Non Overlapping Domain Decomposition for Singularly Perturbed Elliptic Boundary Value Problems

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Abstract: We analyze the Funaro-Quarteroni alternative procedure for the solution of singular perturbation problems. We show that for an appropriate choice of the domain decomposition, one obtains a fast convergent iterative scheme with *no relaxation* that resolves the boundary layers. The convergence is superlinear with respect to the singular perturbation parameter ϵ in the following sense: the amplification factor is $o(\epsilon)$. We give sharp estimates of the interface position and convergent rates for an homogeneous domain decomposition in one dimensional space as well as in two dimensional space problems on a disk. We extend our results to heterogeneous domain decomposition arising in a simplified model of an electromagnetic problem. We report on implementation results with finite difference approximations and finite element codes (*Modulef*).

Key-words: singular perturbations, boundary layers, transition layers, asymptotic analysis, domain decomposition, iterative methods

(Résumé : tsvp)

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Méthode de décomposition de domaine sans recouvrement pour des problèmes elliptiques à perturbations singulières

Résumé : Nous analysons l'algorithme de Funaro-Quarteroni pour résoudre des problèmes de perturbations singulières. Nous montrons que pour un choix adapté de la méthode de décomposition de domaine, on obtient un schéma qui converge rapidement sans relaxation et qui permet de résoudre la couche limite. La convergence est superlinéaire, au sens ou le facteur d'amplification du schema est un $o(\epsilon)$, ϵ étant le paramètre de perturbation singulière. Nous donnons une estimation précise de la position optimale de l'interface entre le domaine correspondant à la couche limite et le domaine extérieur, et de la vitesse de convergence dans le cas d'une décomposition de domaine homogène en dimension 1 et en dimension 2 sur un disque. Nous étendons nos résultats au cas de la décomposition de domaine hétérogène que l'on observe dans un modèle simplifié d'électromagnétisme. Enfin, nous analysons les résultats obtenus pour une implémentation utilisant une approximation en *différences finies* puis en *éléments finis* (Modulef)

Mots-clé : perturbation singulière, couche limite, couche de transition, analyse asymptotique, décomposition de domaine, méthodes itératives

1 Introduction

The goal of this paper is to study the convergence rate of the Funaro Quarteroni algorithm ($F.Q.$) [5] applied to the homogeneous and heterogeneous domain decomposition of linear second order elliptic singular perturbation problem with zero order degeneracies. Solutions of singular perturbation problems ($S.P.P$) typically display rapid variations across narrow regions, the so-called *boundary layers*. It is then challenging to design a numerical scheme that gives a uniform approximation of the solution. Singular perturbation problems have been studied extensively in asymptotic analysis but the results of these studies have not yet been fully exploited to enhance numerical computations (see for example [4], [3], [11] and their references). Following the method in the papers [8] [6] [7] [9], we systematically use the concept of singular perturbation to design efficient and accurate domain decomposition solvers for these problems. First, the subdomains of computation correspond to the "*regular domains*" and the "*singular layers*" of the S.P.P. Next, the position of the interfaces depends on the small parameter ϵ, and is determined in such a way that the truncation error is asymptotically of the same order in each subdomain. Finally, the aspect ratio of the mesh changes drastically from a regular subdomain to a singular layer subdomain.
In previously published work [6] [7], we investigated a Schwarz alternating procedure implemented in its most elementary form. It was shown that the rate of convergence of the iterative procedure is superlinear when the domain decomposition is properly designed. Superlinear rate of convergence means here that the global convergence rate depends on ϵ and goes to zero when $\epsilon \to 0$. In this paper we demonstrate similar properties of the F.Q algorithm and emphazise the impact of the choice of the boundary conditions. We consider classical second order linear elliptic S.P.P with zero order degeneracies [3] and a transmission problem that arises in electromagnetic theory. We give rigorous convergence and accuracy estimates in the maximum norm in the finite difference framework but we apply our results to enhance finite element computations.

In Sect.2, we study in details homogeneous and heterogeneous domain decompositions in one space dimension. We generalize these results to a two dimensional space via a comparison lemma in Sect.3. We apply our main results in Sect 4 to enhance the computation of an elementary two dimensional singular perturbed transmission problem that arises in electromagnetic theory with *Modulef.*

2 Boundary layers in a One Dimensional Space

2.1 Homogeneous Domain Decomposition

In this section, we consider a linear second-order singular perturbation problem of the following type:

$$\begin{cases} L_\epsilon \phi = -\epsilon \phi'' + \phi = F \ in \ \Omega = (0,1); \\ \phi(0) = \alpha_0 \ ; \ \phi(1) = \alpha_1, \end{cases} \tag{2.1}$$

Where ϵ is a small positive parameter in $]0, \epsilon_0]$ for some $\epsilon_0 \ > 0$. Problems of this type exhibit boundary layers usually at both ends of the interval. This trivial one dimensional problem will be used as an illustration for our method. In particular we will show how properly design the domain decomposition in order to get fast convergence for the Quarteroni-Funaro iterative solver with *no relaxation* parameter.

We restrict ourselves to the case of a single boundary layer in the neighborhood of 1. According to the asymptotic analysis, we should split the domain Ω into two subdomains $\Omega_{inner} = (a,1)$ and $\Omega_{outer} = (0,a)$ where $a > 0$. Ω_{inner} covers the boundary layer at 1 and Ω_{outer} covers the domain of validity of the regular approximation.

In order to easily get sharp estimates in the maximum norm, we are going to use the finite difference framework.

We keep the mesh regular in each subdomain and adapt the domain decomposition to the boundary layer stiffness. Let us denote h_1 (respt. h_2) the mesh size on Ω_{outer} (respt Ω_{inner}). Let us denote by $L^{h_i}, \ i = 1,2$ the discretized operator that corresponds to L_ϵ. We will also restrict ourselves to the case where we have the same asymptotic order of grid points in each subdomain, i.e

$$h_1 \approx \frac{h_2}{1-a} \approx \frac{1}{N}, \tag{2.2}$$

in order to balance the amount of work in each subdomain.

2.1.1 Dirichlet-Neumann scheme

We introduce the following iterative procedure [5] in order to solve(2.1)

$$\begin{cases} L^{h_1}\phi^p_{outer} = F \ in \ \Omega_{outer}; \\ \phi^p_{outer}(0) = \alpha_0 \ ; \ \phi^p_{outer}(a) = \phi^p_{inner}(a) \\ L^{h_2}\phi^{p+1}_{inner} = F \ in \ \Omega_{inner}; \\ \phi^{p+1}_{inner}(0) = \alpha_1 \ ; \\ \dfrac{\phi^{p+1}_{inner}(a+h_2) - \phi^{p+1}_{inner}(a)}{h_2} = \dfrac{\phi^p_{outer}(a) - \phi^p_{outer}(a-h_1)}{h_1}. \end{cases} \tag{2.3}$$

To start the scheme, we impose an artificial boundary condition at the point a. We use the same finite difference scheme in each subdomain with Dirichlet boundary condition at a in Ω_{outer} and Neumann boundary condition at a in Ω_{inner}. We will consider in the next section the other possible choice, i.e Neumann boundary condition at a in Ω_{outer} and the Dirichlet boundary condition at a in Ω_{inner}.

We will decompose the analysis of this iterative method in *three steps*: first we define the best interface location between the subdomains, based on a truncation error analysis, secondly we derive from the stability property of the discretized operator the rate of damping of the artificial boundary condition error. Lastly we combine these two results to get an estimate of convergence of the iterative solver to the *exact* solution of the differential problem (2.1).

The technique of demonstration is quit elementary but plays with two types of small parameters: first the space steps (2.2), second the small singular perturbation parameter ϵ. Our goal is to find the best path in the parameter space (h, ϵ) which provides superlinear convergence and optimal uniform approximation.

- **First Step: interface position**

We wish to determine the optimal interface position a, which minimizes the maximum error in both subdomains under the constraint that we have the same asymptotic order of mesh points N inside each subdomain. In this part, we neglect the artificial boundary condition error inherent to the Funaro-Quarteroni alternate $(F.Q)$ procedure. This error will be taken care of later on.

Let ϕ_{outer}(respt ϕ_{inner}) be the restriction of ϕ to Ω_{outer} (respt Ω_{inner}). We define the following errors :

$$e_{outer} = \max_{\Omega_{outer}} |\phi_{outer} - \phi^{h_1}_{outer}|$$
$$e_{inner} = \max_{\Omega_{inner}} |\phi_{inner} - \phi^{h_2}_{inner}|.$$

A centered-order finite difference scheme applied to $-\epsilon u'' + u = f$ with exact Dirichlet boundary conditions gives

$$e_{outer} \approx \epsilon h_1^2 a^2 \max_{\Omega_{outer}} |\frac{d^{(4)}\phi}{dx^4}|. \qquad (2.4)$$

The analysis of the inner subdomain approximation with mixed exact boundary conditions gives two truncation errors, which we should consider in addition the discretisation error of the Neumann boundary condition. We have

$$e_{inner} \approx \epsilon h_2^2 (1-a)^2 \max_{\Omega_{inner}} |\frac{d^{(4)}\phi}{dx^4}| + h_2^2 \frac{R}{1-R} \max_{\Omega_{inner}} |\frac{d^{(2)}\phi}{dx^2}|, \qquad (2.5)$$

where $R = 1 + \frac{h_2^2}{2\epsilon} + \frac{h_2}{\sqrt{\epsilon}}(1 + \frac{h_2^2}{2\epsilon})^{\frac{1}{2}}$.

We first notice that the truncation errors defined above depend strongly on the property of the solution that we want to approximate in each subdomain.

Let ϕ_0 be the outer expansion of ϕ and $\Theta(x,\epsilon)$ be the corrector i.e

$$\Theta(x,\epsilon) = \phi(x,\epsilon) - \phi_0(x,\epsilon) \approx exp(-\eta),$$

in the boundary layer with $\eta = \frac{1-x}{\epsilon}$. We will show that the truncation error is dominated by the behavior of the corrector as in ([6]).

Secondly we remark that the error in both subdomains is coupled because the Neumann boundary condition for the *inner* domain is only an approximation of a derivative in the *outer* domain. So we need to compute directly the error between the exact solution of the continuous problem and the formal limit of (2.3) when $p \to \infty$. We have:

Lemma 1 *Let $\tilde{\phi} = (\tilde{\phi}_{i,j})_{i=0,\dots N, j=1,2}$ be the solution of the following linear system.*

$$\left\{\begin{array}{l} L^{h_1}\tilde{\phi}_{i,1} = F \quad i = 1,\dots N-1 \\ L^{h_2}\tilde{\phi}_{i,2} = F \quad i = 1,\dots N-1 \\ \tilde{\phi}_{0,1} = \alpha_0 \ ; \ \tilde{\phi}_{N,1} = \tilde{\phi}_{0,2}; \ \tilde{\phi}_{N,2} = \alpha_1 \\ \dfrac{\tilde{\phi}_{1,2} - \tilde{\phi}_{0,2}}{h_2} = \dfrac{\tilde{\phi}_{N,1} - \tilde{\phi}_{N-1,1}}{h_1}, \end{array}\right. \tag{2.6}$$

where

$$L^h\phi_i = -\epsilon\frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{h} + \phi_i.$$

Let M be the composit grid $M = M_{outer} \cup M_{inner}$, with

$$\left\{\begin{array}{l} M_{outer} = \{x_{i,1} = i(\frac{a}{N}); i = 0,\dots N\} \\ M_{inner} = \{x_{i,2} = a + i(\frac{1-a}{N}); i = 0,\dots N\} \end{array}\right.$$

Let us suppose that

$$N^{-1} \approx \sqrt{\epsilon}\delta \qquad with \qquad \delta \succ\succ 1.$$

Let $\|\ \|_\infty$ be the maximum norm on the composite grid M.

Under the previous hypothesis on the discretization and approximation of the operators in each subdomain, $\|\ \phi - \tilde{\phi}\|_\infty$ is asymptotically minimum when

$$1 - a \sim \sqrt{\epsilon}\log(\epsilon^{-2}).$$

Proof: The existence and uniqueness of $\tilde{\phi}$ follows from the maximum principle. Let ϕ be the exact solution of the Dirichlet problem: $-\epsilon\phi'' + \phi = F$; $\phi(0) = \alpha_0$; $\phi(1) = \alpha_1$.

Let $e_{i,j} = \phi_{i,j} - \tilde{\phi}_{i,j} \quad i = 0,\dots N, j = 1,2$, where $\phi_{i,j}$ is the trace of ϕ on the composit grid $M_{outer} \cup M_{inner}$.

We have

$$\left\{\begin{array}{l} -\epsilon \dfrac{e_{i+1,1}-2e_{i,1}+e_{i-1,1}}{h_1^2}+e_{i,1}=-\epsilon \dfrac{h_1^2}{12}\phi^{(4)}(\xi_i), \\ -\epsilon \dfrac{e_{i+1,2}-2e_{i,2}+e_{i-1,2}}{h_2^2}+e_{i,2}=-\epsilon \dfrac{h_2^2}{12}\phi^{(4)}(\eta_i), \\ e_{0,1}=0 \ ; \ e_{N,2}=0, \\ e_{N,1}=e_{0,2}, \\ \dfrac{e_{1,2}-e_{0,2}}{h_2}=\dfrac{e_{N,1}-e_{N-1,1}}{h_1}+\dfrac{h_1}{2}\phi^{(2)}(\tilde{\xi})+\dfrac{h_2}{2}\phi^{(2)}(\tilde{\eta}), \end{array}\right.$$

where

$$x_{i-1,1}<\xi_i<x_{i,1};\ x_{i-1,2}<\eta_i<x_{i,2};\ x_{N-1,1}<\tilde{\xi}<x_{N,1};\ x_{0,2}<\tilde{\eta}<x_{1,2}.$$

We split the error $e_{i,j}$ into two componants $\tilde{e}_{i,j}$ and $\hat{e}_{i,j}$ solutions of the following linear systems:

$$\left\{\begin{array}{l} -\epsilon \dfrac{\tilde{e}_{i+1,1}-2\tilde{e}_{i,1}+\tilde{e}_{i-1,1}}{h_1^2}+\tilde{e}_{i,1}=-\epsilon \dfrac{h_1^2}{12}\phi^{(4)}(\xi_i), \\ -\epsilon \dfrac{\tilde{e}_{i+1,2}-2\tilde{e}_{i,2}+\tilde{e}_{i-1,2}}{h_2^2}+\tilde{e}_{i,2}=-\epsilon \dfrac{h_2^2}{12}\phi^{(4)}(\eta_i), \\ \tilde{e}_{0,1}=0 \ ; \ \tilde{e}_{N,2}=0 \\ \tilde{e}_{N,1}=\tilde{e}_{0,2} \\ \dfrac{\tilde{e}_{1,2}-\tilde{e}_{0,2}}{h_2}=\dfrac{\tilde{e}_{N,1}-\tilde{e}_{N-1,1}}{h_1}, \end{array}\right.$$

$$\left\{\begin{array}{l} -\epsilon \dfrac{\hat{e}_{i+1,1}-2\hat{e}_{i,1}+\hat{e}_{i-1,1}}{h_1^2}+\hat{e}_{i,1}=0, \\ -\epsilon \dfrac{\hat{e}_{i+1,2}-2\hat{e}_{i,2}+\hat{e}_{i-1,2}}{h_2^2}+\hat{e}_{i,2}=0, \\ \hat{e}_{0,1}=0 \ ; \ \hat{e}_{N,2}=0, \\ \hat{e}_{N,1}=\hat{e}_{0,2}, \\ \dfrac{\hat{e}_{1,2}-\hat{e}_{0,2}}{h_2}=\dfrac{\hat{e}_{N,1}-\hat{e}_{N-1,1}}{h_1}+\dfrac{h_1}{2}\phi^{(2)}(\tilde{\xi})+\dfrac{h_2}{2}\phi^{(2)}(\tilde{\eta}). \end{array}\right.$$

If $|\tilde{e}_{i,1}|$ is not maximal at the boundary $i=N$ then

$$\max_{i=0,\ldots N}|\tilde{e}_{i,1}|\leq E_1=C\epsilon h_1^2\max\{\phi^{(4)}\}$$

Otherwise, we deduce from

$$e_{1,2} - e_{0,2} = \frac{h_2}{h_1}(e_{N,1} - e_{N-1,1}),$$

that $|e_{i,2}|$ cannot be maximal at the boundary $i = 0$. We have therefore

$$\max |e_{i,2}| \leq E_2 = C(1-a)^2 \epsilon h_2^2 \max\{\phi^{(4)}\}.$$

Consequently, we have:

$$\|\tilde{e}\|_\infty \leq \max(E_1, E_2).$$

Now let us consider the error that comes from the discretization of the Neumann boundary condition.
From the maximum principle, we have $|\hat{e}_{i,1}|$ bounded by $|\hat{e}_{N,1}| = |\hat{e}_{0,2}|$. In addition ([6]-lemme 4):

$$\frac{\hat{e}_{N,1} - \hat{e}_{N-1,1}}{h_1} \approx \frac{\hat{e}_{0,2}}{h_1}.$$

We look for $\hat{e}_{i,2}$ in the following form:

$$\hat{e}_{i,2} = C_1 R^i + C_2 R^{-i},$$

where $R(h_2, \epsilon)$ is the larger root of the quadratic polynomial

$$R^2 - (2 + \frac{h_2^2}{\epsilon})R + 1 = 0.$$

We obtain C_1 and C_2 from the boundary conditions. From this explicit formula we can conclude that $\hat{e}_{i,2}$ is a decreasing function of i and that:

$$\hat{e}_{0,2} \approx E_3 = \sqrt{\epsilon}\frac{h_1}{2}\phi^{(2)}(b).$$

We have finally:

$$\|e\|_\infty \leq \max(E_1, E_2, E_3), \quad \textit{with} \tag{2.7}$$

$$E_1 \approx \epsilon h_1^2 (1 + \epsilon^{-2} \exp(-\frac{b}{\sqrt{\epsilon}}), \tag{2.8}$$

$$E_2 \approx \epsilon^{-1} h_1^2 b^4, \tag{2.9}$$

$$E_3 \approx \sqrt{\epsilon} h_1 (1 + \epsilon^{-1} \exp(-\frac{b}{\sqrt{\epsilon}}). \tag{2.10}$$

Hence, if ϵ is sufficiently small then E_2 is an increasing function of b and E_1 as well as E_3 are decreasing functions of b. It is easy to show that the accuracy is best when

$$b \sim \sqrt{\epsilon} \log \epsilon^{-2}.$$

The error is then

$$\|e\|_\infty \approx \epsilon\delta. \tag{2.11}$$

∎

- **Second Step: Damping of artificial boundary errors**

The convergence of the method depends essentially on the way an error which is introduced at the artificial interface propagates inside the subdomain.

The Schwarz iterate procedure applied to the two points boundary value problem $-u^{"} = F,\ u(0) = \alpha_0,\ u(1) = \alpha_1$, for example, has very poor efficiency because for $[a, b] \subset [0, 1]$ any given subdomain of [0,1] used in the Schwarz iterative procedure, any perturbation of the boundary condition at b decreases linearly inside the subdomain [a,b]. It is no longer the case for a second order singular perturbation problem that have boundary layers with fast exponentially decay. In [6] it is shown that we may, in fact, have fast convergence with relatively small overlap even if we apply the straightforward Schwarz alternate procedure with Dirichlet boundary conditions. We will proove that the F.Q procedure may also have fast convergence and we will compare it with the Schwarz alternating procedure.

Let us consider the F.Q iterative procedure applied to the following homogeneous problem

$$\left\{ \begin{array}{c} L_1^h e_{i,1}^p = 0\ ; \\ e_{0,1}^p = 0\ ;\ e_{N,1}^p = e_{0,2}^{p-1}\ ; \\ L_2^h e_{N,2}^p = 0 \\ \dfrac{e_{1,2}^p - e_{0,2}^p}{h_2} = \dfrac{e_{N,1}^p - e_{N-1,1}^p}{h_1}\ ; \\ e_{N,2}^p = 0\ ; \end{array} \right.$$

with a domain decomposition given by (Lemma 1), i.e $b = 1 - a \approx \sqrt{\epsilon} \log \epsilon^{-1}$. The discretized operator satisfies a maximum principle and we can show that:

$$|e_{0,2}^p| = |e_{N,1}^{p+1}| \geq max_{i=0,\ldots N-1} |e_{i,1}^{p+1}|$$

and

$$|e^p_{0,2}| \geq max_{i=0,\dots N}|e^p_{i,2}|.$$

We will call damping factor a real ξ such that: $|e^{p+1}_{0,2}| \leq \xi |e^p_{0,2}|$, $\forall p$.

Lemma 2 *Let a be the interface position between the subdomains such that $1-a \approx \epsilon^{1/2}\log\epsilon^{-1}$. Suppose that $N^{-1} \approx \epsilon^{\frac{1}{2}}\delta$,. with $\delta \succ\succ 1$.*
Then the amplification factor of the iterative scheme is:

$$\xi \approx \delta^{-1}.$$

Proof: let $\tilde{\phi}^p = (\tilde{\phi}^p_{i,j})_{i=0,\dots N, j=1,2}$ be the solution of

$$\begin{cases} L^{h_1}\tilde{\phi}^p_{i,1} = F \quad i = 1, \dots N-1, \\ L^{h_2}\tilde{\phi}^p_{i,2} = F \quad i = 1, \dots N-1, \\ \tilde{\phi}^p_{0,1} = \alpha_0 \ ; \ \tilde{\phi}^p_{N,1} = \tilde{\phi}^{p-1}_{0,2} \ ; \ \tilde{\phi}^p_{N,2} = \alpha_1, \\ \dfrac{\tilde{\phi}^p_{1,2} - \tilde{\phi}^p_{0,2}}{h_2} = \dfrac{\tilde{\phi}^p_{N,1} - \tilde{\phi}^p_{N-1,1}}{h_1}, \end{cases}$$

Let $(V^p_{i,j})_{i=0,\dots N, j=1,2}$ defined as $V^p_{i,j} = \tilde{\phi}^p_{i,j} - \tilde{\phi}_{i,j}$ where $\tilde{\phi} = (\tilde{\phi}_{i,j})_{i=0,\dots N, j=1,2}$ is the solution of (2.6).

We have:

$$\begin{cases} L^{h_1}V^p_{i,1} = 0 \quad i = 1, \dots N-1; \\ L^{h_2}V^p_{i,2} = 0 \quad i = 1, \dots N-1; \\ V^p_{0,1} = 0 \ ; \ V^p_{N,1} = V^{p-1}_{0,2} \ ; \ V^p_{N,2} = 0; \\ \dfrac{V^p_{1,2} - V^p_{0,2}}{h_2} = \dfrac{V^p_{N,1} - V^p_{N-1,1}}{h_1}. \end{cases}$$

Let $\xi_\leftarrow$ be the damping factor for the outer domain with Dirichlet boundary condition defined as in [6], we have $V^p_{N-1,1} = \xi_\leftarrow V^p_{N,1}$ with $\xi_\leftarrow \prec\prec 1$. Then

$$V^p_{1,2} - V^p_{0,2} = \frac{h_2}{h_1}(V^p_{N,1} - V^p_{N-1,1}) = \frac{h_2}{h_1}(1-\xi_\leftarrow)V^p_{N,1} = \frac{h_2}{h_1}(1-\xi_\leftarrow)V^{p-1}_{0,2}. \qquad (2.12)$$

Let K be the RHS of (2.12), we will show that $(V^p_{0,2})_p$ is decreasing in module. $V^p_{i,2}$ can be writed as

$$V^p_{i,2} = C^p_1 R^i(h_2,\epsilon) + C^p_2 R^{-i}(h_2,\epsilon),$$

where $R(h_2, \epsilon)$ is the larger root of the quadratic polynomial

$$R^2 - (2 + \frac{h_2^2}{\epsilon})R + 1 = 0$$

From the boundary conditions, we obtain C_1^p and C_2^p

$$\begin{cases} C_2^p = -C_1^p R^{2N}(h_2, \epsilon) \\ C_1^p = K/(R(h_2, \epsilon) - 1)(1 + R^{2N-1}(h_2, \epsilon)) \end{cases}$$

We can write

$$\begin{aligned} V_{0,2}^p &= \frac{1 - R^{2N}(h_2, \epsilon)}{1 + R^{2N}(h_2, \epsilon)} \times \frac{K}{R(h_2, \epsilon) - 1} \\ &= \xi \times V_{0,2}^{p-1}, \end{aligned}$$

where ξ is *the amplificator factor of the iterative method.* Consequently

$$\xi = \frac{h_2}{h_1}(1 - \xi_{\leftarrow}) \frac{1 - R^{2N}(h_2, \epsilon)}{1 + R^{2N-1}(h_2, \epsilon)} \times \frac{1}{R(h_2, \epsilon) - 1}$$

From $N \approx \epsilon^{\frac{1}{2}} \delta$, we deduce the asymptotic behavior of ξ which is

$$\xi \approx \delta^{-1}.$$

∎

- **third step: Convergence to the solution of the ODE problem and Uniform approximation**

Theorem 1 *Let ϕ be the solution of the Dirichlet problem*

$$L[\phi] = -\epsilon\phi'' + \phi = F \; ; \; \phi(0) = \alpha_0, \; \phi(1) = \alpha_1$$

Let ϕ_{outer}^p and ϕ_{inner}^p defined by the iterative scheme:

$$\begin{cases} L^{h_1}\phi_{outer}^p = F \; in \; \Omega_{outer}, \\ \phi_{outer}^p(0) = \alpha_0 \; ; \; \phi_{outer}^p(a) = \phi_{inner}^p(a), \\ L^{h_2}\phi_{inner}^{p+1} = F \; in \; \Omega_{inner}, \\ \phi_{inner}^{p+1}(0) = \alpha_1 \; , \\ \dfrac{\phi_{inner}^{p+1}(a + h_2) - \phi_{inner}^{p+1}(a)}{h_2} = \dfrac{\phi_{outer}^p(a) - \phi_{outer}^p(a - h_1)}{h_1}, \end{cases}$$

where

$$L^h[\phi] = -\epsilon \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{h^2} + \phi_i.$$

Let

$$\phi^p = \begin{cases} \phi^p_{outer} & on \quad M_{outer} = \{x_{i,1} = i(\frac{a}{N}); i = 0, \dots N\} \\ \phi^p_{inner} & on \quad M_{inner} = \{x_{i,2} = a + i(\frac{1-a}{N}); i = 0, \dots N\} \end{cases}$$

Let $\| \|_\infty$ *be the maximum norm on the composite grid* $M_{outer} \cup M_{inner}$.
Let us suppose that

$$N^{-1} \approx \sqrt{\epsilon}\delta \quad with \quad \delta \succ\succ 1 \qquad and \qquad b \sim \sqrt{\epsilon}\log(\epsilon^{-2}).$$

Then

$$\|\phi - \phi^p\|_\infty \leq C(\xi^p + \epsilon\delta)), \ \xi \approx \delta^{-1}. \tag{2.13}$$

Proof: We have shown in Lemma 1 that $e_{i,j} = \phi_{i,j} - \tilde{\phi}_{i,j} \quad i = 0, \dots N, j = 1, 2,$ where ϕ is the exact solution of the Dirichlet problem (2.1), $\phi_{i,j}$ is the trace of ϕ on the composit grid $M_{outer} \cup M_{inner}$ and $\tilde{\phi} = (\tilde{\phi}_{i,j})_{i=0,\dots N, j=1,2}$ solution of (2.6) satisfies (2.11) .

Let $\tilde{\phi}_{outer} = (\tilde{\phi}_{i,1})_{i=0,\dots N}$ (respt $\tilde{\phi}_{inner} = (\tilde{\phi}_{i,2})_{i=0,\dots N}$) and $V^p_{outer} = \phi^p_{outer} - \tilde{\phi}_{outer}$ (respt $V^p_{inner} = \phi^p_{inner} - \tilde{\phi}_{inner}$.)
We have

$$\begin{cases} L^{h_1}(V^p_{outer}) = 0, & \text{on } \Omega_{outer} \\ V^p_{outer}(a) = V^{p-1}_{inner}(a) \quad ; \quad V^p_{outer}(0) = 0 \\ L^{h_2}(V^p_{inner}) = 0, & \text{on } \Omega_{inner} \\ V^p_{inner}(1) = 0 \; ; \\ \dfrac{V^p_{inner}(a + h_2) - V^p_{inner}(a)}{h_2} = \dfrac{V^p_{outer}(a) - V^p_{outer}(a - h_1)}{h_1}. \end{cases}$$

We conclude from lemma 2 that:

$$\|V^p_{outer}\|_\infty \leq C\,\xi^p \text{ on } \mathrm{M_{outer}}, \ \|\mathrm{V^p_{inner}}\|_\infty \leq \mathrm{C}\,\xi^{\mathrm{p}} \text{ on } \mathrm{M_{inner}},$$

and therefore

$$\|\phi - \phi^p\|_\infty \leq C(\xi^p + \epsilon\delta).$$

∎

According to the estimate of (1), we observe that p must be choosen such that

$$\xi^p \approx \max(e_{outer}, e_{inner}).$$

It is interesting to notice that the convergence speed of the F.Q procedure is not as good as the convergence speed of the straightforward Schwarz alternate procedure with minimum overlap. For $N^{-1} \approx \sqrt{\epsilon}\log(\epsilon^{-1})$, we have $\xi \approx \epsilon log^{-2}(\epsilon^{-1})$ ([6], Theorem 1), instead of $\xi \approx log^{-1}(\epsilon^{-1})$. But on the other side the F.Q iterative procedure is a *non-overlapping* domain decomposition method.

Let us apply the present Dirichlet-Neumann scheme to the simple model problem

$$-\epsilon u'' + u = (1+\epsilon)\cos(x), x \in (0,1),\ u(0) = 1,\ u(1) = 1 + cos(1). \tag{2.14}$$

Figure 1 shows the dependence of the error in maximum norm as a function of the interface position with two times 20 discretization points. We have checked that the optimal interface position corresponds to the balance of the error in both sub-domains. Figure 2 demonstrates that the smaller is ϵ, the faster is the convergence

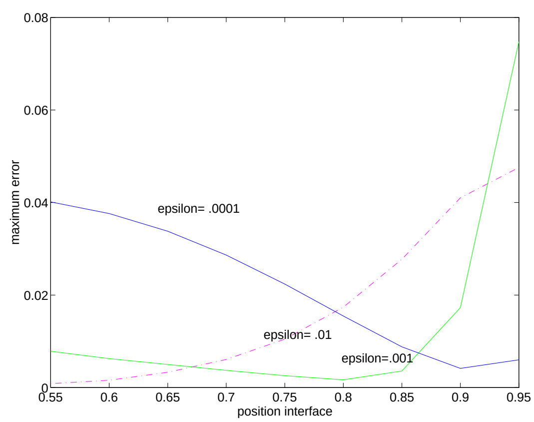

Figure 1: Effect of position interface on maximum error

of the iterative procedure toward the solution of the discretized problem. We have completed our numerical investigation with the original F.Q. scheme which includes a relaxation on the interface condition [5]; Figure 3 shows the effect of this relaxation and we observe that the optimal relaxation factor goes to one as $\epsilon \to 0$.

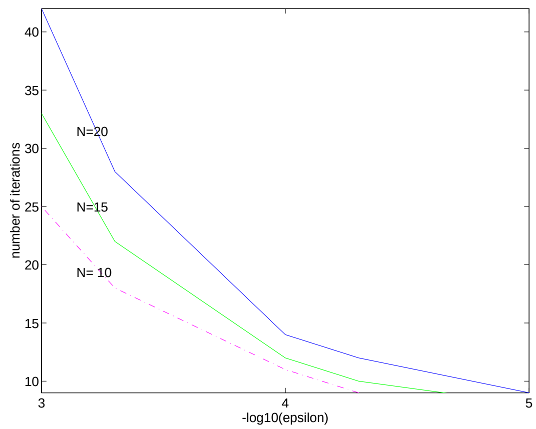

Figure 2: number of iterations (with no relaxation)

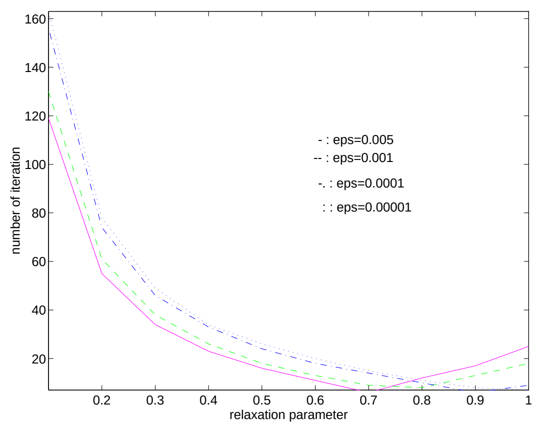

Figure 3: effect of relaxation parameter on number of iterations

2.1.2 Neumann-Dirichlet scheme

We are going to show that the choice of the boundary conditions at the artificial interface a is critical. Let us consider now the F.Q method with Neumann boundary condition at a in Ω_{outer} and the Dirichlet boundary condition at a in Ω_{inner}. The scheme gives:

$$\left\{\begin{array}{l} L^{h_1}\phi^p_{outer} = F \text{ on } \Omega_{\text{outer}}, \\ \dfrac{\phi^{p+1}_{outer}(a) - \phi^{p+1}_{outer}(a-h_1)}{h_1} = \dfrac{\phi^p_{inner}(a+h_2) - \phi^p_{inner}(a)}{h_2} \ ; \ \phi^p_{outer}(0) = \alpha_0 \ , \\ L^{h_2}\phi^p_{inner} = F \text{ on } \Omega_{\text{inner}}, \\ \phi^p_{inner}(1) = \alpha_1 \ ; \ \phi^p_{inner}(a) = \phi^p_{outer}(a). \end{array}\right. \tag{2.15}$$

To start the scheme, we impose an artificial boundary condition at point a. The best choice for the interface position in terms of accuracy is $1 - a \approx \sqrt{\epsilon}\log\epsilon^{-1}$ since the formal limit of (2.15) is identique to the formal limit of (2.3). But, we are going to show that this procedure is then highly unstable.

Theorem 2 *Let us assume that $h_1 \succ\succ h_2$ and $h_1 \succ\succ \sqrt{\epsilon}$.*
Then the amplificator factor of the iterative procedure satisfies

$$\xi \sim \frac{h_1}{h_2},$$

and the F.Q procedure with no relaxation is highly unstable.

Proof: We only need to consider the F.Q procedure applied to the problem with homogeneous boundary condition:

$$\left\{\begin{array}{l} L^{h_1}V^p_{i,1} = 0 \quad i = 1,\dots N-1, \\ L^{h_2}V^p_{i,2} = 0 \quad i = 1,\dots N-1, \\ V^p_{0,1} = 0 \ ; \ V^p_{0,2} = V^p_{N,1} \ ; \ V^p_{N,2} = 0, \\ \dfrac{V^p_{N,1} - V^p_{N-1,1}}{h_1} = \dfrac{V^{p-1}_{1,2} - V^{p-1}_{0,2}}{h_2}. \end{array}\right.$$

We have

$$V^p_{1,2} = \xi_{\leftarrow} V^p_{0,2}$$

with $\xi_{\leftarrow} \prec\prec 1$. Then

$$V^p_{N,1} - V^p_{N-1,1} = \frac{h_1}{h_2}(V^p_{1,2} - V^p_{0,2}) = \frac{h_1}{h_2}(1-\xi_{\leftarrow})V^{p-1}_{0,2} = \frac{h_1}{h_2}(1-\xi_{\leftarrow})V^{p-1}_{N,1}. \tag{2.16}$$

Let K be the RHS of (2.16), we will show that $(V^p_{N,1})_p$ is divergente. $V^p_{i,1}$ can be writed as

$$V^p_{i,1} = C^p_1 R^i(h_1,\epsilon) + C^p_2 R^{-i}(h_1,\epsilon)$$

where $R(h_1, \epsilon)$ is the larger root of the quadratic polynomial

$$R^2 - (2 + \frac{h_1^2}{\epsilon})R + 1 = 0.$$

From the boundaries conditions we determine C_1^p and C_2^p; we have

$$V_{N,1}^p = \frac{1 - R^{2N}(h_1, \epsilon)}{1 + R^{2N}(h_1, \epsilon)} \times \frac{K}{R(h_1, \epsilon) - 1} V_{0,2}^p = \xi V_{N,1}^{p-1}.$$

ξ *the amplificator factor* of the iterative methode is then

$$\xi = \frac{h_1}{h_2}(1 - \xi_{\leftarrow}) \frac{1 - R^{2N}(h_1, \epsilon)}{1 + R^{2N-1}(h_1, \epsilon)} \times \frac{1}{R(h_1, \epsilon) - 1} V_{N,1}^{p-1}.$$

We conclude easily

$$\xi \sim \frac{h_1}{h_2} \succ\succ 1.$$

∎

It is quit possible to introduce a relaxation on the artificial boundary condition as in [5] to retrieve the convergence of F.Q method, however the value of this parameter is not easy to find in general. Our experiment with (2.14: cf Figure 4) shows that this relaxation factor goes to zero as $\epsilon \to 0$.

2.2 Heterogeneous domain decomposition

In this section, we consider a linear second-order transmission problem of the following type:

$$\begin{cases} L_1\phi = -\epsilon\phi'' + \phi = F \ in \ \Omega_1 = (0, A); \\ L_2\psi = \psi'' = G \ in \ \Omega_2 = (A, 1); \\ \phi(A) = \psi(A); \ \phi'(A) = \psi'(A); \\ \phi'(0) = 0 \ ; \ \psi(1) = 0, \end{cases} \tag{2.17}$$

Where ϵ is a small positive parameter, $\epsilon \in]0, \epsilon_0]$ for some $\epsilon_0 > 0$. In addition we assume the compatibility condition: all derivatives of F vanish in 0. This very

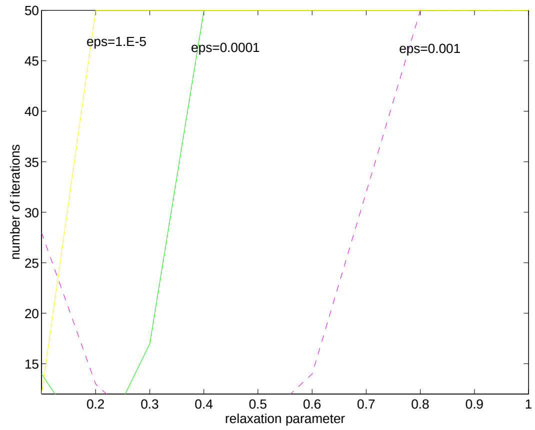

Figure 4: effect of relaxation parameter on convergence

simple model is introduced to study the convergence of an heterogeneous domain decomposition based on the F.Q method. We observe that the domain decomposition is dictated by the definition of the transmission problem and that there is no overlap of the subdomains on A.

2.2.1 Asymptotic analysis

Let us study the boundary layer of (2.17). We define the regular asmptotic expansion of ϕ in Ω_1 as

$$\phi_R = \phi_0 + \sqrt{\epsilon}\phi_1 + \epsilon\phi_2 + \epsilon^{\frac{3}{2}}\phi_3 + ...$$

This regular expansion is completely determined from the PDE equation

$$L_1\phi_R = -\epsilon\phi_R^{"} + \phi_R = F \ in \ \Omega_1 = (0, A);$$

we have formally $\phi_0 = F$, $\phi_1 = F^{(2)}$, ... $\phi_i = F^{(2i)}$, ... ϕ_R satisfies the homogeneous Neumann boundary condition at 0, but not necessarily the transmission condition at A.
We therefore introduce a corrector ϕ_C depending on the local stretched variable $\eta = \frac{x-A}{\sqrt{\epsilon}}$.

At first order we have

$$\phi_0(A) + \phi_C(A) = \psi(A); \ \phi_0^{'}(A) + \epsilon^{-\frac{1}{2}}\phi_C^{'} = \psi^{'}(A).$$

These transmission conditions suggest that the corrector and the solution in domain Ω_2 have the following expansion structures

$$\psi = \psi_0 + \sqrt{\epsilon}\psi_1 + \epsilon\psi_2 + \epsilon^{\frac{3}{2}}\psi_3 +, \dots \textit{ and } \phi_C = \sqrt{\epsilon}\psi_C^0 + \epsilon\psi_C^1 + \epsilon^{\frac{3}{2}}\psi_C^2 + \dots$$

It is then easy to build a uniform formal asymptotic expansion at arbitrary high order with the following chain rule

$$\begin{cases} L_2\psi_i = \delta_i^0 G \textit{ in } \Omega_2; \\ \psi_i(A) = \phi_i(A) - \delta_i^0 \phi_C^{i-1}(0); \\ \psi_i(1) = 0, \end{cases} \tag{2.18}$$

and

$$\begin{cases} -(\phi_C^i)'' + \phi_C^i = 0 \textit{ in } \Omega_1; \\ \phi_C^i(\eta) \to 0, \textit{ when } \eta \to -\infty, \\ (\phi_C^i)'(0) = \psi_i'(A) - \phi_i'(A), \end{cases} \tag{2.19}$$

with $\delta_i^0 = 0(\text{respt}1)$ if $i = 0(\text{respt } i > 0)$ and 1 if $i > 0$. Since each term of these expansions and its derivatives is bounded independantly of ϵ, we conclude that the formal asymptotic expansion is consistant with (2.17) at arbitrary high order. From the energy estimate

$$\int_0^A \epsilon\phi'^2 + \phi^2 dx + \int_A^1 \epsilon\psi'^2 dx = \int_0^A F\phi dx - \int_A^1 \epsilon G dx,$$

we obtain the stability bound

$$\max\,(\max_{(0,A)} \phi, \max_{(A,1)} \psi) \leq \max\,(\epsilon^{-1}||F||_\infty, ||G||_\infty) \tag{2.20}$$

and conclude the validity of the formal asymptotic expansion. (2.17) is consequently a singular perturbation problem with a weak layer of $\sqrt{\epsilon}$ thikness located to the left of A.

2.2.2 Numerical procedure

The asymptotic analysis suggests that the domain computation should be split into three subdomains $\Omega_1 = (O, B), \Omega_2 = (B, A)$ and $\Omega_3 = (A, 1)$ where the intermediate subdomain is used to resolve the boundary layer. We assume that F vanishes in the neighbourhod of zero and that the space step h_i for each subdomain satisfies the following asymptotic relation:

$$\frac{h_1}{B} \approx \frac{h_2}{A-B} \approx h_3 \approx \frac{1}{N}.$$

with $b = A - B \prec\prec 1$. We are going to study the heterogeneous F.Q procedure for such problem.

First scheme:

We choose first the F.Q procedure to resolve the layer and the transmission problem and according to the previous results, we adopt the Dirichlet - Neumann scheme for the layer and the Neumann - Dirichlet scheme for the transmission problem. The iterative procedure gives

$$\left\{ \begin{array}{l} L_1^{h_1} \phi_1^p = F \ in \ \Omega_1; \\ \phi_1^p(0) = \phi_1^p(h_1) \ ; \ \phi_1^p(B) = \phi_2^p(B); \\ L_2^{h_3} \psi^p = G \ in \ \Omega_3; \\ \psi_3^p(A) = \phi_2^p(A) \ ; \ \psi_3^p(1) = 0; \\ L_1^{h_2} \phi_2^{p+1} = F \ in \ \Omega_2; \\ \dfrac{\phi_2^{p+1}(B+h_2) - \phi_2^{p+1}(B)}{h_2} = \dfrac{\phi_1^p(B) - \phi_1^p(B-h_1)}{h_1}; \\ \dfrac{\phi_2^{p+1}(A) - \phi_2^{p+1}(A-h_2)}{h_2} = \dfrac{\psi^p(A+h_3) - \psi^p(A)}{h_3}. \end{array} \right. \quad (2.21)$$

The proof of convergence of this scheme is very similar to Sect 2.1.1. We obtain first the optimal interface position with:

Lemma 3 *Let $(\tilde{\phi}, \tilde{\psi})$ with $\tilde{\phi} = (\tilde{\phi}_{i,j})_{i=0,\dots N, j=1,2}$ and $\tilde{\psi} = (\tilde{\psi}_i)_{i=0,\dots N}$ be the solution of the linear system that is the formal limit of (2.21) when $p \to \infty$.*

Let M be the composite grid $M = M_1 \cup M_2 \cup M_3$, with

$$\left\{ \begin{array}{l} M_1 = \{x_{i,1} = i(\frac{B}{N}); i = 0, \dots N\} \\ M_2 = \{x_{i,2} = B + i(\frac{A-B}{N}); i = 0, \dots N\} \\ M_3 = \{x_{i,2} = A + i(\frac{1-A}{N}); i = 0, \dots N\}. \end{array} \right.$$

Let us suppose that $N^{-1} \approx \sqrt{\epsilon}\delta$, with $\delta \succ\succ 1$. Let $\| \ \|_\infty$ be the maximum norm on the composite grid M.

Under the previous hypothesis on the discretization and approximation of the operators in each subdomain, $\max(\| \phi-\tilde{\phi} \|_\infty, \| \psi-\tilde{\psi} \|_\infty)$ *is asymptotically minimum when*

$$b = A - B \approx \sqrt{\epsilon} \log(\epsilon^{-1}).$$

Proof: The existence and uniqueness of $\tilde{\phi}$ follows from the maximum principle. Let (ϕ, ψ) be the exact solution of the Dirichlet problem (2.17). Let $e_{i,j} = \phi_{i,j} - \tilde{\phi}_{i,j}$ $i = 0, \dots N, j = 1, \dots 2, e_{i,3} = \psi_i - \tilde{\psi}_i$ $i = 0, \dots N$, where $\phi_{i,j}$ (respt ψ_i) is the trace of ϕ (respt ψ) on the composit grid M. We have

$$\begin{aligned}
-\epsilon \frac{e_{i+1,1} - 2e_{i,1} + e_{i-1,1}}{h_1^2} + e_{i,1} &= -\epsilon \frac{h_1^2}{12} \phi^{(4)}(\xi_i), && (2.22)\\
-\epsilon \frac{e_{i+1,2} - 2e_{i,2} + e_{i-1,2}}{h_2^2} + e_{i,2} &= -\epsilon \frac{h_2^2}{12} \phi^{(4)}(\eta_i), && (2.23)\\
\frac{e_{i+1,3} - 2e_{i,2} + e_{i-1,3}}{h_2^3} &= \frac{h_2^3}{12} \psi^{(4)}(\zeta_i), && (2.24)\\
e_{1,1} = e_{0,1} \ ; \ e_{N,3} = 0, & &&\\
e_{N,1} = e_{0,2} \ ; \ e_{N,2} = e_{0,3}, & &&\\
\frac{e_{1,2} - e_{0,2}}{h_2} - \frac{e_{N,1} - e_{N-1,1}}{h_1} &= \frac{h_1}{2} \phi^{(2)}(\tilde{\xi}) + \frac{h_2}{2} \phi^{(2)}(\tilde{\eta}), && (2.25)\\
\frac{e_{1,3} - e_{0,3}}{h_3} - \frac{e_{N,2} - e_{N-1,2}}{h_2} &= \frac{h_2}{2} \phi^{(2)}(\hat{\xi}) + \frac{h_3}{2} \psi^{(2)}(\hat{\eta}), && (2.26)
\end{aligned}$$

where

$$x_{i-1,1} < \xi_i < x_{i,1}; \ x_{i-1,2} < \eta_i < x_{i,2}; \ \dots$$
$$x_{N-1,1} < \tilde{\xi} < x_{N,1}; \ x_{0,2} < \tilde{\eta} < x_{1,2}; \dots$$

Using the superposition principle, we can compute independently the error contribution of each RHS of (2.22), (2.23), (2.24), (2.25) and (2.26). We have

$$\|e\|_\infty \leq \max_{i=1,\dots 5} (E_i),$$

with

$$E_1 \approx \epsilon h_1^2 (1 + \epsilon^{-\frac{3}{2}} \exp(-\frac{b}{\sqrt{\epsilon}})), \quad E_2 \approx \epsilon^{-\frac{1}{2}} h_1^2 b^4, \quad E_3 \approx h_1^2,$$
$$E_4 \approx \sqrt{\epsilon} h_1 (1 + \epsilon^{-\frac{1}{2}} \exp(-\frac{b}{\sqrt{\epsilon}})), \qquad E_5 \approx \sqrt{\epsilon} h_3.$$

It follows that the accuracy is best when $b \approx \sqrt{\epsilon} \log \epsilon^{-1}$. The error is then

$$||e||_\infty \approx \epsilon \delta^2. \tag{2.27}$$

∎

We then have the following convergence property for the iterative scheme (2.21),

Lemma 4 *Let B be the interface position defined as in Lemma 3. Suppose that $N^{-1} \approx \epsilon^{\frac{1}{2}} \delta$, with $\delta \succ\succ 1$.*
Then the amplification factor of the iterative scheme is:

$$\xi \approx \delta^{-1}.$$

Proof: We only need to look at the following homogeneous problem,

$$\begin{cases} L_1^{h_1} e_1^p = 0 \ in \ \Omega_1; \\ e_{1,1}^p = e_{0,1}^p \ ; \ e_{N,1}^p = e_{0,2}^p; \\ L_2^{h_3} e_3^p = 0 \ in \ \Omega_3; \\ e_{0,3}^p = e_{N,2}^p \ ; \ e_{N,3}^p = 0; \\ L_1^{h_2} e_2^{p+1} = 0 \ in \ \Omega_2; \\ \dfrac{e_{1,2}^{p+1} - e_{0,2}^{p+1}}{h_2} = \dfrac{e_{N,1}^p - e_{N-1,1}^p}{h_1}; \\ \dfrac{e_{N,2}^{p+1} - e_{N-1,2}^{p+1}}{h_2} = \dfrac{e_{1,3}^p - e_{0,3}^p}{h_3}. \end{cases} \tag{2.28}$$

We obtain for the first subdomain,

$$e_{i,1}^p = e_{N,1}^p \frac{R^{i-1} + R^{-i}}{R^{N-1} + R^{-N}}, \ \forall i,$$

with $R = 1 + \frac{h_1^2}{2\epsilon} + \frac{h_1}{\sqrt{\epsilon}} \sqrt{1 + \frac{h_1^2}{2\epsilon}} \approx \delta^2$. We have then

$$e_{i,1}^p \prec\prec e_{N,1}^p, \ \forall i < N.$$

We have for the third subdomain,

$$e_{i,3}^p = \frac{N-i}{N} e_{0,3}^p,$$

and then $e_{i,3}^p \leq e_{0,3}$. We have for the second subdomain

$$e_{i,2}^p = h_2 \frac{e_{0,2}^p}{h_1} \left(\frac{R_*^{i-N+1}}{(R_* - 1)(R_*^{-N+1} - R_*^{N-1})} + \frac{R_*^{N-1-i}}{(R_*^{-1} - 1)(R_*^{N-1} - R_*^{-N+1})} \right)$$

$$-h_2 e_{0,3}^p \left(\frac{R_*^i}{(R_* - 1)(R_*^{-N+1} - R_*^{N-1})} + \frac{R_*^{-i}}{(R_*^{-1} - 1)(R_*^{N-1} - R_*^{-N+1})} \right),$$

where $R_* = 1 + \frac{h_2^2}{2\epsilon} + \frac{h_2}{\sqrt{\epsilon}}\sqrt{1 + \frac{h_2^2}{2\epsilon}}$. Using $R_* - 1 \approx \frac{h_2}{\sqrt{\epsilon}}$, we obtain then

$$e_{0,2}^{p+1} \approx \delta^{-1} e_{0,2}^p + 2\sqrt{\epsilon} \exp(-\frac{b}{\sqrt{\epsilon}}) e_{N,2}^p, \ \textit{and} \ e_{N,2}^{p+1} \approx \sqrt{\epsilon} e_{N,2}^p + 2\sqrt{\epsilon} \exp(-\frac{b}{\sqrt{\epsilon}}) \frac{e_{0,2}^p}{h_1}.$$

We conclude that the amplification factor of the method is then asymptotically δ^{-1}.

∎

Combining Lemma 3 and Lemma 4:

Theorem 3 *With the notations defined above,*

$$\|\phi - \phi^p\|_\infty, \|\psi - \psi^p\|_\infty \leq C(\xi^p + \epsilon\delta^2), \tag{2.29}$$

with

$$\xi \sim \delta^{-1}.$$

Proof: The proof is a straightforward application of Lemma 3 and Lemma 4. ∎

Second scheme:

In this scheme, we adopted the F.Q procedure with the Dirichlet-Neumann boundary conditions to resolve the layer and the transmission problem.

$$\left\{\begin{array}{l} L_1^{h_1}\phi_1^p = F \; in \; \Omega_1; \\ \phi_1^p(0) = \phi_1^p(h_1) \; ; \; \phi_1^p(B) = \phi_2^p(B); \\ L_2^{h_3}\psi^p = G \; in \; \Omega_3; \\ \dfrac{\phi_3^p(A+h_3) - \phi^p(A)}{h_3} = \dfrac{\phi_2^p(A) - \phi_2^p(A-h_2)}{h_2}; \\ \psi_3^p(1) = 0; \\ L_1^{h_2}\phi_2^{p+1} = F \; in \; \Omega_2; \\ \dfrac{\phi_2^{p+1}(B+h_2) - \phi_2^{p+1}(B)}{h_2} = \dfrac{\phi_1^p(B) - \phi_1^p(B-h_1)}{h_1}; \\ \phi_2^{p+1}(A) = \psi_3^p(A). \end{array}\right. \tag{2.30}$$

With the same principle as in the previous section, we prove the following lemma.

Lemma 5 *Let $(\tilde{\phi}, \tilde{\psi})$ with $\tilde{\phi} = (\tilde{\phi}_{i,j})_{i=0,\dots N, j=1,2}$ and $\tilde{\psi} = (\tilde{\psi}_i)_{i=0,\dots N}$ be the solution of the linear system obtained as the formal limit of (2.30) when $p \to \infty$.*

Let M be the composite grid defined above. Let us suppose that $N^{-1} \approx \sqrt{\epsilon}\delta$, with $\delta \succ\succ 1$. Let $\| \; \|_\infty$ be the maximum norm on the composite grid M.

Under the previous hypothesis on the discretization and approximation of the operators in each subdomain,
$\max(\| \phi - \tilde{\phi} \|_\infty, \| \psi - \tilde{\psi} \|_\infty)$ *is asymptotically minimum when*

$$b = A - B \approx \sqrt{\epsilon} \log(\epsilon^{-1}).$$

We then have the following property of the iterative scheme.

Lemma 6 *Let B be the interface position defined above.Suppose that $N^{-1} \approx \sqrt{\epsilon} \times \delta$ with $\delta \gg 1$ Then the iterative scheme cannot converge.*

Proof: We look at the following homogeneous problem,

$$\left\{\begin{array}{l} L_1^{h_1} e_1^p = 0 \ in\ \Omega_1; \\ e_{1,1}^p = e_{0,1}^p \ ; \ e_{N,1}^p = e_{0,2}^p; \\ L_2^{h_3} e_3^p = 0 \ in\ \Omega_3; \\ \dfrac{e_{1,3}^p - e_{0,3}^p}{h_3} = \dfrac{e_{N,2}^p - e_{N-1,2}^p}{h_2}; \\ e_{N,3}^p = 0; \\ L_1^{h_2} e_2^{p+1} = 0 \ in\ \Omega_2; \\ \dfrac{e_{1,2}^{p+1} - e_{0,2}^{p+1}}{h_2} = \dfrac{e_{N,1}^p - e_{N-1,1}^p}{h_1}; \\ e_{N,2}^{p+1} = e_{0,3}^p. \end{array}\right. \tag{2.31}$$

As for the previous model, we can easily prove:
$e_{i,1}^p \prec\prec e_{N,1}^p$, $\forall i < N$ and $e_{i,3}^p \leq e_{0,3}^p$ and we have:

$$\left\{\begin{array}{l} e_{0,2}^{p+1} \approx \frac{\sqrt{\epsilon}}{h_1} e_{0,2}^p + \exp\left(-\frac{b}{\sqrt{\epsilon}}\right) e_{N,2}^p \\ e_{N,2}^{p+1} = e_{N,2}^p. \end{array}\right.$$

Since the error at the point A of Ω_2 stays constant, the scheme does not converge.

Third scheme:

Let us use now the *Schwarz alternate procedure* to solve the layer. We keep the F.Q scheme with N - D boundary conditions, to solve the transmission condition in A.
We restrict ourselves to an overlap minimum i.e one cell of step h, between $\Omega_1 = [0, a]$ and $\Omega_2 = [b, 1]$ $\quad (with \quad 0 < b < a < 1)$.
Furthermore, to simplify the proof, we impose that the grids of the subdomains Ω_1 and Ω_2 coincide at the boundary points. Assume that $m \times h_2 = h_1$.
The iterative scheme writes:

$$\left\{\begin{array}{l} L_1^{h_1}\phi_1^p = F \ in \ \Omega_1; \\ \phi_1^p(0) = \phi_1^p(h_1) \ ; \ \phi_1^p(a) = \phi_2^p(b + m * h_2); \\ L_2^{h_3}\psi^p = G \ in \ \Omega_3; \\ \phi_3^p(A) = \phi_2^p(A); \psi_3^p(1) = 0; \\ L_1^{h_2}\phi_2^{p+1} = F \ in \ \Omega_2; \\ \dfrac{\phi_2^{p+1}(A) - \phi_2^{p+1}(A - h_2)}{h_2} = \dfrac{\psi^p(A + h_3) - \psi^p(A)}{h_3}. \end{array}\right. \tag{2.32}$$

It can be proved that $\max(\| \phi - \tilde{\phi}\|_\infty, \| \psi - \tilde{\psi}\|_\infty)$ is asymptotically minimum when

$$A - b \approx \sqrt{\epsilon}\log(\epsilon^{-1}).$$

We then have the following convergence property of the iterative scheme.

Lemma 7 *Let B be the interface position defined above. Suppose that $N^{-1} \approx \epsilon^{\frac{1}{2}}\delta$, with $\delta \succ\succ 1$.*
Then the amplification factor of the iterative scheme is:

$$\xi \approx \sqrt{\epsilon}\log(\epsilon^{-1})\delta.$$

Proof: We look at the following homogeneous problem,

$$\left\{\begin{array}{l} L_1^{h_1}e_1^p = 0 \ in \ \Omega_1; \\ e_{1,1}^p = e_{0,1}^p \ ; \ e_{N,1}^p = e_{m,2}^p; \\ L_2^{h_3}e_3^p = 0 \ in \ \Omega_3; \\ e_{0,3}^p = e_{N,2}^p \ ; \ e_{N,3}^p = 0; \\ L_1^{h_2}e_2^{p+1} = 0 \ in \ \Omega_2; \\ e_{0,2}^{p+1} = e_{N-1,1}^p; \\ \dfrac{e_{N,2}^{p+1} - e_{N-1,2}^{p+1}}{h_2} = \dfrac{e_{1,3}^p - e_{0,3}^p}{h_3}. \end{array}\right. \tag{2.33}$$

Using the same principle as for lemma4, we obtain:

∘ For the first and the third subdomain error:

$$e_{i,1}^p \prec\prec e_{N,1}^p \quad \text{and} \quad e_{i,3}^p \leq e_{0,3} \quad \forall i < N$$

◦ For the second subdomain error:

$$e_{0,2}^{p+1} \approx \epsilon e_{m,2}^{p} \quad \textit{and} \quad e_{N,2}^{p+1} \approx \epsilon \log(\epsilon^{-1})\delta e_{N,2}^{p} + \epsilon^2 e_{m,2}^{p}$$

$$\text{we have:} \quad e_{m,2}^{p} \approx R_*^{m-N} e_{N,2}^{p} + R_*^{-m} e_{0,2}^{p} \quad \text{with} \quad R_* = 1 + \tfrac{h_2^2}{2\epsilon} + \tfrac{h_2}{\sqrt{\epsilon}}\sqrt{1 + \tfrac{h_2^2}{2\epsilon}}.$$

We easily conclude that the amplificator factor of the method is asymptotically $\delta\epsilon\log(\epsilon^{-1})$. ∎

Finally we have

Theorem 4 *With the notations defined above, applying the F.Q and Schwarz mixed method, we have*

$$\max(\|\phi - \phi^p\|_\infty, \|\psi - \psi^p\|_\infty) \leq C(\xi^p + \epsilon\delta^2), \tag{2.34}$$

with

$$\xi \sim \delta\epsilon\log\epsilon^{-1}.$$

We have implemented each of these schemes on the model problem (2.17) with F=0 and $G = \exp(-20 \times (x - 0.8)^2)$ The criterion to stop the scheme is that the jump at the interface A between two consecutive iterates is less than 10^{-4}. For a fixed N, the first and third schemes approximate the solution with the same order of accuracy. The following table of results is in agreement with our analysis. It shows in particular that the third scheme is the most efficient.

Number of iterations	$\epsilon = 10^{-3}$ N=5	$\epsilon = 10^{-3}$ N=10	$\epsilon = 10^{-3}$ N=20	$\epsilon = 10^{-3}$ N=30	$\epsilon = 10^{-4}$ N=10	$\epsilon = 10^{-4}$ N=20
First scheme	10	14	23	33	6	8
Second scheme	DIV	DIV	DIV	DIV	DIV	DIV
Third scheme	3	4	5	8	3	3

Number of iterations	$\epsilon = 10^{-4}$ N=30	$\epsilon = 10^{-4}$ N=40	$\epsilon = 10^{-5}$ N=10	$\epsilon = 10^{-5}$ N=20	$\epsilon = 10^{-5}$ N=30	$\epsilon = 10^{-5}$ N=40
First scheme	10	12	5	5	6	6
Second scheme	DIV	DIV	DIV	DIV	DIV	DIV
Third scheme	4	5	3	3	3	3

3 Boundary layers in a Two Dimensional Space

Let us consider the analogous of (2.1) in a two dimensional space, that is :

$$\begin{cases} L_\epsilon \phi = -\epsilon \Delta \phi + \gamma \phi = F \ in \ \Omega; \\ \phi = g \text{ on } \partial\Omega \end{cases} \tag{3.35}$$

where Ω is a disc of radius 1. We write this problem in polar coodinates, with $\Omega = \{(r, \theta) \in [0, 1[\times[0, 2\pi]\}$. We assume that γ and F are differentiable functions with respect to r and θ, independent of ϵ, with uniformly bounded derivatives on $\bar{\Omega}$. Where ϵ is a small positive parameter,$\epsilon \in]0, \epsilon_0]$; $\epsilon_0 > 0$,
γ is strictly positive such that:

$$\gamma(r, \theta) \geq \gamma_0 > 0 \ (r, \theta) \in \Omega,$$

and ϕ has a boundary layer in the neighborhood of $\partial\Omega$ of $\sqrt{\epsilon}$ thichness([3]) The domain Ω is split into two subdomains:

$$\begin{aligned} \Omega_{inner} &= [a, 1] \times [0, 2\pi] \\ \Omega_{outer} &= [0, a] \times [0, 2\pi] \ \text{ where } a > 0 \end{aligned}$$

On each subdomain, the differential expressions are discretized by means of a finite difference scheme. We approximate the Laplacian operator by the five-point scheme. We assume the mesh to be regular in polar coordinates in both subdomains.
Let (h_1, h_θ)(respt (h_2, h_θ) be the mesh sizes in Ω_{outer} (respt Ω_{inner})
We assume (2.2) for the space step in the radial direction. The goal of this section is to show how to generalize the results in one space dimension to two space dimensions by means of comparaison lemma. For the sake of simplicity, we will restrict ourselves to the Dirichlet - Neumann scheme analogue to (2.3), that is:

$$\begin{cases} L^{h_1} \phi^p_{outer} = F \ on \ \Omega_{outer}; \\ \phi^p_{outer}(1, .) = \phi^p_{outer}(0, .) \ ; \ \phi^p_{outer}(a, .) = \phi^p_{inner}(a, .) \\ \\ L^{h_2} \phi^{p+1}_{inner} = F \ on \ \Omega_{inner}; \\ \\ \phi^{p+1}_{inner}(0, .) = g(\theta) \ ; \\ \\ \dfrac{\phi^{p+1}_{inner}(a + h_2, .) - \phi^{p+1}_{inner}(a, .)}{h_2} = \dfrac{\phi^p_{outer}(a, .) - \phi^p_{outer}(a - h_1, .)}{h_1}. \end{cases}$$

To start the scheme, we impose an artificial boundary condition at the circle of radius a. We define the interface position, i.e the radius **a**, to be such that the maximum of the truncation error in each subdomain is asymptotically minimum. This interface position is optimal when the error at the interface goes to zero. The truncation error for the outer domain is:

$$e_{outer} \sim C \max\left(h_{r_1}^2 a^2 \max_{\Omega_{outer}}\left(\left|\frac{d^{(3)}\phi}{dr^3}\right|, \left|\frac{d^{(4)}\phi}{dr^4}\right|\right), h_y^2 \max\left|\frac{d^{(4)}\phi}{dy^4}\right|\right).$$

The truncation error for the outer domain is:

$$e_{inner} \sim C \max\left(h_{r_2}(1-a) \max_{\Omega_{inner}}\left|\frac{d^{(2)}\phi}{dr^2}\right|, h_y^2 \max\left|\frac{d^{(4)}\phi}{dr^4}\right|\right).$$

We look now for **a**, such that $\max(e_{outer}, e_{inner})$ is minimum. The interface position defined above depends only on the property of the solution that we want to approximate.

Let ϕ_0 be the outer expansion of ϕ. $\Theta(r, \theta, \epsilon)$ be the corrector, that is:

$$\Theta(r, \theta, \epsilon) = \phi(r, \theta, \epsilon) - \phi_0(r, \theta, \epsilon) \text{ in the boundary layer.}$$

It can be shown as in [7] that

Lemma 8 *Suppose that*

(i) the corrector Θ is strictly of order one

(ii) the inner variable $(\xi, \theta) = (\frac{1-r}{\sqrt{\epsilon}}, \theta)$ and all derivatives of $\tilde{\Theta}(\xi, \theta, \epsilon) = \Theta(r, \theta, \epsilon)$ with respect to ξ are of order one

(iii) the corrector function ξ is exponentially decreasing $\Theta(\xi, \theta, \epsilon) \sim C \exp(-C_0 \xi)$ as $\xi \to +\infty$.

Then the accuracy is optimal in both subdomain if

$$1 - a \approx \sqrt{\epsilon} \log(\epsilon^{-1}).$$

The main task is to extend the results of section 2.1.1 with some comparison lemma.

Lemma 9 *Take* $(\phi_{i,j})_{\substack{i=0,\dots N\\ j=0,\dots N}}$ *solution of the following discretized 2D-Elliptic problem:*

$$\begin{cases} -\epsilon\left(\dfrac{\phi_{i+1,j}-2\phi_{i,j}+\phi_{i-1,j}}{h^2}+\dfrac{1}{(ih)^2}\dfrac{\phi_{i,j+1}-2\phi_{i,j}+\phi_{i,j-1}}{h_\theta^2}+\dfrac{1}{ih}\dfrac{\phi_{i+1,j}-\phi_{i,j}}{h}\right)+\gamma_{i,j}\phi_{i,j}=0 \\ \qquad\qquad \forall i=1,\dots N-1\ \ \forall j=1,\dots N-1 \\ \phi_{1,j}=\phi_{0,j} \qquad \forall j=0,\dots N \\ \phi_{N,j}=\alpha(j) \qquad \forall j=0,\dots N \\ \phi(i,.) \qquad \forall i=0,\dots N-1 \qquad \textit{periodic} \end{cases}$$

Let $(e_i)_{i=0,\dots N}$ *be the solution of the following discretized 1D problem:*

$$\begin{cases} -\epsilon\dfrac{2e_{i+1}-3e_i+e_{i-1}}{h^2}+\gamma_0 e_i=0 \quad \forall i=1,\dots N-1 \\ e_N=\alpha_0 \\ e_1=e_0 \end{cases}$$

With:

$$\gamma_{i,j}\geq\gamma^0>0 \quad \forall i,j=0,\dots N,\ \alpha^0\geq\max_{j=0,\dots N}|\alpha(j)| \tag{3.36}$$

Then:

$$|\phi_{i,j}|\leq e_i \qquad \forall i,j=0,\dots N.$$

Proof:

∘ $E_i=\phi_{i,j_0(i)}=\max_{j=0,\dots N}(\phi_{i,j})$ satisfies

$$\begin{cases} -\epsilon\left(\dfrac{E_{i+1}-2E_i+E_{i-1}}{h^2}+\dfrac{1}{ih}\dfrac{E_{i+1}-E_i}{h}\right)+\gamma_{i,j_0(i)}E_i\leq 0 \quad \forall i=1,\dots N-1 \\ E_N=\max_{j=0,\dots N}(\alpha(j))=\beta \quad ; E_1=E_0 \end{cases} \tag{3.37}$$

Let us denote

$$\tilde\gamma_i=\gamma_{i,j_0(i)},\ a_i=\frac{2\epsilon}{h^2}+\frac{\epsilon}{ih^2}+\tilde\gamma_i,\ b_i=\frac{\epsilon}{h^2}+\frac{\epsilon}{ih^2},\ c_i=\frac{\epsilon}{h^2}.$$

(3.37) rewrites:

$$a_i E_i \le b_i E_{i+1} + c_i E_{i-1},\ E_N = \beta,\ E_1 = E_0,$$

with $\qquad a_i > b_i + c_i \quad a_i > 0\ ,\ b_i > 0\ ,\ c_i > 0.$

We show first that:

$$E_i \le \bar{E}_i \qquad \forall i = 0,\dots N \tag{3.38}$$

where

$$a_i \bar{E}_i = b_i \bar{E}_{i+1} + c_i \bar{E}_{i-1},\ \bar{E}_N = \alpha^0 \ge |\beta|,\ \bar{E}_1 = \bar{E}_0.$$

∘ Now let $(\hat{e}_i)_{i=0,\dots N}$ be the solution of the following problem:

$$\begin{cases} -\epsilon\left(\dfrac{\hat{e}_{i+1} - 2\hat{e}_i + \hat{e}_{i-1}}{h^2} + \dfrac{1}{ih}\dfrac{\hat{e}_{i+1} - \hat{e}_i}{h}\right) + \gamma^0 \hat{e}_i = 0 \quad \forall i = 1,\dots N-1 \\ \hat{e}_N = \alpha^0 \\ \hat{e}_1 = \hat{e}_0. \end{cases}$$

Let us suppose that $\hat{e}_0 < 0$; then from the equality:

$$\epsilon\left(\frac{1}{h^2} + \frac{1}{ih^2}\right)(\hat{e}_{i+1} - \hat{e}_i) = \frac{\epsilon}{h^2}(\hat{e}_i - \hat{e}_{i-1}) + \gamma^0 \hat{e}_i \quad \forall i = 1,\dots N-1,$$

we conclude by finite induction that $(\hat{e}_i)_{i=0,\dots N}$ decreases and stays strictly negative, this is in contradiction with $\hat{e}_N = \alpha^0 \ge 0$.

We therefore have $\hat{e}_0 \ge 0$ and show by finite induction that $\hat{e}_i$ is increasing and stay positive. Now let us compare $(\hat{e}_i)$ and $(\bar{E}_i)$; $e^\star_i = \bar{E}_i - \hat{e}_i$ satisfies:

$$\begin{cases} -\epsilon\left(\dfrac{e^\star_{i+1} - 2e^\star_i + e^\star_{i-1}}{h^2} + \dfrac{1}{ih}\dfrac{e^\star_{i+1} - e^\star_i}{h}\right) + \gamma^0 e^\star_i + (\tilde{\gamma}_i - \gamma^0)\hat{e}_i = 0 \quad \forall i = 1,\dots N-1 \\ e^\star_N = 0 \\ e^\star_1 = e^\star_0. \end{cases}$$

Take $a^0_i = \dfrac{2\epsilon}{h^2} + \dfrac{\epsilon}{ih^2} + \gamma^0$ $(e^\star_i)$ satisfies

$$a^0_i e^\star_i \le b_i e^\star_{i+1} + c_i e^\star_{i-1} \qquad \forall i = 1,\dots N-1,\ e^\star_N = 0,\ e^\star_1 = e^\star_0.$$

Since $a^0_i > b_i + c_i$, we have

$$\bar{E}_i \le \hat{e}_i \qquad \forall i = 0,\dots N. \tag{3.39}$$

∘ Because $(\hat{e}_i)$ is increasing, we have

$$\begin{cases} -\epsilon\dfrac{2\hat{e}_{i+1}-3\hat{e}_i+\hat{e}_{i-1}}{h^2}+\gamma^0\hat{e}_i \leq 0 \qquad \forall i=0,\dots N-1 \\ \hat{e}_N=\alpha^0 \\ \hat{e}_1=\hat{e}_0. \end{cases}$$

We notice that

$$\hat{e} \leq e_i \qquad \forall i=0,\dots N. \tag{3.40}$$

with (e_i) solution of:

$$\begin{cases} -\epsilon\dfrac{2e_{i+1}-3e_i+e_{i-1}}{h^2}+\gamma^0 e_i = 0 \qquad \forall i=1,\dots N-1 \\ e_N=\alpha^0 \geq \max\limits_{j=0,\dots N}|\alpha(j)| \\ e_1=e_0. \end{cases}$$

We conclude using (3.38), (3.39), (3.40) that: $\quad \phi_{i,j} \leq e_i \qquad \forall i=0,\dots N.$

With an entirely similar analysis, we have also: $\quad -\phi_{i,j} \leq e_i \qquad \forall i=0,\dots N.$

∎

Lemma 10 *Let* $(\phi_{i,j})_{\substack{i=0,\dots N\\ j=0,\dots N}}$ *be the solution of the following discretized 2D-Elliptic problem:*

$$\begin{cases} -\epsilon\left(\dfrac{\phi_{i+1,j}-2\phi_{i,j}+\phi_{i-1,j}}{h^2}+\dfrac{1}{(i(h+a))^2}\dfrac{\phi_{i,j+1}-2\phi_{i,j}+\phi_{i,j-1}}{h_\theta^2}+\dfrac{1}{i(h+a)}\dfrac{\phi_{i+1,j}-\phi_{i,j}}{h}\right)+\gamma_{i,j}\phi_{i,j}=0 \\ \qquad\qquad \forall i=1,\dots N-1 \;\; \forall j=1,\dots N-1 \\ \dfrac{\phi_{1,j}-\phi_{0,j}}{h}=\alpha(j) \qquad \forall j=0,\dots N \\ \phi(N,j)=0 \qquad \forall j=0,\dots N-1 \end{cases}$$

Let $(e_i)_{i=0,\dots N}$ *be the solution of the following discretized 1D problem:*

$$\begin{cases} -\epsilon\dfrac{2e_{i+1}-3e_i+e_{i-1}}{h^2}+\gamma_0 e_i = 0 \quad \forall i=1,\dots N-1 \\ \dfrac{e_1-e_0}{h} = -\alpha^0 \\ e_N = 0 \end{cases}$$

with:

$$\gamma_{i,j} \geq \gamma^0 > 0 \quad \forall i,j=0,\dots N,\ \alpha^0 \geq \max_{j=0,\dots N}|\alpha(j)|.$$

Then:

$$|\phi_{i,j}| \leq e_i \qquad \forall i,j=0,\dots N.$$

Proof: This proof is very similar to the proof of the previous lemma.

- Let $E_i = \phi_{i,j_0(i)} = \max_{j=0,\dots N}(\phi_{i,j})$.

  We have:

$$\begin{cases} -\epsilon\left(\dfrac{E_{i+1}-2E_i+E_{i-1}}{h^2}+\dfrac{1}{ih+a}\dfrac{E_{i+1}-E_i}{h}\right)+\tilde{\gamma}_i E_i \leq 0 \quad \forall i=1,\dots N-1 \\ E_N = 0 \\ \dfrac{E_1-E_0}{h} \geq -\alpha^0. \end{cases}$$

  Now, take $(\hat{e}_i)_{i=0,\dots N}$ be the solution of the following problem.

$$\begin{cases} -\epsilon\left(\dfrac{\hat{e}_{i+1}-2\hat{e}_i+\hat{e}_{i-1}}{h^2}+\dfrac{1}{ih+a}\dfrac{\hat{e}_{i+1}-\hat{e}_i}{h}\right)+\gamma^0\hat{e}_i = 0 \quad \forall i=1,\dots N-1 \\ \hat{e}_N = 0 \\ \dfrac{\hat{e}_1-\hat{e}_0}{h} = -\alpha^0. \end{cases}$$

  Let us suppose that $\hat{e}_{N-1} < 0$; then from the equality:

$$\epsilon\left(\frac{1}{h^2}+\frac{1}{ih^2}\right)(\hat{e}_{i+1}-\hat{e}_i) = \frac{\epsilon}{h^2}(\hat{e}_i-\hat{e}_{i-1})-\gamma^0\hat{e}_i \quad \forall i=1,\dots N-1.$$

  We conclude by backward finite induction that $\hat{e}_i$ is strictly increasing. This is in contradiction with $\hat{e}_1-\hat{e}_0=-\alpha^0 h$.

We therefore have $\hat{e}_{N-1} \geq 0$ and conclude by finite induction that $\hat{e}_i$ is decreasing and stay positive. Now let us compare $(\hat{e}_i)$ and (E_i). Let $\tilde{e}_i = E_i - \hat{e}_i$, it satisfies:

$$\begin{cases} -\epsilon\left(\dfrac{\tilde{e}_{i+1} - 2\tilde{e}_i + \tilde{e}_{i-1}}{h^2} + \dfrac{1}{ih+a}\dfrac{\tilde{e}_{i+1} - \tilde{e}_i}{h}\right) + \tilde{\gamma}\tilde{e}_i + (\tilde{\gamma}_i - \gamma^0)\hat{e}_i \leq 0 \quad \forall i = 1, \dots N-1 \\ \tilde{e}_N = 0 \\ \tilde{e}_1 - \tilde{e}_0 \geq 0. \end{cases}$$

Let us denote $\forall i = 1, \dots N-1$

$$a^i = \epsilon\left(\frac{2}{h^2} + \frac{1}{h(ih+a)}\right) + \tilde{\gamma}_i,\ b_i = \epsilon\frac{1}{h^2} + \frac{1}{h(ih+a)},\ c_i = \epsilon\frac{1}{h^2}.$$

We have

$$a_i\tilde{e}_i \leq b_i\tilde{e}_{i+1} + c_i\tilde{e}_{i-1} \qquad \forall i = 1, \dots N-1,\ \tilde{e}_N = 0,\ \tilde{e}_1 \geq \tilde{e}_0.$$

With $a_i^0 > b_i + c_i \qquad \forall i = 1, \dots N-1$, we conclude then:

$$\tilde{e}_i \leq 0 \qquad \forall i = 0, \dots N,\ and\ \bar{E}_i \leq \hat{e}_i \qquad \forall i = 0, \dots N.$$

- Because $(\hat{e}_i)$ is decreasing, we have at last:

$$\begin{cases} -\epsilon\dfrac{\hat{e}_{i+1} - 2\hat{e}_i + \hat{e}_{i-1}}{h^2} + \gamma^0\hat{e}_i \leq 0 \qquad \forall i = 0, \dots N-1 \\ \hat{e}_N = 0 \\ \hat{e}_1 - \hat{e}_0 = -\alpha^0 h. \end{cases}$$

It is straightforward to show that:

$$\hat{e} \leq e_i \qquad \forall i = 0, \dots N.$$

where (e_i) is the solution of:

$$\begin{cases} -\epsilon\dfrac{e_{i+1} - 2e_i + e_{i-1}}{h^2} + \gamma^0 e_i = 0 \qquad \forall i = 1, \dots N-1 \\ e_N = 0 \\ e_1 - e_0 = -\alpha^0 h. \end{cases}$$

We conclude that: $\qquad \phi_{i,j} \leq e_i \qquad \forall i = 0, \dots N$

The same analysis holds for $-\phi_{i,j}$.

▮

Using the previous two lemmas, we can now obtain the following convergence estimate:

Theorem 5 *Let the circle with the equation $r = a$, where $1 - a \approx \sqrt{\epsilon}\log\epsilon^{-1}$, as being the interface position, with ξ the amplificator factor.*
If $N_r \approx \epsilon^{1/2}\delta$ with $\delta \gg 1$ then

$$\xi \approx \delta^{-1}.$$

Proof: Let $\tilde{\phi} = (\tilde{\phi}^k_{i,j})_{\substack{i=0,\ldots N; j=0,\ldots N\\ k=1,2}}$ be the solution of the following system

$$\begin{cases} L_1^{h_1}\tilde{\phi}^1_{i,j} = F^1_{i,j} & i = 1,\ldots N-1\ ;\quad j = 1,\ldots N-1 \\ \tilde{\phi}^1_{1,j} = \phi^1_{0,j}\ ;\ \tilde{\phi}^1_{N,j} = \tilde{\phi}^2_{0,j} & j = 0,\ldots N \\ L_2^{h_2}\tilde{\phi}^2_{i,j} = F^2_{i,j} & i = 1,\ldots N-1\ ;\ j = 1,\ldots N-1 \\ \tilde{\phi}^2_{N,j} = g_j & j = 0,\ldots N \\ \dfrac{\tilde{\phi}^2_{1,j} - \tilde{\phi}^2_{0,j}}{h_2} = \dfrac{\tilde{\phi}^1_{N,j} - \tilde{\phi}^1_{N-1,j}}{h_1}\ ; & j = 0,\ldots N \end{cases}$$

where

$$L^h_k\phi_{i,j} = -\epsilon\left(\frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{h^2} + \frac{1}{r_k(i)}\frac{\phi_{i,j} - \phi_{i-1,j}}{h} + \frac{1}{r_k(i)^2}\frac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}}{h_\theta^2}\right) + \gamma_{i,j}\phi_{i,j}$$

with $r_1(i) = ih$ and $r_2(i) = a + ih$.
The existence and uniqueness of $(\tilde{\phi}_{i,j})$ follows from the *maximum principle*. To prove the convergence of the Dirichlet-Neumann scheme, it is sufficient to restrict ourselves to the homogeneous problem ie $F^k_{i,j} = g_j = 0 \quad \forall i, j, k$.

Let $\tilde{\phi}_{outer} = (\tilde{\phi}^1_{i,j})_{\substack{i=0,\ldots N\\ j=0,\ldots N}}$, $e^p_{outer} = \phi^p_{outer} - \tilde{\phi}_{outer}$ (resp $e^p_{inner} = \phi^p_{inner} - \tilde{\phi}_{inner}$).

Let us denote: $e^p_{outer} = (e^{1,p}_{i,j})_{\substack{i=0,\ldots N_r\\ j=0,\ldots N_\theta}}$ (resp $e^p_{inner} = (e^{2,p}_{i,j})_{\substack{i=0,\ldots N_r\\ j=0,\ldots N_\theta}}$) and $(e^{k,p}_i)_{\substack{i=0,\ldots N\\ k=1,2}}$

such that

$$\left\{\begin{array}{l} -\epsilon\dfrac{2e_{i+1}^{1,p}-3e_i^{1,p}+e_{i-1}^{1,p}}{h1^2}+\gamma_0 e_i^{1,p}=0 \qquad \forall i=1,\dots N-1 \\ e_1^{1,p}=e_0^{1,p}\ ;\ e_N^{1,p}=e_0^{2,p} \\ -\epsilon\dfrac{e_{i+1}^{2,p+1}-2e_i^{2,p+1}+e_{i-1}^{2,p+1}}{h_2^2}+\gamma_0 e_i^{2,p+1}=0 \qquad \forall i=1,\dots N-1 \\ \dfrac{e_1^{2,p+1}-e_0^{2,p+1}}{h_2}=-\dfrac{e_N^{1,p}+e_{N-1}^{1,p}}{h_1} \quad ,e_N^{2,p+1}=0, \end{array}\right.$$

with

$$e_N^{1,p}=\max_{j=0,\dots N}|\phi_{i,j}^{1,0}|.$$

From lemma 9, we have:

$$|\phi_{i,j}^{1,0}|\leq e_i^{1,0} \qquad \forall i,j=0,\dots N,$$

and then

$$-\max\left|\frac{\phi_{N,j}^{1,0}-\phi_{N-1,j}^{1,0}}{h_1}\right|\geq-\frac{e_N^{1,0}+e_{N-1}^{1,0}}{h_1}.$$

We can conclude from lemma 10 that:

$$|\phi_{i,j}^{2,1}|\leq e_i^{2,1} \qquad \forall i,j=0,\dots N.$$

By induction, we show that:

$$\left\{\begin{array}{ll} |\phi_{i,j}^{1,p}|\leq e_i^{1,p} & \forall i,j=0,\dots N\ ;\ \forall p\geq 1; \\ |\phi_{i,j}^{2,p}|\leq e_i^{2,p} & \forall i,j=0,\dots N\ ;\ \forall p\geq 1. \end{array}\right.$$

Let us now show that $\max_{j=0,\dots N}|e_i^{k,p}|$ converges to 0, $\quad \forall k=1,2.$

We use the notation $\hat{\epsilon}=\frac{\epsilon}{\gamma^0}$. An explicit calculus gives: $e_{N-1}^{1,p}\sim 2\frac{\hat{\epsilon}}{h^2}e_N^{1,p}$ and therefore:

$$e_{N-1}^{1,p}\ll e_N^{1,p}.$$

Consequently we have :

$$\begin{array}{rcl} e_1^{2,p+1}-e_0^{2,p+1} & \approx & \frac{h_2}{h_1}e_N^{1,p} \\ & \approx & \frac{h_2}{h_1}e_0^{2,p}. \end{array}$$

We conclude as in the one dimensional case that the damping factor is given by:

$$\xi \approx \frac{h_2}{h_1} \times \frac{1-R^{2N+1}}{1+R^{2N-1}} \times \frac{1}{R-1},$$

where R is the larger root of the quadratic polynomial:

$$R^2 - (2 + \frac{h_2^2}{\hat{\epsilon}})R + 1 = 0.$$

Finally, we have:

$$\xi \approx \delta^{-1}.$$

∎

4 Application

Our objective is to solve efficiently a two dimensional singular perturbed transmission problem that arises in electro-magnetic theory([2]) by using the *Modulef* finite element package. To start, we consider the following model:

$$\left\{ \begin{array}{lll} -\epsilon\Delta u + a(r,\theta)u = 0 & in & \Omega^1 \\ -\Delta u = j(r,\theta) & in & \Omega^2, \\ [u] = [\frac{\partial u}{\partial n}] = 0 & on & \partial\Omega^1 \cap \partial\Omega^2 \\ u(R_\infty, \theta) = 0 & for & \theta \in (0, 2\pi). \end{array} \right. \tag{4.41}$$

Here Ω_1 is a disk of radius one, Ω_2 is a ring for $r \in (1, R_\infty)$, j represents a given current density in the inductor, Ω^1 is the domain of the solid conductor, Ω^2 is the exterior domain of the conductor truncated by (R^∞, θ) and the boundary layer in Ω^1 corresponds to the well-known skin effect. For the numerical resolution, we arbitrarily fixed R^∞ at 20.

This simplified model is only a small part of the real magnetic model. However our first experiments with the F.Q iterative procedure in solving alternatively the equations in Ω_1 and Ω_2 did show a problem, where convergence of the algorithm strongly depended on the radius of the elements of the unstructured grid used as well as on ϵ. To be more specific, the iterative procedure converged slowly and could even diverge if the mesh was not appropriate. Applying the **comparison lemma** as previously (Theorem 5), we could extend our results obtained for heterogeneous domain decomposition from a one dimensional space to a two dimensional space. We have therefore modified our implementation according to the analysis in the following way: we use the "F.Q. and Schwarz mixed method" analogous to the third scheme of Sect 2.2.

The main advantage of this approach is that it is easy to use the *Modulef* library to follow this iterative procedure. We used three subdomains, two located on the metal (Ω^1_{outer} and Ω^1_{inner}) and an external subdomain (Ω_2). For each subdomain we used an automatic mesh generator governed by the algebraic method of a generalized triangle for the mesh of a disc and that of a generalized quadrilateral for the mesh of a ring, for which we have given the boundary descriptions and their discretization. We made a regular discretization along the radius of each subdomain such that the number of finite elements is roughly of the same order in each subdomain. We have then applied the hypotheses of Theorem 5 about the discretization (cf figures 5, 6 and 7) as follows:

let h_1(respectively h_2) be the discretization step along the radius of Ω^1_{outer}(respectively Ω^1_{inner}).

We have the following hypothesis about a,b,h_1, h_2:

$$\left\{ \begin{array}{l} a - b = h_1 \\ 1 - b \approx \sqrt{\epsilon} \log \epsilon^{-1}. \end{array} \right.$$

The variational expression of an iteration of our algorithm is the following:

Find u_1^p, u_2^p, u_3^p such that:

$\circ$ $u_1^p \in u_2^{p-1} / \partial \Omega^1_{outer} + H^1_0(\Omega^1_{outer})$

and:

$$\epsilon \int_{\Omega^1_{outer}} \nabla u_1^p . \nabla v + \int_{\Omega^1_{outer}} u_1^p . v = 0 \qquad \forall v \in H^1_0(\Omega^1_{outer})$$

$\circ$ $u_3^p \in u_2^{p-1} / (\partial \Omega^1_{inner} \cap \Omega^2) + H^1_0(\Omega^2)$

and:

$$\int_{\Omega^2} \nabla u_3^p . \nabla v = \int_{\Omega^2} j.v \qquad \forall v \in H^1_0(\Omega^2)$$

	N=10 NE1=320 NE2=576 NE3=992	N=15 NE1=780 NE2=1456 NE3=2576	N=20 NE1=1440 NE2=2736 NE3=4752
$\epsilon = 10^{-2}$	7	7	11
$\epsilon = 10^{-3}$	5	4	6
$\epsilon = 10^{-4}$	4	4	4
$\epsilon = 10^{-5}$	3	3	3

Table 1: number of iterations

∘ $u_2^p \in u_1^p/\partial\Omega^1_{outer} + V_{inner}$ with $V_{inner} = \{v \in H^1(\Omega^1_{inner}) \; ; \; v/\partial\Omega^1_{outer} = 0\}$

and:

$$\epsilon \int_{\Omega^1_{inner}} \nabla u_2^p . \nabla v + \int_{\Omega^1_{inner}} u_2^p . v = \epsilon \int_\Gamma \frac{\partial u}{\partial n} v \qquad \forall v \in V^1_{inner}$$

We used for current density $j(r,\theta) = \sin(2\theta + \frac{3\pi}{4}) \times \exp(-\frac{\sin(4\theta)+1}{0.1}) \times \exp(-(r-10)^2)$. We refined the mesh in the neighbourhood of the inductors (cf figure7)

To resolve our problem for these meshes, we used *curved finite elements* (triangles with six nodes, *P2-Lagrange-isoparametric*) and a *direct solver (Factorization of Cholesky)*.

In Table 1, one can find the results of our numerical experiment; the criterion to stop the iteration is a jump at the interfaces between Ω^1_{inner} and Ω^2, Ω^1_{outer} less than 10^{-8}. For several values of N (number of points along the radius for each subdomain), we give the number of iterations and the number of elements for each subdomain.

We have checked that our numerical scheme, when applied to (4.41), converged with the same number of iterations.

Figure 9-11 show the isovalues of the solution from each subdomain, with N=10 and $\epsilon = 10^{-3}$. We resolved (4.41) on this mesh for different values of ϵ and observed

	$\epsilon = 1.$	$\epsilon = 0.1$	$\epsilon = 10^{-2}$	$\epsilon = 10^{-3}$	$\epsilon = 10^{-4}$	$\epsilon = 10^{-5}$
meshes adapted to N=10 $\epsilon = 10^{-3}$	169	72	22	7	7	6
regular meshes with arbitrary interface N=10	-	-	DIV	DIV	DIV	DIV
regular meshes with optimal interface position for $\epsilon = 10^{-3}$ N=10	-	DIV	16	12	8	7

Table 2: number of iterations

that we must adapt the mesh to obtain a good rate of convergence(cf Table 2 row 1).
We have also applied an algorithm to a regular mesh with approximatively the same order of discretization for each subdomain. We considered two cases, a first case with an arbitrary interface position and a second case with an interface position adapted for $\epsilon = 10^{-3}$ (cf Table 2 rows 2, 3).

We observe that since this computation was done with *Modulef*, it could be extended to a more general geometry.

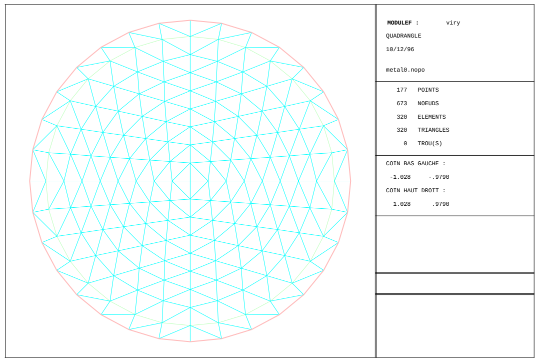

Figure 5: mesh of the outer subdomain on the metal for N=10

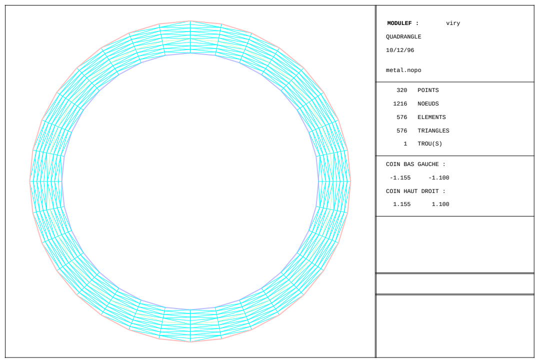

Figure 6: mesh on the inner subdomain on the metal for N=10

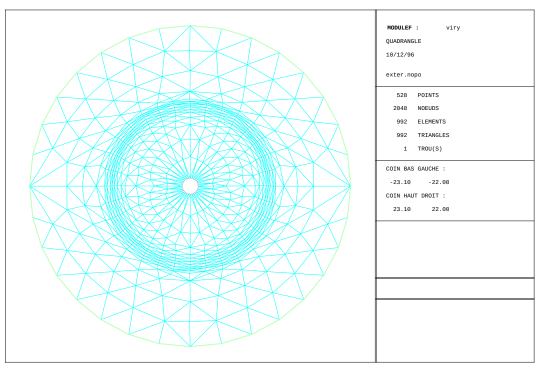

Figure 7: mesh on the external domain for N=10

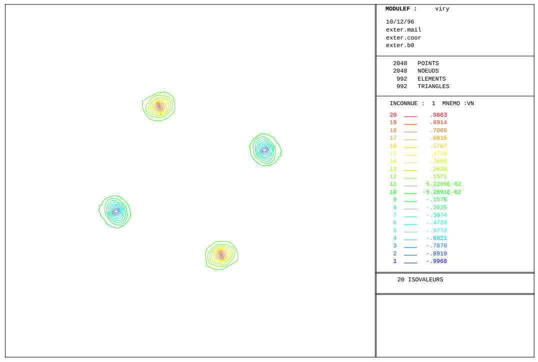

Figure 8: isovalue of j on the external domain for N=10

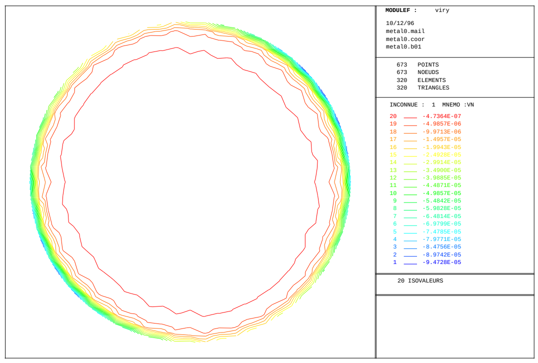

Figure 9: isovalue on Ω^1_{outer} for N=10

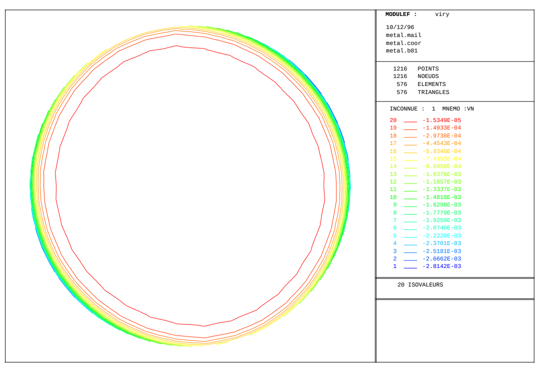

Figure 10: isovalue on Ω^1_{inner} for N=10

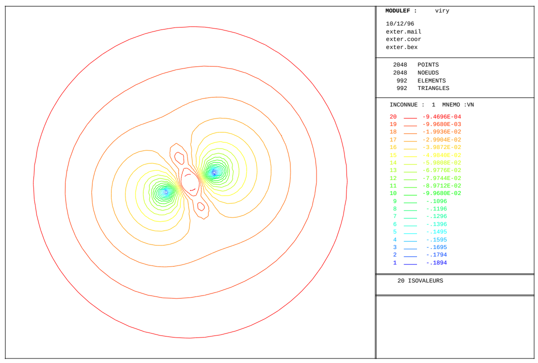

Figure 11: isovalue on Ω^2 for N=10

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