# Control of Nonlinear Chained Systems. From the Routh-Hurwitz Stability Criterion to Time-Varying Exponential Stabilizers

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# Control of Nonlinear Chained Systems. From the Routh-Hurwitz Stability Criterion to Time-Varying Exponential Stabilizers

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**Abstract:** We show how any linear feedback law which asymptotically stabilizes the origin of a linear integrator system of order $(n-1)$ induces a simple continuous time-varying feedback law which exponentially stabilizes the origin of a nonlinear $(2, n)$ single-chain system. The proposed control design method is related to, and extends in the specific case of chained systems, a recent method developed by M'Closkey and Murray [9] for driftless systems in order to transform smooth feedback stabilizers yielding slow polynomial convergence into continuous homogeneous ones which ensure faster exponential convergence.

**Key-words:** Chained system, time-varying control, homogeneous system, continuous feedback.

*(Résumé : tsvp)*

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## Commande de Systèmes Chaînés. Du Critère de Stabilité de Routh-Hurwitz à des Retours d'Etat Exponentiellement Stabilisant

**Résumé :** Nous montrons que tout retour d'état linéaire qui stabilise asymptotiquement l'origine d'un intégrateur linéaire d'ordre $(n-1)$ induit un retour d'état instationnaire continu simple qui stabilise l'origine d'un système chaîné non-linéaire d'ordre $n$ à deux entrées. La méthode proposée pour la construction de la loi de commande est apparentée à, et étend dans le cas particulier des systèmes chaînés, une méthode développée récemment par M'Closkey et Murray [9] pour les systèmes sans dérive afin de transformer des retours d'état stabilisant différentiables conduisant à une convergence polynomiale lente en des retours d'état continus homogènes assurant une convergence exponentielle.

**Mots-clé :** Système chaîné, commande instationnaire, système homogène, retour d'état continu.

# 1 Introduction

Control systems in the so-called *chained form* have been extensively studied in the past recent years. This research interest partly stems from the fact that the kinematic equations of many nonholonomic mechanical systems, such as these arising in mobile robotics (unicycle-type carts, car-like vehicles with trailers,...), can be converted into this form [13, 17, 19]. Systems in the chained form thus offer a general framework for studying the control of these mechanical systems. The present paper addresses the problem of asymptotic stabilization of a given equilibrium point (which corresponds to a fixed desired configuration for a mechanical system).

Since chained systems (with state and control vectors denoted as $x$ and $u$ respectively) do not satisfy Brockett's necessary condition [1], they cannot be asymptotically stabilized, with respect to any equilibrium point, by means of a continuous pure state feedback $u(x)$. In [16], one of the authors proposed and derived smooth *time-varying* feedback laws $u(x,t)$ for the stabilization of a unicycle-type vehicle, the equations of which can be converted into a three-dimensionnal chained system. This showed how the topological obstruction raised by Brocket could be dodged, and was the starting point of other studies about time-varying feedbacks. In [3, 4], Coron established that most controllable systems can be asymptotically stabilized with this type of feedback. The literature on the subject has since then mostly focused on the problem of explicit design of such stabilizing control laws. Smooth feedback laws, yielding slow (polynomial) asymptotic convergence, have first been developed using either a Lyapunov approach ([14, 16, 17],...) or center manifold techniques ([10, 18, 20, 21],...). In order to obtain a faster (exponential) rate of convergence, which cannot be achieved via smooth feedback for systems whose linearization is not controllable, M'Closkey and Murray have used in [7] the properties associated with homogeneous systems. This yields time-varying feedback laws which are only continuous everywhere. This approach has since then been further investigated by other authors, in the specific case of chained systems ([7, 12, 15],...) and for more general driftless controllable systems [8, 11].

Recently, M'Closkey and Murray have also presented in [9] a method for transforming smooth time-varying stabilizers into homogeneous continuous ones. The method is best suited for driftless systems for which it applies systematically. The construction of the exponential stabilizer relies upon the initial knowledge of an adequate Lyapunov function coupled with a smooth stabilizing feedback law. More precisely, the exponential stabilizer is obtained by "scaling" the size of the smooth control inputs on a level set of the Lyapunov function. The continuous time-varying feedbacks derived in the present paper have been obtained by adaptating and com-

bining the core of this method to the control design method earlier proposed by Samson in [17] for the smooth feedback stabilization of chained systems. Although our approach is specific to chained systems, and therefore, in some respect, less general than the work reported in [9], it also carries with it two important improvements with respect to this work.
The first one is that the knowledge of a (definite negative) Lyapunov function coupled with a smooth stabilizing feedback is not needed. In fact, instead of going thru the intermediary stage consisting of finding a stabilizing smooth time-varying feedback and a corresponding Lyapunov function for the controlled system, we go one step further and show that the knowledge of a linear feedback which stabilizes a linear integrator system whose struture is reminiscent of the one of the chained system, and of a Lyapunov function the derivative of which is only semi-negative along the solutions of the stabilized linear integrator, are sufficient. This makes a significant difference because finding a "good" Lyapunov function for a chained system of order larger than three is not such a simple task. Moreover, for general controllable driftless systems, the design of a smooth time-varying stabilizer may in fact be more difficult than the direct construction of a continuous homogeneous time-varying stabilizer. As a matter of fact, no general control design method has so far been developed in the smooth case, while one already exists in the continuous homogeneous case [11].
The second improvement is related to the "scaling factor" used to transform the smooth feedback into a continuous exponentially stabilizing one. In [9], this factor is implicitely defined as the positive real solution to an equation involving the considered Lyapunov function. The uniqueness of this solution along the controlled system's trajectories is required and depends on a *transversality condition* the satisfaction of which itself depends on the candidate Lyapunov function and has to be checked beforehand. Solving such an equation will usually have to be performed numerically. The first continuous time-varying feedback law proposed in the present study is of this type. However, we also show in a second result that this scaling factor may in fact be replaced by an adequate explicit function. The implementation of the resulting control law is consequently simplified.

This paper is organized as follows. In Section 2, a few technical results used further for the design of the control laws are recalled. In particular, useful relationships between the Routh-Hurwitz stability criterion for linear systems and the transformation of a companion matrix into the so-called "Schwartz matrix" are reviewed. The two main results and proposed control laws are presented in Section 3 in the form of two propositions. In the first one, the aforementionned scaling factor is still implicitely defined. The second proposition is an adaptation of the first one in order to get rid of the implicit definition of the scaling factor. The proofs of these results are reported Section 4, and the proofs of the technical lemmas introduced along the paper are given in the paper's Appendix.

# 2 Preliminary recalls

## 2.1 Stabilization of a multi-order integrator and the Routh-Hurwitz criterion

Consider the following linear $(n-1)$-order integrator:

$$\frac{d}{dt}^{(n-1)} x_2 = u \tag{1}$$

whose equivalent controllable state realization is:

$$\left\{ \begin{array}{lcl} \dot{x}_2 & = & x_3 \\ \dot{x}_3 & = & x_4 \\ & \vdots & \\ \dot{x}_{n-1} & = & x_n \\ \dot{x}_n & = & u \end{array} \right. \tag{2}$$

Any linear feedback control

$$u = -\sum_{i=2}^{i=n} a_i x_i \tag{3}$$

asymptotically (and exponentially) stabilizes the origin of this system provided that all roots of the characteristic polynomial $p(s) = s^{n-1} + a_n s^{n-2} + \ldots + a_3 s + a_2$ associated with the closed-loop system have strictly negative real parts. The *Routh-Hurwitz table* associated with this polynomial is

$$\begin{array}{ccccc} 1 & a_{n-1} & a_{n-3} & \ldots & \ldots \\ a_n & a_{n-2} & a_{n-4} & \ldots & 0 \\ b_n & b_{n-2} & \ldots & \ldots & 0 \\ c_n & c_{n-2} & \ldots & \ldots & 0 \\ d_n & d_{n-2} & \ldots & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 \\ \ldots & 0 & 0 & 0 & 0 \end{array} \tag{4}$$

with

$$
\begin{aligned}
a_k &= 0 \quad \text{for } k < 2 \\
b_k &= -\frac{1}{a_n}(a_{k-2} - a_n a_{k-1}) \;=\; -\frac{1}{a_n}\begin{vmatrix} 1 & a_{k-1} \\ a_n & a_{k-2} \end{vmatrix}, \\
c_k &= -\frac{1}{a_n b_n}(a_n b_{k-2} - b_n a_{k-2}) \;=\; -\frac{1}{a_n b_n}\begin{vmatrix} a_n & a_{k-2} \\ b_n & b_{k-2} \end{vmatrix}, \\
d_k &= -\frac{1}{b_n c_n}(b_n c_{k-2} - c_n b_{k-2}) \;=\; -\frac{1}{b_n c_n}\begin{vmatrix} b_n & b_{k-2} \\ c_n & c_{k-2} \end{vmatrix} \\
&\vdots
\end{aligned}
\tag{5}
$$

Let $k \triangleq (k_2, \ldots, k_n)$ be defined from the first column of the Routh-Hurwitz table as follows:

$$
\begin{aligned}
k_n &= a_n \\
k_{n-1} &= b_n \\
k_{n-2} &= c_n \\
&\vdots
\end{aligned}
\tag{6}
$$

Then, we have the following two lemmas whose proofs, which may be found in several control textbooks (see [2] for example), are given in the Appendix for the sake of completeness.

**Lemma 1** *Let $X_2 = (x_2, x_3, \ldots, x_n)^T$ and consider the linear change of coordinates $X_2 \longmapsto Z_2 = (z_2, z_3, \ldots, z_n)^T = \Phi_k X_2$ defined by*

$$
\begin{aligned}
z_2 &= x_2 \\
z_3 &= x_3 \\
z_{j+3} &= k_{j+1} z_{j+1} + L_f z_{j+2} \quad for\ j = 1, \ldots, n-3\,,
\end{aligned}
\tag{7}
$$

*where $L_f z_i = \frac{\partial z_i}{\partial X_2} f$ stands for the Lie-derivative of the function $z_i(X_2)$ along $f(X_2) = (x_3, x_4, \ldots, x_n, 0)^T$. Then, in the coordinates $Z_2$, the controlled system (2)-(3) becomes*

$$
\left\{
\begin{aligned}
\dot z_2 &= z_3 \\
\dot z_3 &= -k_2 z_2 + z_4 \\
&\vdots \\
\dot z_{j+1} &= -k_j z_j + z_{j+1} \\
&\vdots \\
\dot z_{n-1} &= -k_{n-2} z_{n-2} + z_n \\
\dot z_n &= -k_{n-1} z_{n-1} - k_n z_n
\end{aligned}
\right.
\tag{8}
$$

Using the fact that the time-derivative of the quadratic function

$$V_z(Z_2) = Z_2^T diag(1, \frac{1}{k_2}, \frac{1}{k_2 k_3}, \ldots, \frac{1}{\prod_{i=2}^{i=n-1} k_i}) Z_2 \tag{9}$$

along any solution of the system (8) is

$$\dot{V}_z(Z_2) = -2 \frac{k_n}{\prod_{i=2}^{i=n-1} k_i} z_n^2 \,, \tag{10}$$

one easily establishes:

**Lemma 2** *The origin $Z_2 = 0$ of the linear system (8) is asymptotically stable if and only if $k_i > 0$ for $i = 2, \ldots, n$.*

A corollary of the above two lemmas is the well-known Routh-Hurwitz stability criterion:

**Corollary 1 (Routh-Hurwitz stability criterion)** *All roots of the polynomial $p(s) = s^{n-1} + a_n s^{n-2} + \ldots + a_3 s + a_2$ have strictly negative real parts if and only if $k_i > 0$ for $i = 2, \ldots, n$.*

### 2.2 Non-exponential time-varying feedback stabilization of chained systems

Beyond the interest of recalling a rather simple method for proving the Routh-Hurwitz stability criterion, the prime objective of the previous section was to point out the algebraic operations which transform the *chain* of integrators involved in the system (2)-(3) into the *skew-symmetric* representation (8) to which the simple Lyapunov function (9) can be associated. The objective was also to recall the one-to-one correspondance between the two sets of control parameters $a_i \, (i = 2, \ldots, n)$ and Routh-Hurwitz parameters $k_i \, (i = 2, \ldots, n)$ respectively involved in these two equivalent system's representations.

In [17], the structural similitude between the linear $n$-order integrator system 2 and the following nonlinear $(2, n)$ single-chain system:

$$\left\{ \begin{array}{lcl} \dot{x}_1 & = & u_1 \\ \dot{x}_2 & = & u_1 x_3 \\ \dot{x}_3 & = & u_1 x_4 \\ & \vdots & \\ \dot{x}_{n-1} & = & u_1 x_n \\ \dot{x}_n & = & u_2 \end{array} \right. \tag{11}$$

has been used, with the aforementionned transformations, to prove the following stabilization result.

**Proposition 1** *([17, Prop. 2.2]) Let $a_i \, (i = 2, \ldots, n)$ be a set of parameters for which the origin of the linear system (2)-(3) is asymptotically stable. Then, the continuous time-varying feedback control*

$$\left\{ \begin{array}{rcl} u_1(x,t) & = & -k_1 x_1 + g(X_2) \sin t \\ u_2(x,t) & = & -u_1(x,t) \sum_{i=2}^{i=n} a_i \; sign(u_1)^{n+1-i} \; x_i \, , \end{array} \right. \tag{12}$$

*with $k_1 > 0$ and $g(X_2)$ a continuous function which vanishes at $X_2 = 0$ (i.e. $g(0) = 0$) and is strictly positive elsewhere, applied to the chained system (11)*

*i) makes the positive function*

$$V_x(X_2) \equiv X_2^T \Phi_k^T diag(1, \frac{1}{k_2}, \frac{1}{k_2 k_3}, \ldots, \frac{1}{\prod_{i=2}^{i=n-1} k_i}) \Phi_k X_2 \quad (= V_z(Z_2)) \tag{13}$$

*non-increasing along any solution of this system,*

*ii) globally asymptotically stabilizes the origin $x = 0$ of this system.*

This result clearly indicates how any linear feedback control which, by application of the Routh-Hurwitz stability criterion, asymptotically stabilizes the origin of the linear $(n-1)$-order integrator system (2) induces a simple continuous time-varying feedback law which globally asymptotically stabilizes the origin the corresponding chained system (11). However, as pointed out in [17], a shortcoming of the feedback law (12) is that it yields slow (polynomial) asymptotic convergence to zero for most of the system's solutions. The main contribution of this paper is to show how this time-varying control may itself be simply modified in order to render the controlled chained system homogeneous of degree zero with respect to some dilation and ensure uniform exponential convergence. Note that the method proposed by M'Closkey and Murray in [9] to transform a smooth stabilizer into a continuous homogeneous one does not apply directly in the present case because i) the control (12) is not smooth since it is not differentiable on the set $X_2 = 0$, ii) Lyapunov functions for the controlled system are not known, and iii) the degrees of homogeneity of the two control inputs $u_1$ and $u_2$ are not equal.

## 2.3 Homogeneity and exponential stabilization

The set of nonlinear systems which are homogeneous of degree zero with respect to some dilation constitutes a fairly natural extension of the set of linear systems. Some properties of these systems, that will be used in the sequel, are briefly recalled hereafter. For more details, the reader is referred to [5] or [6], for example.

For any $\lambda > 0$ and any set of real parameters $r_i > 0$ ($i = 1, \dots, n$), one defines a "dilation" operator $\delta(\lambda, .) : \mathbb{R}^n \longmapsto \mathbb{R}^n$ by

$$\delta(\lambda, x_1, \dots, x_n) = (\lambda^{r_1} x_1, \dots, \lambda^{r_n} x_n)$$

A function $f \in C^0(\mathbb{R}^n \times \mathbb{R}; \mathbb{R})$ is homogeneous of degree $\tau \geq 0$ with respect to the (family of) dilations $\delta(\lambda, .)$ if :

$$\forall \lambda > 0, \quad f(\delta(\lambda, x), t) = \lambda^\tau f(x, t).$$

An *homogeneous norm* $\rho$ associated with this dilation operator is a function from $\mathbb{R}^n$ to $\mathbb{R}$, homogeneous of degre one with respect to the dilation, positive ($\rho(x) \geq 0$, $\forall x$), and proper ($\rho(x)$ tends to infinity when $|x|$, the euclidean norm of $x$, tends to infinity). A consequence of this definition is that $\rho(x)$ tends to zero only when $|x|$ tends itself to zero. An example of homogeneous norm is:

$$\rho_p(x) = (\sum_{j=1}^{n} |x_j|^{\frac{p}{r_j}})^{\frac{1}{p}} \qquad with\ p > 0. \tag{14}$$

A differential system $\dot{x} = f(x, t)$ (or a vector field $f$), with $f \in C^0(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^n)$, is homogeneous of degree $\tau \geq 0$ with respect to the dilation $\delta(\lambda, .)$ if for any $i = 1, \dots, n$, the $i$th component $f_i$ of the vector field $f$ is homogeneous of degree $\tau + r_i$.

Finally, let $f \in C^0(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^n)$, with $f(x, .)$ $T$ periodic, define an homogeneous vector field of degree 0 with respect to the dilation $\delta(\lambda, .)$. Then, the two following properties are equivalent:

*i)* the origin $x = 0$ of the system $\dot{x} = f(x, t)$ is asymptotically stable,

*ii)* $x = 0$ is globally exponentially stable in the sense that there exists $\gamma > 0$ and, for any homogeneous norm $\rho$, a value $K$ such that along any trajectory $x(t)$ ($t \geq t_0$) of the system $\dot{x} = f(x, t)$,

$$\rho(x(t)) \leq K \rho(x(t_0))\, e^{-\gamma(t - t_0)}$$

# 3 Main results

Let us consider the chained system (11) and define a family of dilations $\delta_q(\lambda, X_2) = (\lambda^{r_2} x_2, \ldots,$
$\lambda^{r_n} x_n)$ indexed by the integer $q \in I\!N$ via the *dilation weights* $r_i$ chosen as follows:

$$r_i = n - i + q \qquad for\ i = 2, \ldots, n\,. \tag{15}$$

Let us also consider a set of parameters $a_i$ $(i = 2, \ldots, n)$ chosen so that the linear control (3) asymptotically stabilizes the origin of the linear system 2. The corresponding positive Routh-Hurwitz parameters are denoted as before as $k_i$ $(i = 2, \ldots, n)$, and the regular square matrix associated with the change of coordinates defined in Lemma 1 is again denoted as $\Phi_k$.

The first result involves a specific homogeneous norm $\rho_q(X_2)$ which satisfies the following equality:

$$V_x(\delta_q(\rho_q(X_2)^{-1}, X_2)) = C\ ,\ \forall X_2 \neq 0 \tag{16}$$

where $C$ is a positive real number and $V_x$ is the quadratic positive function introduced in Proposition 1.

The next lemma asserts that $\rho_q(X_2)$ is uniquely defined by the polynomial equation (16), provided that $q$ is chosen large enough.

**Lemma 3** *There exists $q_0 > 1$ such that, for any $q \geq q_0$ $(q \in I\!N)$,*

*i) $\forall X_2 \neq 0$, the equation $V_x(\delta_q(\lambda, X_2)) = C$ admits a unique positive solution $\lambda(X_2)$,*

*ii) the function $\rho_q$, from $I\!R^{n-1}$ to $I\!R^+$, defined by $\rho_q(X_2) = \lambda(X_2)^{-1}$ when $X_2 \neq 0$ and $\rho_q(0) = 0$, is smooth on $I\!R^{n-1} - \{0\}$ and homogeneous of degree one with respect to the family of dilations $\delta_q(\lambda, .)$.*

In view of the previous notations and definitions we are now ready to state the first main result in the following proposition.

**Proposition 2** *The continuous time-varying feedback control:*

$$\left\{\begin{array}{lcl} u_1(x,t) & = & -k_1 x_1 + \rho_q(X_2)\sin(t) \qquad k_1 > 0,\ q \geq q_0 \\ u_2(x,t) & = & -u_1(x,t) \displaystyle\sum_{i=2}^{i=n} a_i\ sign(u_1)^{n+1-i} x_i / \rho_q(X_2)^{n+1-i} \end{array}\right. \tag{17}$$

*applied to the chained system (11)*

*i) makes the controlled system homogeneous of degree zero with respect to the dilation* $\bar{\delta}_q(\lambda, x) = (\lambda x_1, \delta_q(\lambda, X_2))$,

*ii) makes* $\rho_q(X_2(t))$ *non-increasing along any solution of the controlled system,*

*iii) globally exponentially stabilizes the origin* $x = 0$ *of this system.*

The proof of this Proposition is given in Section 4.

**Remarks**

- By imposing $q$ to be larger than one, although the inverse of $\rho_q(X_2)$ is not defined for $X_2 = 0$, each term $\frac{x_i}{\rho_q(X_2)^{n+1-i}}$ involved in the control $u_2(x,t)$ is homogeneous of positive degree and tends to zero when $X_2$ tends to zero. Therefore, $u_2(x,t)$ is, by continuity, well defined on $I\!R^n \times I\!R$.

- In the control expression (17), $q$ may be seen as a design parameter whose value equals the degree of homogeneity of the control input function $u_2(x,t)$, while the degree of homogeneity of $u_1(x,t)$ is equal to one. The possibility of assigning non-equal degrees of homogeneity for the control inputs results from the specific structure of the chained system and represents an extra degree of freedom at the control design level which had not been considered in [9].

- The minimal value $q_0$ of $q$, for which the homogeneous norm $\rho_q(X_2)$ is uniquely defined, depends *a priori* on the constant $C$, the system's dimension $n$, and the set of parameters $a_i$. The existence of a value of $q_0$ which, for given values of $C$ and $n$, would not depend on the choice of the parameters $a_i$ is a pending question which we have not yet explored.

- The condition imposed on the size of $q_0$ is directly related to the satisfaction of the *transversality condition* described in [9, Th. 5.1]. The connection appears explicitely in the proof of Proposition 2.

- The homogeneous norm $\rho_q(X_2)$ plays the same role as the quadratic function $V_x(X_2)$ in the case of the non-homogeneous controls (12). In particular, the asymptotic stability of the origin of the controlled system stems from the non-increase of this function along any system's solution.

A practical difficulty with the control (17) is that the calculation of $\rho_q(X_2)$ requires solving the polynomial equation $V_x(\delta_q(\rho_q^{-1}, X_2)) = C$. In general, this will have to be done numerically. However, this difficulty can be avoided by considering another homogeneous norm such as

$$\rho_{p,q}(X_2) = (\sum_{i=2}^{i=n} |x_i|^{\frac{p}{n-i+q}})^{\frac{1}{p}}, \qquad \text{with } p > 0 \tag{18}$$

and using this function in the control expression, instead of $\rho_q(X_2)$. This statement is precised in the following proposition which is the second result of this paper.

**Proposition 3** *There exists $q_0 > 1$ such that if $q \geq q_0$ and $p > n-2+q$ then the continuous time-varying feedback control*

$$\left\{ \begin{array}{rcl} u_1(x,t) & = & -k_1 x_1(\sin^2 t + sign(x_1)\sin t) - k_{n+1}\rho_{p,q}(X_2)\sin t, \quad k_1 > 0,\ k_{n+1} > 0 \\ u_2(x,t) & = & -u_1(x,t) \displaystyle\sum_{i=2}^{i=n} a_i\ sign(u_1)^{n+1-i} x_i / \rho_{p,q}(X_2)^{n+1-i} \end{array} \right. \tag{19}$$

*applied to the chained system (11)*

*i) ensures that along any solution of the controlled system,*

$$V_x(Y_2((k+1)\pi)) \leq \alpha(q) V_x(Y_2(k\pi)) \quad \forall k \in I\!N, \tag{20}$$

*where $\alpha(q) < 1$ and* $Y_2 = (\frac{x_2}{\rho_{p,q}(X_2)^{n-2}}, \frac{x_3}{\rho_{p,q}(X_2)^{n-3}}, \ldots, \frac{x_{n-1}}{\rho_{p,q}(X_2)}, x_n)^T$ ,

*ii) globally exponentially stabilizes the origin $x = 0$ of this system.*

The proof of this proposition is given in Section 4.

**Remark**:
Contrary to Propositions 1 and 2, the stability proof no longer relies upon the knowledge of a positive function which is non-increasing along the system's solutions. It uses instead the fact that $V_x(Y_2(t))$, evaluated at periodic time-instants, is decreasing. As shown in the proof of the proposition, this property itself comes from the particular choice of the control $u_1(x,t)$ which is such that $|u_1(x,t)| \geq k_{n+1}\rho_{p,q}(X_2)|\sin t|$, with the sign of $u_1(x,t)$ changing periodically as the sign of $\sin t$. Although the slightly more simple control $u_1(x,t) = -k_1 x_1 - k_{n+1}\rho_{p,q}(X_2)\sin(t)$ does not satisfy this inequality, so that the stability proof does not hold without modification in this case, we conjecture that this control, combined with the control $u_2(x,t)$ of 19, also ensures that the origin of the control system is *g.a.s.* Exponential convergence of the solutions to zero would follow all the same since the controlled systems remains homogeneous of degree zero with respect to the family of dilations defined by $\bar{\delta}_q(\lambda, x) \equiv (x_1, \delta_q(\lambda, X_2))$.

# 4 Proofs of the main results

We report in this section the main steps of the proofs of Propositions 2 and 3. For the sake of conciseness the proofs of a few intermediary technical lemmas are omited. They are of course available from the authors to the interested reader.

## 4.1 Proof of Proposition 2

Let us assume that $q > q_0$ so that, according to Lemma 3, the equation:

$$V_x(\delta_q(\lambda, X_2)) = C \qquad (X_2 \neq 0) \tag{21}$$

has a unique positive solution denoted, from now on, as $\lambda(X_2)$. Differentiating with respect to time both members of the above equality, one obtains

$$V_{x,\lambda}(y)\dot{\lambda} + V_{x,x}(y)\delta_q(\lambda, \dot{X}_2) = 0 \tag{22}$$

with

$$\begin{array}{l} y = (y_2, y_3, \ldots, y_n)^T = \delta_q(\lambda(X_2), X_2)\,, \\ V_{x,x}(y) = (\frac{\partial V_x}{\partial x_2}(y), \ldots, \frac{\partial V_x}{\partial x_n}(y))\,, \\ V_{x,\lambda}(y) = \frac{1}{\lambda} V_{x,x}(y) \begin{pmatrix} r_2 y_2 \\ \vdots \\ r_n y_n \end{pmatrix}. \end{array}$$

From the chained system equations (11) and the definition of $y$, one easily verifies that

$$\delta_q(\lambda, \dot{X}_2) = \lambda \begin{pmatrix} u_1(x,t) y_3 \\ \vdots \\ u_1(x,t) y_n \\ \lambda^{q-1} u_2(x,t) \end{pmatrix}$$

so that one deduces from (22), taking the expression of $u_2(x,t)$ into account, that

$$\dot{\lambda} = -\lambda u_1(x,t) V_{x,x}(y) \begin{pmatrix} y_3 \\ \vdots \\ y_n \\ -\sum_{i=2}^{i=n} sign(u_1)^{n+1-i} a_i y_i \end{pmatrix} / V_{x,\lambda}(y)\,. \tag{23}$$

In the proof of Lemma 3, it is shown that for any $q$ large enough (i.e., $q \geq q_0$), there exists two strictly positive numbers $C_1$ and $C_2$ such that

$$\frac{C_1}{\lambda(X_2)} \leq V_{x,\lambda}(y) \leq \frac{C_2}{\lambda(X_2)} \quad (\forall X_2) \tag{24}$$

Therefore $V_{x,\lambda}(y)$ is strictly positive away from the origin. This corresponds to the tranversality condition introduced in [9, Th. 5.1]. Let us now distinguish two cases.

**Case 1**: $u_1 \geq 0$
In this case,

$$V_{x,x}(y)\begin{pmatrix} y_3 \\ \vdots \\ y_n \\ -\sum_{i=2}^{i=n} sign(u_1)^{n+1-i} a_i y_i \end{pmatrix} = V_{x,x}(y) f_x(y) \tag{25}$$

with

$$f_x(y) = \left(y_3, \ldots, y_n, -\sum_{i=2}^{i=n} a_i y_i\right)^T . \tag{26}$$

In view of Lemma 1, and using the fact that $V_x(y) = V_z(\Phi_k y)$,

$$V_{x,x}(y) f_x(y) = V_{z,z}(w) f_z(w) = -2\frac{k_n}{\prod_{i=2}^{i=n-1} k_i} w_n^2 \quad (\leq 0) \tag{27}$$

with $w = (w_2, \ldots, w_n)^T = \Phi_k y$. Therefore, in view of (23)-(27),

$$\dot{\lambda} = 2\lambda|u_1|\frac{k_n}{\prod_{i=2}^{i=n-1} k_i} w_n^2 / V_{x,\lambda}(y) \quad (\geq 0) \tag{28}$$

This establishes that $\lambda(X_2(t))$ is non-decreasing when $u_1(x(t), t)$ is positive.

**Case 2**: $u_1 < 0$
Let us define the change of coordinates $\psi : I\!R^{n-1} \mapsto I\!R^{n-1}$ as follows:

$$\psi(y) \equiv (y_2, -y_3, \ldots, (-1)^i y_i, \ldots, (-1)^n y_n)^T . \tag{29}$$

From (26) and (29), it is simple to verify that if $u_1$ is negative,

$$\psi[f_x(\psi(y))] = -\begin{pmatrix} y_3 \\ \vdots \\ y_n \\ -\sum_{i=2}^{i=n} sign(u_1)^{n+1-i} a_i y_i \end{pmatrix} . \tag{30}$$

Therefore, using (30) in (23),

$$\dot{\lambda} = -\lambda |u_1(x,t)| V_{x,x}(y) \psi[f_x(\psi(y))] / V_{x,\lambda}(y) \,. \tag{31}$$

Using the definition of the matrix $\Phi_k$ of change of coordinates introduced in Lemma 1, it can also be shown that

$$\Phi_k \psi(y) = \psi(\Phi_k y) \,. \tag{32}$$

Since $V_z$ is a quadratic polynomial function with zero cross terms,

$$V_z(\psi[\Phi_k y]) = V_z(\Phi_k y) \quad (= V_x(y)) \tag{33}$$

By using (32) and (33), it is not difficult to show that

$$V_{x,x}(y) \psi[f_x(\psi(y))] = V_{x,x}(y) f_x(y) \tag{34}$$

so that, in view of (27), (31), and (34),

$$\dot{\lambda} = 2\lambda |u_1| \frac{k_n}{\prod_{i=2}^{i=n-1} k_i} w_n^2 / V_{x,\lambda}(y) \quad (\geq 0) \,. \tag{35}$$

This establishes that $\lambda(X_2(t))$ is non-decreasing when $u_1(x(t),t)$ is negative.

Therefore, whatever the sign of $u_1$, the relation (35) is satisfied and $\dot{\lambda}$ is non-decreasing along any solution of the controlled system. This implies that the positive function $\rho_q(X_2)^p$ $(= \lambda(X_2)^{-p})$, with $p > n - 2 + q$ so as to ensure that this function is of class $C^1$ on $\mathbb{R}^{n-1}$, is non-increasing along any solution of the controlled system. This in turn implies, from the expression of $u_1(x,t)$, that $x_1$ is bounded along any trajectory of the system. Thus, solutions of the controlled system exist for $t \in [0, +\infty)$.

In order to prove that $x = 0$ is asymptotically stable in the sense of Lyapunov, there only remains to show that every solution $x(t)$ asymptotically converges to zero. To this purpose, one can apply Lasalle's invariance principle for time-periodic systems.

**Application of Lasalle's invariance principle**:

One deduces from what precedes that all solutions converge to the largest invariance set $M$ contained in the set

$$E = \{ \ x \ : \ \frac{d}{dt} \rho_q(X_2)^p = 0 \ \} \tag{36}$$

with, in view of (35),

$$\frac{d}{dt}\rho_q(X_2)^p = -2p|u_1(x,t)|\rho_q(X_2)^p \frac{k_n}{\prod_{i=2}^{i=n-1} k_i} w_n^2 / V_{x,\lambda}(y)\,. \tag{37}$$

Using the fact that $C_1\rho_q(X_2) \leq V_{x,\lambda}(y) \leq C_2\rho_q(X_2)$ and that all coefficients $k_i$ are strictly positive:

$$E = \{\ x\ :\ |u_1(x,t)|\rho_q(X_2)^{p-1}w_n^2 = 0\ \}\,. \tag{38}$$

Let us consider a solution $x(t)$ within the set $E$. If $X_2(0) = 0$, then $\rho_q(X_2(t)) = \rho_q(X_2(0)) = 0$, $\forall t \geq 0$, and therefore $X_2(t) = 0$, $\forall t \geq 0$. If $X_2(0) \neq 0$, then $\rho_q(X_2(t))$ is constant and different from zero, so that $X_2(t) \neq 0$, $\forall t \geq 0$. From the expression of $u_1(x,t)$, this in turn implies that $u_1(x(t),t)$ cannot be identically equal to zero. Let $(t_1,t_2)$ denote a non-empty time interval on which $u_1 \neq 0$. Without loss of generality, one can assume that $u_1(x(t),t)$ is positive on $(t_1,t_2)$. Then it comes that $w_n(t) = 0$ for $t \in (t_1,t_2)$, since $x(t)$ belongs to the set $E$. Now, since $\dot{X}_2 = u_1 f_x(X_2)$ when $u_1$ is positive, and since $\lambda(X_2(t))$ $(= \rho_q(X_2(t)^{-1})$ is constant and different from zero, one also has on the interval $(t_1,t_2)$

$$\dot{y} = \lambda u_1 f_x(y) \tag{39}$$

and

$$\dot{w} = \lambda u_1 f_z(w) \tag{40}$$

with

$$f_z(w) = \begin{pmatrix} w_3 \\ -k_2w_2 + w_4 \\ \vdots \\ -k_{n-2}w_{n-2} + w_n \\ -k_{n-1}w_{n-1} - k_nw_n \end{pmatrix} \tag{41}$$

In particular,

$$\dot{w_n} = \lambda u_1(-k_{n-1}w_{n-1} - k_nw_n)\,. \tag{42}$$

Since $w_n(t) = 0$ when $t \in (t_1,t_2)$, with $\lambda$ and $u_1$ strictly positive, one deduces from 42 that $w_{n-1}(t) = 0$ when $t \in (t_1,t_2)$.

By repeating the same reasoning for the other components of the vector $w$, one iteratively establishes that $w_i(t) = 0$ for $i = 2,\ldots,n$ and $t \in (t_1,t_2)$. Therefore, $w(t) = y(t) = X_2(t) = 0$ on the interval $(t_1,t_2)$, so that $\rho_q(X_2(t)) = 0$ on this interval, thus yielding a contradiction with the initial assumption according to which $\rho_q(X_2(t))$ is constant and different from zero for $t \in [0,+\infty)$. The largest invariant set within $E$ is thus contained in the set $\{x : X_2 = 0\}$, so that any solution $x(t)$ is such that $X_2(t)$ asymptotically converges to zero. From the expression of $u_1(x,t)$ and the system's equation $\dot{x}_1 = u_1$, it follows that $x_1(t)$ also converges to zero.

Finally, the exponential rate of convergence of $x(t)$ to zero simply comes from the fact that the controlled system is homogeneous of degree zero with respect to the dilation $\bar{\delta}_q(\lambda, x) = (\lambda x_1, \delta_q(\lambda, X_2))$.

## 4.2 Proof of Proposition 3

This proof makes use of the following three technical lemmas.

**Lemma 4** *The application* $Y_2 \; : \; X_2 \mapsto (\frac{x_2}{\rho_{p,q}(X_2)^{n-2}}, \frac{x_3}{\rho_{p,q}(X_2)^{n-3}}, \dots, \frac{x_{n-1}}{\rho_{p,q}(X_2)}, x_n)^T$, *with* $p > n-2+q$ *and* $q > 0$, *is an homeomorphism on* $\mathbb{R}^{n-1}$, *and a* $C^1$ *diffeomorphism on* $\mathbb{R}^{n-1} - \{0\}$ *provided that* $q$ *is large enough.*
*Moreover,* $Y_2(X_2) = 0$ *if and only if* $X_2 = 0$, *and* $lim_{|X_2| \to +\infty} |Y_2(X_2)| = +\infty$.

**Lemma 5** *Let* $h(u_1, X_2) = (u_1 x_3, u_1 x_4, \dots, u_1 x_n, u_2(u_1, X_2))^T$.
*There exist two continuous functions* $\epsilon_{q,1}(X_2)$ *and* $\epsilon_{q,1}(X_2)$ *from* $\mathbb{R}^{n-1} - \{0\}$ *to* $\mathbb{R}$ *such that*

*i)* $\forall X_2 \neq 0,\ L_{h(u_1,X_2)} \rho_{p,q}(X_2) = \begin{cases} u_1 \epsilon_{q,1}(X_2) & if\ u_1 > 0 \\ u_1 \epsilon_{q,2}(X_2) & if\ u_1 < 0 \end{cases}$

*ii)* $\limsup_{\substack{q \to +\infty \\ X_2 \neq 0}} |\epsilon_{q,i}(X_2)| = 0, \quad (i = 1, 2)$.

**Lemma 6** *Consider the system*

$$\dot{y} = \gamma(t)(A + \epsilon(y,t)B)y \tag{43}$$

*with* $y \in \mathbb{R}^{n-1}$, $\gamma(.)$ *a continuous function from* $\mathbb{R}$ *to* $\mathbb{R}^+$, $A$ *a Hurwitz-stable matrix,* $\epsilon(.,.)$ *a uniformly bounded continuous function from* $(\mathbb{R}^{n-1} - \{0\}) \times \mathbb{R}$ *to* $\mathbb{R}$.

*Let* $P$ *denote a symmetric positive definite (s.p.d.) matrix such that* $PA + A^T P \leq 0$ *(such a matrix exists since* $A$ *is stable), and* $y(t)$ *denote a maximal solution of (43).*

*Then, given:*
*i) a function* $\gamma_0(.)$ *from* $\mathbb{R}$ *to* $\mathbb{R}^+$ *such that* $\gamma_0(t) > 0$ *on some non-empty interval* $(t_1, t_2)$,

*ii) a real number* $\eta \in (0, t_2 - t_1]$,
*there exists* $\alpha \in (0,1)$ *and* $\beta > 0$ *such that:*

$$\left.\begin{array}{r}\gamma(t) \geq \gamma_0(t)\ ,\ \forall t \in (t_1, t_2) \\ \|\epsilon\| \leq \beta\end{array}\right\} \Longrightarrow y(t)^T P y(t) \leq (1-\alpha) y(t_1)^T P y(t_1)\,, \quad \forall t \in [t_1+\eta, t_2] \tag{44}$$

*with* $\|\epsilon\| \stackrel{\Delta}{=} Sup\{\epsilon(x,t) : (x,t) \in (I\!R^{n-1} - \{0\}) \times I\!R\}$.

The proof of Proposition 3 involves two steps. In the first one, we show that if $q$ is large enough and if $x(t)$ is a solution such that $X_2(t) \neq 0$ on some time interval, then there exist two quadratic positive functions $V_+(Y_2)$ and $V_-(Y_2)$ such that, at any time-instant of this time interval, one of them is non-increasing. This implies that any other solution which crosses, at some time, the set $X_2 = 0$ (the same as the set $Y_2 = 0$, in view of Lemma 4) remains in this set everafter. For such a solution, the first state variable satisfies, after a finite time, the equation $\dot{x}_1 = -k_1 x_1(\sin^2 t + sign(x_1)\sin t)$, and this implies that $x_1(t)$ asymptotically converges to zero (see [12], for example). Therefore the only solutions which may not converge to zero are those which never cross the set $X_2 = 0$. The second step of the proof thus consists in showing that any of these solutions asymptotically converges to zero. Exponential stability then simply results from the (easily verifiable) fact that the controlled system is homogeneous of degree zero with respect to the dilation $\bar{\delta}_q(\lambda, x) = (\lambda x_1, \delta_q(\lambda, X_2))$.

**Step 1**:
If $X_2(t) \neq 0$, the derivative of $Y_2(X_2)$ at time $t$ is well defined and such that

$$\dot{Y}_2 = \frac{u_1}{\rho_{p,q}} \begin{pmatrix} y_3 \\ y_4 \\ \vdots \\ y_n \\ -\sum_{i=2}^{i=n} a_i sign(u_1)^{n+1-i} y_i \end{pmatrix} + \frac{L_h \rho_{p,q}}{\rho_{p,q}} \begin{pmatrix} -(n-2)y_2 \\ -(n-3)y_3 \\ \vdots \\ -y_{n-1} \\ 0 \end{pmatrix} \tag{45}$$

with $y_{i+1} = \frac{x_{i+1}}{\rho_{p,q}^{n-i-1}}$ denoting the $ith$ component of the vector $Y_2$.
Let us assume that $X_2(t) \neq 0$ on some interval $[t_0, t_1)$. The function $\frac{u_1}{\rho_{p,q}}$ is well defined on this interval. Moreover, in view of the expression of the control $u_1(x,t)$, $|\frac{u_1}{\rho_{p,q}}| \geq k_{n+1}|\sin t|$ with the sign of $u_1$ being the opposite of the sign of $\sin(t)$. The sign of $u_1$ thus changes periodically. The time interval $[t_0, t_1)$ is also the union of intervals $\Delta_j$ $(j = 0, 1, \ldots)$ such that $|\sin t| \neq 0$ when $t$ belongs to the interior of $\Delta_j$. Without lack of generality, we may assume that $\sin t$ is negative on the intervals $\Delta_{2k}$

and positive on the intervals $\Delta_{2k+1}$.
Let us distinguish two cases.

*First case*: $t \in \Delta_{2k}$ $(\Rightarrow u_1(x(t),t) \geq 0)$.
In this case, using the result of Lemma 5, the equation (45) may be rewritten as

$$\dot{Y}_2 = \frac{|u_1(x,t)|}{\rho_{p,q}(X_2)}(A_+ + \epsilon_{q,1}(X_2)B)Y_2 \tag{46}$$

with

$$A_+ = \begin{pmatrix} 0 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 & 0 \\ -a_2 & -a_3 & \dots & \dots & -a_{n-1} & a_n \end{pmatrix}, \tag{47}$$
$$B = -diag\{n-2, n-3, \dots, 1, 0\}.$$

Since the matrix $A_+$ is stable, there exists two s.p.d. matrices $P_+$ and $Q_+$ such that $P_+A_+ + A_+^T P_+ = -Q_+$. Let us then consider the positive quadratic function $V_+(Y_2) = Y_2^T P_+ Y_2$. Its time-derivative is

$$\dot{V}_+ = \frac{u_1(x,t)}{\rho_{p,q}(X_2)}\; Y_2^T(-Q_+ + \epsilon_{q,1}(X_2)(P_+B + B^T P_+))Y_2\,. \tag{48}$$

Clearly, this time-derivative is non-positive when $\epsilon_{q,1}$ is small enough. In view of Lemma 5, this can be achieved by choosing $q$ large enough. Therefore, $V_+(Y_2(t))$ is non-increasing on $\Delta_{2k}$ when $q$ is large enough.

*Second case*: $t \in \Delta_{2k+1}$ $(\Rightarrow u_1(x(t),t) \leq 0)$.
In this case, we obtain

$$\dot{Y}_2 = \frac{|u_1(x,t)|}{\rho_{p,q}(X_2)}(A_- - \epsilon_{q,2}(X_2)B)Y_2 \tag{49}$$

with

$$A_- = \begin{pmatrix} 0 & -1 & 0 & \dots & \dots & 0 \\ 0 & 0 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -1 & 0 \\ (-1)^{n-1}a_2 & (-1)^{n-2}a_3 & \dots & \dots & (-1)^2 a_{n-1} & -a_n \end{pmatrix}. \tag{50}$$

Since the matrices $A_+$ and $A_-$ share the same characteristic polynomial, the matrix $A_-$ is also Hurwitz-stable and there exists two s.p.d. matrices $P_-$ and $Q_-$ such that $P_-A_- + A_-^T P_- = -Q_-$. Proceeding as in the first case, we obtain that the quadratic positive function $V_+(Y_2) = Y_2^T P_- Y_2$ is non-increasing on the time-interval $\Delta_{2k+1}$ when $q$ is large enough.

**Step 2**:
We now consider a solution such that $X_2(t) \neq 0$, $\forall t \geq 0$.
In this case, the interval $[t_0, t_1)$ considered in Step 1 coincides with $[0, +\infty)$, itself the union of intervals $\Delta_{i+1} = [i\pi, (i+1)\pi)$, $i \in I\!N$. Let us thus again consider two cases, according to whether $u_1$ is positive or negative.

*First case*: $t \in \Delta_{2k}$ ($\Rightarrow u_1(x(t), t) \geq 0$).
The equation of evolution of $Y_2(t)$ is given by (46). Note that $\gamma(t) = \frac{|u_1(x(t),t)|}{\rho_{p,q}(X_2(t))}$ is strictly positive and continuous inside $\Delta_{2k}$, and that it is larger than $\gamma_0(t) = k_{n+1}|\sin(t)|$. In view of Lemma 1, it is also simple to verify that $PA_+ + A_+^T P = diag\{0, \ldots, 0, -\frac{2k_n}{\prod_{i=2}^{i=n-1} k_i}\}$, with $P = \Phi_k^T diag\{1, \frac{1}{k_2}, \ldots, \frac{1}{\prod_{i=2}^{i=n-1} k_i}\}\Phi_k$. Therefore $PA_+ + A_+^T P$ is a semi-negative matrix. Note also that the quadratic function $V_x$ defined in (13) is such that $V_x(Y_2) = Y_2^T P Y_2$. Therefore, by application of Lemma 6, there exists $q_1 > 0$ and $\alpha_{q_1} \in (0, 1)$ such that, if $q \geq q_1$,

$$V_x(Y_2(2k\pi)) \leq (1 - \alpha_{q_1}) V_x(Y_2((2k-1)\pi))\,. \tag{51}$$

*Second case*: $t \in \Delta_{2k+1}$ ($\Rightarrow u_1(x(t), t) \leq 0$).
The equation of evolution of $Y_2(t)$ is given by (49). It has previously been established (see proof of Proposition 2) that $PA_- + A_-^T P = PA_+ + A_+^T P$. By applying again Lemma 6, one deduces that there exists $q_2 > 0$ and $\alpha_{q_2} \in (0, 1)$ such that, if $q \geq q_2$,

$$V_x(Y_2((2k+1)\pi)) \leq (1 - \alpha_{q_2}) V_x(Y_2((2k)\pi))\,. \tag{52}$$

From (51) and (52), we obtain that

$$V_x(Y_2((i+1)\pi)) \leq \alpha V_x(Y_2(i\pi)) \quad \forall i \in I\!N\,, \tag{53}$$

with $\alpha = sup(1-\alpha_{q_1}, 1-\alpha_{q_2})$ ($\in (0,1)$), provided that $q \geq sup(q_1, q_2)$. This relation, plus the fact, proved in Step 1, that $V_+(Y_2(t))$ is non-increasing on the intervals $\Delta_{2k}$ and that $V_-(Y_2(t))$ is non-increasing on the intervals $\Delta_{2k+1}$, clearly imply that $Y_2(t)$ asymptotically converges to zero. The convergence of $x_1(t)$ to zero then easily follows from the first system's equation $\dot{x}_1 = u_1$ and the expression of the control $u_1(x, t)$ (see [12] for example).

# Appendix

## Proof of Lemma 1

The $(n-2)$ first equations of the system (8) directly stem from the definition of the change of coodinates between $X_2$ and $Z_2$, after remarking that $\dot{z}_{j+1} = \frac{\partial z_{j+1}}{\partial X_2}\dot{X}_2 = L_f z_{j+1}$.
Let $p(s) = s^{n-1} + a_n s^{n-2} + \ldots + a_3 s + a_2$ denote the characteristic polynomial associated with the linear system (2)(3). Interpreting $s$ as the time-derivative operator, one has $p(s)x_2 = 0$. In order to prove that the last equation of (8) is correct, one only has to show that the characteristic polynomial $q(s)$ associated with the system (8), i.e. the polynomial with leading coefficient equal to one and such that $q(s)z_2 = 0$, coincides with the polynomial $p(s)$.
Let $B_i$ $(i = 2, \ldots, n)$ denote the operator between $z_2$ and $z_i$, i.e. the operator such that $z_i = B_i z_2$. In view of (8), one has:

$$\begin{aligned} B_2 &= 1 \\ B_3 &= s \\ B_i &= sB_{i-1} + k_{i-2}B_{i-2} \qquad i = 4, \ldots, n \end{aligned} \tag{54}$$

From (54) it is simple to verify that each $B_i$ is a polynomial in the following form, according to whether $i$ is par or odd:

$$\begin{aligned} B_{2j} &= s^{2j-2} + \sum_{k=0}^{j-2} b_{2j,k} s^{2k} \\ B_{2j+1} &= s^{2j-1} + \sum_{k=0}^{j-2} b_{2j+1,k} s^{2k+1} \end{aligned} \tag{55}$$

Using the last equation of (8), one also has:

$$q(s) = (s + k_n)B_n(s) + k_{n-1}B_{n-1}(s) \qquad (= B_{n+1}(s) + k_n B_n(s)) \tag{56}$$

In order to show that $q(s) = p(s)$, one only has to show that the Routh-Hurwitz tables associated with these two polynomials are the same. To this purpose, it is sufficient to prove that the first column of both tables are identical. Indeed, as easily seen from the triangular structure of the Routh-Hurwitz table, each column of the table can be calculated form the previous column, starting with the bottom element of the column and continuing with the upper elements of this column.
Let $R_q$ (resp. $R_p$) denote the Routh-Hurwitz table associated with the polynomial

$q(s)$ (resp. $p(s)$). In view of (6), and since the leading coefficient of the polynomials $B_i(s)$ $(i = 2, \dots, n)$ is equal to one, it is sufficient to prove that, for $j = 2, \dots, n$, the elements of the $jth$ row of $R_q$, denoted as $R_{q,j}$, are equal to the coefficients of the polynomial $k_{n-j+2}B_{n-j+2}$ ordered by decreasing power of $s$. Let us proceed by induction. By construction of $R_q$, and since $q = B_{n+1} + k_n B_n$, the elements of the second row of $R_q$ are equal to the coefficients of $k_n B_n$. The property is thus satisfied for $j = 2$. Assume now that it is satisfied for $j = 2, \dots, m$ with $m < n$, and define $k_{n+1} = 1$. This assumption implies, in view of (54), that the elements of $R_{q,j}$ $(j = 2, \dots, m)$ are equal to the coefficients of the polynomial $k_{n-j+2}(sB_{n-j+1} + k_{n-j}B_{n-j})$. In particular, the elements of $R_{q,m-1}$ are equal to the coefficients of the polynomial $k_{n-m+3}(sB_{n-m+2} + k_{n-m+1}B_{n-m+1})$ so that the $ith$ coefficient of the polynomial $k_{n-m+1}B_{n-m+1}$ is equal to:

$$\frac{1}{k_{n-m+3}} R^{i+1}_{q,m-1} - \frac{1}{k_{n-m+2}} R^{i+1}_{q,m} = -\frac{1}{k_{n-m+3}k_{n-m+2}}(k_{n-m+3}R^{i+1}_{q,m} - k_{n-m+2}R^{i+1}_{q,m-1})$$

where $R^i_{q,j}$ denotes the $i$th term of the row $R_{q,j}$. In view of (5), this coefficient is also, by construction of the Routh-Hurwitz table, equal to $R^i_{q,m+1}$. Therefore the elements of $R_{q,m+1}$ are equal to the coefficients of $k_{n-m+1}B_{n-m+1}$, and the property is satisfied for $j = m + 1$. ∎

## Proof of Lemma 2

Consider the set of quadratic functions:

$$\begin{aligned} V_2(Z_2) &= \tfrac{1}{2} z_2^2 \\ V_k(Z_2) &= \tfrac{1}{2}\left(\textstyle\sum_{i=2}^{k-1} \left(\prod_{j=i}^{k-1} k_j\right) z_i^2 + z_k^2\right) \qquad , \ k = 3, \dots, n \end{aligned}$$

One easily verifies that along any solution $Z_2(t)$ of the system (8):

$$\begin{aligned} \dot{V}_k(Z_2(t)) &= z_k(t) z_{k+1}(t) \qquad , \ k = 2, \dots, n-1 \\ \dot{V}_n(Z_2(t)) &= -k_n z_n(t)^2 \end{aligned}$$

**a) ($k_i > 0$ for $i = 2, \dots n$)$\Rightarrow$(the origin of (8) is asymptotically stable)**
This part of the lemma easily follows from the application of Lasalle's invariance principle, with $V_n(Z_2)$ taken as a Lyapunov function for the system (8).

**b) (the origin of (8) is asymptotically stable)$\Rightarrow$($k_i > 0$ for $i = 2, \dots n$)**
Assume that $k_n \leq 0$, then $\dot{V}_n(Z_2(t)) \geq 0$ so that $V_n(Z_2(t)) > 0$ $\forall t$ if $z_n(0) \neq 0$ and $z_i(0) = 0$ for $i = 2, \dots, n-1$. This implies that the coresponding solution $Z_2(t)$ does

not converge to zero and contradicts the asymptotic stability of $Z_2 = 0$. Therefore $k_n > 0$.
Assume now that at least one coefficient $k_i$, with $i \in (2, \ldots, n-1)$, is equal to zero, and let $m$ be the smallest integer such that $k_m = 0$. Consider a solution with initial conditions such that $z_{m+1}(0) = z_{m+2}(0) = \ldots = z_n(0) = 0$ and $V_m(Z_2(0)) \neq 0$. Since $(z_{m+1} = 0, z_{m+2} = 0, \ldots, z_n = 0)$ is an equilibrium point of the subsystem:

$$\begin{array}{rcl} \dot{z}_{m+1} & = & z_{m+2} \\ \dot{z}_{m+2} & = & -k_{m+1} z_{m+1} + z_{m+3} \\ & \vdots & \\ \dot{z}_{n-1} & = & -k_{n-2} z_{n-2} + z_n \\ \dot{z}_n & = & -k_{n-1} z_{n-1} - k_n z_n \end{array}$$

one deduces that, along this solution, $\dot{V}_m(Z_2(t)) = 0$. Since $V_m(Z_2(t))$ is constant and different from zero, this implies that $Z_2(t)$ does not converge to zero and contradicts the asymptotic stability of $Z_2 = 0$. Therefore none of the coefficients $k_i$ can be equal to zero.
Assume finally that some coefficient $k_m$, with $m \in (2, \ldots, n-1)$, is negative. Then there exists an integer $l \in (m, \ldots, n-1)$ such that $\prod_{j=l}^{n-1} k_j$ is negative. Consider a solution $Z_2(t)$ with initial conditions such that $z_n(0) \neq 0$, $z_n(0)^2 + (\prod_{j=l}^{n-1} k_j) z_l(0)^2 = 0$, and $z_i(0) = 0$ if $i \neq n$ and $i \neq l$. Then $V_n(0) = 0$, $\dot{V}_n(Z_2(t)) \leq 0$, and there exists $\tau > 0$ such that $\dot{V}_n(Z_2(t)) < -\frac{k_n z_n(0)^2}{2}$ for $t \in [0, \tau]$. Therefore $V_n(Z_2(t)) < -\frac{k_n z_n(0)}{2}\tau < 0$ for $t \geq \tau$, implying that $Z_2(t)$ does not converge to zero. Since this contradicts the asymptotic stability of $Z_2 = 0$, one deduces that none of the $k_i$ can be negative. ∎

## Proof of Lemma 3

We first proceed with the proof of part $i$).
In view of (13), the function $V_x$ is a quadratic positive function which vanishes only at the origin. As a consequence, for any $X_2 \neq 0$, $V_x(\delta_q(\lambda, X_2))$ tends to zero as $\lambda$ tends to zero, and to $+\infty$ as $\lambda$ tends to $+\infty$. This implies the existence of a solution $\lambda(X_2)$ to the equation $V_x(\delta_q(\lambda, X_2)) = C$ with $C > 0$. Let us show that, for $q$ large enough, this solution is unique. To this purpose, let us assume that there exists, for some $X_2 \neq 0$, two different values $\lambda_1$ and $\lambda_2$ such that $V_x(\delta_q(\lambda_1, X_2)) = V_x(\delta_q(\lambda_2, X_2)) = C$. Without loss of generality, we can assume that $0 < \lambda_1 < \lambda_2$. By application of the mean value theorem, there exists $\lambda_0 \in [\lambda_1, \lambda_2]$ such that $\dfrac{\partial V_x(\delta_q(\lambda, X_2))}{\partial \lambda}(\lambda_0) = 0$.

Since:

$$\begin{aligned}\frac{\partial V_x(\delta_q(\lambda, X_2))}{\partial \lambda}(\lambda_0) &= \tfrac{\partial V_x}{\partial X_2}(\delta_q(\lambda_0, X_2))\tfrac{\partial \delta_q}{\partial \lambda}(\lambda_0, X_2) \\ &= \tfrac{\partial V_x}{\partial X_2}(\delta_q(\lambda_0, X_2))\tfrac{1}{\lambda_0}(r_2\lambda_0^{r_2}x_2, \ldots, r_n\lambda_0^{r_n}x_n)^T\end{aligned}$$

one also has, in view of (15):

$$\begin{aligned}\tfrac{\partial V_x(\delta_q(\lambda, X_2))}{\partial \lambda}(\lambda_0) &= \tfrac{q}{\lambda_0}\tfrac{\partial V_x}{\partial X_2}(\delta_q(\lambda_0, X_2))(\lambda_0^{r_2}x_2, \ldots, \lambda_0^{r_n}x_n)^T \\ &\quad + \tfrac{q}{\lambda_0}\tfrac{\partial V_x}{\partial X_2}(\delta_q(\lambda_0, X_2))(\tfrac{n-2}{q}\lambda_0^{r_2}x_2, \ldots, \tfrac{1}{q}\lambda_0^{r_{n-1}}x_{n-1}, 0)^T\end{aligned} \tag{57}$$

with, using the fact that $V_x$ is a quadratic function:

$$\frac{\partial V_x}{\partial X_2}(\delta_q(\lambda_0, X_2))(\lambda_0^{r_2}x_2, \ldots, \lambda_0^{r_n}x_n)^T = 2V_x(\delta_q(\lambda_0, X_2)) \tag{58}$$

and:

$$|\frac{\partial V_x}{\partial X_2}(\delta_q(\lambda_0, X_2))(\frac{n-2}{q}\lambda_0^{r_2}x_2, \ldots, \frac{1}{q}\lambda_0^{r_{n-1}}x_{n-1}, 0)^T| \leq \frac{K}{q}V_x(\delta_q(\lambda_0, X_2)) \tag{59}$$

for some positive constant $K$ whose value depends on the matrix involved in the quadratic function $V_x$, and thus on the coefficients $k_i$, $(i = 2, \ldots, n-1)$.
Therefore, in view of (57)-(59):

$$\frac{q}{\lambda_0}(2 - \frac{K}{q})V_x(\delta_q(\lambda_0, X_2)) \leq \frac{\partial V_x(\delta_q(\lambda, X_2))}{\partial \lambda}(\lambda_0) \leq \frac{q}{\lambda_0}(2 + \frac{K}{q})V_x(\delta_q(\lambda_0, X_2)) \tag{60}$$

so that $\frac{\partial V_x(\delta_q(\lambda, X_2))}{\partial \lambda}(\lambda_0)$ (with $X_2 \neq 0$) is strictly positive when $q$ is larger than $\frac{K}{2}$. Since this contradicts the existence of two distincts values $\lambda_1$ and $\lambda_2$ such that $V_x(\delta_q(\lambda_1, X_2)) = V_x(\delta_q(\lambda_2, X_2)) = C$, we have proved that the equation $V_x(\delta_q(\lambda, X_2)) = C$, when $X_2 \neq 0$, has a unique positive real solution.
Note that (60) implies that, for $\lambda_0 = \lambda(X_2)$:

$$\frac{(2q-K)C}{\lambda(X_2)} \leq \frac{\partial V_x(\delta_q(\lambda(X_2), X_2))}{\partial \lambda} \leq \frac{(2q+K)C}{\lambda(X_2)} \tag{61}$$

The positivity of $\frac{\partial V_x(\delta_q(\lambda(X_2), X_2))}{\partial \lambda}$, when $q$ is large enough, corresponds to the *transversality condition* considered in [9].

The smoothness of $\rho_q(X_2) = \lambda(X_2)^{-1}$ on $\mathbb{R}^{n-1} - \{0\}$ simply results from the implicit function theorem, using that $V_x$ and $\delta_q$ are smooth and, as shown above, that $\frac{\partial V_x(\delta_q(\lambda, X_2))}{\partial \lambda}(\lambda(X_2)) \neq 0$ when $X_2 \neq 0$.
Finally, it is simple to verify that $\rho_q(X_2)$ is homogeneous of degree one with respect to the dilation $\delta_q(\lambda, X_2)$ by using the part i) of the lemma and the fact that $\delta_q(\rho_q(Y_2)^{-1}, Y_2) = \delta_q(\frac{\lambda}{\rho_q(Y_2)}, X_2)$ for $Y_2 = \delta_q(\lambda, X_2)$. ∎

### Proof of Lemma 4

We first note that the mapping $Y_2$ is such that:

$$Y_2(\delta_q(\lambda, X_2)) \;=\; \lambda^q Y_2(X_2) \tag{62}$$

This already implies that $Y_2$ is continuous at the origin. This also implies that $Y_2$ is onto since this mapping transforms any element of the set $S_\rho = \{X_2 : \rho_{p,q}(X_2) = 1\}$ into itself.
In order to show that $Y_2$ is injective, let us proceed by contradiction and assume the existence of two distinct non-zero vectors $Z_2 = (z_2, \ldots, z_n)^T$ and $W_2 = (w_2, \ldots, w_n)^T$ such that $Y_2(Z_2) = Y_2(W_2)$. Let us denote as $\lambda_z, \lambda_w$ and $\bar{Z}_2, \bar{W}_2$ the coefficients and vectors defined by:

$$\begin{array}{rl} Z_2 = \delta_q(\lambda_z, \bar{Z}_2) & \quad \rho_{p,q}(\bar{Z}_2) = 1 \\ W_2 = \delta_q(\lambda_w, \bar{W}_2) & \quad \rho_{p,q}(\bar{W}_2) = 1 \end{array}$$

Using (62) and the fact that $Y_2(Z_2) = Y_2(W_2)$ one deduces that $\bar{W}_2 \;=\; (\dfrac{\lambda_z}{\lambda_w})^q \bar{Z}_2$ so that:

$$\rho_{p,q}(\alpha_0 \bar{Z}_2) \;=\; \rho_{p,q}(\bar{Z}_2) \;= 1 \tag{63}$$

with $\alpha_0 = (\dfrac{\lambda_z}{\lambda_w})^q \neq 1$. By application of the mean value theorem, this implies the existence of $\alpha > 0$ such that:

$$\frac{\partial \rho_{p,q}}{\partial X_2}(\alpha \bar{Z}_2)\bar{Z}_2 \;=\; 0 \tag{64}$$

On the other hand, and in view of the definition of $\rho_{p,q}$:

$$\frac{\partial \rho_{p,q}}{\partial X_2}(\alpha \bar{Z}_2)\bar{Z}_2 \;=\; \frac{1}{\rho_{p,q}(\alpha \bar{Z}_2)^{p-1}} \sum_{i=2}^{n} \frac{1}{r_i} \alpha^{\frac{p}{r_i}-1} |\bar{z}_i|^{\frac{p}{r_i}} \tag{65}$$

can be equal to zero only if $\bar{Z}_2 = 0$. Since, $\bar{W}_2 \;=\; (\frac{\lambda_z}{\lambda_w})^q \bar{Z}_2$, this would imply that $W_2 = Z_2 = 0$, thus yielding a contradiction. Therefore $Y_2$ is a continuous one-to-one function on $\mathbb{R}^{n-1}$.
From (62), it is simple to prove that $|Y_2(X_2)|$ tends to infinity when $|X_2|$ tends itself to infinity, and from there that $Y_2^{-1}$ is also continuous.
The fact that $Y_2$ and the inverse mapping are of class $C^1$ on $\mathbb{R}^{n-1} - \{0\}$ simply results from the implicit function theorem and the fact that the jacobian $\dfrac{\partial Y_2}{\partial X_2}$ is regular on this set for $q$ large enough. ∎

## Proof of Lemma 5

By definition of the Lie derivative:

$$L_{h(u_1,X_2)}\rho_{p,q}(X_2) = \frac{\partial \rho_{p,q}}{\partial X_2}(X_2)h(u_1,X_2) \tag{66}$$

with, in view of the definition of $\rho_{p,q}$:

$$\frac{\partial \rho_{p,q}}{\partial X_2}(X_2) = (\frac{1}{r_2}\frac{|x_2|^{\frac{p}{r_2}-1}sign(x_2)}{\rho_{p,q}(X_2)^{p-1}},\dots,\frac{1}{r_n}\frac{|x_n|^{\frac{p}{r_n}-1}sign(x_n)}{\rho_{p,q}(X_2)^{p-1}}) \tag{67}$$

and, in view of the expression of $u_2(u_1,X_2,t)$:

$$h(u_1,X_2) = u_1\ (x_3,\dots,x_n,-\sum_{i=2}^{n}a_i sign(u_1)^{n+1-i}\frac{x_i}{\rho_{p,q}(X_2)^{n+1-i}})^T \tag{68}$$

It is then simple to verify that the functions:

$$\begin{array}{l} \epsilon_{q,1}(X_2) = \frac{\partial \rho_{p,q}}{\partial X_2}(X_2)(x_3,\dots,x_n,-\sum_{i=2}^{n}a_i\frac{x_i}{\rho_{p,q}(X_2)^{n+1-i}})^T \\ \epsilon_{q,2}(X_2) = \frac{\partial \rho_{p,q}}{\partial X_2}(X_2)(x_3,\dots,x_n,\sum_{i=2}^{n}a_i(-1)^{n-i}\frac{x_i}{\rho_{p,q}(X_2)^{n+1-i}})^T \end{array} \tag{69}$$

are homogeneous of degree zero with respect to the dilation $\delta_q(\lambda,X_2)$ and, using the fact that $r_i$ tends to infinity when $q$ tends to infinity, that they satisfy the properties of the lemma. ∎

## Proof of Lemma 6

Define:

$$g(t) = \int_{t_1}^{t}\gamma(s)ds \qquad \text{for } t\in[t_1,t_2] \tag{70}$$

Since $\gamma(t)$, the derivative of $g(t)$, is strictly positive on $(t_1,t_2)$, $g(t)$ monotically increases on the time interval $(t_1,t_2)$ so that the inverse of $g$, denoted as $g^{-1}$, is well defined on $[0,g(t_2)]$. It is also differentiable on $(0,g(t_2))$.
Now, define:

$$z(\tau) \equiv y(g^{-1}(\tau)) \qquad \text{for } \tau\in[0,g(t_2)] \tag{71}$$

Differentiating $z(\tau)$ with respect to $\tau$, one obtains in view of (43):

$$\frac{d}{d\tau}z(\tau) = Az(\tau)+\epsilon(z(\tau),g^{-1}(\tau))Bz(\tau) \qquad \text{for } \tau\in[0,g(t_2)] \tag{72}$$

Let $\bar{z}(\tau)$ denote the solution of the linear system $\frac{d}{d\tau}\bar{z} = A\bar{z}$ with $\bar{z}(0) = z(0)$. One has $\bar{z}(\tau) = exp^{A\tau}z(0)$, with $|exp^{A\tau}| \leq K_1 < +\infty$ (since $A$ is a stable matrix).
Introducing the vector $\tilde{z} = z(\tau) - \bar{z}(\tau)$, one deduces that:

$$\frac{d}{d\tau}\tilde{z}(\tau) = A\tilde{z}(\tau) + \epsilon(z(\tau), g^{-1}(\tau))B(\tilde{z}(\tau) + exp^{A\tau}z(0)) \tag{73}$$

with $\tilde{z}(0) = 0$.
Since $A$ is stable, there exists a s.p.d. matrix $Q$ such that $QA + A^TQ = -I$, where $I$ stands for the unit matrix. Therefore, using (73):

$$\frac{d}{d\tau}(\tilde{z}(\tau)^TQ\tilde{z}(\tau)) = -|\tilde{z}(\tau)|^2 + 2\epsilon(z(\tau), g^{-1}(\tau))\tilde{z}(\tau)^TQB(\tilde{z}(\tau) + exp^{A\tau}z(0)) \tag{74}$$

and, denoting the upperbound of $|\epsilon(x,t)|$ as $\beta$:

$$\frac{d}{d\tau}(\tilde{z}(\tau)^TQ\tilde{z}(\tau)) \leq -(1 - 2\beta|Q||B|)|\tilde{z}(\tau)|^2 + 2\beta K_1|Q||B||z(0)||\tilde{z}(\tau)| \tag{75}$$

Using the fact that $\tilde{z}(0) = 0$, one deduces from the above inequality that, whenever $\beta$ is smaller than $\frac{1}{4|Q||B|}$:

$$\tilde{z}(\tau)^TQ\tilde{z}(\tau) \leq \beta^2K_2|z(0)|^2 \qquad \forall\tau \in [0, g(t_2)] \tag{76}$$

with $K_2 = 16|Q|^3|B|^2K_1^2$.
The above inequality in turn implies the existence of a finite positive number $K_3$ such that:

$$\frac{\tilde{z}(\tau)^TP\tilde{z}(\tau)}{z(0)^TPz(0)} \leq K_3\beta^2 \tag{77}$$

Since $\bar{z}(\tau) \leq K_1|z(0)|$, it also results from (76) that there exists a positive number $K_4$ such that:

$$\frac{\tilde{z}(\tau)P\bar{z}(\tau)}{z(0)^TPz(0)} \leq K_4\beta \tag{78}$$

Now, let us consider the function $a(\tau) = \frac{\bar{z}(\tau)^TP\bar{z}(\tau)}{\bar{z}(0)^TP\bar{z}(0)}$ which is analytic, since $\bar{z}(\tau) = exp^{A\tau}\bar{z}(0)$, and such that $a(0) = 1$. Since $PA + A^TP$ is semi-negative, $a(\tau)$ is non increasing. In fact, $a(\tau) < 1$, $\forall\tau > 0$. Otherwise there would exist $\tau_1 > 0$ such that $a(\tau_1) = 1$, and this would imply that $a(\tau) = 1$ on the interval $[0, \tau_1]$, and consequently that $a(\tau) = 1$, $\forall\tau > 0$ (by the analyticity of $a(\tau)$). This would in turn contradict the fact that the origin of the system $\frac{d}{d\tau}\bar{z} = A\bar{z}$ is asymptotically stable. Setting:

$$\bar{\eta} = \int_{t_1}^{t_1+\eta} \gamma_0(s)ds \quad (> 0) \tag{79}$$

one thus has $a(\bar{\eta}) < 1 - 2\alpha$ for some $\alpha \in (0,1)$, and:

$$\frac{\bar{z}(\tau)^T P \bar{z}(\tau)}{z(0)^T P z(0)} \leq 1 - 2\alpha \qquad \text{for } \tau \geq \bar{\eta} \tag{80}$$

Using that $z^T P z = \tilde{z}^T P \tilde{z} + 2\tilde{z}^T P \bar{z} + \bar{z}^T P \bar{z}$ it comes from (77)-(80) that:

$$\frac{z(\tau)^T P z(\tau)}{z(0)^T P z(0)} \leq K_3\beta^2 + 2K_4\beta + 1 - 2\alpha \qquad \forall \tau \in [\bar{\eta}, g(t_2)] \tag{81}$$

Therefore, if $\beta$ is small enough, then:

$$\frac{z(\tau)^T P z(\tau)}{z(0)^T P z(0)} \leq 1 - \alpha \qquad \forall \tau \in [\bar{\eta}, g(t_2)] \tag{82}$$

Recalling that $z(0) = y(t_1)$ and $z(\tau) = y(g^{-1}(\tau))$, the above inequality is the same as:

$$\frac{y(t)^T P y(t)}{y(t_1)^T P y(t_1)} \leq 1 - \alpha \qquad \forall t \in [g^{-1}(\bar{\eta}), t_2] \tag{83}$$

By assumption $\gamma(t) \geq \gamma_0(t)$, so that $\bar{\eta} \leq g(t_1 + \eta)$ and, since $g^{-1}$ is a monotonic increasing function, $g^{-1}(\bar{\eta}) \leq t_1 + \eta$. It is then clear that (83) implies the inequality in the right-hand side of (44). ∎

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