w + \rho$ and so we arrive at

$$\|(Pw)_{dj}(x) - (Pw)_{dj}(\overline{x})\| \leq L_{Pw} \|x - \overline{x}\|.$$

□

Remark 3.3 *The following relation between the Lipschitz constants of $P^\nu w$ and w holds*

$$L_{P^\nu w} \leq (L_w + \rho)(\eta + 1)^\nu \quad \forall \nu \in N.$$

Corollary 3.1 $P : C(\overline{Q})^{(m+1)\times J} \to C(\overline{Q})^{(m+1)\times J}$.

Proof. Let $w \in C(\overline{Q})^{(m+1)\times J}$ and w^ν a sequence of elements of $\left(W^{1,\infty}(Q)\right)^{(m+1)\times J}$ that approximates w in the following sense

$$\lim_{\nu \to \infty} \|w - w^\nu\|_\infty = 0. \tag{30}$$

It is easy to check that

$$\|Pw - P\overline{w}\|_\infty \leq \|w - \overline{w}\|_\infty \tag{31}$$

and then by virtue of (30) and (31) Pw is the uniform limit of a sequence of Lipschitz continuous functions. In consequence Pw itself is a continuous function.

□

3.3.2 A fixed point problem. Existence and uniqueness of solution.

We consider now the fixed point problem

$$\textit{Find } W \in B(\overline{Q})^{(m+1)\times J} \textit{ such that } W = PW. \tag{32}$$

We prove for this problem, the existence and uniqueness of solution.

Definition 3.5 :
$\underline{w}$ is a subsolution of the operator P if it verifies

$$\underline{w}_{d\,j}(x) \le (P\underline{w})_{d\,j}(x) \quad \forall d \in D, \quad \forall j \in \mathcal{J}, \quad \forall x \in \overline{Q},$$

$\overline{s}$ is a supersolution of the operator P if it verifies

$$\overline{s}_{d\,j}(x) \ge (P\overline{s})_{d\,j}(x), \quad \forall d \in D, \quad \forall j \in \mathcal{J}, \quad \forall x \in \overline{Q} \ .$$

Remark 3.4 *For any $u, v \in B(\overline{Q})^{(m+1)\times J}$ we write $u \le v$ if $u_{d\,j} \le v_{d\,j}$ $\forall x \in \overline{Q}$, $\forall d \in D$, $\forall j \in \mathcal{J}$.*

Lemma 3.2 *The set of subsolutions of operator P is not empty.*

Proof. We prove that if the constant value $\underline{w}$ verifies $\underline{w} \le \dfrac{-M_f}{\alpha}$, then $\underline{w}$ is a subsolution. Applying the definition (27) we get

$$\begin{aligned}
P\underline{w}_{d\,j} &= \int_0^{\overline{\tau}} e^{-\widetilde{\alpha}_j\, s} \left(f(y(s), d) + \sum_{i\neq j} \lambda_{j\,i} \underline{w}_{d\,i} \right) ds + e^{-\widetilde{\alpha}_j\,\overline{\tau}}\, S\underline{w}_{d\,j} \\
&= \int_0^{\overline{\tau}} e^{-\widetilde{\alpha}_j\, s} \left(f(y(s), d) + \sum_{i\neq j} \lambda_{j\,i} \underline{w}_{d\,i} \right) ds + e^{-\widetilde{\alpha}_j\,\overline{\tau}}\, (\underline{w}_{d\,j} + q(d, \widetilde{d})) \\
&\ge \int_0^{\overline{\tau}} e^{-\widetilde{\alpha}_j\, s} (-M_f + \sum_{i\neq j} \lambda_{j\,i} \underline{w}_{d\,i})\, ds + e^{-\widetilde{\alpha}_j\,\overline{\tau}}\, (\underline{w}_{d\,j} + q(d, \widetilde{d})) \\
&= \frac{(1 - e^{-\widetilde{\alpha}_j\,\overline{\tau}})}{\widetilde{\alpha}_j} (-M_f + \underline{w}_{d\,j} \sum_{i\neq j} \lambda_{j\,i}) + e^{-\widetilde{\alpha}_j\,\overline{\tau}} (\underline{w}_{d\,j} + q(d, \widetilde{d})) \\
&\ge \frac{(1 - e^{-\widetilde{\alpha}_j\,\overline{\tau}})}{\widetilde{\alpha}_j} (-M_f - \alpha\, \underline{w}_{d\,j}) + \underline{w}_{d\,j} + e^{-\widetilde{\alpha}_j\,\overline{\tau}} q_0 \\
&\ge \underline{w}_{d\,j} + e^{-\widetilde{\alpha}_j\,\overline{\tau}} q_0 \ge \underline{w}_{d\,j}.
\end{aligned}$$

And so $\underline{w}$ is a subsolution.

$\square$

Lemma 3.3 *The set of supersolutions of the operator P is not empty.*

Proof. Let

$$\rho = \max_{j\in\mathcal{J}}(\frac{\widetilde{\alpha}_j A + M_f\ (1-e^{c\widetilde{\alpha}_j})}{\alpha(1-e^{c\widetilde{\alpha}_j})}),$$

$$c = \min_{j\in\mathcal{J}}\min_{d\in D}(\tau(e,d,j)).$$

We define $\forall d \in D, \forall j \in \mathcal{J}$

$$\overline{s}_{d\,j}(x) = \begin{cases} \rho & x \neq e, \\ \\ \rho - A & x = e. \end{cases} \tag{33}$$

With this definition, it is easy to check that $\overline{s}$ is a supersolution.

$\square$

Lemma 3.4 *Given $\rho > 0$, $\exists\, \underline{w} = \underline{w}(\rho)$, $\overline{w}(\rho)$, $0 < \mu(\rho) \leq 1$ such that $\forall u,\ v \in B(\overline{Q})^{(m+1)\times J}$ verifying $\|u\| \leq \rho$, $\|v\| \leq \rho$ it holds that*

1. $\underline{w} \leq u \leq \overline{w}$,
2. $\underline{w} \leq v \leq \overline{w}$,
3. $\underline{w} + \mu\ (\overline{w} - \underline{w}) \leq P\underline{w}$,
4. $P\overline{w} \leq \overline{w}$.

Proof. We consider

$$\underline{w}_{d\,j}(x) = \min\left(-\rho, \frac{-M_f}{\alpha}\right),\ \ \forall d \in D,\ \ \ \forall j \in \mathcal{J},\ \ \ \forall x \in Q, \tag{34}$$

$$\overline{w}_{d\,j}(x) = \overline{s}_{d\,j}(x) + \rho, \tag{35}$$

where $\overline{s}_{d\,j}(x)$ is given by (33) .

As $\|u\| \leq \rho,\ \|v\| \leq \rho$, it becomes evident that properties 1) and 2) hold.

$\overline{s}_{d\,j}$ is a supersolution then if we add a constant to it, the new function is also a supersolution. Therefore $(P\overline{w})_{d\,j} \leq \overline{w}_{d\,j}$ and then property 4) holds.

By (34) we have

$$\underline{w}_{d\,j} \leq \frac{-M_f}{\alpha} \qquad \Rightarrow \qquad \underline{w}_{d\,j} + e^{-\widetilde{\alpha}_j\,\overline{\tau}} q_0 \leq (P\underline{w})_{d\,j} \quad \forall d \in D, \quad \forall j \in \mathcal{J}.$$

To obtain 3 it should be $\mu(\overline{w} - \underline{w}) \leq e^{-\widetilde{\alpha}_j\,\overline{\tau}} q_0$ or in an equivalent expression

$$\mu \leq \frac{e^{-\widetilde{\alpha}_j\,\overline{\tau}} q_0}{(\overline{w} - \underline{w})},$$

because $\overline{w} - \underline{w} \neq 0$. Besides

$$\overline{w} - \underline{w} \geq \|\overline{s}\| + \rho + \frac{M_f}{\alpha},$$

as for any ρ

$$\frac{e^{-\widetilde{\alpha}_j\,\overline{\tau}} q_0}{\|\overline{s}\| + \rho + \frac{M_f}{\alpha}} < 1,$$

because $\|\overline{s}\| \geq e^{-\widetilde{\alpha}_j\,\overline{\tau}} q_0$ there exists $0 < \mu(\rho) \leq 1$ such that 3 holds

$$0 < \mu \leq \frac{e^{-\widetilde{\alpha}_j\,\overline{\tau}} q_0}{\|\overline{s}\| + \rho + \frac{M_f}{\alpha}} \leq \frac{e^{-\widetilde{\alpha}_j\,\overline{\tau}} q_0}{(\overline{w} - \underline{w})}\,.$$

□

Lemma 3.5 *Let* $u,\ v \in,\ \underline{w},\ \overline{w}$ *and* $0 < \mu \leq 1$ *given by the Lemma 3.4. Let* $\zeta, \xi \in [0,1]$ *such that*

$$\xi\ (\underline{w} - u) \leq v - u \leq \zeta\ (v - \underline{w}),$$

then

$$(1-\mu)\,\xi\,(\underline{w} - Pu) \leq Pv - Pu \leq (1-\mu)\,\zeta\,(Pv - \underline{w}).$$

Proof.

$$u \geq v - \zeta\ (v - \underline{w}) = (1-\zeta)v + \zeta\,\underline{w}.$$

By the properties of monotony and concavity of operator the P, we have

$$Pu \geq (1-\zeta)Pv + \zeta\, P\underline{w}.$$

Using property 3) from Lemma 3.4, we have $Pu \geq (1-\zeta)Pv + \zeta\,(\underline{w} + \mu(\overline{w} - \underline{w}))$. Then

$$\begin{aligned} Pu - Pv &\geq -\zeta Pv + \zeta\,(\underline{w} + \mu(\overline{w} - \underline{w})) \\ &\geq -\zeta Pv + \zeta\,(\underline{w} + \mu(Pv - \underline{w})) \\ &= \zeta\,(-Pv + \underline{w} - \mu(\underline{w} - Pv)) \\ &= \zeta(1-\mu)(\underline{w} - Pv)\,. \end{aligned}$$

In consequence

$$Pv - Pu \leq \zeta(1-\mu)(Pv - \underline{w}).$$

In the same form, as

$$v \geq u + \xi(\underline{w} - u) = (1-\xi)u + \xi\underline{w}$$

we obtain

$$Pv \geq (1-\xi)Pu + \xi P\underline{w}$$

and from here, it results

$$(1-\mu)\xi(\underline{w} - Pu) \leq Pv - Pu.$$

□

Theorem 3.3

1. *There exists* $W \in C(\bar{\Omega})$ *such that* $PW = W$
2. $\forall w \in C(\bar{\Omega})$ *it results* $\lim\limits_{\nu \to \infty} P^\nu w = W$
3. *The following rate of convergence holds*

$$\|P^\nu w - W\| \leq K(\rho)(1-\mu(\rho))^\nu,$$

where $0 < \mu(\rho) \leq 1,\ K(\rho) = \|\overline{w}\| + \|\underline{w}\|$ *are given by Lemma 3.4 with* $\rho = \|w\|$.

Proof. Let u, $v \in C(\bar{\Omega})$, $\rho = \max(\|u\|, \|v\|)$ and $\overline{w}$, $\underline{w}$ given by Lemma 3.4, i.e.

$$\underline{w} \leq u \leq \overline{w},$$

$$\underline{w} \leq v \leq \overline{w}.$$

If we define $v' = Pv,\ u' = Pu,\ \zeta' = \zeta(1-\mu)$ by Lemma 3.5 we have

$$\begin{aligned} v' - u' &\leq (1-\mu)\zeta(\,v' - \underline{w}), \\ v' - u' &\leq \zeta'(\,v' - \underline{w}). \end{aligned}$$

In consequence, again by Lemma 3.5 we have

$$\begin{aligned} Pv' - Pu' &\leq (1-\mu)\zeta'\,(\,Pv' - \underline{w}), \\ P^2 v - P^2 u &\leq (1-\mu)^2\zeta(P^2\,v - \underline{w}), \end{aligned}$$

finally by induction we obtain

$$P^\nu v - P^\nu u \quad \leq (1-\mu)^\nu \zeta(P^\nu\,v - \underline{w}). \tag{36}$$

As $P\bar{w} \leq \bar{w}$ and $Pv \leq P\bar{w}$ by the monotony of P, we have $P^\nu v \leq \overline{w}$ and considering (36) it results

$$\limsup_{\nu \to \infty} (P^\nu v - P^\nu u) \leq 0.$$

In a similar way, we can obtain

$$\liminf_{\nu \to \infty} (P^\nu v - P^\nu u) \geq 0,$$

then

$$\lim_{\nu \to \infty} (P^\nu v - P^\nu u) = 0.$$

We shall prove now that both sequences $v^\nu = P^\nu v$ and $u^\nu = P^\nu u$ are convergent. We take, in particular
$\widehat{u} = \underline{w}$, $\widehat{v} = \overline{w}$ and we define

$$\begin{aligned} \widehat{v}^\nu &= P^\nu \overline{w}, \\ \widehat{u}^\nu &= P^\nu \underline{w}. \end{aligned}$$

As $P\overline{w} \leq \overline{w}$, $\widehat{v}^\nu$ is decreasing due to the monotony of P and by induction we have

$$P^{\nu+1}\overline{w} \leq P^\nu \overline{w} \leq \overline{w}.$$

In a similar way $\widehat{u}^\nu$ is increasing and in consequence it is verified that

$$\underline{w} \leq \widehat{u}^\nu \leq \widehat{u}^{\nu+1} \leq \widehat{v}^{\nu+1} \leq \widehat{v}^\nu \leq \overline{w}.$$

Then v^ν and u^ν are convergent to the same point W because using (36) with $\widehat{u}$ and $\widehat{v}$ we obtain

$$0 \leq \widehat{v}^\nu - \widehat{u}^\nu \leq (1-\mu)^\nu \left(\|\overline{w}\| + \|\underline{w}\|\right).$$

Moreover as

$$|(P^\nu v)_{d\,j}(x) - (P^\nu u)_{d\,j}(x)| \leq \|\widehat{v}^\nu - \widehat{u}^\nu\|, \quad \forall d \in D, \quad \forall j \in \mathcal{J},$$

is verified, the following rate of convergence holds

$$\|P^\nu w - W\| \leq K(\rho)(1-\mu(\rho))^\nu,$$

with $0 < \mu(\rho) \leq 1,\ K(\rho) = \|\overline{w}\| + \|\underline{w}\|$ given by Lemma 3.4 with $\rho = \|w\|$.

As u and v are arbitrary, we obtain the uniqueness of the limit and of the fixed point of P.

As the convergence of $P^\nu w$ to W is uniform, the fixed point of P is a continuous function because (choosing as initial point a continuous function) it is the uniform limit of a sequence of continuous functions.

$\square$

3.3.3 Equivalence between the DPP and the fixed point problem

Remark 3.5 *¿From the definition of $\bar{\tau}$ and the properties that V verifies (Theorem 3.1) it is easy to prove that the unique fixed point of P is V.*

Theorem 3.4 *If V verifies (DPP) then V is the fixed point of P.*

Proof.

- Case $V_{d\,j}(x) = (SV)_{d\,j}(x)$

Let $0 < t \leq \tau(x,d,j)$ by 1 and 2 of (DPP) (Theorem 3.1) we have

$$\begin{aligned} V_{d\,j}(x) &\leq \int_0^t e^{-\widetilde{\alpha}_j\, s}(f(y(s),d) + \sum_{i\neq j} \lambda_{j\,i} V_{d\,i}(y(s)))ds + e^{-\widetilde{\alpha}_j\, t} V_{d\,j}(x + t\, g(d,j)) \\ &\leq \int_0^t e^{-\widetilde{\alpha}_j\, s}(f(y(s),d) + \sum_{i\neq j} \lambda_{j\,i} V_{d\,i}(y(s)))ds + e^{-\widetilde{\alpha}_j\, t} (SV)_{d\,j}(x + t\, g(d,j)), \end{aligned} \tag{37}$$

then

$$V_{d\,j}(x) = (PV)_{d\,j}(x). \tag{38}$$

- Case $V_{d\,j}(x) < (SV)_{d\,j}(x)$

  By 3 of (DPP) we have

$$\begin{aligned} V_{d\,j}(x) = & \int_0^{t_{x\,d\,j}} e^{-\widetilde{\alpha}_j\, s}(f(y(s),d) + \sum_{i\neq j} \lambda_{j\,i} V_{d\,i}(y(s)))ds \\ & + e^{-\widetilde{\alpha}_j\, t_{x\,d\,j}} V_{d\,j}(x + t_{x\,d\,j}\, g(d,j)). \end{aligned}$$

  By definition of $t_{x\,d\,j}$ we have $V_{d\,j}(x + t_{x\,d\,j}\, g(d,j) = (SV)_{d\,j}(x + t_{\,xdj}\, g(d,j))$ and so

$$\begin{aligned} V_{d\,j}(x) = & \int_0^{t_{x\,d\,j}} e^{-\widetilde{\alpha}_j\, s}(f(y(s),d) + \sum_{i\neq j} \lambda_{j\,i} V_{d\,i}(y(s)))ds \\ & + e^{-\widetilde{\alpha}_j\, t_{x\,d\,j}} (SV)_{d\,j}(x + t_{x\,d\,j}\, g(d,j)). \end{aligned} \tag{39}$$

  In addition, by 1 and 2 of (DPP) it is verified

$$\begin{aligned} V_{d\,j}(x) \leq & \int_0^{t} e^{-\widetilde{\alpha}_j\, s}(f(y(s),d) + \sum_{i\neq j} \lambda_{j\,i} V_{d\,i}(y(s)))ds \\ & + e^{-\widetilde{\alpha}_j\, t} (SV)_{d\,j}(x + t\, g(d,j)), \end{aligned} \tag{40}$$

then from (39) and (40) we get

$$\begin{aligned} V_{d\,j}(x) &= \min_{t\,\le \tau(x,d,j)} \Big(\int_0^t e^{-\widetilde{\alpha}_j\, s} (f(y(s),d) + \sum_{i\neq j} \lambda_{j\,i} V_{d\,i}(y(s))) ds \\ &\quad + e^{-\widetilde{\alpha}_j\, t} (SV)_{d\,j}(x + t\, g(d,j)) \Big) \\ &= (PV)_{d\,j}(x)\,. \end{aligned} \tag{41}$$

¿From (38) and (41) we conclude that V is a fixed point of P.

□

3.3.4 Hölder continuity of the optimal cost function

The solution V of the problem is the fixed point of the operator

$$P : C(\overline{\Omega})^{(m+1)\times J} \to C(\overline{\Omega})^{(m+1)\times J},$$

given by ($\forall d \in D, \forall j \in \mathcal{J},\ \forall x \in Q$),

$$(Pw)_{d\,j}(x) =$$

$$\inf_{\tau\,\le \tau(x,d,j)} \Big(\int_0^\tau e^{-\widetilde{\alpha}_j\, s} (f(y(s),d) + \sum_{i\neq j} \lambda_{j\,i}\, w_{d\,i}(y(s))) ds + e^{-\widetilde{\alpha}_j\, \tau} (Sw)_{d\,j}(y(\tau)) \Big).$$

Theorem 3.5 *The fixed point of operator P is a Hölder-continuous function.*

Proof. We know from Theorem 1 that the following rate of convergence holds for the sequence $u^\nu = P^\nu w$

$$\|V - w^\nu\| \le K\xi^\nu \ \text{ with } \ 0 \le \xi < 1,\ \forall \nu \in N.$$

We start the sequence w^ν from a Lipschitz function w (with associated Lipschitz constant L_w), hence

$$\begin{aligned} |V(x) - V(\overline{x})| &\le |V(x) - w^\nu(x)| + |w^\nu(x) - w^\nu(\overline{x})| + |w^\nu(\overline{x}) - V(\overline{x})| \\ &\le K\xi^\nu + (L_w + \rho)(\eta+1)^\nu\, \|x - \overline{x}\| + K\xi^\nu \\ &\le 2K\, e^{\nu \log(\xi)} + (L_w + \rho)\, e^{\nu \log(\eta+1)}\, \|x - \overline{x}\| \qquad \forall \nu \in N. \end{aligned}$$

Let us consider on the entire $\mathbb{R}$ the function

$$F(t) = 2K\, e^{t\log(\xi)} + (L_w + \rho)\, e^{t\log(\eta+1)}\, \|x - \overline{x}\|\,.$$

We search the minimum of this function and we will denote with $\overline{t}$ the value which realizes that minimum (to simplify this presentation we will suppose, without losing of generality, that the value $\overline{t}$ given by this optimization is a natural number; the general case can be carried out using similar arguments and obtaining bounds of the same form). So, for the minimizing $\overline{t}$ we have

$$2K\log(\xi)\, e^{\overline{t}\log(\xi)} + (L_w + \rho)\,\log(\eta+1) e^{\overline{t}\log(\eta+1)}\, \|x - \overline{x}\| = 0$$

hence

$$e^{\overline{t}(\log(\eta+1)-\log(\xi))} = \frac{-2K\log(\xi)}{(L_w+\rho)\,\log(\eta+1)\,\|x-\overline{x}\|}.$$

We define

$$C = \frac{-2K\log(\xi)}{(L_w+\rho)\,\log(\eta+1)},$$

$$\gamma = \frac{-\log(\xi)}{\log(\eta+1) - \log(\xi)}\,.$$

Obviously it is valid that

$$0 < \gamma < 1.$$

It results

$$e^{\overline{t}\log(\xi)} = \left(C\,\|x-\overline{x}\|^{-1}\right)^{-\gamma} = C^{-\gamma}\,\|x-\overline{x}\|^{\gamma}\,,$$

$$e^{\overline{t}\log(\eta+1)} = \left(C\,\|x-\overline{x}\|^{-1}\right)^{1-\gamma}.$$

Hence, for $\overline{t}$, the following equality holds

$$2K\, e^{\overline{t}\log(\xi)} + (L_w+\rho)\, e^{\overline{t}\log(\eta+1)}\,\|x-\overline{x}\| = 2KC^{-\gamma}\,\|x-\overline{x}\|^{\gamma} + (L_w+\rho)C^{1-\gamma}\,\|x-\overline{x}\|^{\gamma}$$

and then, defining c as

$$c = 2KC^{-\gamma} + (L_w+\rho)C^{1-\gamma},$$

we get

$$\|V(x) - V(\overline{x})\| \quad \leq \quad c\,\|x-\overline{x}\|^{\gamma}\ .$$

We conclude that V is Hölder continuous with parameter γ.

$\square$

4 Discrete problem

4.1 Elements of the discrete problem

To define the discrete problem, we introduce an approximation which comprises a discretization of the space $W^{1,\infty}(\Omega)$ and equation (32). We use the techniques and results presented in [5]-[7] and [12, 13].

4.1.1 Domain Approximation

We identify the discretization of the space variables with the parameter k, which also indicates the size of discretization. Let $h > 0$; for each $j \in \mathcal{J}$, we define the uniform discretization $B^{j\,k}$ of $\mathbb{R}^m$ given by:

- $B^{j\,k} = \left\{ \sum_{d=0}^{m} \zeta_d \, e_j^d : \zeta_d \text{ is integer} \right\}$,
- $h_j^0 = \left(1 - \sum_{d=1}^{m} \frac{r_{d\,j}}{p_d} \right) h$,
- $h_j^d = \frac{r_{d\,j}}{p_d} \, h$,
- $e_j^0 = (-r_{1\,j}, ..., -r_{d\,j}, ..., -r_{m\,j}) \, h_j^0$,
- $e_j^d = \left(-r_{1\,j}, ..., -r_{(d-1)\,j}, p_d - r_{d\,j}, -r_{(d+1)\,j}, ..., -r_{m\,j} \right) h_j^d$.

For each state of demand j we approximate Q with $Q_{j\,k} = \bigcup_{\mu} S_\mu^{j\,k}$, where $\left\{ S_\mu^{j\,k} \right\}$ is the maximum set of polyhedrons of $\mathbb{R}^m$ which verify

$$S_\mu^{j\,k} = x_\mu^j + \left\{ \sum_{d=1}^{m} \xi_d \, e_j^d \; : \; \xi_d \in [0,1] \right\}, x_\mu^j \in B^{j\,k}, \; S_\mu^{j\,k} \subset Q \, .$$

We define

$$k_j = \max_{\mu} (\text{diam } S_\mu^{j\,k}), \qquad\qquad k = \max_{j \in \mathcal{J}} (k_j) \, .$$

We denote $\mathcal{V}^{j\,k} = \left\{ x_\mu^j, \; \mu = 1, ..., N_j \right\}$ the set of nodes of $Q_{j\,k}$, where N_j is the cardinal of $\mathcal{V}^{j\,k}$.

Remark 4.1 *If k is small enough, for any two vertices of $\mathcal{V}^{j\,k}$, there always exists a path given by a system trajectory, (in fact, an especial trajectory without demand changes) which joins the first vertex to the second one.*

4.1.2 Approximation of the boundary

We define $\forall d = 1, ..., m,\ \forall j \in \mathcal{J}$,

$$\gamma^{+}_{k,\,d,\,j} = \left\{ x^j_\mu \in \mathcal{V}^{j\,k} : x^j_\mu + h^d\, g(d,j) \notin Q_{j\,k} \right\},$$

$$\gamma^{-}_{k,\,d,\,j} = \left\{ x^j_\mu \in \mathcal{V}^{j\,k} : x^j_\mu + h^{\widetilde{d}}\, g(\widetilde{d},j) \notin Q_{j\,k}, \forall \widetilde{d} \neq d \right\}.$$

4.1.3 Approximation space $\overline{F}_k$

We divide each quadrilateral element $S^{j\,k}_\mu$ in simplices such that the edges are coincident with lines of the form $x^j_\mu + s\ g(d,\ j),\ s \geq 0$. We consider the set $\overline{F}_k$ of functions $w : \left(\prod_{j\,\in J} Q_{j\,k} \right) \times D \to \mathbb{R},\ w_{d\,j}(\cdot) \in W^{1,\infty}(Q_{j\,k})$, such that in each simplex $w_{d\,j}(\cdot)$ is an affine function of the type

$$a_0 + \sum_{i=1}^{m} a_i\, x_i\,.$$

It is obvious that any $w \in \overline{F}_k$ is entirely characterized by the values $w_{d\ j}(x^j_\mu), x^j_\mu \in \mathcal{V}^{j\,k}$,
$d \in D,\ j \in \mathcal{J},\ \mu = 1, ..., N_j$.

4.2 The discrete HJB equation

4.2.1 A discrete fixed point problem

We define

$$(\mathcal{L}_k w)_{d\,j}(x^j_\mu) = \begin{cases} \frac{1}{1+\widetilde{\alpha}_j\, h^d_j} & \left(w_{d\,j}(x^j_\mu + h^d_j\, g(d,\,j)) + h^d_j \left(f(x^j_\mu, d) + \sum_{i \neq j} \lambda_{j\,i}\; w_{d\,i}(\pi_{Q_{i\,k}}(x^j_\mu)) \right) \right), \\ & \forall x^j_\mu \in \left(\mathcal{V}^{j\,k} \setminus \left(\gamma^+_{k,\,d,\,j} \cup \left(\bigcup_{r\,\neq\,d} \gamma^-_{k,\,r,\,j} \right) \right) \right), \\ +\infty & \forall x^j_\mu \in \left(\gamma^+_{k,\,d,\,j} \cup \left(\bigcup_{r\,\neq\,d} \gamma^-_{k,\,r,\,j} \right) \right), \end{cases}$$

where $\pi_{Q_{i\,k}}(x^j_\mu)$ is the projection of x^j_μ over the set $Q_{i\,k}$ and

$$(S_k w)_{d\,j}(x^j_\mu) = \min(\min_{\widetilde{d} \neq d}(w_{\widetilde{d}\,j}(x^j_\mu) + q(d, \widetilde{d})), w_{d\,j}(\pi_{Q_{i\,k}}(e)) + A),$$

where $\pi_{Q_{i\,k}}(e)$ is the projection of e over the set $Q_{i\,k}$.
In this form, $(\mathcal{L}_k w)_{d\,j}(x^j_\mu)$ is a natural discretization of (32) and it includes the boundary conditions (26).
We define $P_k : \overline{F}_k \to \overline{F}_k$ such that $\forall x^j_\mu \in \mathcal{V}^{j\,k}, \forall d \in D,\ \forall j \in \mathcal{J}$

$$(P_k w)_{d\,j}(x^j_\mu) = \min\left((\mathcal{L}_k w)_{d\,j}(x^j_\mu), (\mathcal{S}_k w)_{d\,j}(x^j_\mu) \right). \tag{42}$$

and the following discrete problem:

$$\boxed{\text{Problem P}^k\text{: Find the fixed point of operator } P_k.}$$

4.2.2 Existence and uniqueness of the discrete solution

Using the techniques introduced in [16] we can prove the following characterization of the unique solution U^k of problem P^k.

Theorem 4.1 *There exists a unique fixed point of operator P_k, i.e. $\exists!\, U^k$ such that $U^k = P_k U^k$. Moreover $\forall w \in \overline{F}_k$ we have $U^k = \lim_{\nu \to \infty} (P_k)^\nu w$ and the following estimation of the convergence rate holds*

$$\left\| P^\nu_k w - U^k \right\| \leq K(\rho)(1 - \mu(\rho))^\nu, \tag{43}$$

where $0 < \mu(\rho) \leq 1$ and $K(\rho) > 0$, ρ depends on $\|w\|$.

Proof. To prove this theorem we use the techniques given in [13]. To use this techniques we should prove that the set of supersolutions and the set of subsolutions are not empty, i.e. there exist, at least, two functions $\overline{s}$ and $\overline{w}$ such that:

$$a)\ P_k \overline{s} \leq \overline{s} \qquad\qquad b)\ P_k \overline{w} \geq \overline{w}$$

a) Let $\overline{s}$ be the function given by

$$\overline{s}_{d\,j}(x^j_\mu) = \begin{cases} \dfrac{M_f}{\alpha} + \dfrac{M_q}{\eta\alpha}, & x^j_\mu \in \left(\mathcal{V}^{j\,k} \setminus \left(\gamma^+_{k,\,d,\,j} \cup \left(\bigcup_{r\,\neq d} \gamma^-_{k,\,r,\,j}\right)\right)\right), \\ \dfrac{M_f}{\alpha} + \dfrac{M_q}{\eta\alpha} + M_q, & x^j_\mu \in \left(\gamma^+_{k,\,d,\,j} \cup \left(\bigcup_{r\,\neq d} \gamma^-_{k,\,r,\,j}\right)\right), \end{cases} \qquad \forall d, \forall j$$

where η verifies

$$\eta \leq \min_{d\,j}\left(h^d_j\right). \tag{44}$$

We want to show that $\overline{s}$ is a supersolution.

We consider two cases:

(i) $x^j_\mu \in \left(\mathcal{V}^{j\,k} \setminus \left(\gamma^+_{k,\,d,\,j} \cup \left(\bigcup_{r\,\neq d} \gamma^-_{k,\,r,\,j}\right)\right)\right)$.

(ii) $x^j_\mu \in \left(\gamma^+_{k,\,d,\,j} \cup \left(\bigcup_{r\,\neq d} \gamma^-_{k,\,r,\,j}\right)\right)$.

Case (i)

$$\begin{aligned} (P_k\overline{s})_{d\,j}(x^j_\mu) &\leq \tfrac{1}{1+\widetilde{\alpha}_j\,h^d_j}\left(\overline{s}_{d\,j}(x^j_\mu + h^d_j\,g(d,\,j)) + h^d_j(\,f(x^j_\mu, d) + \sum_{i\neq j} \lambda_{j\,i}\ \overline{s}_{d\,i}(\pi_{Q_{i\,k}}(x^j_\mu)))\right) \\ &\leq \tfrac{1}{1+\widetilde{\alpha}_j\,h^d_j}(\frac{M_f}{\alpha} + \frac{M_q}{\eta\alpha} + M_q + h^d_j(M_f + \sum_{i\neq j}\lambda_{j\,i}(\frac{M_f}{\alpha} + \frac{M_q}{\eta\alpha}))) \\ &= \tfrac{1}{1+\widetilde{\alpha}_j\,h^d_j}\left(M_f\left(\frac{1}{\alpha} + h^d_j + \frac{h^d_j}{\alpha}\sum_{i\neq j}\lambda_{j\,i}\right) + M_q\left(\frac{1}{\eta\alpha} + 1 + \frac{h^d_j}{\eta\alpha}\sum_{i\neq j}\lambda_{j\,i}\right)\right) \\ &= \tfrac{1}{1+\widetilde{\alpha}_j\,h^d_j}\left(\frac{M_f}{\alpha}\left(1 + \widetilde{\alpha}_j\,h^d_j\right) + \frac{M_q}{\eta\alpha}\left(1 + \eta\alpha + h^d_j\sum_{i\neq j}\lambda_{j\,i}\right)\right) \\ &\leq \frac{M_f}{\alpha} + \frac{M_q}{\eta\alpha} = \overline{s}_{d\,j}(x^j_\mu), \end{aligned}$$

this last inequality holds by virtue of (44).

Case (ii)

$(P_k\overline{s})_{d\,j}(x^j_\mu) = (S\overline{s})_{d\,j}(x^j_\mu) = q(d,\widetilde{d}) + \overline{s}_{\widetilde{d}\,j}(x^j_\mu) \le \dfrac{M_f}{\alpha} + \dfrac{M_q}{\eta\alpha} + M_q = \overline{s}_{d\,j}(x^j_\mu).$
The analysis of cases (i) and (ii) proves that $\overline{s}$ is a supersolution.

b) Let $\overline{w}$ be the function given by

$$\overline{w}_{d\,j} = -\frac{M_f}{\alpha},\ \forall d \in D, \forall j \in \mathcal{J}.$$

We want to show that $\overline{w}$ is a subsolution. Since $q(d,\widetilde{d}) \ge 0$ we have

$$(S_k\overline{w})_{d\,j} \ge \overline{w}_{d\,j},\ \forall d \in D, \forall j \in \mathcal{J}.$$

Let us see that the operator L_k verifies

$$(\mathcal{L}_k\overline{w})_{d\,j} \ge \overline{w},\ \forall d \in D, \forall j \in \mathcal{J}.$$

$$\begin{aligned}
(\mathcal{L}_k\overline{w})_{d\,j}(x^j_\mu) &= \frac{1}{1+\widetilde{\alpha}_j\,h^d_j}\left(\overline{w}_{d\,j}(x^j_\mu + h^d_j\,g(d,\,j)) + h^d_j(\,f(x^j_\mu,d) + \sum_{i\neq j}\lambda_{j\,i}\ \overline{w}_{d\,i}(\pi_{Q_{i\,k}}(x^j_\mu)))\right)\\
&\ge \frac{1}{1+\widetilde{\alpha}_j\,h^d_j}\left(-\frac{M_f}{\alpha} - h^d_j\,M_f - h^d_j\,\frac{M_f}{\alpha}\sum_{i\neq j}\lambda_{j\,i}\right)\\
&= \frac{-\,M_f}{1+\widetilde{\alpha}_j\,h^d_j}\left(\frac{M_f + \alpha\,h^d_j + h^d_j\sum_{i\neq j}\lambda_{j\,i}}{\alpha}\right)\\
&\ge -\frac{M_f}{\alpha} = \overline{w}_{d\,j}(x^j_\mu).
\end{aligned}$$

Then we have $\forall d \in D, \forall j \in \mathcal{J}$

$$\overline{w}_{d\,j} \le \min\Big((\mathcal{L}_k\overline{w})^k_{d\,j}, (S_k\overline{w})_{d\,j}\Big)$$

and so $\overline{w}$ is a subsolution of P_k.
The remaining part of the proof is analogous to what we have made for the continuous operator P and it is here omitted for sake of briefness.

$\square$

4.3 Convergence results

For the convergence of the discretization procedure to the solution of the original continuous problem, we have the following result.

Theorem 4.2 *The following rate of convergence holds*

$$\left\| U^k - V \right\|_\infty \le K\, h^\gamma.$$

Proof. Let h be the positive value associated to the parameter of discretization k of the domain Ω, U^k the discrete solution associated to this parameter, then U^k is given by

$$U^k_{d\,j}(x^j_\mu) = \min\left((\mathcal{L}_k U^k)_{d\,j}(x^j_\mu), (\mathcal{S}_k U^k)_{d\,j}(x^j_\mu) \right) \forall x^j_\mu \in \mathcal{V}^{j\,k}, \forall d \in D,\ \forall j \in \mathcal{J},$$

We want to estimate the difference between this function and the solution of the original problem.

We consider first the difference: $V - U^k$. Let

$$\Delta_1 = \max\{V_{d\,j}(x_i) - U^k_{d\,j}(x_i), x_i \in \mathcal{V}^{j\,k}, j \in \mathcal{J},\ d \in D\}.$$

Then, if we call x_0 the point that realizes this maximum, we have

$$U^k_{d\,j}(x_0) = \min\{(1 - \tilde{\alpha}_j\ h_{d\,j}) U^k_{d\,j}(x_0 + h_{d\,j}\, g(d,\, j)) + h_{d\,j}\ (f(x_0, d)$$
$$+ \sum_{i \neq j} \lambda_{j\,i}\ U^k_{d\,i}(\pi_{Q_{i\,k}} x_0)),\ (\mathcal{S}_k\, U^k)_{d\,j}(x_0)\}.$$

By using a recursive argument we can assume that the minimum is attained by the first component appearing in the min operation, i.e.

$$U^k_{d\,j}(x_0) = (1 - \tilde{\alpha}_j\ h_{d\,j}) U^k_{d\,j}(x_0 + h_{d\,j}\, g(d,\, j)) + h_{d\,j}(\, f(x_0, d) + \sum_{i \neq j} \lambda_{j\,i}\ U^k_{d\,i}(\pi_{Q_i}(x_0))).$$

Then $x_0 \notin \left(\gamma^+_{h,d,j} \cup (\bigcup_{r \neq d} \gamma^-_{h,r,j}) \right)$ and so $x_0 + h_{d\,j}\, g(d, j) \in Q_{j\,k} \subset \Omega$; in consequence

$$h_{d\,j} < \tau\, (x_0, d, j) = \sup\{t : x_0 + t\, g(d, j) \in \Omega\}$$

and $V_{d\,j}$ verifies:

$$V_{dj}(x_0) \leq \int_0^{h_{dj}} e^{-\widetilde{\alpha}_j s}(f(y(s), d) + \sum_{i \neq j} \lambda_{ji} V_{di}(y(s)))ds + e^{-\widetilde{\alpha}_j h_{dj}} V_{dj}(x_0 + h_{dj} g(d,j)).$$

Therefore we have

$$\begin{aligned}
\Delta_1 &= V_{dj}(x_0) - U^k_{dj}(x_0) \\
&\leq \int_0^{h_{dj}} e^{-\widetilde{\alpha}_j s}(f(y(s), d) + \sum_{i \neq j} \lambda_{ji} V_{di}(y(s)))ds + e^{-\widetilde{\alpha}_j h_{dj}} V_{dj}(x_0 + h_{dj} g(d,j)) \\
&\quad - \left((1 - \widetilde{\alpha}_j h_{dj}) U^k_{dj}(x_0 + h_{dj} g(d, j)) + h_{dj}(f(x_0, d) + \sum_{i \neq j} \lambda_{ji} U^k_{di}(\pi_{Q_{ik}}(x_0)))\right) \\
&\leq \int_0^{h_{dj}} e^{-\widetilde{\alpha}_j s}(|f(y(s), d) - f(x_0, d)| + \sum_{i \neq j} \lambda_{ji} |V_{di}(y(s)) - U^k_{di}(\pi_{Q_{ik}}(x_0)))| ds \\
&\quad + V_{dj}(x_0 + h_{dj} g(d,j)) e^{-\widetilde{\alpha}_j h_{dj}} - (1 - \widetilde{\alpha}_j h_{dj}) U^k_{dj}(x_0 + h_{dj} g(d, j)) \\
&\leq \int_0^{h_{dj}} e^{-\widetilde{\alpha}_j s}(L_f h_{dj} + \sum_{i \neq j} \lambda_{ji}(|V_{di}(y(s)) - V_{di}(x_0)| \\
&\quad + |V_{di}(x_0) - V_{di}(\pi_{Q_{ik}}(x_0))| + |V_{di}(\pi_{Q_{ik}}(x_0)) - U^k_{di}(\pi_{Q_{ik}}(x_0))))| ds \\
&\quad + (1 - \widetilde{\alpha}_j h_{dj}) \Big(V_{dj}(x_0 + h_{dj} g(d,j)) - U^k_{dj}(x_0 + h_{dj} g(d, j))\Big) + O(h^2_{dj}) \\
&\leq (L_f h_{dj} + \sum_{i \neq j} \lambda_{ji}(2 L_V h^\gamma_{dj} + \Delta_1)) h_{dj} + (1 - \widetilde{\alpha}_j h_{dj}) \Delta_1 + O(h^2_{dj})
\end{aligned}$$

In consequence we obtain

$$\Delta_1 (1 - h_{dj} \sum_{i \neq j} \lambda_{ji} - 1 + \widetilde{\alpha}_j h_{dj}) \leq (L_f h_{dj} + \sum_{i \neq j} \lambda_{ji} L_V h^\gamma_{dj}) h_{dj} + O(h^2_{dj}),$$

which implies

$$\Delta_1 \leq \frac{1}{\alpha} \left(L_f h_{dj} + \sum_{i \neq j} \lambda_{ji} L_V h^\gamma_{dj} \right) + O(h_{dj}).$$

Then for any other node of $Q_{j\,k}$ $\forall j$, $\forall d$ it is verified the same estimation. For any x in the domain $Q_{j\,k}$ there exists a point, which we denote by $x_i \in Q_{j\,k}$ such that $\|x - x_i\| \leq h_{d\,j}$. Therefore, taking in to account that $V_{d\,j}$ is Hölder continuous and that $U^k_{d\,j}$ is an affine function we get

$$\left|V_{d\,j}(x) - U^k_{d\,j}(x)\right| \leq \Delta_1 + C\,\|x - x_i\|^{\gamma}\,.$$

Then for all $x \in \Omega$

$$V_{d\,j}(x) - U^k_{d\,j}(x) \leq \frac{1}{\alpha}\left(L_f\,h_{d\,j} + \sum_{i \neq j} \lambda_{j\,i} L_V\ h^{\gamma}_{d\,j}\right) + O(h_{d\,j}) + O(h^{\gamma}_{d\,j})\,.$$

Remembering that $h_{0\,j} = \left(1 - \dfrac{r_{d\,j}}{p_d}\right) h$ and $h_{d\,j} = \dfrac{r_{d\,j}}{p_d} h$, if we denote by $K = \max\limits_{d\,j}\left\{\left(1 - \dfrac{r_{d\,j}}{p_d}\right), \dfrac{r_{d\,j}}{p_d}\right\}$ we have

$$V_{d\,j}(x) - U^k_{d\,j}(x) \leq \frac{K}{\alpha}\left(L_f\,h + \sum_{i \neq j} \lambda_{j\,i} L_V\ h^{\gamma}\right) + O(h^{\gamma}).$$

We consider now the difference $u^h - V$. Let $\overline{x}$ be such that

$$\Delta_2 = U^k_{d\,j}(\overline{x}) - V_{d\,j}(\overline{x}) = \max_{d\,j\,x}\{U^k_{d\,j}(x) - V_{d\,j}(x)\}.$$

The function $U^k_{d\,j}$ is given by

$$U^k_{d\,j}(x^j_{\mu}) = \min\left((\mathcal{L}_k U^k)_{d\,j}(x^j_{\mu}), (\mathcal{S}_k u^h)_{d\,j}(x^j_{\mu})\right) \forall x^j_{\mu} \in \mathcal{V}^{j,\,k}, \forall d \in D,\ \forall j \in \mathcal{J},$$

then

$$U^k_{d\,j}(x^j_{\mu}) \leq \frac{1}{1 + \widetilde{\alpha}_j\,h_{d\,j}}(U^k_{d\,j}(x^j_{\mu} + h_{d\,j}\,g(d,\,j)) + h_{d\,j}(\,f(x^j_{\mu}, d) + \sum_{i \neq j} \lambda_{j\,i}\ U^k_{d\,i}(\pi_{Q_{i\,k}}(x^j_{\mu}))))$$

$$
\begin{aligned}
(1+\widetilde{\alpha}_j\, h_{d\,j})U^k_{d\,j}(x^j_\mu) &\leq U^k_{d\,j}(x^j_\mu + h_{d\,j}\, g(d,\, j)) \\
&\quad + h_{d\,j}(\, f(x^j_\mu, d) + \sum_{i\neq j} \lambda_{j\,i}\ U^k_{d\,i}(\pi_{Q_{i\,k}}(x^j_\mu))) \\
\widetilde{\alpha}_j\, h_{d\,j}\, U^k_{d\,j}(x^j_\mu) &\leq U^k_{d\,j}(x^j_\mu + h_{d\,j}\, g(d,\, j)) - U^k_{d\,j}(x^j_\mu) \\
&\quad + h_{d\,j}(\, f(x^j_\mu, d) + \sum_{i\neq j} \lambda_{j\,i}\ U^k_{d\,i}(\pi_{Q_{i\,k}}(x^j_\mu))) \\
\widetilde{\alpha}_j\, h_{d\,j}\, U^k_{d\,j}(x^j_\mu) &\leq D_{d\,j}\, h_{d\,j} + h_{d\,j}(\, f(x^j_\mu, d) + \sum_{i\neq j} \lambda_{j\,i}\ U^k_{d\,i}(\pi_{Q_{i\,k}}(x^j_\mu))).
\end{aligned}
\tag{45}
$$

where $D_{d\,j} = \dfrac{U^k_{d\,j}(x^j_\mu + h_{d\,j}\, g(d,\, j)) - U^k_{d\,j}(x^j_\mu)}{h_{d\,j}}$ is the discrete derivative. The function $U^k_{d\,j}$ is affine along the edges (which coincide with segments of lines of the type $x^j_\mu + s\ g(d,\, j),\ s \geq 0$). So in (45) we can replace $h_{d\,j}$ by $\widehat{h}_{d\,j} \leq h_{d\,j}$ being $\widehat{h}_{d\,j}$ such that

$$
V_{d\,j}(\overline{x}) = \int_0^{\widehat{h}_{d\,j}} e^{-\widetilde{\alpha}_j\, s}(f(y\,(s)\,, d) + \sum_{i\neq j} \lambda_{j\,i} V_{d\,i}(y(s)))ds + V_{d\,j}(\overline{x} + \widehat{h}_{d\,j}\, g(d, j))\, e^{-\widetilde{\alpha}_j\ \widehat{h}_{d\,j}}.
$$

In consequence we have

$$
\widetilde{\alpha}_j\, \widehat{h}_{d\,j}\, U^k_{d\,j}(x^j_\mu) \leq D_{d\,j}\, \widehat{h}_{d\,j} + \widehat{h}_{d\,j}(\, f(x^j_\mu, d) + \sum_{i\neq j} \lambda_{j\,i}\, U^k_{d\,i}(\pi_{Q_{i\,k}}(x^j_\mu)))\,.
$$

Moreover there exists an x_i such that $x_i \in Q_{j\,k}$, $x_i \notin \left(\gamma^+_{h,d,j} \cup (\bigcup_{r\neq d} \gamma^-_{h,r,j})\right)$ and

$$
U^k_{d\,j}(\overline{x}) - U^k_{d\,j}(x_i) = O(h^\gamma_{d\,j})
$$

Then we have

$$
U^k_{d\,j}(\overline{x}) \leq (1-\widetilde{\alpha}_j\, \widehat{h}_{d\,j})\, U^k_{d\,j}(\overline{x}+\widehat{h}_{d\,j}\, g(d,\, j)) + \widehat{h}_{d\,j}(\, f(\overline{x}, d) + \sum_{i\neq j} \lambda_{j\,i}\, U^k_{d\,i}(\pi_{Q_{i\,k}}(\overline{x}))) + \widehat{h}_{d\,j} O(h^\gamma_{d\,j})
$$

$$
\begin{aligned}
\Delta_2 &= U^k_{d\,j}(\overline{x}) - V_{d\,j}(\overline{x}) \\
&\le \int_0^{\widehat{h}_{d\,j}} e^{-\widetilde{\alpha}_j\, s}(|f(\overline{x},d) - f(y\,(s)\,,d)| + \sum_{i\neq j} \lambda_{j\,i}\left|U^k_{d\,i}(\pi_{Q_{i\,k}}(\overline{x})) - V_{d\,i}(y(s))\right|)ds \\
&\quad +(1-\widetilde{\alpha}_j\;\widehat{h}_{d\,j})U^k_{d\,j}(\overline{x} + \widehat{h}_{d\,j}\,g(d,\,j)) - V_{d\,j}(\overline{x} + \widehat{h}_{d\,j}\,g(d,j))\,e^{-\widetilde{\alpha}_j\;\widehat{h}_{d\,j}} + \widehat{h}_{d\,j}O(h^\gamma_{d\,j}) \\
&\le \int_0^{h_{dj}} e^{-\widetilde{\alpha}_j\, s}(L_f\;\widehat{h}_{d\,j}\; + \sum_{i\neq j}\lambda_{j\,i}(\left|U^k_{d\,i}(\pi_{Q_{i\,k}}(\overline{x})) - V_{d\,i}(\pi_{Q_{i\,k}}(\overline{x}))\right| \\
&\quad + |V_{d\,i}(\pi_{Q_{i\,k}}(\overline{x})) - V_{d\,i}(\overline{x})| + |V_{d\,i}(\overline{x}) - V_{d\,i}(y(s))|))ds \\
&\quad +(1-\widetilde{\alpha}_j\;\widehat{h}_{d\,j})\Delta_2 + o(\widehat{h}_{d\,j}\,) + \widehat{h}_{d\,j}O(h^\gamma_{d\,j}) \\
&\le (L_f\;\widehat{h}_{d\,j}\; + \sum_{i\neq j}\lambda_{j\,i}(2L_V\;\widehat{h}^\gamma_{d\,j}\; + \Delta_1))\widehat{h}_{d\,j} + (1-\widetilde{\alpha}_j\;\widehat{h}_{d\,j})\Delta_1 + o(\widehat{h}_{d\,j}\,) + \widehat{h}_{d\,j}O(h^\gamma_{d\,j})\,.
\end{aligned}
$$

Therefore it results

$$
\Delta_2(1-\widehat{h}_{d\,j}\sum_{i\neq j}\lambda_{j\,i} - 1 + \widetilde{\alpha}_j\;\widehat{h}_{d\,j}) \le (L_f\,\widehat{h}_{d\,j} + \sum_{i\neq j}\lambda_{j\,i}L_V\;\widehat{h}^\gamma_{d\,j})\,\widehat{h}_{d\,j} + O(\widehat{h}^2_{d\,j}) + \widehat{h}_{d\,j}O(h^\gamma_{d\,j})
$$

$$
\Delta_2 \le \frac{1}{\alpha}\left(L_f\,\widehat{h}_{d\,j} + \sum_{i\neq j}\lambda_{j\,i}L_V\;\widehat{h}^\gamma_{d\,j}\right) + O(h^\gamma_{d\,j})\,.
$$

Similar to that we have done for Δ_1 we have $\forall x$, $\forall d$, $\forall j$

$$
U^k_{d\,j}(x) - V_{d\,j}(x) \le \frac{K_2}{\alpha}\left(L_f\,h + \sum_{i\neq j}\lambda_{j\,i}L_V\;h^\gamma\right) + O(h^\gamma)\,,
$$

then

$$
\left\|U^k - V\right\|_\infty \le \frac{K_2}{\alpha}\left(L_f\,h + \sum_{i\neq j}\lambda_{j\,i}L_V\;h^\gamma\right) + O(h^\gamma)\,.
$$

$\square$

5 Applications

We have applied the above presented numerical procedure to an example with $m = 2$ items, being the discount rate $\alpha = 0.1$.

Production rates: $p_1 = p_2 = 1$.

Demand's data: $\mathcal{J} = \{1, 2, 3, 4\}$, $J = 4$.

$r_{11} = 0.07415$	$r_{21} = 0.37230$
$r_{12} = 0.07415$	$r_{22} = 0.06741$
$r_{13} = 0.32300$	$r_{23} = 0.37230$
$r_{14} = 0.32300$	$r_{24} = 0.06741$

Commutation rates: λ_{ij}

$i\,/\,j$	1	2	3	4
1	0	0.03	0.031	0
2	0.2	0	0	0.031
3	0.2	0	0	0.03
4	0	0.2	0.2	0

Maximum stocks: $M_1 = 0.525$, $M_2 = 1.67$.

The instantaneous cost function is linear in both variables and it does not depend on the parameter d, i.e.

$$f(x_1, x_2) = 4x_1 + 5x_2$$

Commutation costs:

$d\,/\,\tilde{d}$	0	1	2
0		7	7
1	7		7
2	7	7	

<u>Computational times</u>: The program computes 12 tables of 600 independent values (the values of $u^h_{dj}(x_i)$, $j = 1, .., 4$; $d = 0, 1, 2$; $i = 1, .., 600$). It uses to do that computational task a time of 29,6 seconds in a PC Pentium 133 Mhz. In Figures

1-4 we show the results of simulation corresponding to the use of the sub-optimal control policy given by the computational procedure for an operation of 87 time units.

Conclusions

In this paper we have analyzed the problem of optimal scheduling of a production system comprising a multi-item single machine with piecewise deterministic demands.

We have obtained the solution as a feedback control in terms of the optimal cost function. We have established that this function is Hölder continuous. Using dynamic programming techniques, we have characterized the optimal cost function as the unique solution of a fixed point problem associated to the problem.

We have analyzed the structure of this problem, its solution and a numerical method of approximation. We have presented a discretization procedure for the numerical solution based on the finite element method. We have shown that the solution of the discrete problem is reduced to find the unique fixed point of a contractive non-linear operator $P^h : \mathbb{R}^n \to \mathbb{R}^n$ and we have given explicit error estimates. Finally, we have presented an application of this methodology of approximation.

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FIGURES

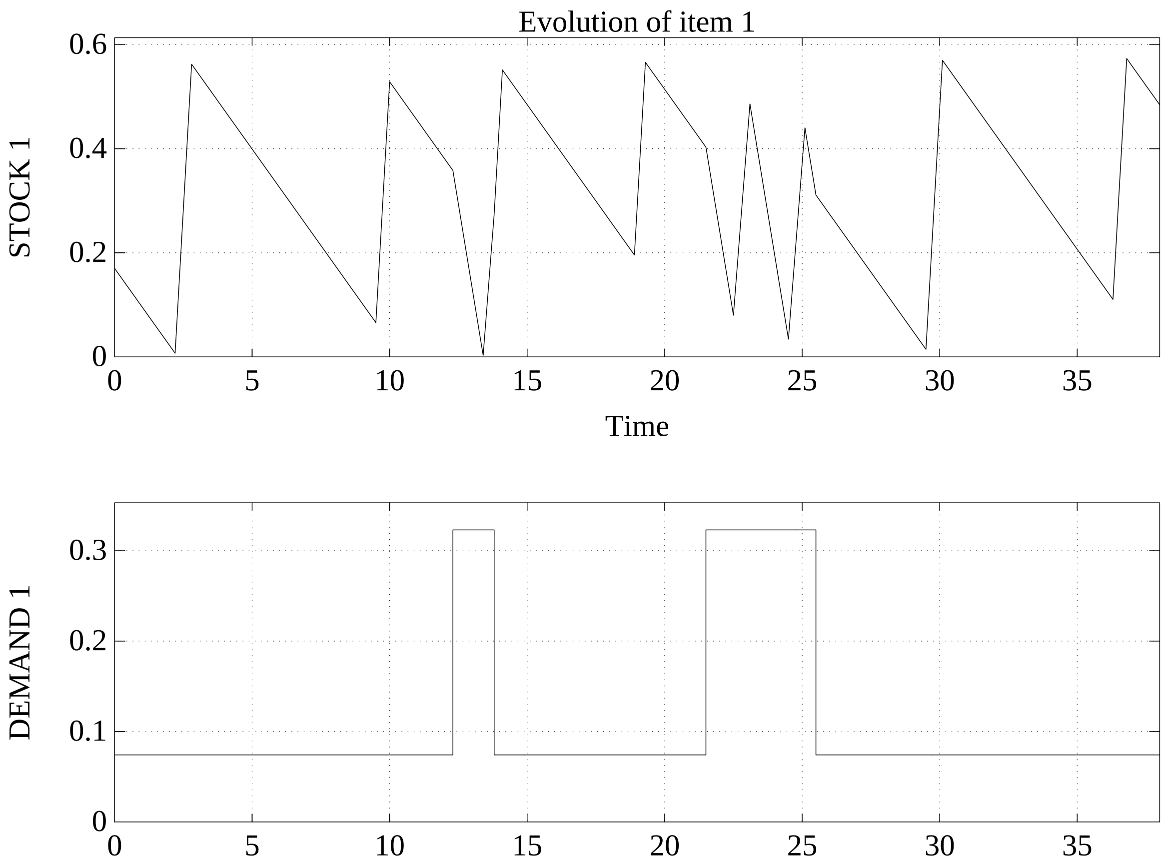

FIGURE 1

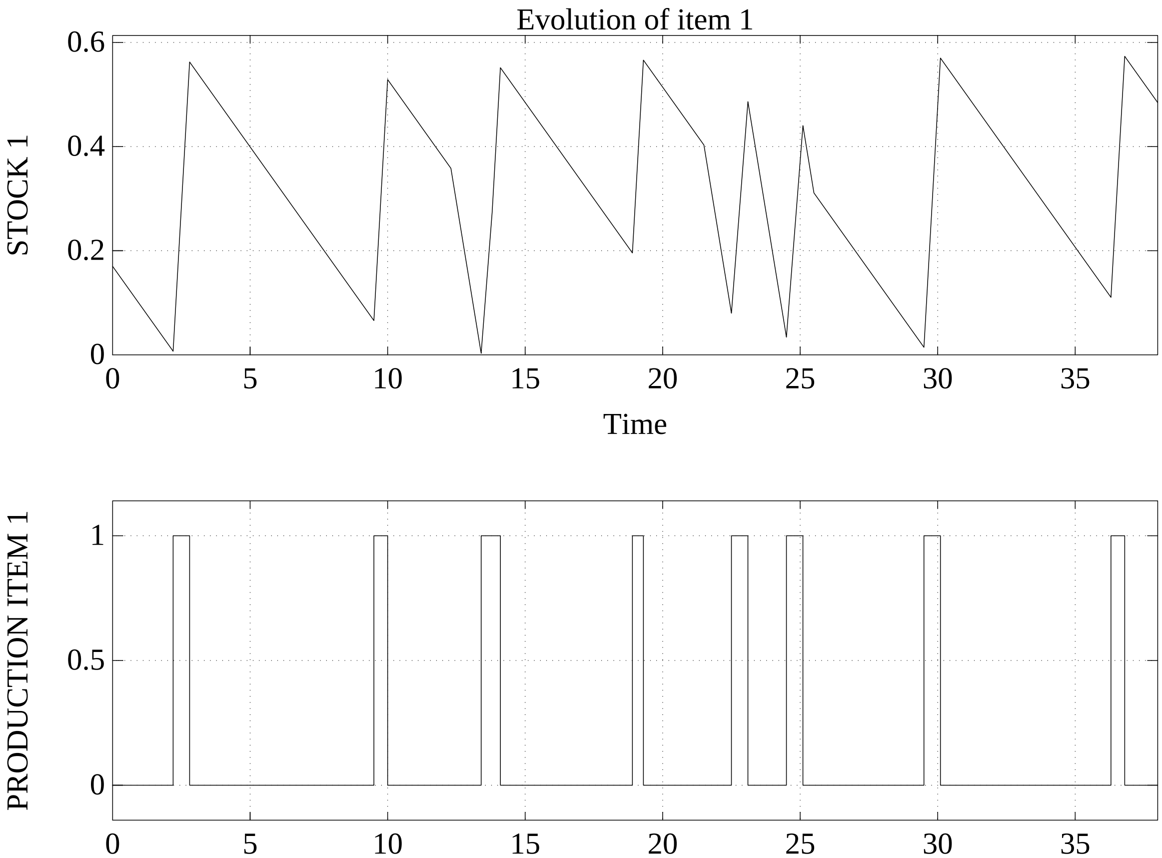
Evolution of item 1
STOCK 1
0.6
0.4
0.2
0
0
5
10
15
20
25
30
35
Time
PRODUCTION ITEM 1
1
0.5
0
0
5
10
15
20
25
30
35

FIGURE 2

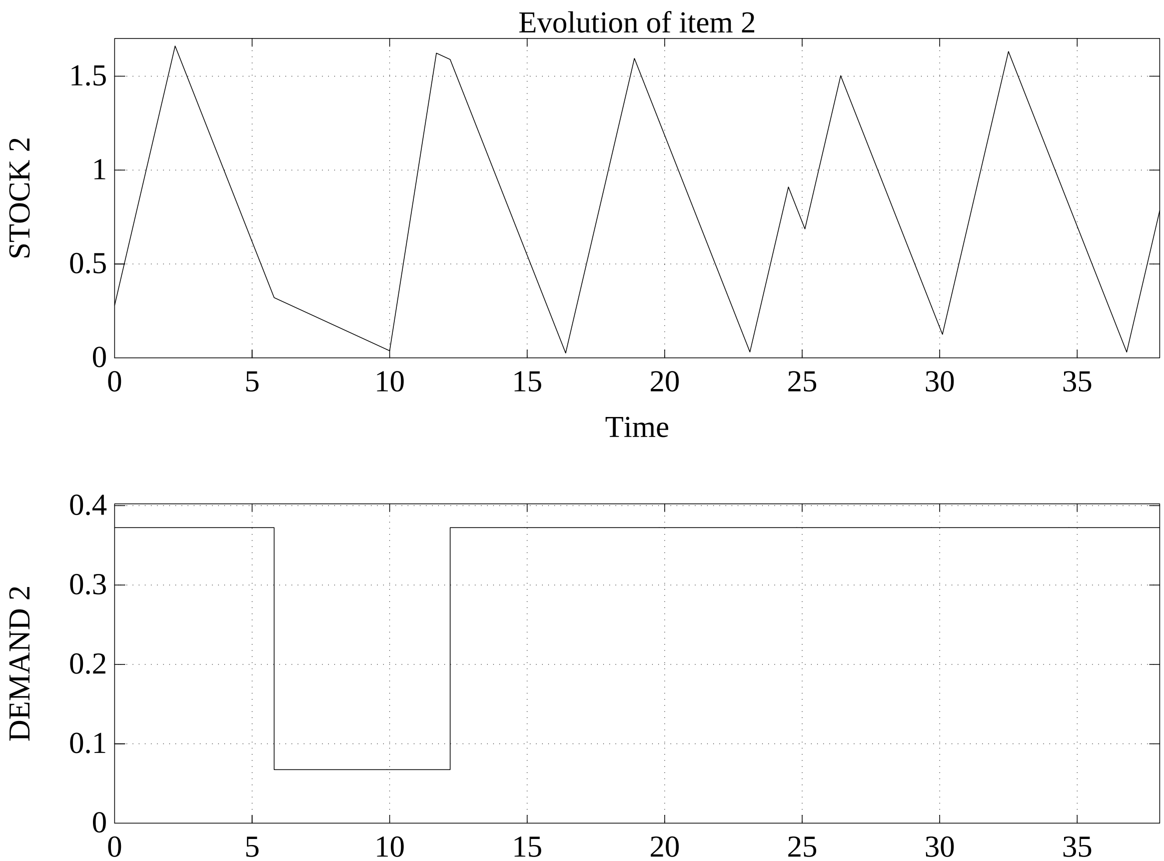
Evolution of item 2
STOCK 2
1.5
1
0.5
0
0
5
10
15
20
25
30
35
Time
DEMAND 2
0.4
0.3
0.2
0.1
0
0
5
10
15
20
25
30
35

FIGURE 3

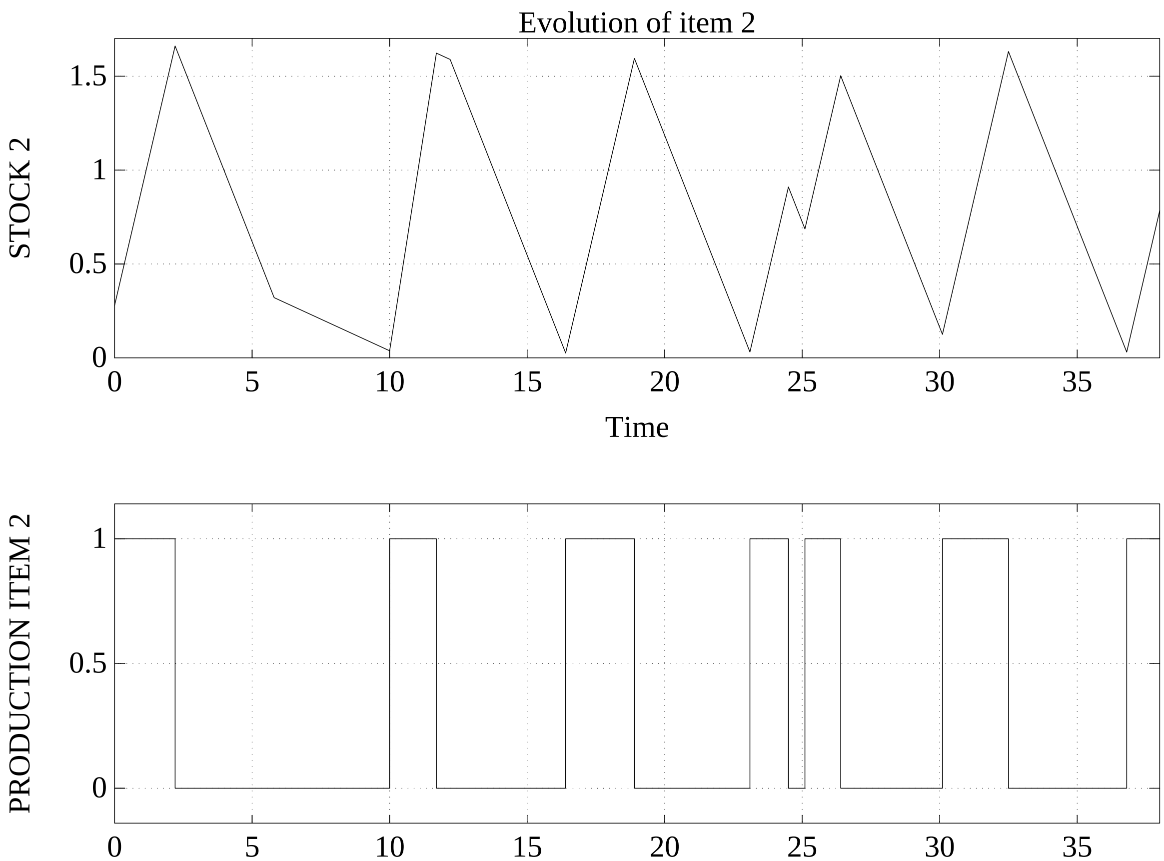
Evolution of item 2
STOCK 2
1.5
1
0.5
0
0
5
10
15
20
25
30
35
Time
PRODUCTION ITEM 2
1
0.5
0
0
5
10
15
20
25
30
35

FIGURE 4

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