On the Prequential Approach for Testing Exponentiality

Mhamed-Ali El Aroui

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

On the Prequential Approach for Testing Exponentiality

Mhamed-Ali El Aroui

N ° 3082

Janvier 1997

THÈME 4

Rapport de recherche

ISSN 0249-6399

On the Prequential Approach for Testing Exponentiality

Mhamed-Ali El Aroui

Thème 4 — Simulation et optimisation
de systèmes complexes
Projet IS2

Rapport de recherche n° 3082 — Janvier 1997 — 13 pages

Abstract: We present a prequential (predictive-sequential) approach for testing the goodness-of-fit of an exponential distribution when the parameter λ is unknown. Instead of using all the available observations, λ is estimated by a prequential approach where at each step i, only the $i-1$ first observations are used. We show that this approach provides a sequence of Kolmogorov-Smirnov type distances whose expressions do not depend on λ and which converge in distribution (under the null hypothesis) to the Kolmogorov-Smirnov distribution. This leads to a simple technique for testing the goodness-of-fit of exponential distributions with unknown parameter using standard quantile tables of the Kolmogorov-Smirnov distribution. Even if Monte Carlo simulations show that the prequential test is less powerful than the standard exponentiality test, the developed results represent a first step in the theoretical study of the *u-plot* which is a prequential empirical tool commonly used for the validation of reliability-growth models.

Key-words: Goodness-of-fit, Prequential approach, Test for exponentiality, Kolmogorov-Smirnov distance and distribution, Sequential estimation, U-plot.

(Résumé : tsvp)

AMS subject classification: 62F03, 62L12, 60B05.

Unité de recherche INRIA Rhône-Alpes
655, avenue de l'Europe, 38330 MONTBONNOT ST MARTIN (France)
Téléphone : (33) 04 76 61 52 00 – Télécopie : (33) 04 76 61 52 52

Approche Préquentielle pour la Validation de Modèles à loi Exponentielle

Résumé : On s'intéresse au problème du test d'adéquation à une loi exponentielle de paramètre λ inconnu. Au lieu d'estimer λ à partir de tout l'échantillon, on utilise l'approche préquentielle (prédictive-séquentielle) où, à chaque itération i, le paramètre λ est estimé à partir des $i-1$ premières observations. On montre que l'approche préquentielle permet d'obtenir des distances de Kolmogorov-Smirnov convergeant en loi, sous l'hypothèse nulle, vers la loi de Kolmogorov-Smirnov , et dont les expressions ne dépendent pas de λ. Ceci permet d'obtenir un test de l'hypothèse nulle considérée où les quantiles de référence sont ceux de la loi de Kolmogorov-Smirnov. Même si les simulations Monte-Carlo semblent montrer que le test préquentiel est moins puissant que le test d'exponentialité standard, les résultats théoriques obtenus ici représentent une première étape dans l'étude théorique de l'*u-plot* qui est un outil préquentiel empirique très utilisé pour le choix de modèles de croissance de fiabilité.

Mots-clé : Test d'adéquation, Approche préquentielle, Lois exponentielles, Loi et distance de Kolmogorov-Smirnov, Estimation séquentielle, U-plot.

1 Introduction

Let $x_1, \ldots, x_n$ be n observations of n i.i.d. r.v. $X_1, \ldots, X_n$ with a continuous unknown distribution function F^*. We consider the problem of testing the exponentiality of this sample, that is testing the null hypothesis:

$$H_0^{(exp)} \equiv \text{“ } F^* = F_{exp}(.,\lambda) \text{ ”} \quad \text{where } \lambda \in \mathbb{R}_+^* \text{ and } F_{exp}(x,\lambda) = 1 - \exp(-\lambda x),\ x \geq 0.$$

In the following, $\widehat{\lambda}(X_1, \ldots, X_n) = n / \sum_{i=1}^n X_i$ denotes the Maximum-Likelihood estimator of λ, and $\mathbb{F}_n$ denotes the empirical distribution function of $X_1, \ldots, X_n$:

$$\forall x \in \mathbb{R}\ ,\ \ \mathbb{F}_n(x) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{X_i \leq x\}}.$$

D_n denotes the Kolmogorov-Smirnov distance:

$$D_n = \sup_{x \geq 0} \mid \mathbb{F}_n(x) - F_{exp}(x,\lambda) \mid = \sup_{t \in [0,1]} \left| \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{U_i \leq t\}} - t \right|, \tag{1}$$

where

$$\forall i \leq n\ ,\quad U_i = F_{exp}(X_i, \lambda).$$

It is well known that the classical Kolmogorov-Smirnov goodness-of-fit test, which compares the realization of the r.v. $\sqrt{n} D_n$ to the quantiles of the Kolmogorov-Smirnov distribution, can not be used here since λ is unknown.

A first solution is to replace in D_n the unknown parameter λ by its estimator $\widehat{\lambda}(X_1, \ldots, X_n)$, but this requires to study, under the null hypothesis $H_0^{(exp)}$, the distribution of the r.v.

$$\sqrt{n} \widehat{D}_n = \sqrt{n} \sup_{x \geq 0} \left| \mathbb{F}_n(x) - F_{exp} \left[x, \widehat{\lambda}(X_1, \ldots, X_n) \right] \right|.$$

David and Johnson (1948) show that $\sqrt{n} \widehat{D}_n$ converges in distribution, under $H_0^{(exp)}$, to a distribution which does not depend on the unknown parameter λ. The quantiles of this distribution were tabulated by Lilliefors (1969) and Stephens (in D'Agostino and Stephens 1986 pp. 134) and can be used to test $H_0^{(exp)}$. This leads to the standard exponentiality test called hereafter the *complete test*.

In this paper we use the prequential (predictive-sequential) approach introduced by Dawid (1984) to construct a r.v. $\sqrt{n} \widetilde{D}_n$ which:

— converges in distribution, under $H_0^{(exp)}$, to the Kolmogorov-Smirnov distribution,

— does not depend on the unknown parameter λ.

This leads to a new goodness-of-fit test for the exponential distribution with unknown parameter. The reference quantiles are those of Kolmogorov-Smirnov distribution, this avoids the use of special quantile tables as those developed by Lilliefors and Stephens. Monte Carlo simulations are used to compare the power of the prequential test to that of the complete test.
Besides the presentation of a new exponentiality test, the results given in this paper represent a first step towards the theoretical study of the *u-plot* which is an empirical criterion widely used to compare predictive qualities of competitive reliability-growth models.

2 The Prequential Approach

The prequential (predictive-sequential) approach, introduced by Dawid (1984) is commonly used to measure goodness-of-fit of software-reliability models (e.g. Abdel-Ghaly, Chen and Littlewood 1986 and Brocklehurst and Littlewood in Lyu *et al.* 1996). In this approach one proceeds as if the data $x_1, x_2, \ldots$ were observed sequentially, that means that at step i, it is supposed that only observations $x_1, \ldots, x_{i-1}$ are available to estimate λ. This leads to a prequential Kolmogorov-Smirnov distance denoted $\widetilde{D}_n$

$$\widetilde{D}_n = \sup_{t\in[0,1]} \left| \frac{1}{n-p} \sum_{i=p+1}^{n} \mathbb{1}_{\{\widetilde{U}_i \leq t\}} - t \right| \tag{2}$$

where p is a fixed integer $(0<p<n)$ and for $i=p+1, \ldots, n$, r.v.'s $\widetilde{U}_i$ are constructed by the prequential approach:

$$\begin{aligned} \widetilde{U}_i &= F_{exp}\left[X_i, \widehat{\lambda}(X_1, \ldots, X_{i-1})\right] \\ &= 1 - \exp\left[-\frac{(i-1)\,X_i}{\sum_{j=1}^{i-1} X_j}\right] \end{aligned} \tag{3}$$

Software-reliability analysts (e.g. Downs and Scott 1992 and Lyu *et al.* 1996) often use this approach (called also *u-plot*) to choose between competitive reliability-growth models. For each model, the prequential Kolmogorov-Smirnov distance $\widetilde{D}_n$ is computed and the realization of the r.v. $\sqrt{n-p}\,\widetilde{D}_n$ is compared to the 5% quantile of the Kolmogorov-Smirnov distribution. This empirical use of the prequential approach would be theoretically justified only if it is proved that, under the assumptions of the tested model, one has:

$$\sqrt{n-p}\,\widetilde{D}_n \overset{n\to+\infty}{\longrightarrow} \mathcal{L}_{ks} \tag{4}$$

where $\mathcal{L}_{ks}$ denotes the Kolmogorov-Smirnov distribution.

Monte Carlo simulations presented by Downs and Scott (1992) (models 1 and 2 given below) and El Aroui (1996) (models 3 and 4) seem to confirm this result for the following models:

1. the X_i's are n i.i.d. r.v. with an exponential distribution,
2. Jelinski-Moranda model where the X_i's are independent with different exponential distributions:
$$\forall i \leq n\ ,\ \ X_i \sim Exp(a + bi),$$
where a and b are two real unknown parameters (with $a + bi > 0$).
3. Moranda Geometric model, where the X_i's are independent with different exponential distributions:
$$\forall i \leq n\ ,\ \ X_i \sim Exp(\alpha \exp(-\theta i)),$$
where $\alpha > 0$ and $\theta \in \mathbb{R}$ are two unknown parameters.
4. The Power-Law model (called also Crow model) where the X_i's are the times between events of a Non-Homogeneous Poisson Process.

In this paper we prove the convergence of $\sqrt{n-p}\widetilde{D}_n$ to $\mathcal{L}_{ks}$ (assertion (4)) for the exponential i.i.d. case (model 1). The obtained theoretical derivations can, in future work, be used to prove assertion (4) for the three other cases (models 2, 3 and 4). This would give theoretical justifications to the present use of the prequential approach (*u-plot* criterion) to measure goodness-of-fit of reliability-growth models.

Remark – Practically the integer p in equation (2) should be chosen large enough so that the first estimators of λ: $\widehat{\lambda}(X_1, \ldots, X_p), \widehat{\lambda}(X_1, \ldots, X_{p+1})$, etc. have reasonable quality. For the Monte Carlo studies presented in section 4, we take $p = n/5$; further theoretical studies are needed to give the optimal choice of p.
For the theoretical results developed in the next section we will take $p = 1$, all the presented results still hold for any other integer p such that $0 < p < n$ and $n - p \stackrel{n \to +\infty}{\longrightarrow} +\infty$.

3 The Prequential Test

The following theorem is used to construct the prequential test measuring the goodness-of-fit of the exponential distribution with unknown parameter:

Theorem – **3.1** *If the $\widetilde{U}_i$ are constructed by the prequential approach described above (with $p = 1$), then the r.v.:*

$$\widetilde{K}_n = \sqrt{n-1}\ \left[\sup_{t \in [0,1]} \left| \frac{1}{n-1} \sum_{i=2}^{n} \mathbb{1}_{\{\widetilde{U}_i \leq t\}} - t \right| \right] \tag{5}$$

converges in distribution, under hypothesis $H_0^{(exp)}$, to the Kolmogorov-Smirnov distribution:

$$\widetilde{K}_n \stackrel{n \to +\infty}{\longrightarrow} \mathcal{L}_{ks}.$$

□

The proof of Theorem 3.1, given in sub-section 3.3, needs some preliminary results presented in sub-sections 3.1 and 3.2.

Using the previous result, one can test the exponentiality of the observed sample by:

— calculating the realizations of the r.v.'s $\widetilde{U}_i$ given by equation (3),

— calculating the realization of the r.v. $\widetilde{K}_n$ given by (5) which can be written:

$$\widetilde{K}_n = \sqrt{n-1}\,\widetilde{D}_n \quad \text{with} \quad \widetilde{D}_n = \max\left[\max_{1\le i\le n-1}\left(\frac{i}{n-1} - \widetilde{U}_i^*\right), \max_{1\le i\le n-1}\left(\widetilde{U}_i^* - \frac{i-1}{n-1}\right)\right], \tag{6}$$

where $\widetilde{U}_1^*, \ldots, \widetilde{U}_{n-1}^*$ denote the order statistics of the r.v.'s $\widetilde{U}_i$.

— In the case of finite samples (which is always the case in practice), replace in (6) $\widetilde{D}_n$ by $\widetilde{D}_n^* = \widetilde{D}_n(\sqrt{n} + 0.12 + 0.11/\sqrt{n})$, and compare $\widetilde{K}_n^* = \sqrt{n-1}\,\widetilde{D}_n^*$ to the quantiles of the Kolmogorov-Smirnov distribution.

The previous transformation of $\widetilde{D}_n$ was proposed by Stephens (in D'Agostino and Stephens 1986 pp. 105) when a fully specified distribution is tested. Monte Carlo simulations presented in section 4 (cf. Tables 1 and 2) show that Stephens transformation still hold for the prequential test.

In the following we present results that will be used to prove theorem 1.

3.1 Properties of r.v. $\widetilde{U}_i$

Proposition – 3.2 *Let $X_1, \ldots, X_n$ be n i.i.d. r.v. with the distribution $Exp(\lambda)$, $\lambda \in \mathbb{R}_+^*$. For $i = 2, \ldots, n$, the following r.v.'s*

$$\widetilde{U}_i = 1 - \exp\left[-\frac{(i-1)\,X_i}{\sum_{j=1}^{i-1} X_j}\right]$$

are independent. Their distribution functions, denoted $F_{\widetilde{U}_i}$, are given for $i=2,\ldots,n$ by:

$$\forall u \in [0,1[\,, \quad F_{\widetilde{U}_i}(u) = 1 - \left[1 - \frac{\ln(1-u)}{i-1}\right]^{-(i-1)} \quad \textit{and} \quad F_{\widetilde{U}_i}(1) = 1. \tag{7}$$

□

Proof – To prove the independence of r.v.'s $\widetilde{U}_i$ we use a result given by Quesenberry (in D'Agostino and Stephens 1986 pp. 254) which proves that under the assumptions of proposition 3.2 the following r.v.'s are independent:

$$\left(\frac{\sum_{j=1}^{i-1} X_j}{\sum_{j=1}^{i} X_j}\right)^{i-1}, \quad \text{for } i = 2, \ldots, n.$$

It follows that for $i=2,\dots,n$, the r.v.'s $\sum_{j=1}^{i-1} X_j / \sum_{j=1}^{i} X_j$ are mutually independent, by noting that

$$\frac{\sum_{j=1}^{i-1} X_j}{\sum_{j=1}^{i} X_j} = \left(1 + \frac{X_i}{\sum_{j=1}^{i-1} X_j}\right)^{-1},$$

we prove the independence of $X_i / \sum_{j=1}^{i-1} X_j$ and consequently that of the r.v.'s $\widetilde{U}_i$.

Let us now compute the distribution functions of the r.v.'s $\widetilde{U}_i$. For $i=2,\dots,n$ and $u \in [0,1[$:

$$\begin{aligned} P(\widetilde{U}_i \leq u) &= P\left[1 - \exp\left[-\frac{(i-1)\,X_i}{\sum_{j=1}^{i-1} X_j}\right] \leq u\right] \\ &= P\left[\frac{X_i}{\sum_{j=1}^{i-1} X_j} \leq -\frac{\ln(1-u)}{i-1}\right] \end{aligned} \tag{8}$$

Moreover for $i=2,\dots,n$

$$X_i \sim Exp(\lambda) \equiv Gamma(1,\lambda) \text{ and } \sum_{j=1}^{i-1} X_j \sim Gamma(i-1,\lambda)$$

this implies that the r.v. $X_i / \sum_{j=1}^{i-1} X_j$ is $Beta(1, i-1)$ distributed with distribution functions:

$$\forall x \in \mathbb{R}_+, \ F_{Beta(1,i-1)}(x) = 1 - (1+x)^{-(i-1)}.$$

Using the previous result in equation (8) one obtains equation (7) given in proposition 3.2. □

Corollary – 1 The r.v. $\widetilde{U}_n$ converges in distribution, when $n \to +\infty$, to the uniform distribution $Unif[0,1]$.

3.2 Shorack Theorem

The following result was given by Shorack (1979):

Theorem – A (Shorack) *Let $U_1,\dots,U_n$ denote independent r.v.'s having distribution functions $G_1,\dots,G_n$ concentrated on $[0,1]$.*

Let:

— *W_n denote the empirical process defined for $t \in [0,1]$ by:*

$$W_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [\, \mathbb{1}_{\{U_i \le t\}} - G_i(t)\,]. \tag{9}$$

— *μ_n denote the covariance function of W_n,*

— *$\bar{G}_n$ denote the function defined on $[0,1]$ by: $\bar{G}_n(t) = \frac{1}{n}\sum_{i=1}^{n} G_i(t)$.*

If there exist two real functions: $\bar{G}$ (continuous) defined on $[0,1]$ and μ defined on $[0,1] \times [0,1]$, such that

(1.) For all $t \in [0,1]$: $\lim_{n \to +\infty} \bar{G}_n(t) = \bar{G}(t)$,

(2.) For all s and $t \in [0,1]$: $\lim_{n \to +\infty} \mu_n(s,t) = \mu(s,t)$,

then there is a Gaussian random process W on $[0,1]$, with means 0 and covariance function μ such that:

$$W_n \overset{n \to +\infty}{\longrightarrow} W \quad \textit{in distribution.}$$

□

3.3 Proof of Theorem 3.1

Let $\widetilde{y}_n$ be the process defined on $[0,1]$ by:

$$\forall t \in [0,1],\ \widetilde{y}_n(t) = \sqrt{n-1}\left[\frac{1}{n-1}\sum_{i=2}^{n} \mathbb{1}_{\{\widetilde{U}_i \le t\}} - t\right]. \tag{10}$$

To prove Theorem 3.1, we first prove the following proposition:

Proposition – 3.3 *Under the hypothesis $H_0^{(exp)}$, the process $\widetilde{y}_n$ converges weakly to the Brownian bridge $\mathbb{B}$:*

$$\widetilde{y}_n \overset{n \to +\infty}{\longrightarrow} \mathbb{B} \quad \textit{in distribution.}$$

Proof – To prove proposition 3.3 we rewrite $\widetilde{y}_n$ as follows:

$$\forall t \in [0,1]\ ,\ \widetilde{y}_n(t) = \widetilde{W}_n(t) + d_n(t),$$

where $\widetilde{W}_n$ is a random process defined on $[0,1]$ by:

$$\widetilde{W}_n(t) = \frac{1}{\sqrt{n-1}} \sum_{i=2}^{n} [\, \mathbb{1}_{\{\widetilde{U}_i \le t\}} - F_{\widetilde{U}_i}(t)\,], \tag{11}$$

and d_n is a function defined on $[0,1]$ by:

$$d_n(t) = \sqrt{n-1}\left[\frac{1}{n-1}\sum_{i=2}^{n} F_{\widetilde{U}_i}(t) - t\right]. \tag{12}$$

Properties of r.v.'s $\widetilde{U}_i$ allow the use of Theorem A to prove the weak convergence of $\widetilde{W}_n$ to the Brownian bridge $\mathbb{B}$. This is obtained by noting that, due to Corollary 1, conditions (1.) and (2.) of Theorem A hold since:

1. For all $u \in [0,1]$, $\lim_{n\to+\infty} \frac{1}{n-1}\sum_{i=2}^{n} F_{\widetilde{U}_i}(u) = u$.
2. For all s and $t \in [0,1]$

   $$\widetilde{\mu}_n(s,t) \equiv \mathrm{Cov}(\widetilde{W}_n(s), \widetilde{W}_n(t)) = \frac{1}{n-1}\sum_{i=2}^{n}\left[F_{\widetilde{U}_i}(\min(s,t)) - F_{\widetilde{U}_i}(s)F_{\widetilde{U}_i}(t)\right]$$

   so using Corollary 1 again one has

   $$\widetilde{\mu}_n(s,t) \overset{n\to+\infty}{\longrightarrow} \min(s,t) - st.$$

It follows that $\widetilde{W}_n \overset{n\to+\infty}{\longrightarrow} \mathbb{B}$ in distribution.

We need now to prove that:

$$\forall t \in [0,1]\,,\ d_n(t) \overset{n\to+\infty}{\longrightarrow} 0. \tag{13}$$

By replacing the variable t by $u = \ln(1-t)$, one defines a function c_n on $\mathbb{R}_-$ by:

$$\begin{aligned} \forall n \geq 1\ ,\ \forall u \leq 0 \qquad c_n(u) &= -d_{n+1}(1-e^u) \\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[(1-\frac{u}{i})^{-i} - e^u\right]. \end{aligned}$$

Thus, (13) will follow from the following result:

$$\forall u \leq 0,\ \ c_n(u) \overset{n\to+\infty}{\longrightarrow} 0.$$

Since

$$\left(1-\frac{u}{i}\right)^{-i} - e^u = \frac{u^2}{i}\,e^u + o(\frac{1}{i}),$$

if a is a non-negative number, there exist an integer n_0 such that $\forall i \geq n_0$:

$$\left(1-\frac{u}{i}\right)^{-i} - e^u \leq \frac{1}{i}(u^2 e^u + a). \tag{14}$$

Writing

$$c_n(u) = \left[\frac{1}{\sqrt{n}} \sum_{i=1}^{n_0-1} \left\{ \left(1-\frac{u}{i}\right)^{-i} - e^u \right\}\right] + \left[\frac{1}{\sqrt{n}} \sum_{i=n_0}^{n} \left\{ \left(1-\frac{u}{i}\right)^{-i} - e^u \right\}\right]$$

we have

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n_0-1} \left\{ \left(1-\frac{u}{i}\right)^{-i} - e^u \right\} \overset{n\to+\infty}{\longrightarrow} 0$$

and from (14)

$$\begin{aligned} \frac{1}{\sqrt{n}} \sum_{i=n_0}^{n} \left\{ \left(1-\frac{u}{i}\right)^{-i} - e^u \right\} &\leq \frac{1}{\sqrt{n}} (u^2 e^u + a) \sum_{i=1}^{n} \frac{1}{i} \\ &\leq \frac{1}{\sqrt{n}} (u^2 e^u + a) \,[\,1 + \ln(n)\,]\,. \end{aligned}$$

As $\forall u \leq 0,\, \forall i \in \mathbb{N}^*,\quad 0 \leq \left(1-\frac{u}{i}\right)^{-i} - e^u$, it follows that $\forall u \leq 0,\ c_n(u) \overset{n\to+\infty}{\longrightarrow} 0$.
This concludes the proof of Proposition 3.3. □

Remark – It is possible to prove that:

$$\forall t \in [0,1]\ ,\ \forall n \geq 1\ ,\quad |\, d_{n+1}(t)\,| \leq [\ln(1-t)\,]^2\,(1-t)\ \frac{1+\ln(n)}{\sqrt{n}}.$$

Proof of Theorem 3.1 is completed by using the following result (Billingsley 1968 pp. 105):

$$\sup_{t\in[0,1]} |\,\mathbb{B}(t)\,| \sim \mathcal{L}_{ks}.$$

Since the function sup is continuous on the space D of cad-lag functions defined on $[0,1]$ with Skorokhod metric (cf. Durbin 1973 pp. 18), Billingsley proposition (Billingsley 1968 pp. 30) allows the use of the following results

$$\begin{cases} \widetilde{y}_n \overset{Dist.}{\longrightarrow} \mathbb{B} \\ \\ \sup_{t\in[0,1]} |\,\mathbb{B}(t)\,| \sim \mathcal{L}_{ks} \end{cases}$$

to prove that

$$\widetilde{K}_n = \sup_{t\in[0,1]} |\ \tilde{y}_n(t)\ | \overset{Dist.}{\longrightarrow} \mathcal{L}_{ks},$$

this concludes the proof of theorem 3.1. □

4 Power of the Prequential Test

We use here Monte Carlo simulations to examine the power of the prequential test against the standard complete test. Four alternative distributions are considered: Uniform (on [0,1]), Lognormal (from the standard normal distribution), χ^2_4 and Weibull (with shape parameter $\beta = 1.5$); for each one 10.000 samples of size n (n = 30, 60, 100 and 200) are simulated. Tables 1 (5% level) and 2 (10% level) give the percentage of Monte Carlo samples for which the exponentiality is not rejected.

	True distribution	Exponential	Uniform	Lognormal	χ^2_4	Weibull
$n = 30$	Prequential Test	94.98	70.13	86.66	80.68	80.10
	Complete Test	95.41	28.75	76.31	43.71	44.35
$n = 60$	Prequential Test	94.41	39.98	78.08	54.95	56.56
	Complete Test	95.34	3.35	60.01	10.94	11.91
$n = 100$	Prequential Test	95.01	13.01	69.45	25.82	29.21
	Complete Test	95.32	0.01	42.37	0.95	1.63
$n = 200$	Prequential Test	94.81	0.01	47.31	1.28	2.69
	Complete Test	95.11	0.00	13.39	0.00	0.00

Table 1: Powers comparison: 5% level.

	True distribution	Exponential	Uniform	Lognormal	χ^2_4	Weibull
$n = 30$	Prequential Test	89.83	56.38	79.33	67.49	67.77
	Complete Test	90.78	16.77	67.65	29.12	30.45
$n = 60$	Prequential Test	89.28	25.54	69.40	38.58	41.00
	Complete Test	90.57	1.06	49.26	5.47	5.87
$n = 100$	Prequential Test	90.00	5.11	59.51	14.04	16.77
	Complete Test	90.53	0.00	31.68	0.29	0.51
$n = 200$	Prequential Test	90.09	0.00	37.70	0.34	0.79
	Complete Test	90.32	0.00	7.93	0.00	0.00

Table 2: Powers comparison: 10% level.

5 Discussion

Similarly to Stephens Half-sample method (1978), the prequential test avoids the use of special quantile tables but is clearly less powerful than the standard complete test. Nevertheless the prequential approach still represents an appropriate tool for comparing reliability-growth models; particularly because:

— According to the Monte Carlo studies (Downs and Scott 1992 and El Aroui 1996), the prequential Kolmogorov-Smirnov distances seem to have the the same distribution (Kolmogorov-Smirnov) for several reliability growth-models.

— The prequential Kolmogorov-Smirnov distances measure the predictive qualities of competitive models, i.e. their capability to predict future observations, this seems to be very useful for practicians.

It is therefore worth generalizing the theoretical results presented above to reliability-growth models, this seems to be possible for Jelinski-Moranda model, Moranda Geometric model and the family of Non-Homogeneous Poisson Process models (these models are presented for example in Xie 1991).

Acknowledgement – I would like to thank Professors J. Diebolt and J.P. Raoult for fruitful discussions.

References

[1] A.A. Abdel-Ghali, P.Y. Chen, and B. Littlewood. Evaluation of Competing Software Reliability Predictions. *IEEE Transactions on Reliability*, SE-12(9):518–524, 1986.

[2] P. Billingsley. *Convergence of Probability Measures.* Wiley series in Probability and Mathematical Statistics. Wiley and Sons, Inc., 1968.

[3] R.B. D'Agostino and M.A. Stephens. *Goodness-of-fit Techniques*, volume 68 of *Statistics, textbooks and monographs.* New York and Basel : Marcel Dekker, inc, 1986.

[4] F.N. David and N.L. Johnson. The Probability Integral Transformation when parameters are estimated from the sample. *Biometrika*, 35:182–190, 1948.

[5] A.P. Dawid. Statistical Theory: The Prequential Approach. *J. R. Statist. Soc. A*, 147:278–292, 1984.

[6] T. Downs and A. Scott. Evaluating the performance of software reliability models. *IEEE Transactions on Reliability*, 41(4):518–524, Dec 1992.

[7] J. Durbin. *Distribution theory for tests based on the sample distribution function.* Number 9 in Statistics textbooks and monographs. SIAM Publications, Philadelphia, 1973.

[8] M.A. El Aroui. *Outils Statistiques pour la Construction et le Choix de Modèles en Fiabilité des Logiciels (Statistical Tools for the Construction and the Choice of Models in Software Reliability).* PhD thesis, Université Joseph Fourier de Grenoble, 1996.

[9] H.W. Lilliefors. On the Kolmogorov-Smirnov test for the exponential distribution with mean unknown. *JASA*, 64:387–389, 1969.

[10] M.R. Lyu and al. *Handbook of Software Reliability Engineering.* IEEE Computer Society Press and McGraw-Hill Book Company, 1996.

[11] G.R. Shorack. The Weighted empirical process of row independent random variables with arbitrary distribution functions. *Statistica Neerlandica*, 33(4):169–189, 1979.

[12] M.A. Stephens. On the Half-sample Method for Goodness-of-fit. *J. R. Statist. Soc. B*, 40(1):64–70, 1978.

[13] M. Xie. *Software Reliability Modelling.* World Scientific, 1991.

Unit´e de recherche INRIA Lorraine, Technopôle de Nancy-Brabois, Campus scientifique,
615 rue du Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY
Unit´e de recherche INRIA Rennes, Irisa, Campus universitaire de Beaulieu, 35042 RENNES Cedex
Unit´e de recherche INRIA Rhône-Alpes, 655, avenue de l'Europe, 38330 MONTBONNOT ST MARTIN
Unit´e de recherche INRIA Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex
Unit´e de recherche INRIA Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex

Éditeur
INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
http://www.inria.fr
ISSN 0249-6399