

# A Proof of Weak Termination of the Simply-Typed $\lambda\sigma$ -Calculus

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INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET AUTOMATIQUE

*A Proof of Weak Termination of the  
Simply-Typed  $\lambda\sigma$ -Calculus*

Jean Goubault-Larrecq

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THÈME 2

A large blue rectangular area at the bottom of the page. On the left side of this area is a large, light grey stylized letter 'R'. To the right of the 'R', the words 'Rapport de recherche' are written in a white serif font. A horizontal white brushstroke underline is positioned below the text.

*Rapport  
de recherche*





## A Proof of Weak Termination of the Simply-Typed $\lambda\sigma$ -Calculus

Jean Goubault-Larrecq \*

Thème 2 — Génie logiciel  
et calcul symbolique

Projet Coq

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**Abstract:** We show that reducing any simply-typed  $\lambda\sigma$ -term by applying the rules in  $\sigma$  eagerly always terminates, by a translation to the simply-typed  $\lambda$ -calculus, and similarly for  $\lambda\sigma_{\uparrow}$ -terms with  $\sigma_{\uparrow}$ -eager rewrites. This holds even with term and substitution meta-variables. In fact, every reduction terminates provided that  $(\beta)$ -redexes are only contracted under so-called safe contexts. The previous results follow because in  $\sigma$ , resp.  $\sigma_{\uparrow}$ -normal forms, all contexts around terms of sort  $T$  are safe.

**Key-words:**  $\lambda\sigma$ -calculus, explicit substitutions, termination,  $\lambda$ -calculus, simple types

*(Résumé : tsvp)*

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## Une preuve de terminaison faible du $\lambda\sigma$ -calcul simplement typé

**Résumé :** Nous montrons que réduire n'importe quel  $\lambda\sigma$ -terme simplement typé en appliquant toujours les règles de  $\sigma$  le plus tôt possible est un processus qui termine. Ceci est prouvé à l'aide d'une traduction vers le  $\lambda$ -calcul simplement typé, et de même pour le  $\lambda\sigma_{\uparrow}$ -calcul avec réduction prioritaire par  $\sigma_{\uparrow}$ . Ceci est valide même en présence de méta-variables de termes et de substitutions. En fait, toute réduction termine dès qu'on ne contracte que les  $(\beta)$ -rédux qui apparaissent sous des contextes dits sûrs. Les résultats énoncés plus haut s'en déduisent car dans toute forme  $\sigma$ -normale, resp.  $\sigma_{\uparrow}$ -normale, tous les contextes autour de termes de sorte  $T$  sont sûrs.

**Mots-clé :**  $\lambda\sigma$ -calcul, substitutions explicites, terminaison,  $\lambda$ -calcul, types simples

# Introduction

Although the simply-typed  $\lambda\sigma$ -calculus does not terminate strongly [Mel94, Mel95], it terminates in the weak sense: every typed term has some (unique) normal form. We present a refined proof of this fact. In fact, we prove the widely believed claim that every reduction where  $\sigma$  steps are applied eagerly is finite.

For the sake of generality, we prove this result even in the presence of term and substitution metavariables, and for a general  $\lambda\sigma$ -calculus that encompasses both the original  $\lambda\sigma$ -calculus [ACCL90] and the  $\lambda\mathcal{E}$ -calculus of [HL89]. The techniques that we use are basically the same as in [GL97], where they are used for a more complicated calculus.

The plan of the paper is as follows. We recapitulate some basic definitions in Section 1 and prove a few preliminaries in Section 2. In Section 3, we prove the main theorem of this paper: every reduction where  $(\beta)$ -contraction occurs only under so-called *safe* contexts always terminates. This result is applied in Section 4 to show that all normalization strategies that apply  $\sigma$  (resp.  $\sigma_{\uparrow}$ ) eagerly in the  $\lambda\sigma$ -calculus (resp. in  $\lambda\sigma_{\uparrow}$ , a.k.a.  $\lambda\mathcal{E}$ ) terminate.

## 1 Definitions

Consider the language  $\lambda\sigma_{\uparrow}$  defined as  $T \cup S$ , where the sublanguage  $T$  and that of *explicit substitutions* or *stacks*  $S$  are given by:

$$\begin{aligned} T &::= \mathcal{V}_T \mid \lambda T \mid TT \mid T[S] \mid 1 \\ S &::= \mathcal{V}_S \mid S \circ S \mid id \mid T \cdot S \mid \uparrow \uparrow S \end{aligned}$$

where  $\mathcal{V}_T$  and  $\mathcal{V}_S$  are disjoint infinite sets of *term variables*  $x, y, z, \dots$ , and *stack variables*  $X, Y, Z$ , etc. The set of  $\lambda\sigma$ -terms is the subset of  $\lambda\sigma_{\uparrow}$ -terms where  $\uparrow$  does not occur.

We call (untyped)  $\lambda\sigma$ -calculus the rewrite system of Figure 1. It is a slight generalization, in the sense that it can simulate every reduction, of both the  $\lambda\sigma$ -calculus of [ACCL90] and of the  $\lambda\sigma_{\uparrow}$ -calculus of [HL89] (where it is named  $\lambda\mathcal{E}$ ). Notice that, because of rule  $(\eta \uparrow)$ , all the other rules with  $\uparrow$  in their names are superfluous. The  $\lambda\sigma_{\uparrow}$ -calculus is the rewrite system defined by all rules of  $\lambda\sigma$  without the three rules with an  $\eta$  in their names. (The latter is basically group (B) of [GL96], while the former is group (B) plus part of group (H)).

$(\beta)$	$(\lambda u)v \rightarrow u[v \cdot id]$	$(\eta \uparrow)$	$\uparrow w \rightarrow 1 \cdot (w \circ \uparrow)$
$([id])$	$u[id] \rightarrow u$	$(\eta \cdot)$	$1 \cdot \uparrow \rightarrow id$
$(\circ id)$	$u \circ id \rightarrow u$	$(\eta \cdot \circ)$	$(1[u]) \cdot (\uparrow \circ u) \rightarrow u$
$(ido)$	$id \circ u \rightarrow u$	$(1 \uparrow)$	$1[\uparrow u] \rightarrow 1$
$([])$	$(u[v])[w] \rightarrow u[v \circ w]$	$(1 \uparrow \circ)$	$1[\uparrow u \circ v] \rightarrow 1[v]$
$(\circ)$	$(u \circ v) \circ w \rightarrow u \circ (v \circ w)$	$(\uparrow \uparrow)$	$\uparrow \circ \uparrow u \rightarrow u \circ \uparrow$
$(1)$	$1[u \cdot v] \rightarrow u$	$(\uparrow \uparrow \circ)$	$\uparrow \circ (\uparrow u \circ v) \rightarrow u \circ (\uparrow \circ v)$
$(\uparrow)$	$\uparrow \circ (u \cdot v) \rightarrow v$	$(\uparrow \uparrow \uparrow)$	$\uparrow u \circ \uparrow v \rightarrow \uparrow (u \circ v)$
$(\cdot)$	$(u \cdot v) \circ w \rightarrow (u[w]) \cdot (v \circ w)$	$(\uparrow \uparrow \circ)$	$\uparrow u \circ (\uparrow v \circ w) \rightarrow \uparrow (u \circ v) \circ w$
$(\lambda)$	$(\lambda u)[w] \rightarrow \lambda(u[\uparrow w])$	$(\uparrow \cdot)$	$\uparrow u \circ (v \cdot w) \rightarrow v \cdot (u \circ w)$
$(app)$	$(uv)[w] \rightarrow (u[w])(v[w])$	$(\uparrow id)$	$\uparrow id \rightarrow id$

Figure 1: The  $\lambda\sigma_{\uparrow}$ -calculus

The purpose of this paper is to show that the simply-typed  $\lambda\sigma$ -calculus terminates, when rewrites are restricted to be  $\sigma$ -eager rewrites, i.e. rewrites of the form:

$$u \xrightarrow{\sigma}^* u_1 \xrightarrow{(\beta)} u'_1 \xrightarrow{\sigma}^* u_2 \xrightarrow{(\beta)} u'_2 \xrightarrow{\sigma}^* \dots \xrightarrow{\sigma}^* u_k \xrightarrow{(\beta)} u'_k \xrightarrow{\sigma}^* \dots$$

where  $u_1, u_2, \dots, u_k, \dots$  are  $\sigma$ -normal. And similarly, that the simply-typed  $\lambda\sigma_{\uparrow}$ -calculus terminates, when rewrites are restricted to be  $\sigma_{\uparrow}$ -eager rewrites, defined similarly.

The typing rules are given in Figure 2. Types are either *term types* or *stack types*. Term types are sequents  $\Gamma \vdash \tau$ , where  $\tau$  is a formula and  $\Gamma$  is a finite list of formulas  $\tau_1, \dots, \tau_n, \cdot$ , where  $\cdot$  marks the end of the list; formulas  $\tau$  are either atoms  $A, B, \dots$ , or implications (arrow types)  $\tau_1 \rightarrow \tau_2$ . Stack types are similar sequents  $\Gamma \vdash \Delta$ , but  $\Delta$  is a finite list of formulas as well. For simplicity, we shall assume that the calculus is given in Church style: variables  $x_{\Gamma \vdash \tau}$  are annotated with their type  $\Gamma \vdash \tau$ ; we write them  $x$  when no confusion arises. (Writing terms in Church style will allow us to translate  $\lambda\sigma$ -terms to  $\lambda$ -terms in Section 3; otherwise, we would need to translate typing derivations of  $\lambda\sigma$ -terms to typing derivations of  $\lambda$ -terms, which is not that different, but would be much less readable.)

$$\begin{array}{c}
\frac{}{X_{\Gamma \vdash \Delta} : \Gamma \vdash \Delta} (Vars) \qquad \frac{}{x_{\Gamma \vdash \tau} : \Gamma \vdash \tau} (Var_T) \\
\\
\frac{}{id : \Gamma \vdash \Gamma} (I) \\
\\
\frac{u : \Gamma \vdash \tau \quad v : \Gamma \vdash \Delta}{u \cdot v : \Gamma \vdash \tau, \Delta} (, I) \qquad \frac{}{1 : \tau, \Gamma \vdash \tau} (, E_1) \\
\\
\frac{u : \Gamma \vdash \Delta}{\uparrow u : \tau, \Gamma \vdash \tau, \Delta} (\uparrow) \qquad \frac{}{\uparrow : \tau, \Gamma \vdash \Gamma} (, E_2) \\
\\
\frac{u : \tau_1, \Gamma \vdash \tau_2}{\lambda u : \Gamma \vdash \tau_1 \rightarrow \tau_2} (\rightarrow I) \qquad \frac{u : \Gamma \vdash \tau_1 \rightarrow \tau_2 \quad v : \Gamma \vdash \tau_1}{uv : \Gamma \vdash \tau_2} (\rightarrow E) \\
\\
\frac{u : \Lambda \vdash \Delta \quad v : \Gamma \vdash \Lambda}{u \circ v : \Gamma \vdash \Delta} (Cut_S) \qquad \frac{u : \Lambda \vdash \tau \quad v : \Gamma \vdash \Lambda}{u[v] : \Gamma \vdash \tau} (Cut_T)
\end{array}$$

Figure 2: Typing rules

To get a calculus in Church style, we also need to annotate the terms  $1, \uparrow, id$  and  $\uparrow u$  with types agreeing with the rules of Figure 2. We shall usually write these terms without their type annotations when context permits, however.

Then, every typed term has a unique type. There is a unique way of lifting the rules of Figure 1 to the typed case so that the resulting typed calculus has the subject reduction property: see Figure 3.

$(\beta)$	$(\lambda u)v \rightarrow u[v \cdot id_{\Gamma \vdash \Gamma}]$ where $v : \Gamma \vdash \tau$	$(\eta \uparrow)$	$(\uparrow w)_{\tau, \Gamma \vdash \tau, \Delta} \rightarrow 1_{\tau, \Gamma \vdash \tau} \cdot (w \circ \uparrow_{\tau, \Gamma \vdash \Gamma})$
$([id])$	$u[id_{\Gamma \vdash \Gamma}] \rightarrow u$	$(\eta \cdot)$	$1_{\tau, \Gamma \vdash \tau} \cdot \uparrow_{\tau, \Gamma \vdash \Gamma} \rightarrow id_{\tau, \Gamma \vdash \tau, \Gamma}$
$(\circ id)$	$u \circ id_{\Gamma \vdash \Gamma} \rightarrow u$	$(\eta \cdot \circ)$	$(1_{\tau, \Gamma \vdash \tau}[u]) \cdot (\uparrow_{\tau, \Gamma \vdash \Gamma} \circ u) \rightarrow u$
$(ido)$	$id_{\Gamma \vdash \Gamma} \circ u \rightarrow u$	$(1 \uparrow)$	$1_{\tau, \Delta \vdash \tau} [(\uparrow u)_{\tau, \Gamma \vdash \tau, \Delta}] \rightarrow 1_{\tau, \Gamma \vdash \tau}$
$([])$	$(u[v])[w] \rightarrow u[v \circ w]$	$(1 \uparrow \circ)$	$1_{\tau, \Delta \vdash \tau} [(\uparrow u)_{\tau, \Gamma \vdash \tau, \Delta} \circ v] \rightarrow 1_{\tau, \Gamma \vdash \tau}[v]$
$(\circ)$	$(u \circ v) \circ w \rightarrow u \circ (v \circ w)$	$(\uparrow \uparrow)$	$\uparrow_{\tau, \Delta \vdash \tau} \circ (\uparrow u)_{\tau, \Gamma \vdash \tau, \Delta} \rightarrow u \circ \uparrow_{\tau, \Gamma \vdash \Gamma}$
$(1)$	$1_{\Gamma \vdash \tau}[u \cdot v] \rightarrow u$	$(\uparrow \uparrow \circ)$	$\uparrow_{\tau, \Delta \vdash \tau} \circ ((\uparrow u)_{\tau, \Gamma \vdash \tau, \Delta} \circ v) \rightarrow u \circ (\uparrow_{\tau, \Gamma \vdash \Gamma} \circ v)$
$(\uparrow)$	$\uparrow_{\Gamma \vdash \Delta} \circ (u \cdot v) \rightarrow v$	$(\uparrow \uparrow \uparrow)$	$(\uparrow u)_{\tau, \Lambda \vdash \tau, \Delta} \circ (\uparrow v)_{\tau, \Gamma \vdash \tau, \Lambda} \rightarrow (\uparrow (u \circ v))_{\tau, \Gamma \vdash \tau, \Delta}$
$(\cdot)$	$(u \cdot v) \circ w \rightarrow (u[w]) \cdot (v \circ w)$	$(\uparrow \uparrow \circ)$	$(\uparrow u)_{\tau, \Lambda \vdash \tau, \Delta} \circ ((\uparrow v)_{\tau, \Gamma \vdash \tau, \Lambda} \circ w) \rightarrow (\uparrow (u \circ v))_{\tau, \Gamma \vdash \tau, \Delta} \circ w$
$(\lambda)$	$(\lambda u)[w] \rightarrow \lambda(u[(\uparrow w)_{\tau, \Gamma \vdash \tau, \Delta}])$ where $w : \Gamma \vdash \Delta, u : \tau, \Delta \vdash \tau'$	$(\uparrow \cdot)$	$(\uparrow u)_{\tau, \Gamma \vdash \tau, \Delta} \circ (v \cdot w) \rightarrow v \cdot (u \circ w)$
$(app)$	$(uv)[w] \rightarrow (u[w])(v[w])$	$(\uparrow id)$	$(\uparrow id_{\Gamma \vdash \Gamma})_{\tau, \Gamma \vdash \tau, \Gamma} \rightarrow id_{\tau, \Gamma \vdash \tau, \Gamma}$

Figure 3: The typed reduction rules

## 2 Other Preliminaries

**Lemma 1**  $\sigma$  is a terminating rewrite system.

**Proof:** We use Zantema's distribution elimination technique [Zan94]. (Refer to this paper for notations and concepts; our  $\sigma$  is slightly more complicated than the  $\sigma$  that Zantema considers, but the argument is similar.) We actually show that  $\sigma$  is totally terminating, i.e. that the reduction relation for  $\sigma$  can actually be extended to a *total* well-founded quasi-ordering on terms. Total termination implies *simple* termination, which is the fact that  $\sigma$  plus all rules of the form  $f(\dots, u, \dots) \rightarrow u$ , for every function symbol  $f$ , terminates.

Now consider the following rewrite system  $\sigma'$ , which is obtained from  $\sigma$  by replacing  $u[v]$  by  $u \circ v$ ,  $uv$  by  $u \cdot v$ ,  $id$  by  $1 \cdot \uparrow$ , merging duplicate rules, eliminating rule  $(\eta \cdot)$  and adding rule  $(d \uparrow)$ :

$$\begin{array}{ll}
(d \uparrow) & \uparrow u \circ (v \cdot w) \rightarrow (\uparrow u \circ v) \cdot (\uparrow u \circ w) & (\eta \uparrow) & \uparrow w \rightarrow 1 \cdot (w \circ \uparrow) \\
(\circ id) & u \circ (1 \cdot \uparrow) \rightarrow u & (\eta \cdot \circ) & (1 \circ u) \cdot (\uparrow \circ u) \rightarrow u \\
(id \circ) & (1 \cdot \uparrow) \circ u \rightarrow u & (1 \uparrow) & 1 \circ \uparrow u \rightarrow 1 \\
& & (1 \uparrow \circ) & 1 \circ (\uparrow u \circ v) \rightarrow 1 \circ v \\
(\circ) & (u \circ v) \circ w \rightarrow u \circ (v \circ w) & (\uparrow \uparrow) & \uparrow \circ \uparrow u \rightarrow u \circ \uparrow \\
(1) & 1 \circ (u \cdot v) \rightarrow u & (\uparrow \uparrow \circ) & \uparrow \circ (\uparrow u \circ v) \rightarrow u \circ (\uparrow \circ v) \\
(\uparrow) & \uparrow \circ (u \cdot v) \rightarrow v & (\uparrow \uparrow) & \uparrow u \circ \uparrow v \rightarrow \uparrow (u \circ v) \\
(\cdot) & (u \cdot v) \circ w \rightarrow (u \circ w) \cdot (v \circ w) & (\uparrow \uparrow \circ) & \uparrow u \circ (\uparrow v \circ w) \rightarrow \uparrow (u \circ v) \circ w \\
(\lambda) & (\lambda u) \circ w \rightarrow \lambda(u \circ \uparrow w) & (\uparrow \cdot) & \uparrow u \circ (v \cdot w) \rightarrow v \cdot (u \circ w) \\
& & (\uparrow id) & \uparrow (1 \cdot \uparrow) \rightarrow (1 \cdot \uparrow)
\end{array}$$

If  $\sigma'$  is totally terminating, then it is clear that  $\sigma$  is totally terminating as well. Let indeed  $>$  be any total well-founded quasi-ordering extending the rewrite relation of  $\sigma'$ . This induces a total well-founded quasi-ordering on  $\lambda\sigma$ -terms that extends all the rules in  $\sigma$  except  $(\eta \cdot)$  (for which both sides are equivalent). Let  $>'$  be the quasi-ordering such that  $u >' v$  if and only if  $u$  has more occurrences of  $\cdot$  than  $v$ : then the lexicographic product of  $>$  and  $>'$  is a well-founded total quasi-ordering extending the rewrite relation of  $\sigma$ .

Now call a rule *embedding* if and only if its right-hand side is homeomorphically embedded in its left-hand side (i.e., it can be simulated by a sequence of rewrite steps of the form  $f(\dots, u, \dots) \rightarrow u$ ). The rules  $(\circ id)$ ,  $(id \circ)$ ,  $(1)$ ,  $(\uparrow)$ ,  $(\eta \cdot \circ)$ ,  $(1 \uparrow)$ ,  $(1 \uparrow \circ)$ ,  $(\uparrow id)$  are embedding. Since total termination implies simple termination,  $\sigma'$  is totally terminating if and only if  $\sigma'$  minus these rules is. Moreover,  $(\uparrow \cdot)$  can be simulated by  $(d \uparrow)$  plus embedding, so we can ignore the former for purposes of total termination. To sum up,  $\sigma'$  is totally terminating if and only if the following system  $\sigma''$  is totally terminating:

$$\begin{array}{ll}
(d \uparrow) & \uparrow u \circ (v \cdot w) \rightarrow (\uparrow u \circ v) \cdot (\uparrow u \circ w) & (\eta \uparrow) & \uparrow w \rightarrow 1 \cdot (w \circ \uparrow) \\
(\circ) & (u \circ v) \circ w \rightarrow u \circ (v \circ w) & (\uparrow \uparrow) & \uparrow \circ \uparrow u \rightarrow u \circ \uparrow \\
& & (\uparrow \uparrow \circ) & \uparrow \circ (\uparrow u \circ v) \rightarrow u \circ (\uparrow \circ v) \\
& & (\uparrow \uparrow) & \uparrow u \circ \uparrow v \rightarrow \uparrow (u \circ v) \\
(\cdot) & (u \cdot v) \circ w \rightarrow (u \circ w) \cdot (v \circ w) & (\uparrow \uparrow \circ) & \uparrow u \circ (\uparrow v \circ w) \rightarrow \uparrow (u \circ v) \circ w \\
(\lambda) & (\lambda u) \circ w \rightarrow \lambda(u \circ \uparrow w) & &
\end{array}$$

Now, in  $\sigma''$  the rules  $(d \uparrow)$  and  $(\cdot)$  are *distribution rules* for  $\cdot$ , and no other rule features  $\cdot$  on the left-hand side. Zantema's theorem [Zan94] then states that the rewrite system  $\sigma'''$  obtained from  $\sigma''$  by dropping these distribution rules and replacing each rule with some subterm  $u \cdot v$  on the right-hand side by two rules, one with  $u \cdot v$  replaced by  $u$ , the other with  $u \cdot v$  replaced by  $v$ , is totally terminating if and only if  $\sigma''$  is. Here is  $\sigma'''$ :

$$\begin{array}{ll}
\uparrow w \rightarrow 1 & \uparrow w \rightarrow w \circ \uparrow \\
(u \circ v) \circ w \rightarrow u \circ (v \circ w) & \uparrow \circ \uparrow u \rightarrow u \circ \uparrow \\
\uparrow \circ (\uparrow u \circ v) \rightarrow u \circ (\uparrow \circ v) & \uparrow u \circ \uparrow v \rightarrow \uparrow (u \circ v) \\
\uparrow u \circ (\uparrow v \circ w) \rightarrow \uparrow (u \circ v) \circ w & (\lambda u) \circ w \rightarrow \lambda(u \circ \uparrow w)
\end{array}$$

$\sigma'''$  is then proved totally terminating by using two polynomial interpretations  $P_1$  and  $P_2$ , ordered lexi-



graphically, as in [HL89]; we let:

	$P_1$	$P_2$
$u \circ v$	$uv$	$u(v + 1)$
$\uparrow u$	$u$	$3u$
$1$	$2$	$1$
$\uparrow$	$2$	$1$
$\lambda u$	$u + 1$	$u$

Using the fact that  $P_1(u) \geq 2$  for every term  $u$ , and  $P_2(u) \geq 1$ , a quick calculation shows that for every rule  $l \rightarrow r$  but the last one (i.e.,  $(\lambda)$ ),  $P_1(l) \geq P_1(r)$  and  $P_2(l) > P_2(r)$ , while in the case of  $(\lambda)$   $P_1(l) > P_1(r)$ . It follows that  $\sigma$  is totally terminating.  $\square$

In fact,  $\sigma$  is locally confluent, as a Knuth-Bendix test shows, whether as the subsystem of Figure 1 or Figure 3, so it is convergent; but we shall not be interested in this property here.

### 3 Reduction Under Safe Contexts Terminates

We translate each  $\lambda\sigma$ -term with Church-style typing into a family of  $\lambda$ -terms, typed à la Church as well. Let  $\top$  be a given arbitrary type. We assume that we have infinitely many variables of each type.

With each term variable  $x_{\tau_1, \dots, \tau_n, \top, \tau}$  of  $\lambda\sigma$ , we associate a variable  $\hat{x}_{\tau_1 \rightarrow \dots \rightarrow \tau_n \rightarrow \top \rightarrow \tau}$  of the  $\lambda$ -calculus of the indicated type. Similarly, we map each stack variable  $X_{\tau_1, \dots, \tau_n, \top, \tau'_1, \dots, \tau'_{n'}}$  to a list  $\hat{X}_{\tau_1 \rightarrow \dots \rightarrow \tau_n \rightarrow \top \rightarrow \tau'_1} \dots \hat{X}_{\tau'_{n'}} \tau_1 \rightarrow \dots \rightarrow \tau_n \rightarrow \top \rightarrow \tau'_1 \dots \hat{X}_{n'+1} \tau_1 \rightarrow \dots \rightarrow \tau_n \rightarrow \top \rightarrow \tau'$  of  $\lambda$ -variables of the indicated types. (The cons operator  $::$  is just a meta-linguistic way of forming lists; it is assumed to be right-associative.) We do not assume the mapping to be injective in any sense.

Our translation maps every  $\lambda\sigma$ -term  $u$  of type  $\tau_1, \dots, \tau_n, \top, \tau$  to a function  $\llbracket u \rrbracket$  taking a list of  $n + 1$   $\lambda$ -terms  $t_1 :: \dots :: t_n :: t_{n+1}$  of respective types  $\tau_1, \dots, \tau_n$  and  $\top$ , and returns a  $\lambda$ -term  $\llbracket u \rrbracket(t_1 :: \dots :: t_n :: t_{n+1})$  of type  $\tau$ . Similarly, our translation maps every  $\lambda\sigma$ -term  $u$  of type  $\tau_1, \dots, \tau_n, \top, \tau'_1, \dots, \tau'_{n'}$  to a function  $\llbracket u \rrbracket$  taking a list of  $n + 1$   $\lambda$ -terms  $t_1 :: \dots :: t_n :: t_{n+1}$  of respective types  $\tau_1, \dots, \tau_n$  and  $\top$ , and returns a list of  $n' + 1$   $\lambda$ -terms of respective types  $\tau'_1, \dots, \tau'_{n'}$  and  $\top$ . The translation is given in Figure 4.

$\llbracket x_{\tau_1, \dots, \tau_n, \top, \tau} \rrbracket(t_1 :: \dots :: t_n :: t_{n+1})$	$= \hat{x}_{\tau_1 \rightarrow \dots \rightarrow \tau_n \rightarrow \top \rightarrow \tau} t_1 \dots t_n t_{n+1}$
$\llbracket X_{\tau_1, \dots, \tau_n, \top, \tau'_1, \dots, \tau'_{n'}} \rrbracket(t_1 :: \dots :: t_n :: t_{n+1})$	$= (\hat{X}_{\tau_1 \rightarrow \dots \rightarrow \tau_n \rightarrow \top \rightarrow \tau'_1} t_1 \dots t_n t_{n+1})$
	$:: \dots$
	$:: (\hat{X}_{\tau'_{n'}} \tau_1 \rightarrow \dots \rightarrow \tau_n \rightarrow \top \rightarrow \tau'_1} t_1 \dots t_n t_{n+1})$
	$:: (\hat{X}_{n'+1} \tau_1 \rightarrow \dots \rightarrow \tau_n \rightarrow \top \rightarrow \tau'_{n'}} t_1 \dots t_n t_{n+1})$
$\llbracket \lambda u \rrbracket(s)$	$= \lambda z_{\tau} \cdot \llbracket u \rrbracket(z :: s)$ where $u : \tau, \Gamma \vdash \tau'$
$\llbracket uv \rrbracket(s)$	$= (\llbracket u \rrbracket(s))(\llbracket v \rrbracket(s))$
$\llbracket u[v] \rrbracket(s)$	$= \llbracket u \rrbracket(\llbracket v \rrbracket(s))$
$\llbracket 1_{\tau, \Gamma \vdash \tau} \rrbracket(t :: s)$	$= t$
$\llbracket u \circ v \rrbracket(s)$	$= \llbracket u \rrbracket(\llbracket v \rrbracket(s))$
$\llbracket id_{\Gamma \vdash \Gamma} \rrbracket(s)$	$= s$
$\llbracket u \cdot v \rrbracket(s)$	$= \llbracket u \rrbracket(s) :: \llbracket v \rrbracket(s)$
$\llbracket \uparrow_{\tau, \Gamma \vdash \Gamma} \rrbracket(t :: s)$	$= s$
$\llbracket \uparrow u \rrbracket(t :: s)$	$= t :: \llbracket u \rrbracket(s)$

Figure 4: Translation to the simply-typed  $\lambda$ -calculus

The  $\beta$  rule  $(\lambda x \cdot t)t' \rightarrow t[t'/x]$  defines a rewrite relation on  $\lambda$ -terms that we again write  $\rightarrow$ . Recall that the simply-typed  $\lambda$ -calculus terminates [Kri92]. We write  $\rightarrow^+$ , resp.  $\rightarrow^*$ , its (resp. reflexive) transitive closure. These notions are extended to lists:  $t_1 :: \dots :: t_n :: t_{n+1} \rightarrow^* t'_1 :: \dots :: t'_n :: t'_{n+1}$  if and only if  $t_i \rightarrow^* t'_i$  for every  $i$ ,  $1 \leq i \leq n + 1$ ; and  $t_1 :: \dots :: t_n :: t_{n+1} \rightarrow^+ t'_1 :: \dots :: t'_n :: t'_{n+1}$  if in addition  $t_i \rightarrow^+ t'_i$  for some  $i$ ,  $1 \leq i \leq n + 1$ .

We define the quasi-ordering  $\succeq$  and the strict ordering  $\succ$  on typed  $\lambda\sigma$ -terms by:  $u \succeq v$  if and only if  $u$  and  $v$  have the same type and  $\llbracket u \rrbracket(s) \longrightarrow^* \llbracket v \rrbracket(s)$  for every list  $s$  of  $\lambda$ -terms of the right types. (Observe that  $\succ$  is an ordering, in particular it is irreflexive precisely because there are  $\lambda$ -terms of any given type, namely variables.) Similarly,  $u \succ v$  if and only if  $u$  and  $v$  have the same type and  $\llbracket u \rrbracket(s) \longrightarrow^+ \llbracket v \rrbracket(s)$  for every list  $s$  of  $\lambda$ -terms of the right types. We also let  $u \approx v$  if and only if  $\llbracket u \rrbracket(s) = \llbracket v \rrbracket(s)$  for every list  $s$  of  $\lambda$ -terms of the right types.

The interpretation is monotonic in the following sense:

**Lemma 2** *If  $s \longrightarrow^* s'$ , then  $\llbracket u \rrbracket(s) \longrightarrow^* \llbracket u \rrbracket(s')$ .*

**Proof:** We claim that for any terms  $t_1, \dots, t_n, t_{n+1}$  of the right types, for any fresh variables  $x_1, \dots, x_n, x_{n+1}$  of the right types:

$$\llbracket u \rrbracket(t_1 :: \dots :: t_n :: t_{n+1}) = \llbracket u \rrbracket(x_1 :: \dots :: x_n :: x_{n+1})[t_1/x_1, \dots, t_n/x_n, t_{n+1}/x_{n+1}]$$

where substitution is extended componentwise when  $u$  is a stack. The lemma follows immediately from the claim, while the claim is proved by structural induction on  $u$ .

If  $u = v[w]$ , then  $\llbracket u \rrbracket(t_1 :: \dots :: t_n :: t_{n+1}) = \llbracket v \rrbracket(t'_1 :: \dots :: t'_{n'} :: t'_{n'+1})$ , where  $t'_1 :: \dots :: t'_{n'} :: t'_{n'+1} = \llbracket w \rrbracket(t_1 :: \dots :: t_n :: t_{n+1})$ . By induction hypothesis,  $t'_1 :: \dots :: t'_{n'} :: t'_{n'+1} = \llbracket w \rrbracket(x_1 :: \dots :: x_n :: x_{n+1})[t_1/x_1, \dots, t_n/x_n, t_{n+1}/x_{n+1}]$ . Let: (1)  $t''_1 :: \dots :: t''_{n'} :: t''_{n'+1}$  be  $\llbracket w \rrbracket(x_1 :: \dots :: x_n :: x_{n+1})$ , then for each  $i'$ ,  $1 \leq i' \leq n' + 1$ : (2)  $t'_{i'} = t''_{i'}[t_1/x_1, \dots, t_n/x_n]$ .

Then:

$$\begin{aligned} & \llbracket u \rrbracket(t_1 :: \dots :: t_n :: t_{n+1}) \\ &= \llbracket v \rrbracket(t'_1 :: \dots :: t'_{n'} :: t'_{n'+1}) \\ &= \llbracket v \rrbracket(x'_1 :: \dots :: x'_{n'} :: x'_{n'+1})[t'_1/x'_1, \dots, t'_{n'}/x'_{n'}, t'_{n'+1}/x'_{n'+1}] \quad (\text{by induction hypothesis again}) \\ &= \llbracket v \rrbracket(x'_1 :: \dots :: x'_{n'} :: x'_{n'+1}) \\ & \quad [t''_1[t_1/x_1, \dots, t_n/x_n, t_{n+1}/x_{n+1}]/x'_1, \dots, t''_{n'+1}[t_1/x_1, \dots, t_n/x_n, t_{n+1}/x_{n+1}]/x'_{n'+1}] \quad (\text{by (2)}) \\ &= \llbracket v \rrbracket(x'_1 :: \dots :: x'_{n'} :: x'_{n'+1})[t''_1/x'_1, \dots, t''_{n'+1}/x'_{n'+1}][t_1/x_1, \dots, t_n/x_n, t_{n+1}/x_{n+1}] \\ &= \llbracket v \rrbracket(t''_1 :: \dots :: t''_{n'} :: t''_{n'+1})[t_1/x_1, \dots, t_n/x_n, t_{n+1}/x_{n+1}] \quad (\text{by induction hypothesis again}) \\ &= \llbracket u \rrbracket(x_1 :: \dots :: x_n :: x_{n+1})[t_1/x_1, \dots, t_n/x_n, t_{n+1}/x_{n+1}] \quad (\text{by (1)}) \end{aligned}$$

If  $u = v \circ w$ , then the argument is identical, while all other cases are trivial.  $\square$

Recall that a *context*  $\mathcal{C}$  is any term with a distinguished occurrence  $\_$ , which we call the *hole*. For any term  $t$ ,  $\mathcal{C}\{t\}$  denotes  $\mathcal{C}$  with the hole replaced by  $t$ . Similarly if  $t$  is itself a context, in which case we get a new context.

The interpretation is then also monotonic in the following sense:

**Lemma 3** *If  $u \succeq v$ , then  $\mathcal{C}\{u\} \succeq \mathcal{C}\{v\}$  for any context  $\mathcal{C}$ . If  $u \approx v$ , then  $\mathcal{C}\{u\} \approx \mathcal{C}\{v\}$  for any context  $\mathcal{C}$ .*

**Proof:** The case of  $\approx$  is trivial, because our definition of  $\llbracket \_ \rrbracket$  is compositional. That is, for any context  $\mathcal{C}$ , there is a functional  $\llbracket \mathcal{C} \rrbracket$  such that, for every  $u$ ,  $\llbracket \mathcal{C}\{u\} \rrbracket = \llbracket \mathcal{C} \rrbracket(\llbracket u \rrbracket)$ : this is an easy structural induction on  $\mathcal{C}$ .

We prove the first claim by structural induction on the context  $\mathcal{C}$  as well. The base case, when  $\mathcal{C}$  is the empty context  $\_$ , is clear. Otherwise, the induction case reduces to the elementary cases that whenever  $u \succeq v$ , then  $f(\dots, u, \dots) \succeq f(\dots, v, \dots)$  for each function symbol  $f$  in the language of  $\lambda\sigma$  and each argument position.

For example, if  $f = \lambda$ :  $\llbracket \lambda u \rrbracket(s) = \lambda z \cdot \llbracket u \rrbracket(z :: s) \longrightarrow^* \lambda z \cdot \llbracket v \rrbracket(z :: s)$  (since  $u \succeq v$ ) =  $\llbracket \lambda v \rrbracket(s)$ . Since  $s$  is arbitrary,  $\lambda u \succeq \lambda v$ .

The cases when  $f$  is the application symbol or the  $\cdot$  pair operator (whatever the argument position), or the  $\uparrow$  lift operator are equally easy.

When  $f$  is the  $\_[-]$  operator, we have two cases. At argument position 1, we must show that  $u \succeq v$  entails  $u[w] \succeq v[w]$ . But this is trivial, since  $\llbracket u[w] \rrbracket(s) = \llbracket u \rrbracket(\llbracket w \rrbracket(s)) \longrightarrow^* \llbracket v \rrbracket(\llbracket w \rrbracket(s))$  (by assumption) =  $\llbracket v[w] \rrbracket(s)$ . At argument position 2, we must show that  $u \succeq v$  entails  $w[u] \succeq w[v]$ . But  $\llbracket w[u] \rrbracket(s) = \llbracket w \rrbracket(\llbracket u \rrbracket(s)) \longrightarrow^* \llbracket w \rrbracket(\llbracket v \rrbracket(s))$  (since  $u \succeq v$  and by Lemma 2) =  $\llbracket w[v] \rrbracket(s)$ . The case of  $\circ$  is entirely analogous.  $\square$

Observe that the interpretation is not strictly monotonic, in that  $u \succ v$  does not imply  $\mathcal{C}\{u\} \succ \mathcal{C}\{v\}$ , only  $\mathcal{C}\{u\} \succeq \mathcal{C}\{v\}$ : consider the context  $\mathcal{C} = \uparrow \circ (\_ \cdot w)$  for some  $w$ . This leads to the following definition:

**Definition 1** A context  $\mathcal{C}$  is called safe if and only if  $u \succ v$  implies  $\mathcal{C}\{u\} \succ \mathcal{C}\{v\}$  for every terms  $u$  and  $v$  of the same type such that  $\mathcal{C}\{u\}$  and  $\mathcal{C}\{v\}$  are typable.

We shall see in Section 4 that there are enough safe contexts to all typed  $\lambda\sigma$ -terms to be reduced to normal form by reductions under safe contexts.

**Lemma 4** For each rule  $l \rightarrow r$  except  $(\beta)$  in Figure 3,  $l \approx r$ . For rule  $(\beta)$ ,  $l \succ r$ .

**Proof:** We omit all type information unless strictly necessary.

Rule  $(\beta)$ :

$$\begin{aligned} & \llbracket l \rrbracket(s) \\ &= (\llbracket \lambda u \rrbracket(s))(\llbracket v \rrbracket(s)) \\ &= (\lambda z \cdot \llbracket u \rrbracket(z :: s))(\llbracket v \rrbracket(s)) \\ &\rightarrow (\llbracket u \rrbracket(z :: s))(\llbracket v \rrbracket(s)/z) \\ &= \llbracket u \rrbracket(\llbracket v \rrbracket(s) :: s) \end{aligned}$$

by Lemma 3, while  $\llbracket r \rrbracket(s) = \llbracket u \rrbracket(\llbracket v \cdot id \rrbracket(s)) = \llbracket u \rrbracket(\llbracket v \rrbracket(s) :: s)$ .

The cases of rules  $(id)$ ,  $(oid)$  and  $(ido)$  are obvious:  $\llbracket l \rrbracket(s) = \llbracket r \rrbracket(s) = \llbracket u \rrbracket(s)$  in all three cases.

Rule  $(\circ)$ :  $\llbracket l \rrbracket(s) = \llbracket u[v] \rrbracket(\llbracket w \rrbracket(s)) = \llbracket u \rrbracket(\llbracket v \rrbracket(\llbracket w \rrbracket(s))) = \llbracket u \rrbracket(\llbracket v \circ w \rrbracket(s)) = \llbracket u[v \circ w] \rrbracket(s)$ , and similarly for rule  $(\circ)$ .

Rule  $(1)$ :  $\llbracket l \rrbracket(s) = \llbracket 1 \rrbracket(\llbracket u \cdot v \rrbracket(s)) = \llbracket 1 \rrbracket(\llbracket u \rrbracket(s) :: \llbracket v \rrbracket(s)) = \llbracket u \rrbracket(s) = \llbracket r \rrbracket(s)$ .

Rule  $(\uparrow)$ :  $\llbracket l \rrbracket(s) = \llbracket \uparrow \rrbracket(\llbracket u \cdot v \rrbracket(s)) = \llbracket \uparrow \rrbracket(\llbracket u \rrbracket(s) :: \llbracket v \rrbracket(s)) = \llbracket v \rrbracket(s) = \llbracket r \rrbracket(s)$ .

Rule  $(\cdot)$ :  $\llbracket l \rrbracket(s) = \llbracket u \cdot v \rrbracket(\llbracket w \rrbracket(s)) = \llbracket u \rrbracket(\llbracket w \rrbracket(s)) :: \llbracket v \rrbracket(\llbracket w \rrbracket(s)) = \llbracket u[w] \rrbracket(s) :: \llbracket v \circ w \rrbracket(s) = \llbracket (u[w] \cdot (v \circ w)) \rrbracket(s) = \llbracket r \rrbracket(s)$ .

Rule  $(\lambda)$ :  $\llbracket l \rrbracket(s) = \llbracket \lambda u \rrbracket(\llbracket w \rrbracket(s)) = \lambda z_\tau \cdot \llbracket u \rrbracket(z :: \llbracket w \rrbracket(s))$ , while  $\llbracket r \rrbracket(s) = \lambda z_\tau \cdot \llbracket u[\uparrow w] \rrbracket(z :: s) = \lambda z_\tau \cdot \llbracket u \rrbracket(\llbracket \uparrow w \rrbracket(z :: s)) = \lambda z_\tau \cdot \llbracket u \rrbracket(z :: \llbracket w \rrbracket(s))$ .

Rule  $(app)$ : similar argument as for rule  $(\cdot)$ .

Rule  $(\eta \uparrow)$ :  $\llbracket l \rrbracket(t :: s) = t :: \llbracket w \rrbracket(s)$ , while  $\llbracket r \rrbracket(t :: s) = \llbracket 1 \rrbracket(t :: s) :: \llbracket w \circ \uparrow \rrbracket(t :: s) = t :: \llbracket w \circ \uparrow \rrbracket(t :: s) = t :: \llbracket w \rrbracket(\llbracket \uparrow \rrbracket(t :: s)) = t :: \llbracket w \rrbracket(s)$ .

Rule  $(\eta \cdot)$ :  $\llbracket l \rrbracket(t :: s) = \llbracket 1 \rrbracket(t :: s) :: \llbracket \uparrow \rrbracket(t :: s) = t :: s = \llbracket r \rrbracket(t :: s)$ ; and  $t :: s$  is the most general form of argument to  $\llbracket l \rrbracket$  by typing, so  $l \approx r$ .

Rule  $(\eta \circ)$ :  $l = (1[u]) \cdot (\uparrow ou) \approx (1 \cdot \uparrow) \circ u$  (by case  $(\circ)$ )  $\approx id \circ u$  (by case  $(\eta \cdot)$ )  $\approx u = r$  (by case  $(ido)$ ). Observe that we have used Lemma 3 (the  $\approx$  part) implicitly throughout. All the other rules are dealt with similarly:

Rule  $(1 \uparrow)$ :  $l = 1[\uparrow u] \approx 1[1 \cdot (u \circ \uparrow)]$  (by  $(\eta \uparrow)$ )  $\approx 1 = r$  (by  $(1)$ ).

Rule  $(1 \uparrow \circ)$ :  $l = 1[\uparrow u \circ v] \approx 1[(1 \cdot (u \circ \uparrow)) \circ v]$  (by  $(\eta \uparrow)$ )  $\approx 1[1[v] \cdot ((u \circ \uparrow) \circ v)]$  (by  $(\cdot)$ )  $\approx 1[v] = r$  (by  $(1)$ ).

Rule  $(\uparrow \uparrow)$ :  $l = \uparrow \circ \uparrow u \approx \uparrow \circ (1 \cdot (u \circ \uparrow))$  (by  $(\eta \uparrow)$ )  $\approx u \circ \uparrow = r$  (by  $(\uparrow)$ ).

Rule  $(\uparrow \uparrow \circ)$ :  $l = \uparrow \circ (\uparrow u \circ v) \approx \uparrow \circ ((1 \cdot (u \circ \uparrow)) \circ v)$  (by  $(\eta \uparrow)$ )  $\approx \uparrow \circ (1[v] \cdot ((u \circ \uparrow) \circ v))$  (by  $(\cdot)$ )  $\approx (u \circ \uparrow) \circ v$  (by  $(\uparrow)$ )  $\approx u \circ (\uparrow \circ v) = r$  (by  $(\circ)$ ).

Rule  $(\uparrow \uparrow \uparrow)$ :  $l = \uparrow \circ \uparrow u \circ \uparrow v \approx (1 \cdot (u \circ \uparrow)) \circ \uparrow v$  (by  $(\eta \uparrow)$ )  $\approx (1[\uparrow v]) \cdot ((u \circ \uparrow) \circ \uparrow v)$  (by  $(\cdot)$ )  $\approx 1 \cdot ((u \circ \uparrow) \circ \uparrow v)$  (by  $(1 \uparrow)$ )  $\approx 1 \cdot (u \circ (\uparrow \circ \uparrow v))$  (by  $(\circ)$ )  $\approx 1 \cdot (u \circ (v \circ \uparrow))$  (by  $(\uparrow \uparrow)$ )  $\approx 1 \cdot ((u \circ v) \circ \uparrow)$  (by  $(\circ)$  used in the opposite direction)  $\approx \uparrow (u \circ v) = r$  (by  $(\eta \uparrow)$  used in the opposite direction).

Rule  $(\uparrow \uparrow \circ)$ :  $l = \uparrow \circ \uparrow u \circ (\uparrow v \circ w) \approx (\uparrow \circ \uparrow u \circ \uparrow v) \circ w$  (by  $(\circ)$  used in the opposite direction)  $\approx \uparrow (u \circ v) \circ w = r$  (by  $(\uparrow \uparrow)$ ).

Rule  $(\uparrow \cdot)$ :  $l = \uparrow u \circ (v \cdot w) \approx (1 \cdot (u \circ \uparrow)) \circ (v \cdot w)$  (by  $(\eta \uparrow)$ )  $\approx (1[v \cdot w]) \cdot ((u \circ \uparrow) \circ (v \cdot w))$  (by  $(\cdot)$ )  $\approx v \cdot ((u \circ \uparrow) \circ (v \cdot w))$  (by  $(1)$ )  $\approx v \cdot (u \circ (\uparrow \circ (v \cdot w)))$  (by  $(\circ)$ )  $\approx v \cdot (u \circ w) = r$  (by  $(\uparrow)$ ).

Rule  $(\uparrow id)$ :  $l = \uparrow id \approx 1 \cdot (ido \uparrow)$  (by  $(\eta \uparrow)$ )  $\approx 1 \cdot \uparrow$  (by  $(ido)$ )  $\approx id = r$  (by  $(\eta \cdot)$ ).  $\square$

It follows:

**Theorem 5 (Main Theorem)** Every reduction in the typed  $\lambda\sigma$ -calculus, where every contracted  $(\beta)$ -redex occurs under a safe context, is finite.

**Proof:** Any step in such a reduction is either a  $(\beta)$ -contraction  $\mathcal{C}\{(\lambda u)v\} \longrightarrow \mathcal{C}\{u[v \cdot id]\}$ , then  $\mathcal{C}\{(\lambda u)v\} \succ \mathcal{C}\{u[v \cdot id]\}$  by Lemma 4 and the fact that  $\mathcal{C}$  is safe; or a  $\sigma$ -contraction  $u \longrightarrow_{\sigma} v$ , where  $u \approx v$  by Lemma 4 and Lemma 3. Therefore each rewrite step is strictly decreasing in the lexicographic product of  $\succ$  and  $\longrightarrow_{\sigma}^+$ , which is well-founded since the typed  $\lambda$ -calculus terminates (for  $\succ$ ) and  $\sigma$  terminates (by Lemma 1).  $\square$

## 4 Some Safe Contexts

It remains to produce some non-trivial safe contexts:

**Definition 2** *Let the syntactically safe contexts be defined as those in  $\mathcal{S}_T \cup \mathcal{S}_S$ , where:*

$$\begin{aligned} \mathcal{S}_T &::= \_ \mid \lambda \mathcal{S}_T \mid \mathcal{S}_T T \mid T \mathcal{S}_T \mid \mathcal{S}_T[S] \mid \mathcal{V}_T[\mathcal{S}_S] \mid 1[\mathcal{S}'_S] \\ \mathcal{S}_S &::= \mathcal{S}'_S \mid \mathcal{S}_T \cdot S \mid T \cdot \mathcal{S}_S \mid \uparrow \mathcal{S}_S \mid \uparrow S \circ \mathcal{S}'_S \mid \mathcal{S}_S \circ S \\ \mathcal{S}'_S &::= \mathcal{V}_S \circ \mathcal{S}_S \mid \uparrow \circ \mathcal{S}'_S \mid \mathcal{S}'_S \circ S \end{aligned}$$

Before we show Lemma 7, we need the following technical lemma. Let  $tl$  be defined by  $tl(t :: s) = s$ , and let  $tl^n$  be defined by:  $tl^0(s) = s$ ,  $tl^{n+1} = tl \circ tl^n$ .

**Lemma 6** *For every context  $\mathcal{C}'$  in  $\mathcal{S}'_S$ , there is a stack variable  $X$ , a proper subcontext  $\mathcal{C}$  of  $\mathcal{C}'$  in  $\mathcal{S}_S$  and an integer  $m \in \mathbb{N}$  such that, for every term  $u$  such that  $\mathcal{C}'\{u\}$  is well-typed, for every  $s'$  of the right type, for some  $s$ ,  $\llbracket \mathcal{C}'\{u\} \rrbracket(s') = tl^m(\llbracket X \rrbracket(\llbracket \mathcal{C}\{u\} \rrbracket(s)))$ .*

**Proof:** By structural induction on  $\mathcal{C}'$ . This is obvious if  $\mathcal{C}'$  has the form  $X \circ \mathcal{C}$ , with  $\mathcal{C} \in \mathcal{S}_S$ :  $m = 0$  and  $s = s'$ .

If  $\mathcal{C}' = \uparrow \circ \mathcal{C}''$  with  $\mathcal{C}'' \in \mathcal{S}'_S$ , then for every  $u$  such that  $\mathcal{C}'\{u\}$  is well-typed, in particular  $\mathcal{C}''\{u\}$  is well-typed, so by induction hypothesis  $\llbracket \mathcal{C}''\{u\} \rrbracket(s') = tl^m(\llbracket X \rrbracket(\llbracket \mathcal{C}\{u\} \rrbracket(s)))$  for some  $m \geq 0$  and some  $s$ . Then  $\llbracket \mathcal{C}'\{u\} \rrbracket(s') = \llbracket \uparrow \rrbracket(\llbracket \mathcal{C}''\{u\} \rrbracket(s')) = tl(\llbracket \mathcal{C}''\{u\} \rrbracket(s')) = tl^{m+1}(\llbracket X \rrbracket(\llbracket \mathcal{C}\{u\} \rrbracket(s)))$ .

If  $\mathcal{C}' = \mathcal{C}'' \circ w$ , where  $\mathcal{C}'' \in \mathcal{S}'_S$ , then for every  $u$  such that  $\mathcal{C}'\{u\}$  is well-typed, in particular  $\mathcal{C}''\{u\}$  is well-typed, so by induction hypothesis, for every  $s'_1$  of the right type,  $\llbracket \mathcal{C}''\{u\} \rrbracket(s'_1) = tl^m(\llbracket X \rrbracket(\llbracket \mathcal{C}\{u\} \rrbracket(s_1)))$  for some  $m \geq 0$  and some  $s_1$ . Then  $\llbracket \mathcal{C}'\{u\} \rrbracket(s') = \llbracket \mathcal{C}''\{u\} \rrbracket(\llbracket w \rrbracket(s'))$ ; letting  $s'_1$  be  $\llbracket w \rrbracket(s')$ ,  $\llbracket \mathcal{C}'\{u\} \rrbracket(s') = \llbracket \mathcal{C}''\{u\} \rrbracket(s'_1) = tl^m(\llbracket X \rrbracket(\llbracket \mathcal{C}\{u\} \rrbracket(s_1)))$ .  $\square$

**Lemma 7** *Every syntactically safe context is safe.*

**Proof:** We have to show that if  $u \succ v$ , then  $\mathcal{C}\{u\} \succ \mathcal{C}\{v\}$  for any syntactically safe context  $\mathcal{C}$ . This is by structural induction on  $\mathcal{C}$ , using Definition 1. The base case, when  $\mathcal{C}$  is the empty context  $\_$ , is clear. Otherwise:

If  $\mathcal{C} = \lambda \mathcal{C}_1$ , where  $\mathcal{C}_1 \in \mathcal{S}_T$ , then  $\llbracket \mathcal{C}\{u\} \rrbracket(s) = \lambda z \cdot \llbracket \mathcal{C}_1\{u\} \rrbracket(z :: s) \longrightarrow^+ \lambda z \cdot \llbracket \mathcal{C}_1\{v\} \rrbracket(z :: s) = \llbracket \mathcal{C}\{v\} \rrbracket(s)$  by induction hypothesis. Similarly when  $\mathcal{C} = \mathcal{C}_1 w$  or  $\mathcal{C} = w \mathcal{C}_1$ , or  $\mathcal{C} = \mathcal{C}_1 \cdot w$  for some term  $w$  and some  $\mathcal{C}_1 \in \mathcal{S}_T$ , or  $\mathcal{C} = \uparrow \mathcal{C}_1$  or  $\mathcal{C} = w \cdot \mathcal{C}_1$  for some term  $w$  and some  $\mathcal{C}_1 \in \mathcal{S}_S$ .

If  $\mathcal{C} = \mathcal{C}_1[w]$  for some  $\mathcal{C}_1 \in \mathcal{S}_T$  and some stack  $w$ , then  $\llbracket \mathcal{C}\{u\} \rrbracket(s) = \llbracket \mathcal{C}_1\{u\} \rrbracket(\llbracket w \rrbracket(s)) \longrightarrow^+ \llbracket \mathcal{C}_1\{v\} \rrbracket(\llbracket w \rrbracket(s)) = \llbracket \mathcal{C}\{v\} \rrbracket(s)$  by induction hypothesis. Similarly when  $\mathcal{C} = \mathcal{C}_1 \circ w$  for some  $\mathcal{C}_1$  in  $\mathcal{S}_S$  and some stack  $w$ .

If  $\mathcal{C} = x[\mathcal{C}_1]$  where  $x$  is an term variable and  $\mathcal{C}_1 \in \mathcal{S}_S$ , then  $\llbracket \mathcal{C}\{u\} \rrbracket(s) = \hat{x}t_1 \dots t_n t_{n+1}$  where  $t_1 :: \dots :: t_n :: t_{n+1} = \llbracket \mathcal{C}_1\{u\} \rrbracket(s) \longrightarrow^+ \llbracket \mathcal{C}_1\{v\} \rrbracket(s)$  by induction hypothesis. Let  $(t'_1, \dots, t'_n, t'_{n+1})$  be  $\llbracket \mathcal{C}_1\{v\} \rrbracket(s)$ : this means that  $t_i \longrightarrow^* t'_i$  for every  $i$ ,  $1 \leq i \leq n+1$ , and that  $t_i \longrightarrow^+ t'_i$  for some  $i$ . So  $\hat{x}t_1 \dots t_n t_{n+1} \longrightarrow^+ \hat{x}t'_1 \dots t'_n t'_{n+1} = \llbracket \mathcal{C}\{v\} \rrbracket(s)$ .

It remains to deal with the cases when  $\mathcal{C}$  is in  $\mathcal{S}'_S$ , and when  $\mathcal{C}$  is of the form  $1[\mathcal{C}']$  or  $\uparrow w \circ \mathcal{C}'$  with  $\mathcal{C}' \in \mathcal{S}'_S$ .

Consider first the case where  $\mathcal{C}$  is some context  $\mathcal{C}'$  in  $\mathcal{S}'_S$ . By Lemma 6, there is a stack variable  $X$ , a proper subcontext  $\mathcal{C}_1$  in  $\mathcal{S}_S$  and an integer  $m \geq 0$  such that for every  $u$  of the right type, for every  $s'$  of the right type, there is a  $\lambda$ -term  $s$  such that  $\llbracket \mathcal{C}'\{u\} \rrbracket(s') = tl^m(\llbracket X \rrbracket(\llbracket \mathcal{C}_1\{u\} \rrbracket(s)))$ . Let  $X$  be  $X_{\Gamma \vdash \tau_1, \dots, \tau_p}$ , where

by typing  $p \geq m$ . Let also  $t_1 :: \dots :: t_n :: t_{n+1}$  be  $\llbracket \mathcal{C}_1\{u\} \rrbracket(s)$ , and  $t'_1 :: \dots :: t'_n :: t'_{n+1}$  be  $\llbracket \mathcal{C}_1\{v\} \rrbracket(s)$ . Then:

$$\begin{aligned}
& \llbracket \mathcal{C}'\{u\} \rrbracket(s') \\
&= tl^m \left( (\hat{X}_1 t_1 \dots t_n t_{n+1}) :: \dots :: (\hat{X}_m t_1 \dots t_n t_{n+1}) :: (\hat{X}_{m+1} t_1 \dots t_n t_{n+1}) :: \dots :: (\hat{X}_{p+1} t_1 \dots t_n t_{n+1}) \right) \\
&= (\hat{X}_{m+1} t_1 \dots t_n t_{n+1}) :: \dots :: (\hat{X}_{p+1} t_1 \dots t_n t_{n+1}) \\
&\longrightarrow^+ (\hat{X}_{m+1} t'_1 \dots t'_n t'_{n+1}) :: \dots :: (\hat{X}_{p+1} t'_1 \dots t'_n t'_{n+1}) \quad (\text{by induction hypothesis}) \\
&= tl^m \left( (\hat{X}_1 t'_1 \dots t'_n t'_{n+1}) :: \dots :: (\hat{X}_m t'_1 \dots t'_n t'_{n+1}) :: (\hat{X}_{m+1} t'_1 \dots t'_n t'_{n+1}) :: \dots :: (\hat{X}_{p+1} t'_1 \dots t'_n t'_{n+1}) \right) \\
&= \llbracket \mathcal{C}'\{v\} \rrbracket(s')
\end{aligned}$$

so  $\mathcal{C}\{u\} \succ \mathcal{C}\{v\}$ .

If  $\mathcal{C} = 1[\mathcal{C}']$ , then taking the same notation as above:

$$\begin{aligned}
& \llbracket \mathcal{C}\{u\} \rrbracket(s') \\
&= \hat{X}_{m+1} t_1 \dots t_n t_{n+1} \\
&\longrightarrow^+ \hat{X}_{m+1} t'_1 \dots t'_n t'_{n+1} \quad (\text{by induction hypothesis}) \\
&= \llbracket \mathcal{C}\{v\} \rrbracket(s')
\end{aligned}$$

(observe that now  $p > m$  by typing) so again  $\mathcal{C}\{u\} \succ \mathcal{C}\{v\}$ .

And if  $\mathcal{C} = \uparrow w \circ \mathcal{C}'$ , then:

$$\begin{aligned}
& \llbracket \mathcal{C}\{u\} \rrbracket(s') \\
&= (\hat{X}_{m+1} t_1 \dots t_n t_{n+1}) :: \llbracket w \rrbracket(tl(\llbracket \mathcal{C}'\{u\} \rrbracket(s'))) \\
&\longrightarrow^* (\hat{X}_{m+1} t_1 \dots t_n t_{n+1}) :: \llbracket w \rrbracket(tl(\llbracket \mathcal{C}'\{v\} \rrbracket(s'))) \quad (\text{by Lemma 3}) \\
&\longrightarrow^+ (\hat{X}_{m+1} t'_1 \dots t'_n t'_{n+1}) :: \llbracket w \rrbracket(tl(\llbracket \mathcal{C}'\{v\} \rrbracket(s'))) \quad (\text{by induction hypothesis}) \\
&= \llbracket \mathcal{C}\{v\} \rrbracket(s')
\end{aligned}$$

so  $\mathcal{C}\{u\} \succ \mathcal{C}\{v\}$ .  $\square$

**Lemma 8** *Let  $\mathcal{C}\{u\}$  be a  $\sigma$ -normal simply-typed term, where  $u \in T$ . Then  $\mathcal{C}$  is a syntactically safe context.*

**Proof:** Recall that the  $\sigma$ -normal forms ([Río93], p.76) are all elements of the languages described by the following grammar:

$$\begin{aligned}
\text{In } T: \quad & t ::= 1 \mid \lambda t \mid tt \mid \mathcal{V}_T \mid \mathcal{V}_T[s] \mid 1[s'] \\
\text{In } S: \quad & s ::= s' \mid id \mid t \cdot s \\
& s' ::= \uparrow \mid \mathcal{V}_S \mid \mathcal{V}_S \circ s \mid \uparrow \circ s'
\end{aligned}$$

(Although our  $\sigma$  is different from Ríos', it is easy to see that it has exactly the same normal forms.) Therefore the contexts having a hole accepting terms of  $T$  are elements of the languages defined by the grammar:

$$\begin{aligned}
\text{In } T: \quad & C_t ::= \_ \mid \lambda C_t \mid C_t t \mid t C_t \mid C_t[s] \mid \mathcal{V}_T[C_s] \mid 1[C'_s] \\
\text{In } S: \quad & C_s ::= C'_s \mid C_t \cdot s \mid t \cdot C_s \\
& C'_s ::= \mathcal{V}_S \circ C_s \mid \uparrow \circ C'_s
\end{aligned}$$

which clearly defines sublanguages of  $\mathcal{S}_T$ ,  $\mathcal{S}_S$  and  $\mathcal{S}'_S$  respectively, proving the claim.  $\square$

**Corollary 9** *In the simply-typed  $\lambda\sigma$ -calculus, every  $\sigma$ -eager rewrite is finite.*

**Proof:** In every  $\sigma$ -eager rewrite, every  $(\beta)$  is performed under a syntactically safe context by Lemma 8. This context is safe by Lemma 7, so Theorem 5 applies.  $\square$

We can do the same thing with the  $\lambda\sigma_{\uparrow}$ -calculus and  $\sigma_{\uparrow}$ -eager rewrites:

**Lemma 10** *Let  $\mathcal{C}\{u\}$  be a  $\sigma_{\uparrow}$ -normal simply-typed term, where  $u \in T$ . Then  $\mathcal{C}$  is a syntactically safe context.*

**Proof:** An easy argument, similar to those found in [Río93], shows that the  $\sigma_{\uparrow}$ -normal forms are all in the languages described by the following grammar:

$$\begin{aligned} \text{In } T: \quad t &::= 1 \mid \lambda t \mid tt \mid \mathcal{V}_T \mid \mathcal{V}_T[s] \mid 1[s'] \\ \text{In } S: \quad s &::= s' \mid id \mid t \cdot s \mid \uparrow s \mid \uparrow s \circ s' \\ s' &::= \uparrow \mid \mathcal{V}_S \mid \mathcal{V}_S \circ s \mid \uparrow \circ s' \end{aligned}$$

Indeed, we show by structural induction on the  $\sigma_{\uparrow}$ -normal term  $u$  that: (1) if  $u$  is in  $T$ , then  $u$  is in language  $t$ , (2) if  $u$  is a stack, then  $u$  is in language  $s$ , (3) if  $u$  is a stack and  $1[u]$  is  $\sigma_{\uparrow}$ -normal, then  $u$  is in language  $s'$ , (4) if  $u$  is a stack and  $\uparrow \circ u$  is  $\sigma_{\uparrow}$ -normal, then  $u$  is in language  $s'$ , and (5) if  $u$  is a stack and  $\uparrow v \circ u$  is  $\sigma_{\uparrow}$ -normal, then  $u$  is in language  $s'$ .

First, observe that (2) implies (3): since  $1[u]$  is  $\sigma_{\uparrow}$ -normal,  $u$  cannot be  $id$  (by rule ( $[id]$ )), a cons  $v \cdot w$  (rule (1) would be applicable) or a lift  $\uparrow v$  (rule (1  $\uparrow$ ) would be applicable), or of the form  $\uparrow v \circ w$  (rule (1  $\uparrow \circ$ ) would be applicable).

Similarly, (2) implies (4), because of rules ( $\circ id$ ), ( $\uparrow$ ), ( $\uparrow\uparrow$ ) and ( $\uparrow\uparrow \circ$ ). And (2) implies (5) because of rules ( $\circ id$ ), ( $\uparrow \cdot$ ), ( $\uparrow\uparrow$ ) and ( $\uparrow\uparrow \circ$ ).

We now show (2). Let  $u$  be a  $\sigma_{\uparrow}$ -normal stack. If  $u$  is in  $\mathcal{V}_S$  or is  $\uparrow$ , then  $u$  is clearly in  $s'$ , hence in  $s$ . If  $u = id$  or if  $u$  is a cons  $v \cdot w$  or a lift  $\uparrow v$ , then  $u$  is also in  $s$ . Finally, if  $u$  is a composition  $v \circ w$ , then in particular  $v$  is in  $s$ ; but  $v$  cannot be  $id$  (rule ( $ido$ ) would apply), or a cons  $v_1 \cdot v_2$  (rule ( $\cdot$ ) would apply), or a composition  $v_1 \circ v_2$  (rule ( $\circ$ ) would apply); so  $v$  must be of the form  $\uparrow v_1$  (then  $u$  has the form  $\uparrow v_1 \circ w$  with  $v_1 \in s$  and  $w \in s$  by induction hypothesis; since (2) implies (5),  $w$  is actually in  $s'$ , so  $u$  is in  $s$ ), or  $v = \uparrow$  (then  $u = \uparrow \circ w$ , where  $w \in s$  by induction hypothesis; since (2) implies (4),  $w$  is actually in  $s'$ , so  $u$  is in  $s'$ , hence in  $s$ ), or  $v$  is a variable  $X$  in  $\mathcal{V}_S$  (then  $u = X \circ w$  with  $w \in s$  by induction hypothesis, so  $u$  is in  $s'$ , hence in  $s$ ).

We now show (1). Let  $u$  be a  $\sigma_{\uparrow}$ -normal term in  $T$ . If  $u$  is a variable in  $\mathcal{V}_T$ , an abstraction  $\lambda v$ , an application  $vw$  or  $1$ , then  $u$  is in  $t$ . Finally, if  $u$  is of the form  $v[w]$ , then  $v$  must be a variable in  $\mathcal{V}_T$  or be equal to  $1$ : the other cases would be reducible by rules ( $\lambda$ ), ( $app$ ) or ( $[]$ ). Moreover, by induction hypothesis  $w$  is in  $s$ . If  $v$  is a variable  $x$  in  $\mathcal{V}_T$ , then  $u = x[w]$  is in  $t$ . If  $v = 1$ , then  $u = 1[w]$  where  $w$  is in  $s$ , and since (2) implies (3),  $w$  is actually in  $s'$ ; so  $u$  is in  $t$  again.

It follows that the contexts having a hole accepting terms in  $T$  are in the languages defined by the grammar:

$$\begin{aligned} \text{In } T: \quad C_t &::= - \mid \lambda C_t \mid C_t t \mid t C_t \mid C_t[s] \mid \mathcal{V}_T[C_s] \mid 1[C'_s] \\ \text{In } S: \quad C_s &::= C'_s \mid C_t \cdot s \mid t \cdot C_s \mid \uparrow C_s \mid \uparrow C_s \circ s' \mid \uparrow s \circ C'_s \\ C'_s &::= \mathcal{V}_S \circ C_s \mid \uparrow \circ C'_s \end{aligned}$$

and it is easy to see that  $C_t$ ,  $C_s$ ,  $C'_s$  are respective sublanguages of  $\mathcal{S}_T$ ,  $\mathcal{S}_S$  and  $\mathcal{S}'_S$ .  $\square$

It follows:

**Corollary 11** *In the simply-typed  $\lambda\sigma_{\uparrow}$ -calculus, every  $\sigma_{\uparrow}$ -eager rewrite is finite.*

**Proof:** In every  $\sigma_{\uparrow}$ -eager rewrite, every ( $\beta$ ) is performed under a syntactically safe context by Lemma 10. This context is safe by Lemma 7, so Theorem 5 applies.  $\square$

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