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Abstract: We consider systems with Poisson arrivals and a deterministic delay structure given by some increasing sequence. We characterize the distribution of waiting times in the transient regime, and describe effective ways to evaluate it. We also compute the stationary waiting time distribution in the case where the deterministic structure becomes ultimately periodic.

This analysis is strongly connected to the theory of linear (max, +) systems and has the same range of applications: we give examples from stochastic Petri Net theory and stochastic task graph theory. It also applies in general to single server queues with Poisson arrivals and known service times, in particular to the $M/D/1$ queue with periodic service durations and the $E/D/1$ queue.

Key-words: Queueing Networks, Petri Networks, Event Graphs, (max,+) Systems, Deterministic Service, Periodic Service, Waiting Time, M/D/1, E/D/1.

(Résumé : tsvp)

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La Distribution du Temps d'Attente dans les Systèmes Déterministes avec Arrivées Poisson

Résumé : Nous considérons des systèmes où les arrivées de clients sont selon un processus de Poisson, et les services donnés par une structure de délai déterministe. Nous caractérisons la distribution du temps d'attente en régime transitoire, et décrivons des algorithmes efficaces pour la calculer. Nous calculons également la distribution stationnaire du temps de réponse quand la structure de délai devient périodique.

Cette analyse est intimement liée à la théorie des systèmes $(max, +)$-linéaires, et a le même domaine d'application: nous donnons des exemples provenant de la théorie des réseaux de Petri et de la théorie des graphes de tâches stochastiques. Elle s'applique aussi en général aux systèmes de files d'attente mono-serveur avec des temps de service connus, et en particulier à la $M/D/1$ avec services périodiques, et la file $E/D/1$.

Mots-clé : Files d'Attente, Réseaux de Petri, Graphes d'Événements, Systèmes (max,+), Services Déterministes, Services Périodiques, Temps d'Attente, M/D/1, E/D/1.

1 Introduction

The motivation for this work originates in the paper of Baccelli & Schmidt [5] on the analysis of waiting times in Poisson-driven $(\max, +)$ systems. Under very general conditions, these authors obtain expansions of the expected stationary waiting times with respect to λ, the intensity of the arrivals:

$$\mathbb{E}W(\lambda) \;=\; \sum_{k=0}^{m} \lambda^k \;\mathbb{E}\; p_{k+1}(D_0, \ldots, D_k) \;+\; \mathcal{O}(\lambda^{m+1}) \;,$$

where p_k are known polynomials, and $\{D_n\}$ a sequence of random variables that may be interpreted as longest paths in task graphs. The technique used is based on the perturbation analysis of functionals of Poisson processes. This analysis was later extended to the computation of Laplace Transforms [3], and is being applied to transient measures [4].

The authors demonstrate with many examples that in the particular case where the sequence $\{D_n\}$ is deterministic, this expansion gives an effective way of constructing approximations of the stationary waiting time. Unfortunately, the complexity of the polynomials p_k increases very fast, which limits so far the applicability of the method, even when the growth of the sequence $\{D_n\}$ is known to become periodic after a finite rank.

In this paper, we address the same problem using an entirely different approach. Our principal objective is the direct computation of the stationary distributions in the case where the sequence $\{D_n\}$ is deterministic (or known) and has periodic increments when n is large. For this, we consider generally a system where arrivals occur according to a Poisson process, and where delays are induced by an arbitrary increasing sequence $\{D_n\}$. In the process, we establish formulas for the distribution of the *transient* waiting times and its Laplace transform. Although these expressions can barely be considered as "closed form" because they involve terms solution of recurrences, they are perfectly suited for numerical evaluation. In addition, they allow an asymptotic analysis which leads to a characterization of the stationary distribution in the form of finite formulas involving computable coefficients. Our stationary analysis reveals the strong ties these systems have with the $M/D/1$ queue.

In [4], the authors also derive finite formulas for the transient waiting times, as well as for stationary waiting times when increments of the sequence $\{D_n\}$ are constant after a certain rank. These formulas appear to be of a different nature than ours, and the comparison of the two approaches is left outside the scope of this report.

The organization of the paper is as follows. In Section 2, we introduce our approach and our notation through basic examples of queueing theory and stochastic Petri Net analysis. The conclusion of this section is that the distributions of interest are given by a certain family of functions $\{H_n, n \in \mathbb{N}\}$ defined from the sequence $\{D_n, n \in \mathbb{N}\}$. In Section 3, we show that these may be computed recursively and we give expressions for their Laplace transforms and moments. Next, we consider particular sequences $\{D_n\}$ is ultimately periodic. We establish in Section 4 the Laplace transform of the stationary waiting time in this case. We also describe how to effectively compute the distribution itself. Algorithms and computational complexities are summarized in Section 5.

We provide in Section 6 examples of applications to several queueing systems ($M/GI/1$, $M/D/1$ with periodic service times, $E/D/1$), Petri nets and periodic PERT graphs originating in the performance analysis of parallel programs. Finally, we conclude in Section 7 by discussing several interesting directions for future work.

2 Preliminaries

This paper is mainly devoted to the study of a certain family of functions $H_n, n \in \mathbb{N}$. In this section, we motivate this by showing that these functions appear naturally in stochastic discrete event systems. We begin with the classical $G/G/1$ queue, and proceed with $(\max, +)$ systems.

The queue $G/G/1$. It is well known that in the $G/G/1$ queue with inter-arrivals $\{\tau_n, n \geq 1\}$, service times $\{\sigma_n, n \geq 0\}$, and zero initial workload, the waiting time of the n-th customer is given by Loynes' formula:

$$W_n = \max_{i=0..n} \{ \sum_{j=i}^{n-1} \sigma_j - \sum_{j=i+1}^{n} \tau_j \} , \quad n \in \mathbb{N} ,$$

where by convention an empty sum is zero. If the initial workload is not zero, it has to be added to σ_0 in this expression for $n \geq 1$. Using the notation $d_i^{(n)} = \sum_{j=n-i}^{n-1} \sigma_j$, $0 \leq i \leq n$, we get: $\forall x \in \mathbb{R}$,

$$\begin{aligned} \mathbb{P}(W_n \leq x) = \mathbb{P}(& d_0^{(n)} \leq x, d_1^{(n)} \leq x + \tau_n; d_2^{(n)} \leq x + \tau_{n-1} + \tau_n; \ldots ; \\ & d_n^{(n)} \leq x + \tau_1 + \ldots + \tau_n) . \end{aligned} \tag{1}$$

If inter-arrivals are i.i.d., the variables τ_i may be renumbered. It follows that the distributions of the transient response times in the $GI/G/1$ queue, conditionally on the service times, are given in terms of the function:

$$H_n(b_0, b_1, b_2, \ldots, b_n; x) = \mathbb{P}(b_0 \leq x, b_1 \leq x + \tau_1; b_2 \leq x + \tau_1 + \tau_2; \ldots ; b_n \leq x + \tau_1 + \ldots + \tau_n) .$$

Here, $\{b_n, n \in \mathbb{N}\}$ is an increasing sequence, such that $b_0 = 0$.

Event Graphs and $(\max, +)$ systems. Event Graphs are a class of Petri nets. Stochastic event graphs have received much attention in the recent literature due in particular to the fact that their evolution is governed by equations using $(\max, +)$ operators [1]. They can actually be seen as a particular case of *linear* $(\max, +)$ *systems*, in that this evolution obeys equations of the type:

$$X(n+1) = A(n) \otimes X(n) ,$$

where $\otimes$ is a matrix/vector multiplication in the $(\max, +)$ pseudo-algebra. The reader is referred to [1, 5] for details on this algebraic representation of discrete event systems.

For the purpose of this paper, we shall rather represent these systems with *task graphs* (or PERT graphs), which are commonly used in operations research.

Consider a (stochastic) event graph (EG) containing $1 + \alpha$ transitions. Assume that the system contains at least one *source*, that is, a recycled transition without other inputs.

Without loss of generality, denote this transition by t_0. Let $\{\tau_n, n \geq 1\}$ be the sequence of inter-arrivals. Assume that the n-th firing of transition t_i takes a time $\sigma_i(n)$. For simplicity, assume also that there are no holding times in places.

In Figure 1 a/, we have represented a Petri Net of the event graph family. Transition t_0 is a source of tokens (customers) which then participate in the firings of transitions t_1, t_2 and t_3.

If we consider that each firing is a "task", then the precedence constraints between firings can be displayed in a task graph such as the one of Figure 1 b/.

In this graph, the node in row n of column t_i, which we shall denote (n, i) represents the n-th firing of transition i and has the duration $\sigma_i(n)$ if $1 \leq i \leq \alpha$, and τ_n if $i = 0$. The special node "$\perp$" represents the origin of times. The arcs between nodes correspond to the places of the EG. There is an arc from node (n, i) to node $(n + \mu, j)$ if there is a place with μ tokens between transitions t_i and t_j of the EG. We shall not consider here that arcs are weighted, although this could be possible if places of the EG had holding times and/or the initial tokens had time lags (see [1] for the signification of these terms).

For $0 \leq i \leq \alpha$, let us call $X_i(n)$ and $Y_i(n)$ the time at which transition t_i starts and completes its n-th firing, respectively. The n-th *waiting time* at transition t_i is defined as:

$$W_i(n) \;=\; X_i(n) - Y_0(n) \; . \tag{2}$$

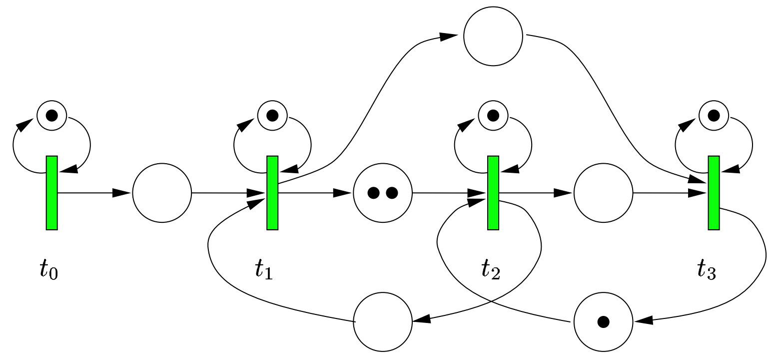
Remark 2.1 The quantity $Y_0(n)$ is the time at which the n-th token is released from the source. It is noted T_n in [5, 3]. Obviously, $Y_0(n) = \tau_1 + \ldots + \tau_n$, $n \geq 1$.

Remark 2.2 The interpretation of $W_i(n)$ as a "waiting time" is valid when the n-th token of the source is actually "responsible" for the n-th firing of transition i. In the network of Figure 1, this interpretation fails for t_2 because the two initial tokens of the place between t_1 and t_2 introduce a "delay" in the influence of tokens on firings. This possibly undesirable effect can be taken into account by re-defining $W_2(n)$ as $X_2(n+2) - Y_0(n)$ for instance. The theory below can be easily adapted to this modification. When the Petri network is *input connected* [5], there is always a path between nodes $(n, 0)$ and (n, i), $1 \leq i \leq \alpha + 1$, and $W_i(n) \geq 0$. We need not impose this restriction here.

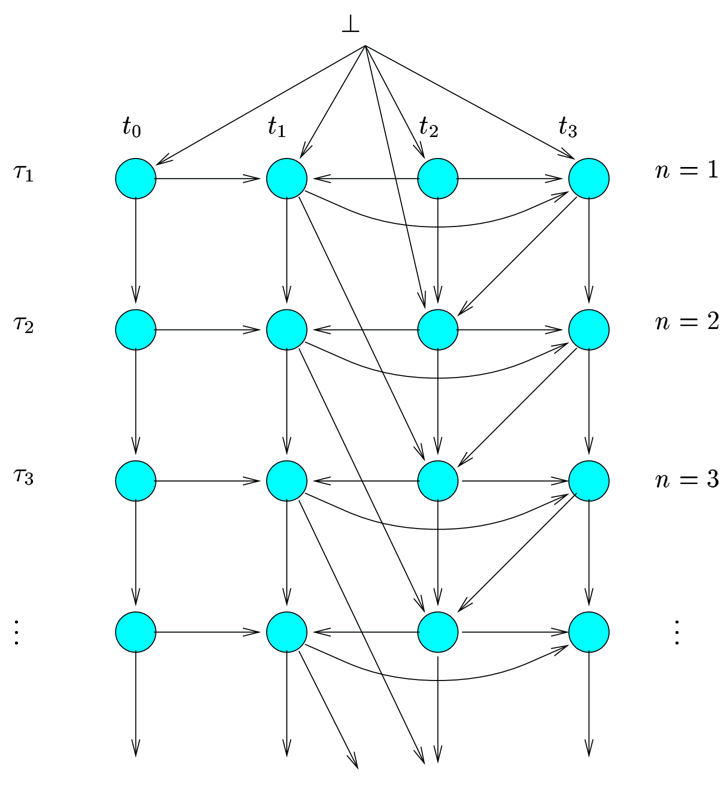
Remark 2.3 Task graphs associated with event graphs have a periodic topology, as in Figure 1 b/. It is possible to associate task graphs to general $(\max, +)$ linear systems. These graphs will have a random topology in general, but the theory below still applies. Note also that many problems are set directly in terms of task graphs. See for instance Section 6.3.2.

We now turn to the main point of this section. Let:

- $\hat{X}_i(n)$, $1 \leq i \leq \alpha$ be the longest path in the task graph going from $\perp$ to node (n, i), this one excluded, while never going through nodes $(m, 0)$. This can be interpreted as the time at which the n-th firing would begin at t_i if the source t_0 were removed (or saturated).
- $d^{0,i}_{m,n}$ the longest path in the graph going from node $(m, 0)$ to node (n, i), extremities excluded. If there are no such path, $d^{0,i}_{m,n} = -\infty$.



a/ Petri Net representation



b/ Task Graph representation

Figure 1: An Event Graph and its Task Graph representation

From the fact that $X_i(n)$ is the longest path from $\perp$ to (n,i), the following equality is easily verified:

$$\begin{aligned}\mathbb{P}(W_i(n) \le x) &= \mathbb{P}(X_i(n) \le x + Y_0(n)) \\ &= \mathbb{P}(\hat{X}_i(n) \le x + \sum_{j=1}^{n} \tau_j, \tau_1 + d_{1,n}^{0,i} \le x + \sum_{j=1}^{n} \tau_j, \dots , \\ &\qquad \sum_{j=1}^{k} \tau_j + d_{k,n}^{0,i} \le x + \sum_{j=1}^{n} \tau_j, \dots , \sum_{j=1}^{n} \tau_j + d_{n,n}^{0,i} \le x + \sum_{j=1}^{n} \tau_j) \\ &= \mathbb{P}(\hat{X}_i(n) \le x + \sum_{j=1}^{n} \tau_j, d_{1,n}^{0,i} \le x + \sum_{j=2}^{n} \tau_j, \dots , \\ &\qquad d_{k,n}^{0,i} \le x + \sum_{j=k+1}^{n} \tau_j, \dots , d_{n-1,n}^{0,i} \le x + \tau_n, d_{n,n}^{0,i} \le x) \ .\end{aligned}$$

As in the $GI/G/1$ case, if we renumber the variables τ_i, and let:

$$b_n^{(n)} = \hat{X}_i(n), \qquad b_k^{(n)} = d_{n-k,n}^{0,i}, \ 0 \le k < n,$$

we can express this distribution as:

$$\mathbb{P}(W_i(n) \le x) \ = \ \mathbb{P}(b_0^{(n)} \le x, b_1^{(n)} \le x + \tau_1; b_2^{(n)} \le x + \tau_1 + \tau_2; \dots ; b_n^{(n)} \le x + \tau_1 + \dots + \tau_n) \ .$$

Here, $\{b_m^{(n)}, n \ge m \ge 0\}$ is an increasing sequence, because $\hat{X}_i(n) \ge d_{1,n}^{0,i}$. It may contain values equal to $-\infty$.

A particularly interesting case will be that in which the system is deterministic, except for the source of customers. In this case, the value of $d_{m,n}^{0,i}$ depends only on $n-m$. Moreover, the results of the deterministic $(\max,+)$ theory (see *e.g.* [1]) allow to conclude that for each i, $1 \le i \le \alpha$:

$$\exists T_i, \gamma_i, d_i, \qquad \forall m \ge T_i, \quad d_{m+n+d_i,n}^{0,i} \ = \ d_{m+n,n}^{0,i} \ + \ d_i \gamma_i \ . \tag{3}$$

In other words, after a transient period of length T_i, the sequence grows in a periodic manner.

This regular asymptotic behavior motivates the special attention we give to ultimately periodic sequences in section 4.

3 The Transient Waiting Time

In both examples of the previous section, the distribution of the waiting time has been expressed in the form:

$$\begin{aligned}\mathbb{P}(W_n \le x) &= \mathbb{P}(b_0 \le x, b_1 \le x + \tau_1; b_2 \le x + \tau_1 + \tau_2; \dots ; b_n \le x + \tau_1 + \dots + \tau_n) \\ &= \mathbb{E} \ H_n(b_0, b_1, b_2, \dots , b_n; x) \ ,\end{aligned} \tag{4}$$

where $H_n(b_0, b_1, b_2, \ldots, b_n; x)$ is the distribution of W_n conditionally on the value of $(b_0, \ldots, b_n)$. In the following we shall assume that the numbers b_n are deterministic, while keeping in mind that the analysis may be extended whenever de-conditioning is possible (see Section 6.1).

The purpose of this section is the study $H_n(x)$, the (conditional) distribution function of W_n, when arrivals form a Poisson process. It is formally defined in Definition 3.1. We provide in paragraph 3.1 some general properties of this function. We then give in Theorem 3.3 a precise description of this distribution, which has a piecewise-like behavior.

The function $H_n(x)$ is expressed using certain coefficients $\chi^{(n)}_{u,v}$. In paragraph 3.3, we study the generating functions of these numbers.

This is then used in paragraph 3.4 to compute the Laplace transform (Theorem 3.5) and the moments of W_n (Corollary 3.6).

Each function H_n is defined from a finite set of coefficients $\{b_m, 0 \le m \le n\}$. However, we will sometimes have to consider all members of this family at once. For this reason, we shall consider throughout the analysis that there is a bi-infinite sequence $\{b_m, m \in \mathbb{Z}\}$ of numbers in $[-\infty, +\infty)$. Although final formulas will involve only the numbers $b_m, m \in \mathbb{N}$, completing the sequence in an arbitrary manner simplifies greatly some computations. Also, the sequence $\{b_m, m \in \mathbb{Z}\}$, will be assumed to be increasing, but not necessarily strictly.

3.1 Elementary properties of H_n

Definition 3.1 *Let $\{\tau_n, n \ge 1\}$ be a sequence of i.i.d. random variables with exponential distribution of parameter λ. The function H_n is defined on $[-\infty, +\infty)^n \times \mathbb{R}$ by:*

$$H_n(b_0, b_1, b_2, \ldots, b_n; x) \;=\; \mathbb{P}(b_0 \le x, b_1 \le x + \tau_1; b_2 \le x + \tau_1 + \tau_2; \ldots; b_n \le x + \tau_1 + \ldots + \tau_n) \;.$$

As a function of x, H_n is a distribution function, and is therefore increasing and right continuous. The following properties are easily proven. The three first of them do not depend on the nature of arrivals.

Lemma 3.2 *We have:*

i/ H_n is the distribution of a random variable which is bounded above by b_n and below by b_0 (if $b_0 \ne -\infty$), that is:

$$\forall x < b_0,\; H_n(b_0, b_1, b_2, \ldots, b_n; x) = 0, \qquad \forall x \ge b_n,\; H_n(b_0, b_1, b_2, \ldots, b_n; x) = 1.$$

ii/ The function H_n is homogeneous, *that is: $\forall \delta \in \mathbb{R}$,*

$$H_n(b_0 + \delta, b_1 + \delta, b_2 + \delta, \ldots, b_n + \delta; x + \delta) \;=\; H_n(b_0, b_1, b_2, \ldots, b_n; x) \;. \tag{5}$$

iii/ For any fixed x, The sequence $\{H_n(b_0, b_1, b_2, \ldots, b_n; x), n \in \mathbb{N}\}$ is decreasing.

iv/ For any j, $0 < j \le n$ and any $x \ge b_{j-1}$:

$$H_n(b_0, b_1, b_2, \ldots, b_n; x) \;=\; \int_0^\infty H_{n-j}(b_j, \ldots, b_n; x + t)\, dE_j(t) \;, \tag{6}$$

where $dE_j(t)/dt$, is the density of the Gamma distribution with parameters λ and j, that is: $\lambda e^{-\lambda t}(\lambda t)^{j-1}/(j-1)!$.

Note that expression (6) does not depend on the values of $b_0, \ldots, b_{j-1}$.

These properties have useful probabilistic interpretations. Denote by $W(b_0, \ldots, b_n)$ a random variable with distributions $H_n(b_0, \ldots, b_n; \cdot)$. By *iii/* of Lemma 3.2, the sequence $\{W(b_0, \ldots, b_n), n \in \mathbb{N}\}$ is stochastically increasing. The homogeneity property (5) states that:

$$W(\delta + b_0, \ldots, \delta + b_n) \ =_d \ \delta \ + \ W(b_0, \ldots, b_n) \ , \tag{7}$$

and (6) is the analytic formulation of:

$$\mathbb{P}(W(b_0, \ldots, b_n) \leq x) \ = \ \mathbb{P}(W(b_j, \ldots, b_n) \leq E_j + x) \ , \quad x \geq b_{j-1}. \tag{8}$$

In particular,

$$W(-\infty, \ldots, -\infty, b_p, \ldots, b_n) \ =_d \ W(b_p, \ldots, b_n) \ - \ E_p \ . \tag{9}$$

In the queueing theory literature, the distribution of E_p is known as an Erlang distribution with p phases.

Remark 3.1 Formula (6) provides a recurrence which is essential to the analysis to come. However, the fact that the sequence $\{b_n\}$ is reduced to the *left* introduces technical difficulties. Conditioning on τ_n in the definition of H_n gives the relation:

$$H_n(b_0, b_1, b_2, \ldots, b_n; x) \ = \ \int_0^\infty H_{n-1}(b_0, \ldots, b_{n-2}, b_{n-1} \vee (b_n - t)^+; x) \lambda e^{-\lambda t} \mathrm{d}t \ .$$

This recurrence "from the right" is far less natural than (6).

3.2 Characterization of H_n

The distribution of W_n is characterized by the following theorem.

Theorem 3.3 *Assume that $b_n > -\infty$. The function H_n writes:*

$$H_n(b_0, b_1, b_2, \ldots, b_n; x) \ = \ e^{-\lambda(b_n - x)^+} \ L_n(b_0, b_1, b_2, \ldots, b_n; x) \ , \tag{10}$$

where L_n has the properties:

i/ $L_n(b_0, b_1, b_2, \ldots, b_n; x) = 1, \ \forall x \geq b_n$, and $L_n(b_0, b_1, b_2, \ldots, b_n; x) = 0, \ \forall x < b_0$.

ii/ For all $0 \leq j \leq n-1$, L_n is a polynomial of degree j on the interval $[b_j, b_{j+1}]$, given by:

$$L_n(b_0, b_1, b_2, \ldots, b_n; x) \ = \ \sum_{s=0}^{j} \frac{\lambda^s}{s!} (b_{j+1} - x)^s L_{n-s}(b_s, \ldots, b_n; b_{j+1}) \ . \tag{11}$$

iii/ If $b_j > -\infty$, the numbers $\xi_{n,m,j} = L_{n-m}(b_m, \ldots, b_n; b_j)$, $0 \le m \le j < n$ are given by the recurrence:

$$\xi_{n,m,j} \;=\; \sum_{s=0}^{j-m} \frac{\lambda^s}{s!}(b_{j+1}-b_j)^s \xi_{n,m+s,j+1}\;, \quad 0 \le m \le j < n, \tag{12}$$

with $\xi_{n,m,n} = 1$, $\forall 0 \le m \le n$.

Proof Property *i/* is a direct consequence of *ii/* in Lemma 3.2.

Point *ii/* may be proved by a recurrence based on (6). We use here a direct probabilistic argument. Consider that the sequence $\{\tau_m, m \in \mathbb{N}\}$ defines a Poisson process $\mathcal{P}$ on the half line $[x, \infty)$, with intensity λ. Let A denote the event: $\{b_{j+1} \le x + \tau_1 + \ldots + \tau_{j+1}, \ldots, b_n \le x + \tau_1 + \ldots + \tau_n\}$. For $x \in [b_j, b_{j+1}]$, we have: $H_n(b_0, b_1, b_2, \ldots, b_n; x) = \mathbb{P}(A)$. But

$$\mathbb{P}(A) \;=\; \sum_{s=0}^{\infty} \mathbb{P}(A \mid \mathcal{P} \text{ has } s \text{ points in } [x, b_{j+1}))\; \frac{\lambda^s}{s!}(b_{j+1}-x)^s e^{-\lambda(b_{j+1}-x)}\;. \tag{13}$$

If $j+1$ points of $\mathcal{P}$, or more, fall in $[x, b_{j+1})$, then A cannot be true for then: $\tau_1 + \ldots + \tau_{j+1} \le b_{j+1} - x$. Take therefore $s \le j$. Conditionally to the fact that s points are in $[x, b_{j+1})$, the variable $\tau'_{s+1} = \tau_1 + \ldots + \tau_{s+1} - b_{j+1}$ is exponentially distributed. Then, (13) rewrites as (see Fig. 2):

$$\begin{aligned}
&\mathbb{P}(A \mid s \text{ points in } [x, b_{j+1}))\\
&\quad= \mathbb{P}(\tau'_{s+1} + \tau_{s+2} + \ldots + \tau_{j+1} \ge b_{j+2} - b_{j+1}, \ldots, \tau'_{s+1} + \tau_{s+2} + \ldots + \tau_n \ge b_n - b_{j+1})\\
&\quad= \mathbb{P}(b_{j+2} \le b_{j+1} + \tau'_{s+1} + \tau_{s+2} + \ldots + \tau_{j+1}, \ldots, b_n \le b_{j+1} + \tau'_{s+1} + \tau_{s+2} + \ldots + \tau_n)\\
&\quad= \mathbb{P}(b_s \le b_{j+1}, b_{s+1} \le b_{j+1} + \tau'_{s+1}, \ldots,\\
&\qquad\quad b_{j+2} \le b_{j+1} + \tau'_{s+1} + \tau_{s+2} + \ldots + \tau_{j+1}, \ldots, b_n \le b_{j+1} + \tau'_{s+1} + \tau_{s+2} + \ldots + \tau_n)\\
&\quad= H_{n-s}(b_s, \ldots, b_n; b_{j+1})\;.
\end{aligned} \tag{14}$$

Replacing in (13) yields (11).

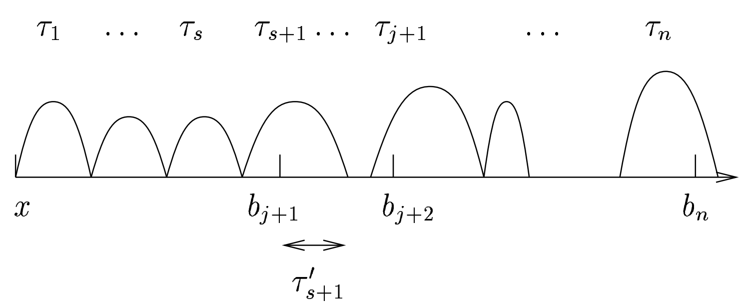


Figure 2: Illustration of the proof of Theorem 3.3

Point *iii/* follows from *ii/* using the right continuity of $H_n(x)$ at point $x = b_j$. ∎

In terms of the function H_n, Theorem 3.3 gives:

Corollary 3.4 *The function H_n satisfies: for all $0 \le j \le n-1$, $x \in [b_j, b_{j+1}]$:*

$$H_n(b_0, b_1, b_2, \ldots, b_n; x) \;=\; \sum_{s=0}^{j} \frac{\lambda^s}{s!}(b_{j+1}-x)^s e^{-\lambda(b_{j+1}-x)} H_{n-s}(b_s, \ldots, b_n; b_{j+1}) \;. \tag{15}$$

If $b_j > -\infty$, the numbers $\zeta_{n,m,j} = H_{n-m}(b_m, \ldots, b_n; b_j)$, $0 \le m \le j < n$ are given by the recurrence:

$$\zeta_{n,m,j} \;=\; \sum_{s=0}^{j-m} \frac{\lambda^s}{s!}(b_{j+1}-b_j)^s e^{-\lambda(b_{j+1}-b_j)} \zeta_{n,m+s,j+1} \;, \quad 0 \le m \le j < n, \tag{16}$$

with $\zeta_{n,m,n} = 1$, $\forall 0 \le m \le n$.

Remark 3.2 A consequence of Corollary 3.4 is that the function H_n is continuous, except at b_0 (if $b_0 > -\infty$), differentiable except at b_0, b_1, and so on.

Remark 3.3 It may seem to be redundant to introduce the two families of numbers $\{\xi_{n,m,i}\}$ and $\{\zeta_{n,m,i}\}$. Actually, both have their own advantages, depending on whether formal or numerical procedures are in mind. For instance, from point *iii/* of Theorem 3.3, if is seen that the $\xi_{n,m,i}$ are polynomials in λ. On the other hand, the $\zeta_{n,m,i}$ are probabilities and are easier to interpret in terms of the original problem. We shall use them only in the remainder of the paper.

According to Corollary 3.4, the computation of the function H_n is reduced to that of the numbers $\{\zeta_{n,m,i}\}$ corresponding to the boundaries of the intervals defined by the b_ms. For this reason, we have called these quantities "hinge probabilities".

3.3 Hinge probabilities and their generating functions

The dependence structure of the recurrence (16) is depicted in Figure 3. One sees that, n being fixed, coefficients are computed "backwards", from right to left. This particularity is the cause of the complexity of the notation and of forthcoming calculations.

Recurrence (16) may be reduced through usual generating function techniques. For this a few technical manipulations are needed. Fix an integer n. Let, for any $0 \le v \le u$,

$$\chi_{u,v}^{(n)} \;=\; \zeta_{n,n-u,n-v} \;=\; H_u(b_{n-u}, \ldots, b_n; b_{n-v}) \;. \tag{17}$$

With this notation, (16) rewrites as: if $b_{n-v} > -\infty$,

$$\chi_{u,v}^{(n)} \;=\; \sum_{s=0}^{u-v} \frac{\lambda^s}{s!}(b_{n-v+1}-b_{n-v})^s e^{-\lambda(b_{n-v+1}-b_{n-v})} \chi_{u-s,v-1}^{(n)} \qquad 1 \le v \le u \;, \tag{18}$$

$$\chi_{u,0}^{(n)} \;=\; 1 \qquad u \ge 0 \;. \tag{19}$$

If $b_{n-v} = -\infty$, then $\chi_{u,v}^{(n)} = 0$. These equations define for each $v \in \mathbb{N}$ an infinite sequence $\{\chi_{u,v}^{(n)}, u \ge v\}$. Let, for $v \in \mathbb{N}$,

$$\Xi_v^{(n)}(z) = \sum_{u=v}^{\infty} \chi_{u,v}^{(n)} z^u \;.$$

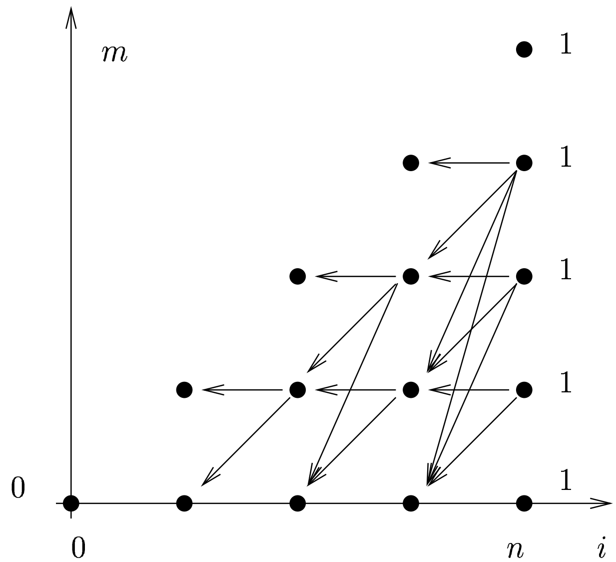


Figure 3: Illustration of the recurrence on $\zeta_{n,m,i}$

Each $\chi^{(n)}_{u,v}$ being a probability, it is clear that $\Xi^{(n)}_v(z)$ is analytic inside the unit disc. Moreover, $|\Xi^{(n)}_v(z)| \le |z|^v/(1-|z|)$ when $|z|<1$.

From (18), standard computations give:

$$\Xi^{(n)}_v(z) \;=\; e^{\lambda(z-1)(b_{n-v+1}-b_{n-v})} \left[\Xi^{(n)}_{v-1}(z) \;-\; z^{v-1}\chi^{(n)}_{v-1,v-1}\right], \quad v \ge 1, \tag{20}$$

and from (19): $\Xi^{(n)}_0(z) \;=\; 1/(1-z)$.

Applying repeatedly this recurrence, one obtains: for any $m \le n$,

$$\Xi^{(n)}_v(z) \;=\; e^{\lambda(z-1)(b_{n-m}-b_{n-v})}\Xi^{(n)}_m - \sum_{j=m}^{v-1} z^j e^{\lambda(z-1)(b_{n-j}-b_{n-v})}\chi^{(n)}_{j,j} \;.$$

Extracting the coefficient of z^u, and taking into account the fact that $[z^u]\Xi^{(n)}_v(z)=0, u<v$, we obtain the general relationship:

$$\begin{aligned}\chi^{(n)}_{u,v} \;&=\; \sum_{\ell=0}^{u-m}\frac{\lambda^\ell}{\ell!}(b_{n-m}-b_{n-v})^\ell e^{-\lambda(b_{n-m}-b_{n-v})}\chi^{(n)}_{u-\ell,m} \\ &\quad - \sum_{\ell=u-v+1}^{u-m}\frac{\lambda^\ell}{\ell!}(b_{n-u+\ell}-b_{n-v})^\ell e^{-\lambda(b_{n-u+\ell}-b_{n-v})}\chi^{(n)}_{u-\ell,m-\ell} \;.\end{aligned} \tag{21}$$

Several instances of this formula will be used in the sequel.

3.4 Laplace Transforms and moments

We now turn to the computation of the Laplace transform of H_n, which is useful for deriving synthetic formulas for the moments of W_n, and is also the cornerstone of the stationary analysis of Section 4.

Let

$$\begin{aligned} H_n^*(b_0, b_1, b_2, \ldots, b_n; s) &= \int_{-\infty}^{\infty} e^{-sx} \mathrm{d} H_n(b_0, b_1, b_2, \ldots, b_n; x) \\ &= s \int_{b_0}^{\infty} e^{-sx} H_n(b_0, b_1, b_2, \ldots, b_n; x) \mathrm{d}x \end{aligned}$$

The support of the random variable W_n is bounded above by b_n (Lemma 3.2, *i/*). This ensures that $H_n^*(s)$ is defined and analytic for $\Re(s) \leq 0$. Moreover, for any $p \leq n$, and any s in this domain,

$$|H_{n-p}^*(b_p, b_{p+1}, \ldots, b_n; s)| \leq e^{-b_n \Re(s)}.$$

Theorem 3.5 *We have:*

$$\begin{aligned} & H_n^*(b_0, b_1, b_2, \ldots, b_n; s) \\ &= e^{-b_n s} \left(\frac{\lambda}{\lambda - s} \right)^{n+1} - \frac{s}{\lambda - s} \sum_{p=0}^{n} \chi_{p,p}^{(n)} \, e^{-s b_{n-p}} \left(\frac{\lambda}{\lambda - s} \right)^{n-p} \\ &= e^{-b_n s} \left(\frac{\lambda}{\lambda - s} \right)^{n+1} - \frac{s}{\lambda - s} \sum_{p=0}^{n} H_{n-p}(b_p, \ldots, b_n; b_p) \, e^{-s b_p} \left(\frac{\lambda}{\lambda - s} \right)^{p} , \qquad (22) \end{aligned}$$

with the convention that $\chi_{p,p}^{(n)} \, e^{-s b_{n-p}}$ *vanishes if* $b_{n-p} = -\infty$. *This function is analytic on* $\mathbb{C}$ *except at point* $s = \lambda$ *if* $b_0 = -\infty$.

Proof Consider the following generating function of Laplace transforms:

$$\begin{aligned} \mathcal{H}^{(n)}(z, s) &= \sum_{p=0}^{\infty} H_p^*(b_{n-p}, \ldots, b_n; s) z^p \,. & (23) \\ &= \sum_{p=0}^{\infty} s z^p \int_{-\infty}^{\infty} e^{-sx} H_p(b_{n-p}, \ldots, b_n; x) dx \\ &= s \int_{-\infty}^{\infty} e^{-sx} \left(\sum_{p=0}^{\infty} z^p \; H_p(b_{n-p}, \ldots, b_n; x) \right) dx & (24) \end{aligned}$$

The function $\mathcal{H}^{(n)}(z, s)$, as defined by (23) is analytic in the domain $\{(z, s), |z| < 1, \Re(s) \leq 0\}$. The integrand is uniformly integrable on $\mathcal{D} = \{(z, s), |z| \leq \alpha < 1, \Re(s) \leq \beta < 0\}$, and Fubini's theorem applies. Let us evaluate the inner series. Assume $x \in [b_{n-v}, b_{n-v+1}]$. Using

Corollary 3.4, the computation goes as follows:

$$
\begin{aligned}
&\sum_{p=0}^{\infty} z^p H_p(b_{n-p}, \ldots, b_n; x) \\
&= \sum_{p=0}^{\infty} z^p \sum_{q=0}^{p-v} \frac{\lambda^q}{q!}(b_{n-v+1}-x)^q e^{-\lambda(b_{n-v+1}-x)} H_{p-q}(b_{n-p+q}, \ldots, b_n; b_{n-v+1}) \\
&= e^{-\lambda(b_{n-v+1}-x)} \sum_{p=v}^{\infty}\sum_{q=0}^{p-v} z^p \frac{\lambda^q}{q!}(b_{n-v+1}-x)^q \chi^{(n)}_{p-q,v-1} \\
&= e^{-\lambda(b_{n-v+1}-x)} \sum_{q=0}^{\infty}\sum_{p=v}^{\infty} z^{p+q} \frac{\lambda^q}{q!}(b_{n-v+1}-x)^q \chi^{(n)}_{p,v-1} \\
&= e^{-\lambda(b_{n-v+1}-x)} \sum_{q=0}^{\infty} \frac{\lambda^q}{q!}(b_{n-v+1}-x)^q z^q \sum_{p=v}^{\infty} z^p \chi^{(n)}_{p,v-1} \\
&= e^{-\lambda(b_{n-v+1}-x)} e^{\lambda z(b_{n-v+1}-x)} (\Xi^{(n)}_{v-1}(z) - \chi^{(n)}_{v-1,v-1} z^{v-1}) \\
&= e^{\lambda(z-1)(b_{n-v}-x)} \Xi^{(n)}_v(z) \ . \qquad (25)
\end{aligned}
$$

In the last equation, we have made use of (20). On the other hand, this series is clearly $1/(1-z)$ if $x \geq b_n$. If $x < \lim_{m \to -\infty} b_m$, the series vanishes. Replacing (25) in (24), we obtain:

$$
\begin{aligned}
\mathcal{H}^{(n)}(z,s) &= s \sum_{v=1}^{\infty} \int_{b_{n-v}}^{b_{n-v+1}} e^{-sx} \ e^{\lambda(z-1)(b_{n-v}-x)} \Xi^{(n)}_v(z) dx \ + \ s \int_{b_n}^{\infty} e^{-sx} \frac{1}{1-z} dx \\
&= s \sum_{v=1}^{\infty} \Xi^{(n)}_v(z) e^{\lambda(z-1)b_{n-v}} \left[\frac{e^{x(\lambda(1-z)-s)}}{\lambda(1-z)-s} \right]_{b_{n-v}}^{b_{n-v+1}} \ + \ \frac{e^{-sb_n}}{1-z} \\
&= \frac{s}{\lambda(1-z)-s} \sum_{v=1}^{\infty} \Xi^{(n)}_v(z) e^{\lambda(z-1)b_{n-v}} (e^{b_{n-v+1}(\lambda(1-z)-s)} - e^{b_{n-v}(\lambda(1-z)-s)}) \\
&\quad + \frac{e^{-sb_n}}{1-z} \qquad (26)
\end{aligned}
$$

This computation is justified because the series are uniformly convergent in the domain $\mathcal{D}$. Let us first reduce the series in (26). We have, using again (20):

$$
\begin{aligned}
\sum_{v=1}^{\infty} \cdots &= \sum_{v=1}^{\infty} e^{-sb_{n-v+1}} (\Xi^{(n)}_{v-1}(z) - z^{v-1}\chi^{(n)}_{v-1,v-1}) - \sum_{v=1}^{\infty} \Xi^{(n)}_v(z) e^{-sb_{n-v}} \\
&= e^{-sb_n} \Xi^{(n)}_0(z) - \sum_{p=0}^{\infty} z^p \chi^{(n)}_{p,p} e^{-sb_{n-p}} \ . \qquad (27)
\end{aligned}
$$

After replacing (27) in (26) and remembering that $\Xi_0^{(n)}(z) = 1/(1-z)$, simplifications lead to the formula:

$$\mathcal{H}^{(n)}(z,s) \;=\; \frac{1}{\lambda(1-z)-s}\left(\lambda e^{-sb_n} - s\sum_{p=0}^{\infty}\chi_{p,p}^{(n)} z^p e^{-sb_{n-p}}\right) . \tag{28}$$

Developing (28) in series of z, and identifying the coefficients z^n yields the first form of (22). The second follows using the definition (17). From this formula, it is obvious that the only possible singularities of $H_n^*(s)$ are a pole at $s = \lambda$. On the other hand, if b_0 is finite, W_n is a bounded random variable, and $H_n^*(s)$ is entire. ■

Corollary 3.6 *The moments of W_n can be computed as:*

$$\begin{aligned}\mathbb{E}W_n^k \;=\; & \sum_{j=0}^{k-1}\frac{(-1)^j}{\lambda^j}\binom{k}{j}\left(\frac{(n+j)!}{n!}\,b_n^{k-j} \;+\; \frac{1}{\lambda}\sum_{p=0}^{n}\chi_{p,p}^{(n)}\frac{(n-p+j)!}{(n-p)!}\,b_{n-p}^{k-j-1}\right)\\ & +\frac{(-1)^k}{\lambda^k}\frac{(n+k)!}{n!} .\end{aligned} \tag{29}$$

In particular:

$$\mathbb{E}W_n \;=\; b_n \;+\; \frac{1}{\lambda}\sum_{p=0}^{n}\chi_{p,p}^{(n)} - \frac{n+1}{\lambda} . \tag{30}$$

Proof Equation (29) derives easily from (22) using the fact that for $\beta \geq 1$,

$$\frac{d^k}{ds^k}\, e^{\alpha s}\left(\frac{\lambda}{\lambda - s}\right)^{\beta}\Bigg|_{s=0} \;=\; \sum_{j=0}^{k}\binom{k}{j}\frac{(\beta+j-1)!}{(\beta-1)!}\frac{\alpha^{k-j}}{\lambda^j} .$$

■

4 Stationary Analysis

In this section, we consider the limit behavior of the distribution W_n when $n \to \infty$, knowing some properties on the sequence $\{b_m, m \in \mathbb{Z}\}$. More precisely, we shall assume that the sequence is *ultimately pseudo periodic*, that is:

$$\exists T, d \in \mathbb{N}, \gamma \in \mathbb{R}_+, \quad \forall i \geq T,\; b_{i+d} \;=\; b_i + d\gamma .$$

The number d is the *period* of the sequence, while γ is its average growth rate. As already discussed in Section 2, it is known that all deterministic (max, +) systems have an evolution of this type. The analysis of this section therefore covers a large class of systems with deterministic internal evolution, driven by Poisson inputs.

We first state general results. We then investigate the case where $d = 1$. This case turns out to be very close to the analysis of the $M/D/1$ queue, which it actually contains. In a second part, we study the general case $d \geq 2$. The Laplace transform of the stationary distribution is obtained in Theorems 4.2 and 4.4.

In order to handle at the same time the transient part and the periodic part of the sequence $\{b_m, m \in \mathbb{N}\}$, it is convenient to introduce a second sequence, $\{c_m, m \in \mathbb{Z}\}$ which is completely periodic, i.e.: $c_{i+d} = c_i + d\gamma$ for any $i \in \mathbb{Z}$, and which coincides with $\{b_m\}$ for values of m larger than T.

The stationary analysis involves the following limits:

$$\begin{aligned}
H^*(s) &= \lim_{n\to\infty} H_n^*(b_0, \ldots, b_n; s) \\
h_p &= \lim_{n\to\infty} H_n(c_p, \ldots, c_{n+p}; c_p) && (31) \\
\mathcal{W}(s) &= \lim_{n\to\infty} H_n^*(c_0, \ldots, c_n; s) \\
\tau_p &= \lim_{n\to\infty} H_n(b_p, \ldots, b_{n+p}; b_p)\ , \quad 0 \leq p \leq T && (32) \\
\phi_k &= \lim_{n\to\infty} H_n(b_{T-k}, \ldots, b_n; b_T)\ , \quad 0 \leq k \leq T. && (33)
\end{aligned}$$

The limits of H_n exist because the sequence is bounded and monotone (Lemma 3.2). This implies the pointwise convergence of Laplace transforms H_n^* as well.

The key observation is that the "irregularity" of the sequence $\{b_m\}$ for $m < T$ can be compensated so that the difference with the periodic case is expressed in finite form.

Lemma 4.1 *The limits $H^*(s)$ and $\mathcal{W}(s)$ are related by:*

$$H^*(s) \;=\; \mathcal{W}(s) \;+\; \frac{s}{\lambda - s} \sum_{p=0}^{T-1} \left(h_p e^{-sc_p} - \tau_p e^{-sb_p}\right) \left(\frac{\lambda}{\lambda - s}\right)^p . \tag{34}$$

Proof From Theorem 3.5, we have, for any $n \geq T$:

$$\begin{aligned}
&H_n^*(b_0, \ldots, b_n; s) \\
&= e^{-sb_n}\left(\frac{\lambda}{\lambda - s}\right)^{n+1} - \frac{s}{\lambda - s}\sum_{p=0}^{T-1} H_{n-p}(b_p, \ldots, b_n; b_p) e^{-sb_p}\left(\frac{\lambda}{\lambda - s}\right)^p \\
&\qquad - \frac{s}{\lambda - s}\sum_{p=T}^{n} H_{n-p}(b_p, \ldots, b_n; b_p) e^{-sb_p}\left(\frac{\lambda}{\lambda - s}\right)^p \\
&= e^{-sc_n}\left(\frac{\lambda}{\lambda - s}\right)^{n+1} - \frac{s}{\lambda - s}\sum_{p=0}^{T-1} H_{n-p}(b_p, \ldots, b_n; b_p) e^{-sb_p}\left(\frac{\lambda}{\lambda - s}\right)^p \\
&\qquad - \frac{s}{\lambda - s}\sum_{p=T}^{n} H_{n-p}(c_p, \ldots, c_n; c_p) e^{-sc_p}\left(\frac{\lambda}{\lambda - s}\right)^p && (35)
\end{aligned}$$

In particular,

$$\begin{aligned} & H_n^*(c_0,\dots,c_n;s) \\ = \; & e^{-sc_n}\left(\frac{\lambda}{\lambda-s}\right)^{n+1} - \frac{s}{\lambda-s}\sum_{p=0}^{n} H_{n-p}(c_p,\dots,c_n;c_p)e^{-sc_p}\left(\frac{\lambda}{\lambda-s}\right)^{p} . \end{aligned} \tag{36}$$

Combining (35) and (36) gives immediately:

$$\begin{aligned} & H_n^*(b_0,\dots,b_n;s) = H_n^*(c_0,\dots,c_n;s) \\ + \; & \frac{s}{\lambda-s}\sum_{p=0}^{T-1}\left(H_{n-p}(c_p,\dots,c_n;c_p)e^{-sc_p} - H_{n-p}(b_p,\dots,b_n;b_p)e^{-sb_p}\right)\left(\frac{\lambda}{\lambda-s}\right)^{p} . \end{aligned}$$

The result follows, letting $n\to\infty$ and using (31) and (32). ∎

There remains to evaluate $\mathcal{W}(s)$ and the probabilities h_p, τ_p. In the case $d=1$, this is done by reduction to the $M/D/1$ queue. In the case $d=2$, this requires a little more work.

4.1 Laplace Transforms: the case $d = 1$

Assume therefore that:

$$\exists T\in\mathbb{N}, \gamma\in\mathbb{R}_+, \quad \forall i\geq T,\; b_i \;=\; b_T+(T-i)\gamma .$$

In this case, the sequence $\{c_m\}$ is given by:

$$c_0 \;=\; b_T - T\gamma, \qquad c_m \;=\; c_0+m\gamma .$$

Using the homogeneity property (5) of H_n: for all $x\geq 0$,

$$H_n(c_p,\dots,c_{n+p};c_p+x) \;=\; H_n(0,c_{p+1}-c_p,\dots,c_{n+p}-c_p;x) \;=\; H_n(0,\gamma,\dots,n\gamma;x) .$$

This quantity depends only on n and not on p. From Section 2, we know that

$$H_n(0,\gamma,\dots,n\gamma;x) \;=\; \mathbb{P}(W_n^{(D)}\leq x) . \tag{37}$$

where the random variable $W_n^{(D)}$ is the response time of the n-th customer to enter a $M/D/1$ queue with input rate λ and service duration γ. It is well known that if $\rho=\lambda\gamma<1$, $W_n^{(D)}$ converges to the stationary waiting time in the $M/D/1$ queue: $W^{(D)}$. The Laplace transform of its distribution, say, $\mathcal{W}^{(D)}(s)$ is given by the Pollaczek-Khinchine transform formula:

$$\mathcal{W}^{(D)}(s) \;=\; \frac{s(1-\rho)}{s-\lambda(1-e^{-\gamma s})} . \tag{38}$$

Therefore, we know that under the stability condition, the following limit exists:

$$\lim_{n\to\infty} H_n^*(0,\gamma,\dots,n\gamma;s) \;=\; \mathcal{W}^{(D)}(s) . \tag{39}$$

Also, if

$$g_n \triangleq H_n(0, \gamma, \ldots, n\gamma; 0) \;=\; \mathbb{P}(W_n^{(D)} = 0) \;,$$

we have, for any p,

$$h_p \;=\; \lim_{n\to\infty} g_n \;=\; 1-\rho \;. \tag{40}$$

We can now state the principal result of this section.

Theorem 4.2 *Assume that $\lambda\gamma < 1$. The distribution of W_n admits a limit when $n \to \infty$. Its Laplace transform is given by:*

$$H^*(s) \;=\; \mathcal{W}^{(D)}(s)\left(\frac{\lambda}{\lambda - s}\right)^T e^{-sb_T} \;-\; \frac{s}{\lambda - s}\sum_{p=0}^{T-1}\tau_p e^{-sb_p}\left(\frac{\lambda}{\lambda - s}\right)^p \;. \tag{41}$$

Proof The homogeneity property implies:

$$\mathcal{W}(s) \;=\; e^{-sc_0}\mathcal{W}^{(D)}(s) \;=\; e^{-s(b_T - T\gamma)}\mathcal{W}^{(D)}(s) \;.$$

From Lemma 4.1 and (40), we have therefore:

$$\begin{aligned} H_n^*(s) \;&=\; e^{-s(b_T - T\gamma)}\mathcal{W}^{(D)}(s) \\ &\quad+\; \frac{s}{\lambda - s}\left[\sum_{p=0}^{T-1}(1-\rho)e^{-sp\gamma}\left(\frac{\lambda}{\lambda - s}\right)^p \;-\; \sum_{p=0}^{T-1}\tau_p e^{-sb_p}\left(\frac{\lambda}{\lambda - s}\right)^p\right] \;. \end{aligned}$$

After simplification of the middle term, and using (38) this reduces to (41). ∎

The result involves the limiting probabilities τ_p. Their calculation is discussed in Appendix B. Moments follow easily from (41). In particular:

Corollary 4.3 *The first moment of W is given by:*

$$\mathbb{E}\, W \;=\; \frac{1}{2}\,\frac{\rho\gamma}{1-\rho} \;+\; b_T \;-\; \frac{T}{\lambda} \;+\; \frac{1}{\lambda}\sum_{p=0}^{T-1}\tau_p \;. \tag{42}$$

One recognizes in this formula the expected stationary waiting time in the $M/D/1$ queue:

$$\mathbb{E}W^{(D)} \;=\; \frac{1}{2}\,\frac{\rho\gamma}{1-\rho} \;=\; \frac{1}{2\lambda}\,\frac{\rho^2}{1-\rho} \;. \tag{43}$$

4.2 Laplace Transforms: the case $d \geq 2$

We now turn to the general case. The difficulty here is that even for the periodic sequence $\{c_n\}$, the limits:

$$\lim_{n \to \infty} H_n(c_0, \ldots, c_n; c_p)$$

are not known from the literature. In addition, the number

$$H_n(c_p, \ldots, c_{n+p}; c_p + x) \;=\; H_n(0, c_{p+1} - c_p, \ldots, c_{n+p} - c_p; x)$$

does depend on p, more precisely on p modulo d.

The Laplace transform of the stationary distribution of W is provided by the following theorem, which generalizes the Pollaczek-Khinchine formula (38).

Theorem 4.4 *Assume that $\lambda\gamma < 1$. Then the distribution of W_n admits a limit when $n \to \infty$. Its Laplace transform is given by (34), where:*

$$\mathcal{W}(s) \;=\; -\frac{1}{1 - \left(\dfrac{\lambda}{\lambda - s}\right)^d e^{-sd\gamma}} \sum_{q=0}^{d-1} h_q \left(\frac{\lambda}{\lambda - s}\right)^q \frac{s}{\lambda - s}\, e^{-sc_q} \;. \tag{44}$$

Proof We shall establish this result by computing the generating function:

$$\mathcal{G}(z, s) \;=\; \sum_{n=0}^{\infty} H_n^*(c_0, \ldots, c_n; s) z^n \;.$$

The sequence $\{c_m\}$ having periodic increments, c_0 is finite, and therefore $\mathcal{G}(z, s)$ is defined and analytic on the domain $\{(z, s) \mid |z| < 1, \Re(s) \geq 0\}$. Using a classical result (*e.g.* [11, p. 224]), and the convergence of the sequence $H_n^*(\ldots; s)$ for s real and positive, we shall then obtain:

$$\mathcal{W}(s) \;=\; \lim_{z \to 1} (1 - z)\mathcal{G}(z, s) \;. \tag{45}$$

The computation of $\mathcal{G}(z, s)$ proceeds as follows. Let $f = \lambda/(\lambda - s)$. Starting from Theorem (3.5), we obtain:

$$\begin{aligned}
\mathcal{G}(z, s) &= \sum_{n=0}^{\infty} e^{-sc_n} f^{n+1} z^n \;+\; (1 - f) \sum_{n=0}^{\infty} \sum_{p=0}^{n} H_{n-p}(c_p, \ldots, c_n; c_p) e^{-sc_p} f^p z^n \\
&= f \sum_{q=0}^{d-1} \sum_{m=0}^{\infty} e^{-s(c_q + d\gamma)} (zf)^{q+md} \;+\; (1 - f) \sum_{p=0}^{\infty} \sum_{n=0}^{\infty} H_n(c_p, \ldots, c_{n+p}; c_p) e^{-sc_p} f^p z^{n+p} \\
&= \frac{1}{1 - (zf)^d e^{-sd\gamma}} \sum_{q=0}^{d-1} e^{-sc_q} (zf)^q \left(f + (1 - f) \sum_{n=0}^{\infty} H_n(c_q, \ldots, c_{n+q}; c_q) z^n \right) \;.
\end{aligned} \tag{46}$$

The results follows, applying (45), and using

$$\lim_{z\to 1}(1-z)\sum_{n=0}^{\infty} H_n(c_q,\ldots,c_{n+q};c_q)z^n \;=\; h_q \;.$$

■

Appendices C and B discuss the actual computation of the numbers h_q and τ_q. Appendix D investigates the application of (46) to the computation of transient probabilities.

Moments of the stationary waiting time may be computed from the Laplace transform. In particular:

Corollary 4.5 *The expected stationary waiting time is given by:*

$$\mathbb{E}W \;=\; \frac{1}{\lambda(1-\rho)}\left(\frac{1}{2}\,(1+d(1-\rho)^2) \;-\; \frac{1}{d}\sum_{q=0}^{d-1} h_q\,(q+1-\lambda c_q)\right) \tag{47}$$

$$\;=\; \mathbb{E}W^{(D)} \;+\; \frac{1}{\lambda}\left(\frac{d-1}{2}(1-\rho) \;-\; \frac{1}{d(1-\rho)}\sum_{q=0}^{d-1} h_q\,(q-\lambda c_q)\right)\;. \tag{48}$$

Formula (47) is obtained directly, whereas (48) makes appear the expected stationary waiting time in the $M/D/1$ queue (43).

Remark 4.1 The limiting distribution $H^*(s)$ assumes a different form in (34) than in (41) because rearrangements that were possible when $d=1$ are not done here. A form similar to (41) may be obtained when T is a multiple of d, using the periodicity of the sequence $\{h_q\}$.

4.3 The stationary distribution

The distribution of the stationary waiting time W may be obtained by (formal) inversion of the corresponding Laplace Transform given in Theorem 4.2 or 4.4.

In the case $d=1$, this is feasible because equation (41) involves:

- the distribution of $W^{(D)}$, which is known from the literature (*e.g.* [10, p. 273]):

$$\mathbb{P}(W^{(D)}\le x) \;=\; (1-\rho)\sum_{j=0}^{k}(-1)^j\frac{\lambda^j}{j!}(x-j\gamma)^j e^{-\lambda(x-j\gamma)}, \qquad x\in[k\gamma,(k+1)\gamma)\;.$$

- terms of the form:

$$\left(\frac{\lambda}{\lambda-s}\right)^p e^{-s\alpha},$$

which are the transform of a shifted, negated Gamma (or Erlang) distribution.

In the case $d \geq 2$, the inversion of (34) requires first the inversion of $\mathcal{W}(s)$, which is possible from (44) using a series expansion of the denominator.

However, computing the convolutions is quite cumbersome. A simpler way to obtain the stationary distribution is to use directly Corollary 3.4.

Letting $n \to \infty$, (15) becomes: for $x \in [b_j, b_{j+1}]$,

$$\mathbb{P}(W \leq x) \;=\; \sum_{s=0}^{j} \frac{\lambda^s}{s!}(b_{j+1} - x)^s e^{-\lambda(b_{j+1}-x)} u_{j+1,s} \;,$$

where:

$$u_{j,s} \;=\; \lim_{n\to\infty} H_n(b_s, \ldots, b_{n+s}; b_j) \;.$$

When $j \geq T$, we have

$$u_{j,s} \;=\; \lim_{n\to\infty} H_n(c_s, \ldots, c_j, \ldots, c_{n+s}; c_j).$$

Comparing with (33), we see that $u_{j,s} = \phi_{j-s}$, but *for the shifted sequence* $\{c_m^{(j)}\}$ such that $c_T^{(j)} = c_j$. Denote it by ϕ_{j-s}^{T-j}. The remaining coefficients are computed from the limit of (16):

$$u_{j,m} \;=\; \sum_{s=0}^{j-m} \frac{\lambda^s}{s!}(b_{j+1} - b_j)^s e^{-\lambda(b_{j+1}-b_j)} u_{j+1,m+s} \;, \tag{49}$$

starting from $u_{T,s} = \phi_{T-s}^{0}$.

5 Computational Issues

We discuss here the computational complexity of the formulas established in the previous sections. The elementary operations we consider are addition, multiplication, and the evaluation of exponentials and logarithms.

5.1 Transient Measures

Using Theorem 3.3, it is possible to compute the value of $\mathbb{P}(W_n \leq x)$ at a particular $x \in [b_{n-p}, b_{n-p+1}]$. The algorithm for this is:

- Compute the $\zeta_{n,m,j}$ for all $p \leq m \leq j < n$ using the recurrence (16); this step requires $\mathcal{O}(pn^2)$ operations.

- Compute the value of $H_n(x)$ using (11) and (10); this step requires $\mathcal{O}(p)$ operations.

The global complexity is therefore $\mathcal{O}(pn^2)$ operations. Once coefficients $\zeta_{n,m,j}$ have been computed, they can be stored for future use. Computing all the coefficients has a complexity of $\mathcal{O}(n^3)$. Therefore, the computation of a number N of values of $\mathbb{P}(W_n \leq x)$ will take $\mathcal{O}(n^3) + \mathcal{O}(nN)$ elementary operations.

Computing the k-th moment using Theorem 3.6 by evaluation of (29) is of complexity $\mathcal{O}(kn)$. However, this requires first the computation of all $\chi^{(n)}_{p,p}$, $0 \leq p \leq n$, which is of complexity $\mathcal{O}(n^3)$.

This complexity may be improved using the relations we have shown on coefficients $\chi^{(n)}_{u,v}$. First, $\chi^{(n)}_{u,v}$ can be expressed in terms of the "diagonal" coefficients $\chi^{(n)}_{k,k}$. Set $m = 0$ in (21) to obtain:

$$\begin{aligned} \chi^{(n)}_{u,v} &= \sum_{\ell=0}^{u} \frac{\lambda^\ell}{\ell!}(b_n - b_{n-v})^\ell e^{-\lambda(b_n - b_{n-v})} \\ &\quad - \sum_{\ell=u-v+1}^{u} \frac{\lambda^\ell}{\ell!}(b_{n-u+\ell} - b_{n-v})^\ell e^{-\lambda(b_{n-u+\ell} - b_{n-v})} \chi^{(n)}_{u-\ell,u-\ell} \,. \end{aligned} \tag{50}$$

Note that the term corresponding to $\ell = u$ in both sums cancels out.

In particular, the numbers $\chi^{(n)}_{v,v}$ may be computed by the recurrence:

$$\begin{aligned} \chi^{(n)}_{v,v} &= \sum_{\ell=0}^{v-1} \frac{\lambda^\ell}{\ell!}(b_n - b_{n-v})^\ell e^{-\lambda(b_n - b_{n-v})} \\ &\quad - \sum_{\ell=1}^{v-1} \frac{\lambda^\ell}{\ell!}(b_{n-v+\ell} - b_{n-v})^\ell e^{-\lambda(b_{n-v+\ell} - b_{n-v})} \chi^{(n)}_{v-\ell,v-\ell} \,. \end{aligned} \tag{51}$$

This new recurrence is an improvement in that it allows the computation of all the $\chi^{(n)}_{n,v}$ (i.e. the $\xi_{n,0,n-v}$, hence all H_n) in $\mathcal{O}(n^2)$ steps. It has the drawback of involving subtractions, which may cause numerical instabilities.

5.2 Stationary Measures

The steps involved in the computation of the stationary distribution of W are the following.

1. Compute the numbers $\{h_q, 0 \leq q < d\}$, by solving the linear system (68) or (70). The computational complexity of this step is in principle $\mathcal{O}(d^3)$, but computing the matrix itself using the series (57) or (59) may be expensive for large values of ρ because a large number of terms are needed to attain a desired precision. In the unlikely case where the system (70) should be singular (see remark C.2), an approximation of h_p using Corollary 3.4 with a large value of n can be used.

2. Compute the numbers $\{\phi_n, 0 \leq n \leq N\}$, using recurrence (60). Here, N is such that the interval $[b_0, b_N]$ contains all values x at which the distribution is to be computed. According to Section 4.3, this step has to be performed for *all circular permutations* of the sequence $(c_0, \ldots, c_{d-1})$ to obtain all ϕ^q_n, $0 \leq q < d-1$. The total complexity of this step is therefore $\mathcal{O}(dN^3)$.

3. Compute the numbers $\{u_{j,s}, 0 \leq s \leq j \leq T\}$ using recurrence (49). This step takes $\mathcal{O}(T^3)$.

Step 1 is of course simplified when $d = 1$. In step 2, only the ϕ^0 are useful if no value for $x \geq b_T$ is needed. Likewise, step 3 may be avoided if no value for $x \leq b_T$ is needed. Also, if only moments are needed, then this step may be reduced to the computation of the numbers $\{\tau_p, 0 \leq p \leq T\}$ using recurrence (63). In that case, only the first T values of the sequence $\{\phi_n\}$ are needed, and the computation takes $\mathcal{O}(d^3) + \mathcal{O}(T^3)$.

6 Applications and Examples

We give here several applications of our analysis. The first one (Section 6.1) is an example where delays are not *a priori* deterministic, and removing the conditioning in (4) is possible. The second example (Section 6.2) shows that queues with periodic service times and/or with Erlang inter-arrivals fall in the class of systems we have studied. Finally, Section 6.3 discusses apparently more general $(\max, +)$ systems, which turn out to be quite close to the $M/D/1$ queue.

6.1 The transients of the $M/GI/1$ queue

As we have seen in Section 2, the queue $M/GI/1$ fits in the framework of this paper, with the sequence $\{b_m\}$ defined as:

$$b_m \;=\; \sum_{j=0}^{m} \sigma_j \;.$$

Here, the increments $\delta_i = b_i - b_{i-1}$ are i.i.d. random variables with a certain distribution $B(.)$ with Laplace transform $B^*(.)$ and first moment $1/\mu$. Let

$$m_0 = \mathbb{E}(e^{-\lambda\delta}) = B^*(\lambda), \qquad m_1 = \mathbb{E}(\delta e^{-\lambda\delta}) = -\frac{\mathrm{d}B^*}{\mathrm{d}s}(\lambda),$$

$$m_2 = \mathbb{E}(\delta^2 e^{-\lambda\delta}) = \frac{\mathrm{d}^2 B^*}{\mathrm{d}s^2}(\lambda), \qquad m_3 = \mathbb{E}(\delta^3 e^{-\lambda\delta}) = -\frac{\mathrm{d}^3 B^*}{\mathrm{d}s^3}(\lambda) \;.$$

Using Corollary 3.6 and the tables of Appendix E.2, one obtains formulas for the first values of the average waiting time for successive customers:

$$\begin{aligned}
\mathbb{E}W_1 &= \frac{1}{\mu} + \frac{m_0}{\lambda} - \frac{1}{\lambda} \\
\mathbb{E}W_2 &= \frac{2}{\mu} + \frac{m_0}{\lambda}(1 + m_0 + \lambda m_1) - \frac{2}{\lambda} \\
\mathbb{E}W_3 &= \frac{3}{\mu} + \frac{m_0}{\lambda}(1 + m_0 + m_0^2 + \lambda m_1(1 + 2m_0) + \frac{\lambda^2}{2}(m_0 m_2 + 2m_1^2)) - \frac{3}{\lambda} \\
\mathbb{E}W_4 &= \frac{4}{\mu} + \frac{m_0}{\lambda}(1 + m_0 + m_0^2 + m_0^3 + \lambda m_1(1 + 2m_0 + 3m_0^2) \\
&\quad + \frac{\lambda^2}{2}(m_0 m_2 + 2m_1^2 + 2m_0^2 m_2 + 6m_0 m_1^2) + \frac{\lambda^3}{6}(9m_0 m_1 m_2 + 6m_1^3 + m_3 m_0^2)) - \frac{4}{\lambda}
\end{aligned}$$

6.2 Queues with periodic services and the $E/D/1$

Consider a single server queue in which service durations are periodic with period d:

$$\sigma_{q+kd} = \sigma_q, \quad 0 \le q < d, \quad k \in \mathbb{N} .$$

According to the remarks of Section 2, the response time of customer n with $n = r + kd$ and $0 \le r < d$ is characterized by the sequence

$$d_i^{(r+kd)} = \sum_{j=r+kd-i}^{r+kd-1} \sigma_j = \sum_{j=1}^{i} \sigma_{r-j} .$$

This is an increasing sequence, which is pseudo periodic with period d and $T = 0$. Accordingly, results of section 4.2 and Appendix C may be used to compute the distribution of the limiting random variable:

$$Z_r = \lim_{k\to\infty} W_{r+kd} .$$

A particular case of this queue is the queue $E_d/D/1$. Indeed, set $\sigma_{kd} = \sigma_0$ and $\sigma_n = 0$ for $n \ne kd, k \in \mathbb{N}$. Then the waiting time of customers numbered $n = kd$ are that of customers arriving according to an Erlang process with d phases, and a service time of σ_0.

We now give the detail of the computations in the case $d = 2$. There are two alternating service times σ_0 and σ_1. With the notation of section 4.2 and Appendix C, we have, for even-numbered customers ($r = 0$):

$$c_0 = 0, \quad c_1 = \sigma_1, \quad c_2 = 2\gamma = \sigma_0 + \sigma_1, \quad \rho = \frac{\lambda}{2}(\sigma_0 + \sigma_1) .$$

Consequently,

$$\nu_1 = \frac{\lambda}{2}(\sigma_0 - \sigma_1) .$$

Solving (68) gives:

$$h_0 = (1-\rho)\frac{2}{1+e^{\nu_1(w_1-1)}}, \qquad h_1 = (1-\rho)\frac{2}{1+e^{-\nu_1(w_1-1)}},$$

with (see (65) and (54)):

$$1 - w_1 = 1 + \frac{1}{\rho} W(\rho e^{-\rho}) = 1 + e^{-\rho} \sum_{n=0}^{\infty} \frac{((n+1)\rho e^{-\rho})^n}{(n+1)!} (-1)^n .$$

The expected waiting time of even numbered customers is obtained from Corollary 4.5:

$$\mathbb{E}W^{(0)} = \frac{1}{\lambda}\left(\frac{1}{2}\frac{1-2\rho+2\rho^2}{1-\rho} - \frac{1-\lambda\sigma_1}{1+e^{\frac{1}{2}(\sigma_0-\sigma_1)(1-w_1)}}\right) . \tag{52}$$

By symmetry, the waiting time for odd numbered customers is:

$$\mathbb{E}W^{(1)} = \frac{1}{\lambda}\left(\frac{1}{2}\frac{1-2\rho+2\rho^2}{1-\rho} - \frac{1-\lambda\sigma_0}{1+e^{\frac{1}{2}(\sigma_1-\sigma_0)(1-w_1)}}\right) . \tag{53}$$

6.3 Poisson driven $(\max,+)$ systems

6.3.1 The example of Figure 1

Consider now the example used in introduction. Assume that the firing times are deterministic, with values 1, 2, 1 for transitions 1, 2 and 3, respectively.

It is not difficult to show that the vector $\hat{X}(n) = (\hat{X}_1(n), \hat{X}_2(n), \hat{X}_3(n))'$ of distances from $\perp$ to nodes (n,i), $i = 1,2,3$ is given by:

$$\hat{X}(n) \;=\; \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \;+\; 4n\;, \qquad n \geq 0.$$

Likewise, the vector $D(n)$ of the distances $d^{0,i}_{0,n} = d^{0,i}_{k,k+n}$, $i = 1,2,3$, is given by:

$$D(0) \;=\; \begin{pmatrix} 0 \\ -\infty \\ 1 \end{pmatrix}\;, \qquad D(n) \;=\; \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix} \;+\; 4(n-1)\;, \qquad n \geq 1.$$

Assume we want to compute the total response time of the n-th token. Up to the firing time in the last transition, this is: $W_3(n) = X_3(n) - Y_0(n)$. The corresponding sequence is then $\{b_m, 0 \leq m \leq n\} = (1, 5, 9, \ldots, 4(n-1)+1, 4n+1, 4n+3)$. Notice the anomaly at the end of the sequence.

For stationary response times, the anomaly disappears. The sequence is periodic with $d = 1$, $T = 0$, $b_0 = 1$ and $\gamma = 4$. Theorem 4.2 may be applied, but we can also directly conclude from (7) that W_3 is the stationary waiting time in a $M/D/1$ queue with service time $\gamma = 4$, to which $b_0 = 1$ unit of time is added.

Consider now the variable $W_2(n)$. For $\mathbb{P}(W_2(n) \leq x)$, $1 \leq n \leq 4$ we have the table:

$$\begin{aligned}
&\mathbb{P}(W_2(1) \leq x) = H_1(-\infty, 2\,; x) \\
&\quad = \begin{cases} e^{\lambda(x-2)} & x \leq 2 \\ 1 & x \geq 2 \end{cases} \\
&\mathbb{P}(W_2(2) \leq x) = H_2(-\infty, 2, 4\,; x) \\
&\quad = \begin{cases} (1+2\lambda)e^{\lambda(x-4)} & x \leq 2 \\ (1+\lambda(4-x))e^{\lambda(x-4)} & x \in [2,4] \\ 1 & x \geq 4 \end{cases} \\
&\mathbb{P}(W_2(3) \leq x) = H_3(-\infty, 2, 6, 8\,; x) \\
&\quad = \begin{cases} (1+6\,\lambda+10\,\lambda^2)e^{\lambda(x-8)} & x \leq 2 \\ (1+\lambda(8-x)+2\lambda^2(7-x))\,e^{\lambda(x-8)} & x \in [2,6] \\ \left(1+\lambda(8-x)+\frac{1}{2}\lambda^2(8-x)^2\right)e^{\lambda(x-8)} & x \in [6,8] \\ 1 & x \geq 8 \end{cases} \\
&\mathbb{P}(W_2(4) \leq x) = H_4(-\infty, 2, 6, 10, 12\,; x) \\
&\quad = \begin{cases} \left(1+10\,\lambda+42\,\lambda^2+\frac{1}{3}196\,\lambda^3\right)e^{\lambda(x-12)} & x \leq 2 \\ \left(1+\lambda(12-x)+\lambda^2(54-6x)+\frac{1}{3}\lambda^3(256-30x)\right)e^{\lambda(x-12)} & x \in [2,6] \\ \left(1+\lambda(12-x)+\frac{1}{2}\lambda^2(12-x)^2+\frac{1}{3}\lambda^3(364-66x+3x^2)\right)e^{\lambda(x-12)} & x \in [6,10] \\ \left(1+\lambda(12-x)+\frac{1}{2}\lambda^2(12-x)^2+\frac{1}{6}\lambda^3(12-x)^3\right)e^{\lambda(x-12)} & x \in [10,12] \\ 1 & x \geq 12 \end{cases}
\end{aligned}$$

Using (7) and (9), the limiting distribution is given by:

$$\mathbb{P}(W_2 \le x) = \begin{cases} (1-4\lambda)e^{\lambda(x-2)} & x \le 2 \\ \mathbb{P}(W^{(D)} - Y + 2 \le x) & x \ge 2\ , \end{cases}$$

where $W^{(D)}$ is here the distribution of the waiting time in a $M/D/1$ with service time $\gamma = 4$, and Y exponentially distributed with parameter λ.

Figure 4 represents the graphs of the distributions of $W_2(n), n = 1, 2, 3, 4$ and the limiting distribution, when $\lambda = 0.2$, *i.e.* $\rho = 0.8$.

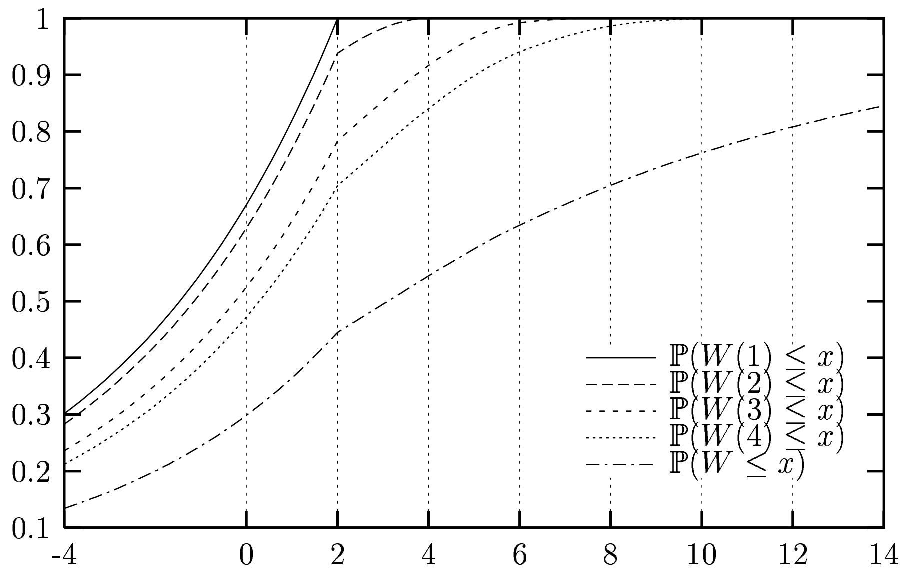


Figure 4: Distributions for waiting times in the network of Fig. 1

6.3.2 An Example from Parallel Computing

Consider the problem of executing a sequence of parallel programs consisting a certain number of tasks. Assume that the structure of each program is the same, and consider that tasks are statically mapped onto processors, is it possible to use the $(\max, +)$ algebraic framework to study performance metrics such as stability domains, throughputs and response times [2]. In particular, when programs arrive according to a Poisson process and have deterministic execution times, the results of this paper may be applied.

For instance, the example reported in [2] is characterized by the task graph of Figure 5. The graph consists in the endless repetition of a set of 15 tasks, mapped onto 5 processors. Their execution times and the constraints between two consecutive executions are given in the figure.

Here, a straightforward deterministic $(\max, +)$ calculus shows that $D = 1$, $T = 5$, $\gamma = 9$, and that the values of $\{b_i, 0 \le i \le T\}$ are:

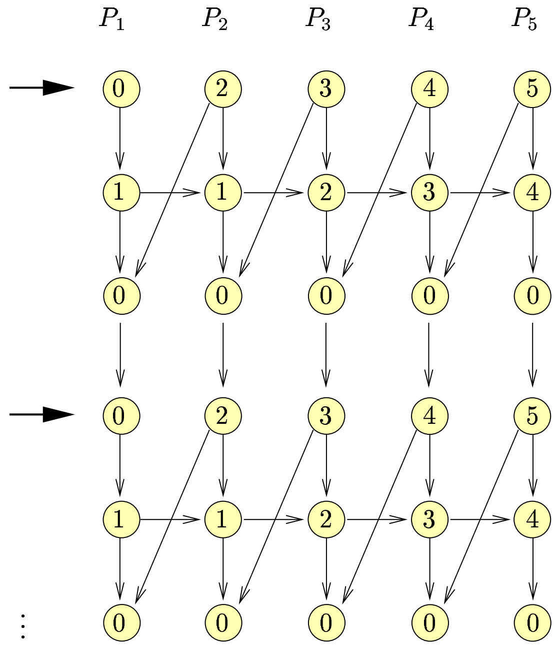


Figure 5: Task graph of the parallel program

	b_0	b_1	b_2	b_3	b_4	b_5
W_0	0	2	5	10	18	26
W_1	0	3	8	15	24	33
W_2	0	5	12	21	30	39
W_3	0	8	17	26	35	44
W_4	0	12	21	30	39	48

In Figure 6, the expected waiting times corresponding to the five tasks at the top of figure 5 are represented as a function of the arrival rate λ. As a point of comparison with the approach of [5], the figure also shows the Taylor approximations obtained by developing the expected waiting time in series of λ up to order 10. Note however that this approximation was obtained from Theorem 4.2 using a symbolic computation package, rather than constructing and instantiating the polynomials $p_{i+1}(x_0, \ldots, x_i)$ of [5]. The approximation appears to be a lower bound in this particular case.

7 Conclusion

We have computed the waiting time distribution in Poisson-driven deterministic systems, both in the transient and the asymptotic regimes. We have provided formulas and efficient algorithms for computing numerically these quantities. The software implementing them is available upon request.

The analysis exposes the ties between this class of systems and queues of the $M/D/1$ type. For instance, we can conclude that when $d = 1$, these systems can be "reduced" to the $M/D/1$ queue in many respects.

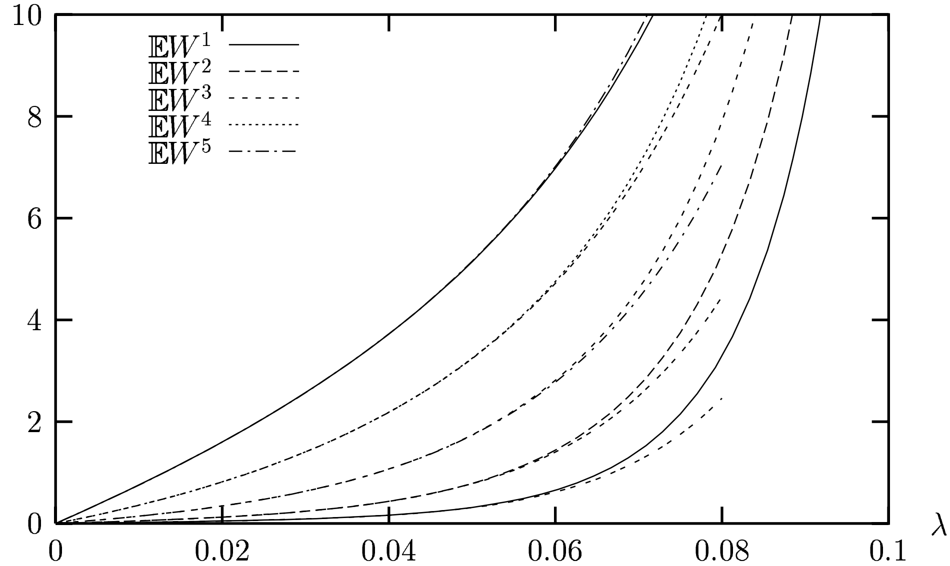


Figure 6: Exact waiting times and Taylor expansions as a function of λ

This work has potential applications in several directions. First, the connections with the method used in [5, 3, 4] is to be closely investigated. Our results involve several sets of recursively computed coefficients which are necessarily connected with the polynomials p_k introduced there. Elucidating this connection may, on the one hand, provide formal expression for our coefficients (hinge probabilities, τ_p, ϕ_p) and on the other hand give ways of mastering the exponential complexity of the p_k. We have focussed in this paper on the numerical applications of our approach, but the results are clearly amenable to a more formal analysis.

Another line of research is the possible generalization of the $E/D/1$ example to applications to the analysis of queues with constants services and arrivals with an embedded Poisson process. More generally, it seems interesting to explore the possible applications of the approach to fully stochastic systems by removing the conditioning on service times. The examples studied references [5, 3] show the way.

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A On Lambert's function

This appendix regroups various facts connected to Lambert's function, which turns out to be ubiquitous in the analysis of our periodic systems. A detailed analysis of this function appears in [6].

Lambert's function W is defined on $\mathbb{C}$ as:

$$w = W(z) \quad \Longleftrightarrow \quad z = w\, e^w \ .$$

This function is multi-valued. Its principal branch W_0 is defined so that $W_0(xe^x) = x$ for all $x \in [-1, \infty)$. This branch admits the series expansion:

$$W_0(z) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} z^n \ , \qquad |z| < e^{-1} \ . \tag{54}$$

Consider now the equation:

$$ze^{\rho(1-z)} = \omega \quad \Longleftrightarrow \quad z = -\frac{1}{\rho}\, W(-\omega\rho e^{-\rho}) \ , \tag{55}$$

where $|\omega| = 1$. We are interested in locating the solutions of this equation. Setting $z = x+iy$ and taking the modulus, it is easy to see that the solutions are located on the curve of equation:

$$y = y(x) \stackrel{\triangle}{=} \pm\sqrt{e^{2\rho(1-x)} - x^2} \ . \tag{56}$$

When $\rho \in (0,1)$, the equation $e^{2\rho(1-x)} = x^2$ has three real roots $x_1 < x_0 = 1 < x_2$, and $y(x)$ is defined for $x \in [x_1, 1] \cup [x_2, \infty)$. When $\rho = 1$, $x = 1$ is a double root, and the curve exists for $x \geq x_1$. In all cases, the part corresponding to $x \in [x_1, 1]$ lies inside the unit disk, while the other part lies in the half plane $\{\Re(z) > 1\}$. See examples in Figure 7.

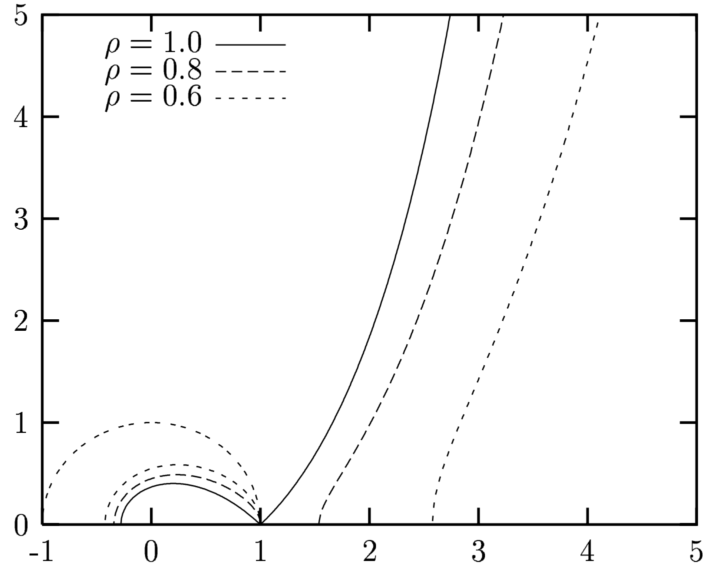

Figure 7: Location of the solutions of (55)

Applying Rouché's theorem to the function $z - \omega e^{\rho(z-1)}$, it follows that for $\rho \in (0,1)$, (55) has a unique root z_1 inside the circle of radius 1. This root is $z = 1$ when $\omega = 1$, for any ρ. From (55), it it seen that this corresponds to the principal branch of W, so that in general:

$$z_1 \;=\; -\;\frac{1}{\rho}\;W_0(-\omega\rho e^{-\rho})\;.$$

When $\omega \neq 1$, this root is strictly inside the disk.

The roots may be found numerically by solving the equation:

$$e^{\rho(1-x)}\;\;(x\cos(\rho y(x)) + y(x)\sin(\rho y(x))) \;\;=\;\; \cos\theta\;,$$

where $y(x)$ is given in (56), and $\omega = e^{i\theta}$.

Finally, consider the case where $\omega = e^{2ij\pi/d}$, where d is a nonnegative integer, and $0 \leq j < d$. We are interested in finding an expression for $e^{\nu(1-z)}$, where z is a solution of (55). Using Lagrange's inversion formula (see *e.g.* [8, p. 146]), we have:

$$e^{\nu(1-z)} \;\;=\;\; -\,\nu e^{\nu}\sum_{n=0}^{\infty}\frac{1}{n!}\omega^n(-\nu+n\rho)^{n-1}e^{-n\rho}. \tag{57}$$

Using the fact that ω is a root of unity, we can proceed with the *multisection* of the series:

$$\begin{aligned} e^{\nu(1-z)} &= -\nu e^{\nu} \sum_{\ell=0}^{d-1} \omega^{\ell} \sum_{m=0}^{\infty} \frac{1}{(\ell+md)!}(-\nu+\ell\rho+md\rho)^{\ell+md-1} e^{-(\ell+md)\rho} \\ &= \sum_{\ell=0}^{d-1} \omega^{\ell} \alpha_{\ell}(\nu,\rho) \end{aligned} \tag{58}$$

where

$$\alpha_{\ell}(\nu,\rho) := -\nu e^{\nu} \sum_{m=0}^{\infty} \frac{1}{(\ell+md)!}(-\nu+\ell\rho+md\rho)^{\ell+md-1} e^{-(\ell+md)\rho} \tag{59}$$

is a real function of ν and ρ. Note the special values:

$$\alpha_0(0,\rho) = 1, \qquad \alpha_{\ell}(0,\rho) = 0, \; \ell \neq 0 \,.$$

B Computation of τ_p and ϕ_p

We provide in this appendix formulas for computing the coefficients τ defined in (32) and involved in Theorems 4.2 and Corollary 4.3. This involves in turn the coefficients ϕ_k defined in (33):

$$\phi_k = \lim_{n\to\infty} H_n(b_{T-k},\ldots,b_n;b_T)\,, \quad 0 \le k \le T.$$

Consider in this section that the sequence $\{c_m\}$ is given by: $c_m = b_{T+m} - b_T$. In particular, $c_0 = 0$. Using (8), we have:

$$\phi_k = \lim_{n\to\infty} \mathbb{P}(W_n(0,c_1,\ldots,c_n) \le E_k)\,.$$

Lemma B.1 *The numbers $\{\phi_n, n \in \mathbb{N}\}$ are given by: $\phi_0 = h_0$ and for $n \ge 1$,*

$$\phi_n = \sum_{j=0}^{n-d-1} \frac{(-d\rho)^j}{j!} e^{d\rho} \phi_{n-d-j} + \sum_{q=0}^{d-1} h_q \frac{(\lambda c_q - d\rho)^{n-d+q}}{(n-d+q)!} e^{-(\lambda c_q - d\rho)} \mathbf{1}_{\{n \ge d-q\}}\,. \tag{60}$$

Proof Assume that X is a *positive* random variable with distribution $F(\cdot)$ and Laplace Transform $X^*(\cdot)$. Let $\phi_n = \mathbb{P}(X \le E_n)$. Then, $\phi_0 = F(0)$, and

$$\phi_n = \int_0^{\infty} F(x)\lambda \frac{(\lambda x)^{n-1}}{(n-1)!} e^{-\lambda x} dx, \quad n \ge 1.$$

The generating function $\Phi(z) = \sum_{n=0}^{\infty} \phi_n z^n$ is therefore:

$$\begin{aligned} \Phi(z) &= \phi_0 + \sum_{n=1}^{\infty} \int_0^{\infty} z^n F(x)\lambda \frac{(\lambda x)^{n-1}}{(n-1)!} e^{-\lambda x} dx \\ &= \phi_0 + z \int_0^{\infty} F(x)\lambda e^{-\lambda(1-z)x} dx \\ &= \phi_0 + \lambda z \frac{1}{\lambda(1-z)} X^*(\lambda(1-z))\,. \end{aligned}$$

In particular, if X is the stationary waiting time corresponding to the periodic sequence $\{c_m\}$ with $c_0 = 0$, then $F(0) = h_0$ and $X^*(s) = \mathcal{W}(s)$ is given by (44). Hence, after simplifications,

$$\Phi(z) \;=\; \phi_0 \;+\; \frac{1}{1 - z^d e^{d\rho(1-z)}} \sum_{q=0}^{d-1} z^{d-q} h_q e^{(d\rho - \lambda c_q)(1-z)} \;. \tag{61}$$

Expanding (61) in series of z gives a closed form, but complicated formula for ϕ_n. Alternately, from the identity:

$$(\Phi(z) - \phi_0)\,\left(1 - z^d e^{d\rho(1-z)}\right) \;=\; \sum_{q=0}^{d-1} z^{d-q} h_q e^{(d\rho-\lambda c_q)(1-z)},$$

one obtains the recurrence (60).

■

Remark B.1 In the particular case $d = 1$, (60) becomes:

$$\phi_n \;=\; \sum_{j=0}^{n-1} \frac{(-\rho)^j}{j!} e^{-\rho} \phi_{n-1-j} \;. \tag{62}$$

Remark B.2 Recurrence (60) involves alternating sums, which causes numerical problems when n is large. To circumvent this problem, a possibility is to perform an asymptotic analysis of these coefficients, following the ideas of [7] for instance. For this, note that the singularities of $\Phi(z)$ are the simple poles located at the zeroes of the denominator $\Psi(z) = 1 - (ze^{\rho(1-z)})^d$. These have been studied in Appendix A.

Lemma B.2 *The numbers $\{\tau_0, \ldots, \tau_T\}$ are given by: $\tau_T = 1 - \rho$, and for $1 \leq p < T$:*

$$\tau_p \;=\; \sum_{j=p}^{T-1} \frac{\lambda^{j-p}}{(j-p)!}(b_T - b_p)^{j-p} e^{-\lambda(b_T - b_p)} \phi_{T-j} - \sum_{j=p+1}^{T-1} \frac{\lambda^{j-p}}{(j-p)!}(b_j - b_p)^{j-p} e^{-\lambda(b_j - b_p)} \tau_j \;. \tag{63}$$

Proof Using (21) with $u = v = n - p$ and $m = n - T$, we get:

$$\begin{aligned}
\chi^{(n)}_{n-p,n-p} \;=\; & \sum_{j=p}^{T} \frac{\lambda^{j-p}}{(j-p)!}(b_T - b_p)^{j-p} e^{-\lambda(b_T-b_p)} \chi^{(n)}_{n-j,n-T} \\
& - \sum_{j=p+1}^{T} \frac{\lambda^{j-p}}{(j-p)!}(b_j - b_p)^{j-p} e^{-\lambda(b_j-b_p)} \chi^{(n)}_{n-j,n-j} \;.
\end{aligned}$$

Here again, terms corresponding to $j = T$ in both sums cancel out. Letting $n \to \infty$ in this identity gives (63).

The τ_j's are computed from (63) in the order $\tau_{T-1}, \ldots, \tau_0$. ■

C Computation of h_q

In this appendix, we describe how to compute the coefficients h_q appearing in (44). It is actually a classical application of the kernel method for generating functions.

It is convenient to perform the change of variable $w = (\lambda - s)/\lambda$ in (44) and to rearrange the formula into:

$$\mathcal{W}(s) \;=\; \widetilde{\mathcal{W}}(w) \;=\; \frac{1-w}{1-w^d e^{d\rho(1-w)}} \sum_{q=0}^{d-1} h_q e^{(d\rho-\lambda c_q)(1-w)} w^{d-1-q} \;. \tag{64}$$

The function $\widetilde{\mathcal{W}}$ is analytic in $\{\Re(w) \leq 1\}$. Therefore, if the denominator of (64) vanishes at some point w_i in this domain, so does the numerator.

The denominator vanishes if and only if:

$$we^{\rho(1-w)} = \omega^j, \quad \omega = e^{2i\pi/d} \text{ and } 0 \leq j < d \;.$$

In terms of Lambert's W function, the solutions are:

$$w_j \;=\; -\frac{1}{\rho}\, W(-\omega^j \rho e^{-\rho}) \;, \quad 0 \leq j < d \;. \tag{65}$$

Among the infinity of solutions of (65), only one is in the half plane $\{\Re(w) \leq 1\}$ (see Appendix A). For $j \neq 0$, setting the numerator to 0 gives, after simplifications:

$$\sum_{q=0}^{d-1} h_q \, \omega^{-jq} \, e^{(\rho q - \lambda c_q + \lambda c_0)(1-w_j)} \;=\; 0. \tag{66}$$

For $w = 1$, the numerator vanishes due to the term $(1 - w)$, but the additional condition $\mathcal{W}(0) = \widetilde{\mathcal{W}}(1) = 1$ implies:

$$\sum_{q=0}^{d-1} h_q \;=\; d\,(1-\rho). \tag{67}$$

Setting:

$$\nu_q \;=\; q\rho - \lambda c_q + \lambda c_0, \qquad 0 \leq q \leq d-1 \;,$$

the vector $\mathbf{h} = (h_0, \ldots, h_{d-1})'$ is (the) solution of the system:

$$\mathbf{B}\;\mathbf{h} \;=\; d(1-\rho) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \;, \tag{68}$$

with

$$\mathbf{B} \;=\; \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & \omega^{-1} e^{\nu_1(1-w_1)} & \ldots & \omega^{-(d-1)} e^{\nu_{d-1}(1-w_1)} \\ \vdots & \vdots & & \vdots \\ 1 & \omega^{-(d-1)} e^{\nu_1(1-w_{d-1})} & \ldots & \omega^{-(d-1)^2} e^{\nu_{d-1}(1-w_{d-1})} \end{pmatrix} \;. \tag{69}$$

Finally, note that the matrix $\mathbf{B}$ has complex coefficients. In the implementation of the calculations, it is more convenient to work with real-valued matrices. Let $\Omega = ((\omega^{\ell c}, 0 \leq \ell, c < d))$. Using the multisection (58), we have:

$$\begin{aligned}
\sum_{j=0}^{d-1} \omega^{\ell j} \omega^{-jc} e^{\nu_c(1-w_j)} &= \sum_{j=0}^{d-1} \omega^{\ell j} \omega^{-jc} \sum_{q=0}^{d-1} \omega^{qj} \alpha_q(\nu_c, \rho) \\
&= \sum_{q=0}^{d-1} \alpha_q(\nu_c, \rho) \sum_{j=0}^{d-1} \omega^{(\ell-c+q)j} \\
&= \sum_{q=0}^{d-1} \alpha_q(\nu_c, \rho)\; d\; \mathbf{1}_{\{\ell-c+q \equiv 0 [d]\}} \\
&= d\; \alpha_{c-\ell}(\nu_c, \rho)\ .
\end{aligned}$$

Therefore, multiplying (68) to the left by the matrix Ω yields:

$$\begin{pmatrix} 1 & \alpha_{1,1} & \alpha_{2,2} & \dots & \alpha_{d-1,d-1} \\ 0 & \alpha_{1,0} & \alpha_{2,1} & \dots & \alpha_{d-1,d-2} \\ 0 & \alpha_{1,d-1} & \alpha_{2,0} & \dots & \alpha_{d-1,d-3} \\ \vdots & & \alpha_{c,c-\ell} & & \vdots \\ 0 & \alpha_{1,2} & \alpha_{2,3} & \dots & \alpha_{d-1,0} \end{pmatrix} \mathbf{h} = (1-\rho) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \tag{70}$$

where $\alpha_{i,j} := \alpha_j(\nu_i, \rho)$. This concludes the computation: the limits h_q are obtained by solving (68) or (70).

Remark C.1 The numbers ν_q represent the offsets of the sequence $\{c_m\}$ with respect to an "ideal" sequence with same growth rate but $d = 1$. When these offsets are actually zero, then the matrix of (70) is the identity, and $\mathbf{h} = (1-\rho)(1, \dots, 1)'$. It can be checked that using this result in Theorem 4.4 gives (38) as expected.

Remark C.2 We conjecture that $\mathbf{B}$ is always invertible for values of $\nu = (\nu_1, \dots, \nu_{d-1})$ given by $\nu_q = q\rho - \lambda c_q + \lambda c_0$ and $c_0 \leq c_1 \leq \dots \leq c_{d-1} \leq c_0 + d\gamma$, and when $\rho \in [0, 1)$. This actually fails to be true for arbitrary values of ν.

The invertibility of $\mathbf{B}$ for $\rho \in [0, 1)$ can be proved for $d = 2$. It is also easily shown to be true when $\nu = (0, \dots, 0)$ or $\nu = \rho(1, \dots, (d-1))$, because $\mathbf{B}$ has then the Vandermonde form. The matrix is also nonsingular in a neighborhood of these points, by continuity.

Proving this conjecture probably requires a closer investigation of the functions $\alpha_j(\nu, \rho)$ and their relationships.

D Closed Form Formulas for the Transients

In this appendix, we briefly study the generating functions of hinge probabilities in the case when the sequence $\{c_n\}$ is periodic. This should be useful for obtaining closed form formulas for transient probabilities, and asymptotic expansions, speed of convergence and so on. However, the applications of these formulas are not fully explored here.

We start from (46) and apply the kernel method. Let:

$$G_q(z) \;=\; \sum_{n=0}^{\infty} H_n(c_q, \ldots, c_{n+q}; c_q) z^n \;.$$

The problem is to evaluate those unknown functions. We know that $\mathcal{G}(z,s)$ is analytic for $\Re(s) \geq 0$ for any z in the unit disc. The denominator of (46) vanishes when

$$z \frac{\lambda}{\lambda - s}\, e^{-s\gamma} \;=\; \omega^j \;, \qquad \omega = e^{2i\pi/d},\; 0 \leq j < d \;.$$

Solving for s gives: $s \;=\; \lambda(1 - w_j(z))$, where

$$w_j(z) \stackrel{\triangle}{=} -\; \frac{1}{\rho}\, W(-z\omega^{-j}\rho e^{-\rho}) \;.$$

Among the many possible values for $w_j(z)$, only the one corresponding to the principal branch of W is has a real part less than 1 (see Appendix A), and corresponds to an s in the right-half plane. Making in turn the numerator vanish for these values of s leads to the condition:

$$\sum_{q=0}^{d-1} \omega^{-jq}\; e^{\nu_q(1-w_j(z))}\; G_q(z) \;=\; \frac{1}{1 - w_j(z)} \sum_{q=0}^{d-1} \omega^{-jq} e^{\nu_q(1-w_j(z))} \;. \tag{71}$$

The parameters ν_q are those introduced in Appendix C. Put in matrix form, this gives:

$$\begin{pmatrix} 1 & e^{\nu_1(1-w_0(z))} & \ldots & e^{\nu_{d-1}(1-w_0(z))} \\ 1 & \omega^{-1} e^{\nu_1(1-w_1(z))} & \ldots & \omega^{-(d-1)} e^{\nu_{d-1}(1-w_1(z))} \\ \vdots & \vdots & & \vdots \\ 1 & \omega^{-(d-1)} e^{\nu_1(1-w_{d-1}(z))} & \ldots & \omega^{-(d-1)^2} e^{\nu_{d-1}(1-w_{d-1}(z))} \end{pmatrix} \mathbf{G}(z) \;=\; \mathbf{E}(z) \;, \tag{72}$$

with

$$\mathbf{G}(z) \;=\; \begin{pmatrix} G_0(z) \\ G_1(z) \\ \vdots \\ G_{d-1}(z) \end{pmatrix} , \qquad \mathbf{E}(z) \;=\; \begin{pmatrix} \sum_{q=0}^{d-1} e^{\nu_q(1-w_0(z))}(1 - w_0(z))^{-1} \\ \sum_{q=0}^{d-1} \omega^{-q} e^{\nu_q(1-w_1(z))}(1 - w_1(z))^{-1} \\ \vdots \\ \sum_{q=0}^{d-1} \omega^{-q(d-1)} e^{\nu_q(1-w_{d-1}(z))}(1 - w_{d-1}(z))^{-1} \end{pmatrix} . \tag{73}$$

An expression for the functions $G_q(z)$ may be obtained by solving this linear system. When d is small enough, the resulting generating functions may be simple enough to be inverted using (76). Also, it is possible to derive (68) again from (72) by multiplying by $(1 - z)$ and letting z go to 1.

In the case $d = 1$, we find the generating function of the sequence $\{g_p, p \in \mathbb{N}\}$, introduced in Section 4.1. This function is known from the literature. Indeed, according to *e.g.* Takács [9, p. 57], we have:

$$G(z) \;=\; \sum_{p=0}^{\infty} g_p z^p \;=\; \sum_{p=0}^{\infty} \mathbb{P}(W_p = 0) z^p \;=\; \frac{1}{1-g(z)}, \tag{74}$$

where $g(z)$ is solution of $x = ze^{-\rho(1-x)}$. This is what we obtain from (71). As in (55), this gives:

$$g(z) \;=\; -\frac{1}{\rho}\, W(-\rho e^{-\rho} z)\;,$$

in a neighborhood of 0. Finally:

$$G(z) \;=\; \frac{1}{1 + W(-\rho e^{-\rho} z)/\rho}\;. \tag{75}$$

This allows to derive both a "closed form" formula for the g_p. Using (54),

$$1 + \frac{1}{\rho} W(-\rho e^{-\rho} z) \;=\; 1 - z e^{-\rho} \sum_{n=0}^{\infty} \frac{((n+1)\rho e^{-\rho})^n}{(n+1)!} z^n\;. \tag{76}$$

Consequently:

$$\begin{aligned} g_p \;=\; [z^p]G(z) \;&=\; [z^p] \sum_{m=0}^{\infty} z^m e^{-\rho m} \left(\sum_{n=0}^{\infty} \frac{((n+1)\rho e^{-\rho})^n}{(n+1)!} z^n \right)^m \\ &=\; \sum_{m=0}^{p} e^{-\rho m} [z^{p-m}] \left(\sum_{n=0}^{m-p} \frac{((n+1)\rho e^{-\rho})^n}{(n+1)!} z^n \right)^m \;. \end{aligned}$$

The final formula involves multinomial coefficient and is omitted...

Instead, we conclude this section with recurrences for computing the g_ns. We already know from (51) that:

$$g_v \;=\; \sum_{\ell=0}^{v-1} \frac{(v\rho)^\ell}{\ell!} e^{-v\rho} \;-\; \sum_{\ell=1}^{v-1} \frac{((v-\ell)\rho)^\ell}{\ell!} e^{-\ell\rho} g_{v-\ell}\;. \tag{77}$$

However, from (75) we have: $G(z)(1 + W(-\rho e^{-\rho} z)/\rho) = 1$. Using (76), and extracting the coefficient of z^v, we obtain a different recurrence:

$$g_v \;=\; \mathbf{1}_{\{v=0\}} \;+\; e^{-\rho} \sum_{\ell=0}^{v-1} \frac{(\rho e^{-\rho}(\ell+1))^\ell}{(\ell+1)!} g_{v-\ell}\;. \tag{78}$$

Observe that $g_n e^{n\rho}$ is a polynomial in ρ. The table of the first values of the sequence is given in Appendix E.3.

E Tables of coefficients

E.1 Coefficients ϕ_n

$$
\begin{aligned}
\phi_0 &= 1-\rho \\
\phi_1 &= (1-\rho)\, e^{\rho} \\
\phi_2 &= (1-\rho)\, e^{\rho}\, (-\rho + e^{\rho}) \\
\phi_3 &= (1-\rho)\, e^{\rho} \left(\frac{1}{2}\rho^2 - 2\, e^{\rho}\rho + e^{2\rho} \right) \\
\phi_4 &= (1-\rho)\, e^{\rho} \left(-\frac{1}{6}\rho^3 + 2\, e^{\rho}\rho^2 - 3\, e^{2\rho}\rho + e^{3\rho} \right) \\
\phi_5 &= (1-\rho)\, e^{\rho} \left(\frac{1}{24}\rho^4 - \frac{4}{3} e^{\rho}\rho^3 + \frac{9}{2}\rho^2 e^{2\rho} - 4\, e^{3\rho}\rho + e^{4\rho} \right)
\end{aligned}
$$

E.2 Coefficients $\chi^{(n)}$

For $n=4$ and the sequence $\{b_0, \ldots, b_4\}$, the diagonal coefficients $\chi^{(4)}_{p,p}$ are given below. The coefficients $\chi^{(m)}_{u,u}$, $0 \le u \le m \le 3$ are easily derived by substitution of variables.

$$
\begin{aligned}
\chi^{(4)}_{0,0} &= 1 \\
\chi^{(4)}_{1,1} &= e^{-\lambda\,(b_4-b_3)} \\
\chi^{(4)}_{2,2} &= e^{-\lambda\,(b_4-b_2)}\,(1+\lambda(b_4-b_3)) \\
\chi^{(4)}_{3,3} &= e^{-\lambda\,(b_4-b_1)} \left(1 + \lambda(b_4-b_2) + \frac{\lambda^2}{2}\left({b_4}^2 - {b_3}^2 - 2\, b_2\, b_4 + 2\, b_2\, b_3 \right) \right) \\
\chi^{(4)}_{4,4} &= e^{-\lambda\,(b_4-b_0)} \left(1 + \lambda(b_4-b_1) + \frac{\lambda^2}{2}\left({b_4}^2 - {b_2}^2 + 2\, b_1\, b_2 - 2\, b_1\, b_4 \right) \right. \\
&\quad \left. + \frac{\lambda^3}{6}\left({b_4}^3 - {b_3}^3 - 3\, b_1\, {b_4}^2 + 3\, b_1\, {b_3}^2 - 3\, {b_2}^2 b_4 + 3\, {b_2}^2 b_3 + 6\, b_1\, b_2\, b_4 - 6\, b_1\, b_2\, b_3 \right) \right)
\end{aligned}
$$

Another useful form is obtained using the *increments* of the sequence. Let $\delta_i = b_{i+1} - b_i$, $i = 0, 1, 2, 3$.

$$
\begin{aligned}
\chi^{(4)}_{0,0} &= 1 \\
\chi^{(4)}_{1,1} &= e^{-\lambda\,\delta_3} \\
\chi^{(4)}_{2,2} &= (1+\lambda\,\delta_3)\, e^{-\lambda\,(\delta_3+\delta_2)} \\
\chi^{(4)}_{3,3} &= \left(1 + \lambda(\delta_3+\delta_2) + \frac{\lambda^2}{2}\left({\delta_3}^2 + 2\,\delta_2\,\delta_3 \right) \right) e^{-\lambda\,(\delta_3+\delta_2+\delta_1)} \\
\chi^{(4)}_{4,4} &= \left(1 + \lambda(\delta_3+\delta_2+\delta_1) + \frac{\lambda^2}{2}\left({\delta_3}^2 + {\delta_2}^2 + 2\,\delta_2\,\delta_3 + 2\,\delta_1\,\delta_3 + 2\,\delta_1\,\delta_2 \right) \right. \\
&\quad \left. \frac{\lambda^3}{6}\left(3{\delta_2}^2\delta_3 + 3\,\delta_1\,{\delta_3}^2 + 3\,\delta_2\,{\delta_3}^2 + 6\,\delta_1\,\delta_2\,\delta_3 + {\delta_3}^3 \right) \right) e^{-\lambda\,(\delta_3+\delta_2+\delta_1+\delta_0)}
\end{aligned}
$$

E.3 Coefficients g_n

$$
\begin{aligned}
g_0 &= 1 \\
g_1 &= e^{-\rho} \\
g_2 &= (1+\rho)\, e^{-2\rho} \\
g_3 &= \left(1 + 2\,\rho + \frac{3\,\rho^2}{2}\right) e^{-3\rho} \\
g_4 &= \left(1 + 3\,\rho + 4\,\rho^2 + \frac{8\,\rho^3}{3}\right) e^{-4\rho} \\
g_5 &= \left(1 + 4\,\rho + \frac{15\,\rho^2}{2} + \frac{25\,\rho^3}{3} + \frac{125\,\rho^4}{24}\right) e^{-5\rho}
\end{aligned}
$$

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