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Abstract: This paper generalizes the traditional Mean-Variance method in portfolio analysis when asset returns are assumed to be jointly stable. An α-stable efficient frontier is computed and compared to the classical Gaussian one. The efficient frontier computed from this analysis model dominates the one defined in terms of the Markowitz portfolio selection model criterion.

Key-words: Stable laws, Stable vector, Spectral measure, Risk, Value at Risk, Portfolio selection.

(Résumé : tsvp)

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Choix optimal de portefeuille dans un marché α-stable

Résumé : Nous définissons en matière de choix de portefeuilles, un modèle d'analyse des portefeuilles efficients dans un contexte où les taux de rentabilité des actifs suivent une loi α-stable multivariée. La représentation des portefeuilles efficients 1.7-stables et gaussiens dans la même espace (rentabilité espérée,paramètre d'échelle), montre que la frontière efficiente 1.7-stable domine son homologue gaussienne.

Mots-clé : Lois stables, vecteur stable, Mesure spectrale, Risque Value at Risque, Choix de portefeuilles.

1. Introduction

Portfolio selection models have been extensively discussed in the finance literature, in particular following the ideas of Markowitz [11] and Sharpe [15].

The Markowitz analysis of efficient portfolio selection can be formulated as a quadratic programming problem of optimization which objective is to build a portfolio that offers the highest level of return for each level of risk or the lowest level of risk for a desired level of return. Sharpe proposed a simplified version of this problem by assuming that the security returns are generated by a single-index model. Interrelationships among security returns are therefore modeled through a relationship to one common index.

In both these works, the joint probability distribution of security returns is assumed to be multivariate normal, and risk is represented as the variance of return. As a consequence, optimal portfolio for all risk averse investors lie on the Markowitz mean-variance admissible frontier. However, as noted in Fama [5] and Bawa([1]), in reference to symmetric stable distributions, it is possible to weaken this condition by considering large classes of probability distributions,for instance a location-scale family. This family includes stable distributions as special cases. Relying on the stochastic dominance concept (Bawa [1]), one can apply the same type of analysis as in the mean-variance case, replacing mean and variance with mean and scale respectively. The problem of estimating multivariate α-stable distributions has thus received increasing attention in recent years in modeling portfolio of financial assets (see Mittnik and Rachev [12] and references therein).

from a general point of view, stable distributions are interesting, because their statistical properties are very useful in economic analysis. There are several reasons for the popularity of stable distributions for modeling asset returns :

1. from a "practical" view point, stable distributions are capable of explaining the observed leptokurtosis and skewness in asset return distributions (see Mandelbrot [9, 10] and Fama [5]) ;
2. as their name indicates, stable distributions are stable by linear combination ;
3. stable distributions, while being fairly flexible, remain tractable, since they are characterized by four parameters.

For these reasons, stable distributions have frequently been viewed as an appropriate model. In particular, in the frame considered here of optimal portfolio selection, stable distributions yield an alternative to normal ones: the randomness observed is still the result of summing many small effects, but now these effects follow a heavy-tailed distribution.

The purpose of this paper is to solve the problem of efficient portfolio optimization in a market where asset returns follow a multivariate stable distribution. The paper is organized as follows. In section 2, we recall some definitions and characteristics of an asset portfolio under "stable" assumptions. Section 3 contains a formulation of the portfolio analysis problem. In section 4 we generate the "stable" efficient frontier from a particular case study and compare it to the "Gaussian" one.

2. Characteristics of "stable" asset portfolio

Recall [14] that a d-dimensional random vector X follows an α-stable multivariate distribution $(0 < \alpha < 2)$ if there exists a finite measure Γ on the unit sphere S_d of $\mathbb{R}^d$ ($S_d = \{s,\ ||s|| = 1\}$), and a shift vector $\mu^0 \in \mathbb{R}^d$ such that the general form of the logarithm of its characteristic function is:

$$\log \Psi_X(\lambda) = -\int_{S_d} |(\lambda, s)|^\alpha (1 - i\ sign((\lambda, s)) W(\alpha, s, \lambda)) \Gamma(ds) + i(\lambda, \mu^0),\ \ \lambda \in \mathbb{R}^d \tag{1}$$

where $(.,.)$ denotes the inner product and

$$W(\alpha, s, \lambda) = \begin{cases} \tan \frac{\pi\alpha}{2} & \text{if } \alpha \neq 1 \\ -\frac{2}{\pi} \log |(\lambda, s)| & \text{if } \alpha = 1 \end{cases} \tag{2}$$

X is a *symmetric* α-stable denoted $S\alpha S$ vector in $\mathbb{R}^d$, $0 < \alpha < 2$ if and only if the shift vector $\mu^0 = 0$ and the spectral measure Γ is symmetric. Thus its characteristic function takes the following form:

$$E \exp i(\lambda, X) = \exp\left\{ -\int_{S_d} |(\lambda, s)|^\alpha \Gamma(ds) \right\},\ \ \lambda \in \mathbb{R}^d \tag{3}$$

From now on, R will denote a d-vector of considered asset returns and $E(R) = \mu^0$. We assume that $R - \mu^0$ follows a $S\alpha S$, $\alpha > 1$, distribution with characteristic function

$$E \exp i(\lambda, R) = \exp\left\{ -\int_{S_d} |(\lambda, s)|^\alpha \Gamma(ds) + i(\lambda, \mu^0) \right\},\ \ \lambda \in \mathbb{R}^d \tag{4}$$

The process of return on an asset is defined by $\log \frac{S_t}{S_{t-1}}$, where S_t denotes the price of the asset at time t (market closes).

A portfolio p, linear combination of d financial assets, will be characterized by a pair (mean,scale) given by:

$$E(R_p) = \sum_{i=1}^{d} x_i E(R_i) \tag{5}$$

$$\Lambda_p(x) = \left(\int_{S_d} |(x, s)|^\alpha \Gamma(ds) \right)^{\frac{1}{\alpha}} \tag{6}$$

where

- x_i is the amount invested in the asset i,
- x is the d-vector of portfolio components,
- $E(R_i)$ is the expected return of the asset i,
- R_p is the portfolio return and
- Γ is the spectral measure of the random vector $(R_1, ..., R_d)$ of asset returns on the unit sphere S_d.

Remark 1.
If $\alpha = 2$, Λ_p^α defines a positive quadratic form. Indeed, let $VarR$ be the variance of R, $Cov(R_i, R_j)$ be

the covariance of R_i and R_j, Ω be the Variance-Covariance matrix of $R = (R_1, ..., R_d)$ and $[R_i, Rj]_\alpha$ denotes the α-covariation [1] of R_i on R_j defined as:

$$[R_i, Rj]_\alpha = \int_{S_2} s_1 s_2^{<\alpha-1>} \Gamma(ds) \tag{7}$$

where $a^{<\alpha-1>}$ is the "signed power" defined by $a^{<\alpha-1>} = |a|^{\alpha-1} sign(a)$, then:

$$\begin{aligned}
\Lambda_p(x)^2 &= \int_{S_d} (\sum_{i=1}^{d} x_i s_i)^2 \Gamma(ds) \\
&= \int_{S_d} (\sum_{i=1}^{d} x_i^2 s_i^2) \Gamma(ds) + \int_{S_d} \sum_{i,j,i\neq j} x_i x_j s_i s_j \Gamma(ds) \\
&= \sum_{i=1}^{d} x_i^2 \int_{S_d} s_i^2 \Gamma(ds) + \sum_{i,j,i\neq j} x_i x_j \int_{S_d} s_i s_j \Gamma(ds) \\
&= \frac{1}{2} \sum_{i=1}^{d} x_i^2 Var R_i + \sum_{i,j,i\neq j} x_i x_j [R_i, R_j]_2 \\
&= \frac{1}{2} \left[\sum_{i=1}^{d} x_i^2 Var X_i + \sum_{i,j,i\neq j} x_i x_j Cov(R_i, R_j) \right] \\
&= \frac{1}{2} x^t \Omega x \\
&= \frac{1}{2} Var(x^t R)
\end{aligned}$$

Under our assumptions ($1 < \alpha < 2$), the variance is infinite. The risk is thus quantified by the scale parameter, given by (6), of a linear combination of an α-stable vector (see Belkacem et al [2]), rather than by a quadratic form (i.e. the Gaussian case).

Given n observations $R_1, ..., R_n$ of the vector R during a period of time, we consider the vectors obtained in polar coordinates by gathering the observations: $\rho = (|R_1|, ..., |R_n|)$ and $\Theta = \theta(R) = (\theta_1(R), ..., \theta_n(R))$ (each $\theta_i(R)$ is a $(d-1)$-vector).

$\Lambda_p(x)$ may be estimated (see Rachev-Xin [13] and Cheng-Rachev [3]) by:

$$\hat{\Lambda}_p^\alpha(x) = \int_{\Omega_d} \xi(x, \vartheta) d\Phi_n(\vartheta) \tag{8}$$

where

- $x = (r \prod_{i=1}^{d-1} \sin \varphi_i, r \prod_{i=1}^{d-2} \sin \varphi_i \cos \varphi_{d-1}, ..., r \sin \varphi_1 \cos \varphi_2, r \cos \varphi_1)$
- $\xi(x, \vartheta) = |x|^{\alpha_n} \left| \cos \varphi_1 \cos \vartheta_1 + \sum_{i=2}^{d-1} \cos \varphi_i \cos \vartheta_i \prod_{k=1}^{i-1} \sin \varphi_k \sin \vartheta_k + \prod_{i=1}^{d-1} \sin \varphi_i \sin(\vartheta_i) \right|^{\alpha_n}$
- $\Omega_d = [0, \pi]^{d-2} \times [0, 2\pi]$;
- α_n is an estimator of the index α chosen using the family of estimators given by:

$$\alpha_n(k) = \frac{\log 2}{\log(\rho_{(\text{n-k+1})}) - \log(\rho_{(\text{n-2k+1})})} \tag{9}$$

where $k = (k_n)_{n\geq 1}$ is a sequence of integers satisfying $1 \leq k_n \leq \frac{n}{2}$, $k \to \infty$, $\frac{k}{n} \to 0$ as $n \to \infty$ and $\rho_{(\text{k})}$ is the $k-th$ order statistic from ρ;

[1] For more facts of this notion see Samorodnitsky-Taqqu [14].

- $\Phi_n(\vartheta)$ is an estimator of the distribution function of Γ on Ω_d, obtained as follows

$$\Phi_n(\vartheta) = \varphi_n(\vartheta)\Phi_n(\Pi) \tag{10}$$

where

- $\Pi = (\pi, ..., \pi, 2\pi) \in \Omega_d,\ \vartheta = (\vartheta_1, ..., \vartheta_{d-1})$,
- $\varphi_n(\vartheta) = \frac{1}{k}\sum_{i=1}^{n} \mathbb{1}_{(\Theta_i \leq \vartheta\ ,\ \rho_i \geq \rho_{(\text{n-k})})}$,
- $\Phi_n(\Pi) = \frac{k}{n}\rho_{(\text{n-k})}^{\alpha_n}$.

Under some regularity assumptions, Cheng and Rachev [3] show that

$$\frac{\log 2\sqrt{k}}{\log n/k}\left(\frac{\hat{\Lambda}_p^\alpha(x)}{\Lambda_p^\alpha(x)} - 1\right) \xrightarrow{d} \mathbf{N}(0,1). \tag{11}$$

3. The "stable" portfolio analysis problem

An efficient portfolio was defined by Markowitz [11] and Sharpe [15] as a portfolio of risky asset which can not achieve greater expected return without increasing risk.

In our frame of stable distributions, and in the absence of a riskless asset, a portfolio P on the efficient frontier is defined as a portfolio solution of the following optimization problem :

$$\min_{x\in\mathbb{R}^d} \Lambda_p(x) = \left(\int_{S_d} |(x,s)|^\alpha \Gamma(ds)\right)^{\frac{1}{\alpha}} \tag{12}$$

subject to :

$$\begin{aligned} (x, \mu^0) &= \bar{R}_p \\ (x, e) &= 1 \text{ (the entire portfolio must be invested)} \\ x_i &\geq 0 \text{ (no short sales)} \end{aligned}$$

where μ^0 is the d-vector of assets expected returns, $\bar{R}_p$ is a fixed level of portfolio return, and e denotes an d-vector of ones.

Remark 2.
If $\alpha = 2$, by remark (1), the optimization problem will reduced to :

$$\min_{x\in\mathbb{R}^d} \frac{1}{2} x^t \Omega x$$

subject to :

$$\begin{aligned} x^t \mu^0 &= \bar{R}_p \\ x^t e &= 1 \\ x_i &\geq 0 \textit{ for } i = 1, ..., d \end{aligned}$$

We thus recover the well known Markowitz [11] *problem of portfolio analysis.*

In the case of $\alpha = 2$, the optimization is a quadratic programming. Efficient portfolios depend upon expected return and variance combinations. A good review of quadratic programming can be found in Dorn [4] and references therein. Ho [8] introduce a new algorithm for the quadratic programming

solution targeting portfolio optimization. However, if the non negativity constraints are inactive, an explicit solution can be obtained by the method of Lagrange multipliers.

Let consider now the case where $1 < \alpha < 2$. This case is much more difficult because it is not clear how to reduces the problem to a quadratic programming one. However, $\Lambda_p(\boldsymbol{x})$ is a convex function of $\boldsymbol{x}$, and the solution can thus be obtained by convex programming.

Having estimated the risk using (9), the subset of efficient portfolios may be obtained by fixing $\bar{R}_p$ and then choosing $\boldsymbol{x}$ to satisfy:

$$\min_{\boldsymbol{x}\in\mathbb{R}^d} \hat{\Lambda}_p(\boldsymbol{x}) \tag{13}$$

subject to :

$$\begin{aligned} \boldsymbol{x}^t\mu^0 &= \bar{R}_p \\ \boldsymbol{x}^t e &= 1 \\ x_i &\geq 0,\ i = 1, \ldots, d\,; \end{aligned}$$

and then changing $\bar{R}_p$ and repeating the minimization. One point in the efficient set is obtained for each $\bar{R}_p$.

In practice (13) is resolved numerically using a Sequential Quadratic Programming (SQP) algorithm. For an overview of SQP methods see for example Fletcher [6] and Gill et *al.* [7].

4. Case study

To illustrate these techniques, we present some results from a particular case study. The study consists in the determination of the "stable" efficient frontier for the combination of three assets and its comparison to the Gaussian one.

The data consist of 2059 observations representing the successive differences of the logarithm of daily closing prices for three chosen stocks [2], ranging from 09/07/87 to 31/05/95.

In order to be able to analyze the risk and the efficient frontier, and to make some meaningful comparisons between the normal and the stable modelings, we need to compute the estimated mean, variance (assuming the data are Gaussian) and scale parameter [3] (assuming the data follow a stable distribution). These estimations are shown in Table 1. For estimating the risk function, we need to estimate the normalized spectral measure φ_n and then the spectral density Φ_n, given by (10).

Assets	Mean (%)	σ^2 (%)	$\frac{\sigma}{\sqrt{2}}$ (%)	$\gamma_{1.7}$ (%)
ACCOR	0.0279	0.028	1.17	0.80
Cie. BANCAIRE	0.0345	0.051	1.60	1.07
CARREFOUR	0.0780	0.026	1.13	0.66

Table 1. Estimates mean, variances, st-deviation over $2^{\frac{1}{2}}$ and scale parameter of daily rate return for different assets.

[2] Accor, Carrefour and Cie. Bancaire.

[3] The scale parameter is estimated with the spectral method [13], [2]

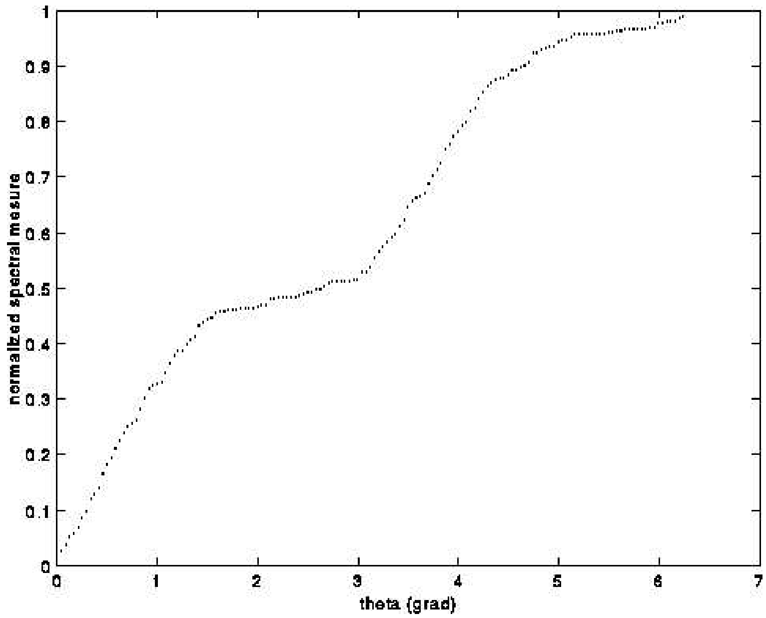

FIGURE 1. The estimated normalized spectral measure φ_n of Accor-Carrefour data.

The plot of the estimated normalized spectral measure of the ACCOR-CARREFOUR data, displayed on figure 1, shows that the two assets are positively associated at a given confidence level because the spectral measure is concentrated on the thirst and the third quadrant (for more fact on this notion see e.g. Rachev and Xin [13]).

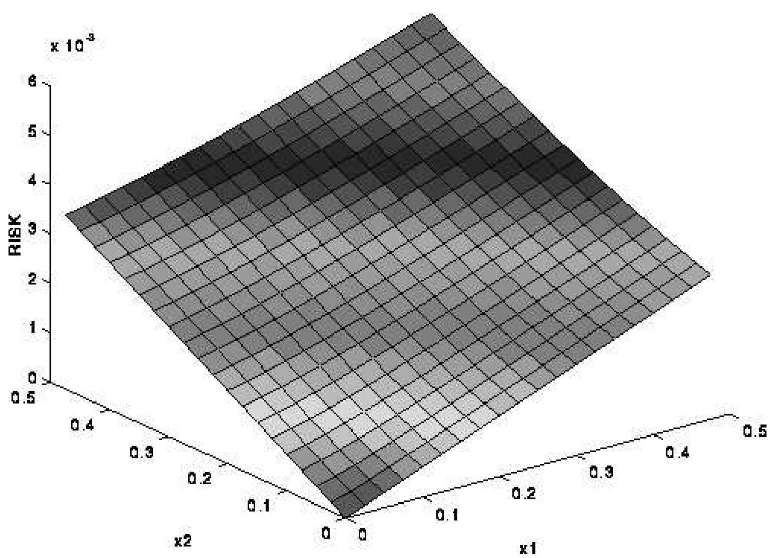

FIGURE 2. The estimated risk function of a portfolio of Accor and Carrefour stocks.

From the estimation of Φ, we compute numerically $\hat{\Lambda}_p$ using Riemann-Stieljes sum approximation. We verify on figure 2, which displays the estimated risk function of different combinations of ACCOR-CARREFOUR, that risk increases with increasing weights attached to each stock. For more than two assets, computation of the risk function is numerically more complicated but results stay consistent.

We now consider, in both cases (Gaussian and stable) several initial investment strategies. Table 2 gives the initial distributions assumed for three portfolios denoted by A, B and C and their respective means, variances (in the Gaussian case) and scale parameters (in the stable case).

Assets	Percentage by asset in portfolios:		
	A	B	C
ACCOR	15	10	10
Cie BANCAIRE	10	20	30
CARREFOUR	75	70	60
Total assets yields (% a day)	100	100	100
Mean rate of return (% a day)	0.066	0.0643	0.0599
Variance of return (% a day)	0.0207	0.0212	0.0217
$\frac{\sigma}{\sqrt{2}}$ (% a day)	1.02	1.03	1.04
$\hat{\Lambda}_p$ (% a day)	0.62	0.64	0.66

TABLE 2. Distribution of assets assumed for each portfolio

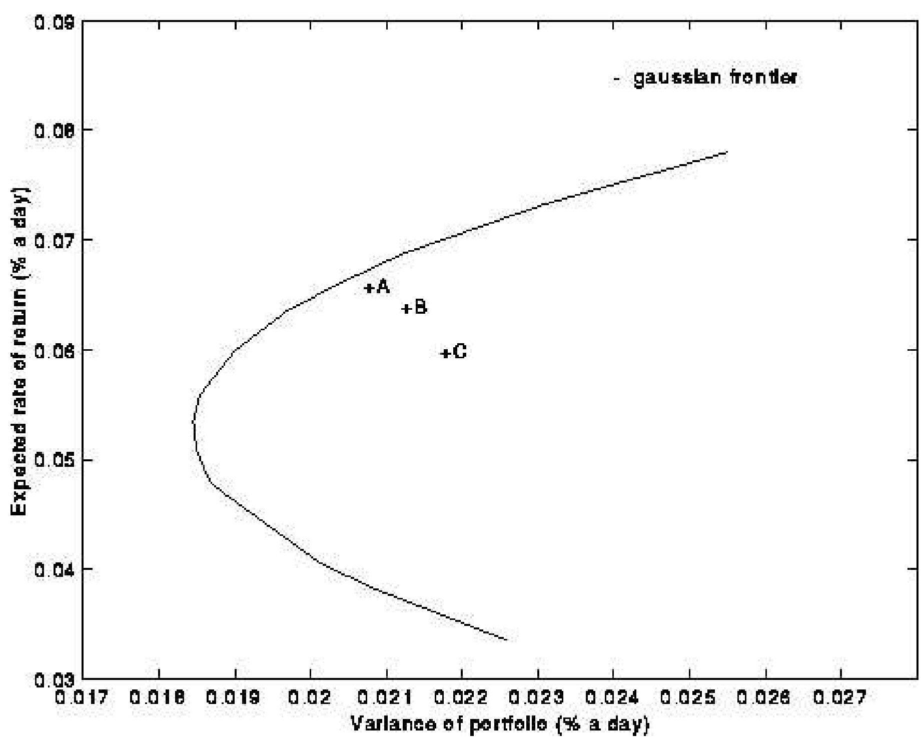

FIGURE 3. Gaussian efficient frontier

Figure 3 shows the "Gaussian" efficient frontier for combinations of the three assets used in table 1 and defined in term of the Markowitz portfolio selection model criterion. As expected, the diagram

shows the trade-off between expected rate of return and the variance of the rate of return. Note that none of points A, B and C, are efficient in traditional sense (MV).

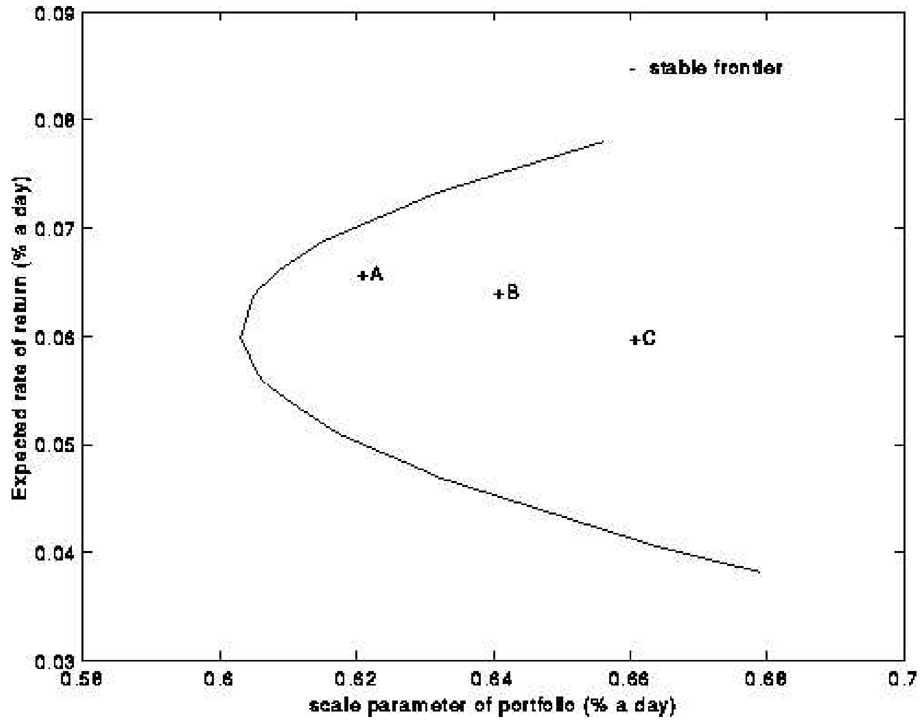

FIGURE 4. 1.7-stable efficient frontier.

Figure 4 shows the 1.7-stable efficient frontier for combinations of the same three assets and defined in term of our portfolio selection model presented in section 3. The diagram shows the trade-off between expected rate of return and scale parameter of the rate of return. A, B and C are the same initial portfolios. All these portfolios are feasible in the mean-scale space but here again none of them is efficient in the sense of our model.

One can note that the Gaussian model is a particular case of α-stable model with $\alpha = 2$. In this case, the scale parameter is equal to the standard deviation divided by $\sqrt{2}$. One may thus represent then the Gaussian efficient frontier, deduced from the Markowitz optimization problem, in the mean-scale space. The plot of the Gaussian and stable efficient frontier in the same mean-scale space, displayed on figure 5, shows that for a given expected portfolio return, the associated risk in the stable model is lower than its counterpart in the Gaussian model. Thus an efficient portfolio in the Gaussian sense is not in general an efficient one in the stable sense. We can conclude from this diagram that the set of 1.7-stable efficient portfolios combinations of the three assets given by table 1 dominates the Gaussian one. The characteristics of stable and Gaussian efficient portfolios are given respectively on Table 3 and Table 4. One can note that the structure of these two optimal portfolios are quite different.

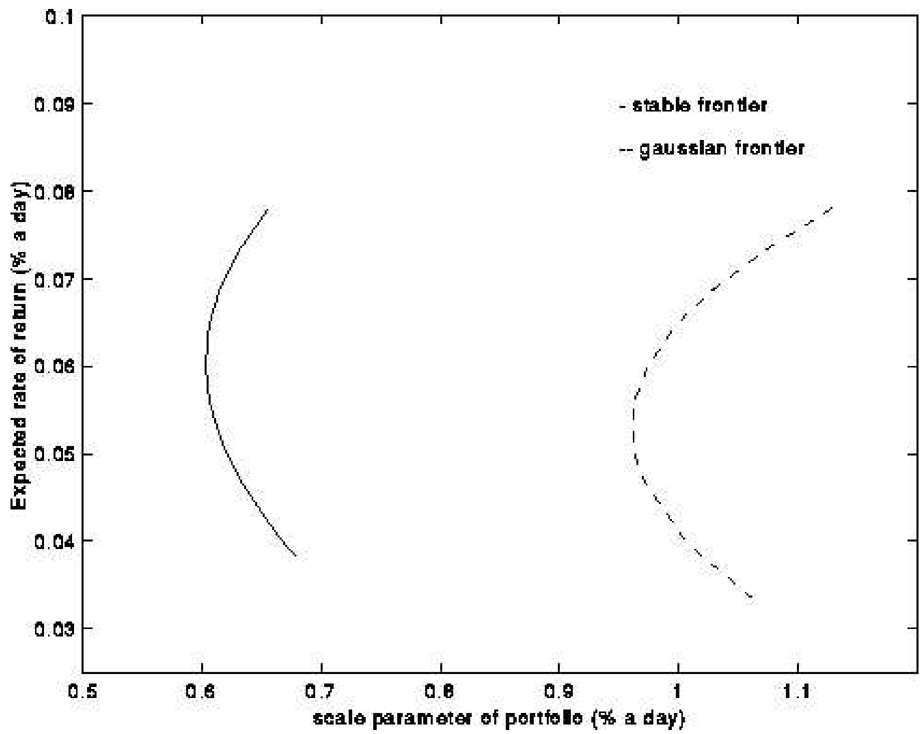

FIGURE 5. 1.7-stable efficient frontier versus Gaussian efficient frontier.

Assets	Optimal percentage by asset in "stable" efficient portfolios :		
	A_s^*	B_s^*	C_s^*
ACCOR	24	27	35
Cie BANCAIRE	-	-	1
CARREFOUR	76	73	64
Total assets yields (% a day)	100	100	100
Mean rate of return (% a day)	0.066	0.0643	0.0599
$\hat{\Lambda}_p$ (% a day)	0.61	0.606	0.603

TABLE 3. The structure of some 1.7-stable efficient portfolios.

Assets	Optimal percentage by asset in Gaussian efficient portfolios :		
	A_g^*	B_g^*	C_g^*
ACCOR	23	26	32
Cie BANCAIRE	1	2	5
CARREFOUR	76	72	63
Total assets yields (% a day)	100	100	100
Mean rate of return (% a day)	0.066	0.0643	0.0599
Variance of portfolio (% a day)	0.0204	0.0199	0.0190
$\frac{\sigma}{\sqrt{2}}$ (% a day)	1.01	0.996	0.974

TABLE 4. The structure of some Gaussian efficient portfolios.

Remark 3.

The stable and Gaussian efficient frontiers computed when short selling is allowed[4] *(figure 6) shows that for a fixed level of risk* 0.10%, *the stable efficient portfolio* S *offers a higher expected rate of return (*0.115%*) than the Gaussian one* G *(*0.0643%*). In this case, the respective weights are* $S = (-53\%, -24\%, 177\%)$, $G = (26\%, 2\%, 72\%)$.

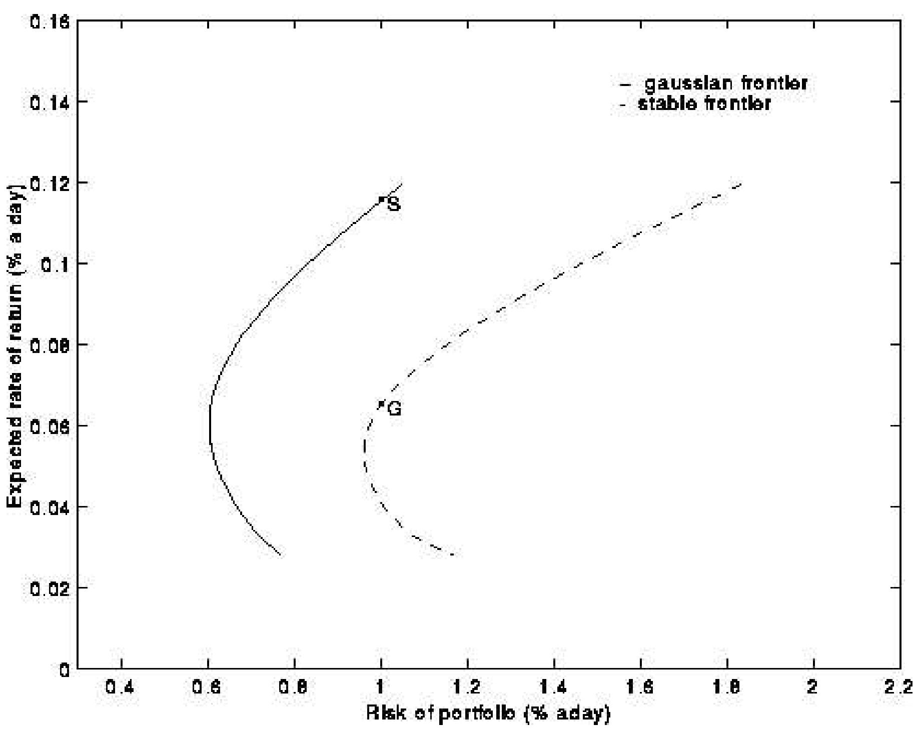

FIGURE 6. Stable efficient frontier versus Gaussian efficient frontier with short selling.

[4] If x_i is the relative weights we place on asset i, $x_i > 0$ for a long position and $x_i < 0$ for a short position.

One way to compare the Gaussian strategy investment and the stable one is to look at the "major" loss expected by both strategies. A recommended measure of the "major" loss is the Value at Risk (VaR). VaR is defined as the amount of money that one may lose within a given time horizon for a fixed confidence interval. VaR at date t with confidence interval p is defined as:

$$Prob(R_t^A < -VaR_t) = p \tag{14}$$

where R_t^A is return of the portfolio A at time t.

If R_t^A follows a univariate α-stable symmetric law with mean μ_A and scale parameter γ_A then

$$VaR_t^A = \mu + \gamma_A Q_{1-p} \tag{15}$$

where Q_{1-p} denotes the $(1-p)$-fractile of a standardized α-stable law ($S_\alpha(0,1)$).

Portfolios	Daily VaR (%) at confidence interval:										
	1%	1.5%	2%	5%	10%	15%	20%	25%	30%	40%	50%
A_g^*	3.39	3.17	3.00	2.42	1.90	1.54	1.27	1.03	0.82	0.43	0.066
A_s^*	3.20	2.69	2.38	1.67	1.24	0.99	0.81	0.65	0.52	0.28	0.066
B_g^*	3.34	3.12	2.96	2.38	1.87	1.52	1.25	1.01	0.80	0.42	0.064
B_s^*	3.19	2.66	2.37	1.66	1.23	0.98	0.80	0.65	0.51	0.28	0.064
C_g^*	3.26	3.05	2.89	2.33	1.83	1.49	1.22	0.99	0.78	0.40	0.060
C_s^*	3.16	2.65	2.35	1.65	1.22	0.97	0.79	0.64	0.50	0.27	0.060

TABLE 5. VaR estimation in Gaussian and stable portfolios.

Table 5 present VaR estimation in 6 cases of optimal portfolios: A_g^*, B_g^*, C_g^*, A_s^*, B_s^* and C_s^*. The index "g" denotes "Gaussian" portfolios and "s" denotes "stable" portfolios. It shows that the major losses measured by VaR for the Gaussian portfolios A_g^*, B_g^* and C_g^* at different level confidence intervals are higher than the ones given by VaR for the stable portfolios A_s^*, B_s^* and C_s^*. To stress the significance of these results, let us take for example a nominal of 10000 to be invested independently in portfolios A_g^* and A_s^*, which have nearly the same structure, but are deduced from different models. The correspondent VaR at 5% confidence interval is 242 for the Gaussian portfolio and 167 for the stable portfolio. The 75 of difference may be viewed as the gain taken up in terms of risk when considering a more adapted model. Table 6 shows that the maximum difference between the Gaussian-VaR and the stable-VaR is reached at a 5% confidence level for the three portfolios.

We would like emphasize, however, that the stable VaR becomes much higher than the Gaussian ones when a confidence level greater than 1% is considered (table 7). This is due to the extreme upper fractiles of the $S\alpha S$ law.

Since it is more practical for fund managers to interpret risks in terms of the VaR measures, it is convenient to set ourself into this broader context and to represent stable and Gaussian efficient frontiers in the *mean-VaR* space. The plots of this more general representation are displayed in

Portfolios	Daily $(VaR_g - VaR_s)$ (%) at confidence interval:										
	1%	1.5%	2%	5%	10%	15%	20%	25%	30%	40%	50%
A^*	0.28	0.51	0.66	0.88	0.75	0.56	0.40	0.36	0.30	0.14	0
A^*	0.24	0.46	0.61	0.65	0.57	0.54	0.43	0.35	0.29	0.14	0
B^*	0.19	0.43	0.56	0.61	0.54	0.52	0.41	0.33	0.28	0.13	0

TABLE 6. Gain in terms of risk taken up by the stable model.

Portfolios	Daily VaR (%) at 0.5 %	Daily VaR (%) at 0.2 %
A_g^*	3.74	4.18
A_s^*	4.51	7.40
B_g^*	3.69	4.12
B_s^*	4.48	7.35
C_g^*	4.14	4.56
C_s^*	4.99	7.85

TABLE 7. Extreme VaR estimation

figures 7 and 8. We may read these figures in two ways.First, at a given probability (greater than 1%) and for a fixed level expected rate of return, the risk of loss is much higher for Gaussian model of portfolio analysis than for the stable model. Second, taking dual point of view, let us fix the level of loss at 3.33% which corresponds to a VaR at 1%. then, the stable efficient portfolio S (c.f. figure 8) has an expected return much higher than the Gaussian one G. Note that the structure of these two portfolios are different as becomes clear from table 8.

Actifs	Optimal percentage by asset in “stabe” efficient portfolios:	
	S	**G**
ACCOR	9	38
Cie BANCAIRE	-	6
CARREFOUR	91	56
Total assets yields (% a day)	100	100
Mean rate of return (% a day)	0.0733	0.0636
VaR à 1% (% a day)	3.33	3.33

TABLE 8. Structure of portfolios S and G at a fixed level of VaR.

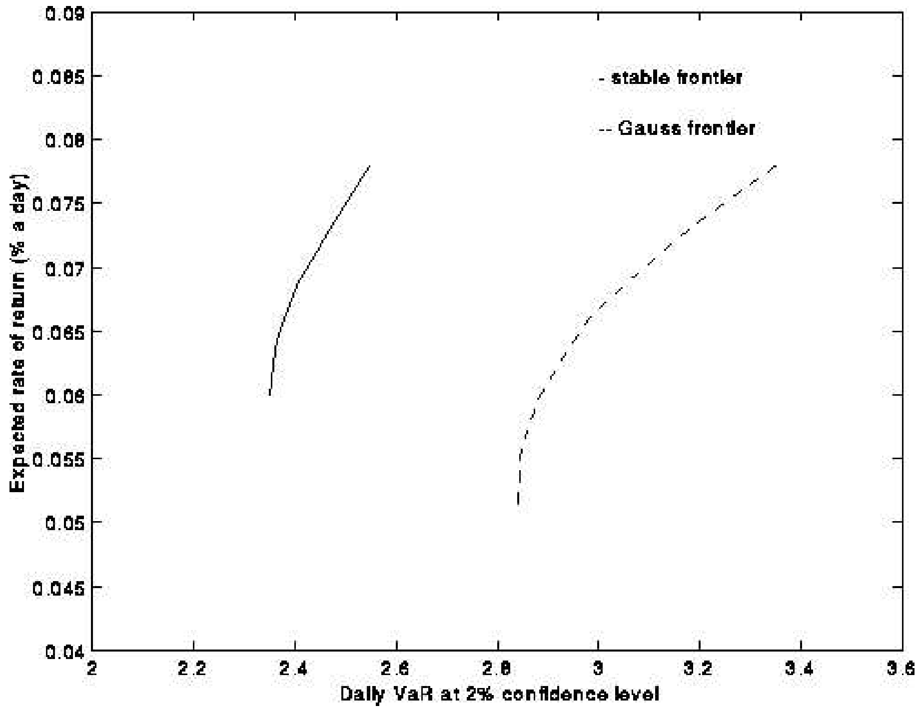

FIGURE 7. Efficient frontiers in (Moyenne $\times$ VaR à 2%) space.

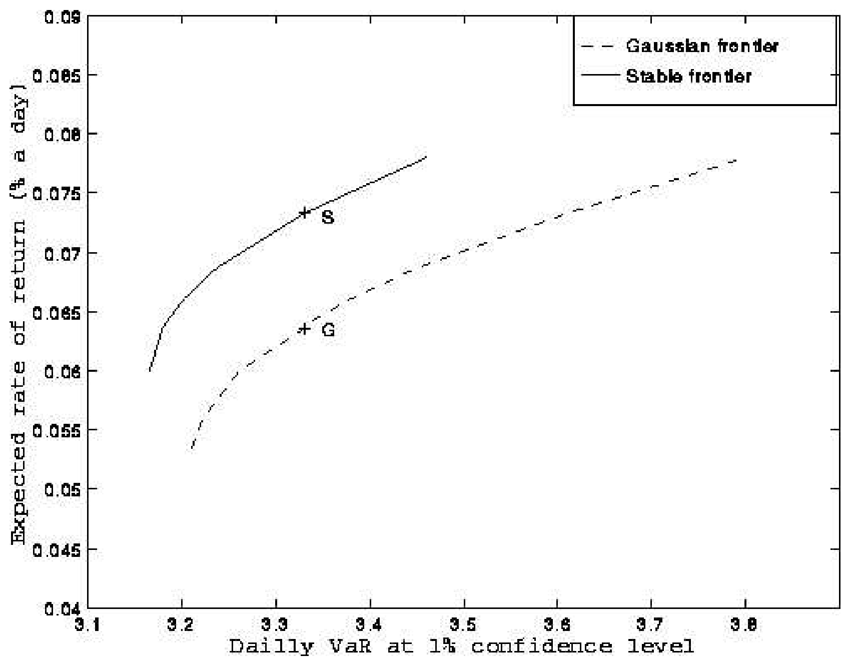

FIGURE 8. Efficient frontiers in (Moyenne $\times$ VaR à 1%) space.

5. Conclusion

The stable portfolio presented here generalizes the standard Mean-Variance model. The efficient frontier computed from this analysis model dominates the one defined in terms of the Markowitz portfolio selection model criterion. Thus an efficient portfolio in the Gaussian sense is not necessary an efficient one in the stable sense. Indeed one may further reduce the optimal risk given by the Gaussian model by taking into account the strong variations of the market, until reaching the optimum given by the most adequate stable model. Our study should then be a useful and convenient tool for fund managers and investors who seek to maximize their trade-off between risk and return. On the other hand, we have shown from some portfolio VaR estimation with different confidence interval, that major loss at a confidence level greater at 1% is much higher in the Gaussian portfolio analysis model than in the stable model. Consequently, we have given evidence that using VaR at confidence level lower than 1% with stable laws leads to a paradox. Indeed, since we considered the extreme upper fractiles of $S\alpha S$ (fractiles of order greater than 99.5%), we obtained much higher VaR in the stable case than in the Gaussian one.

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