Design of Homogeneous Time-Varying Stabilizing Control Laws for Driftless Controllable Systems Via Oscillatory Approximation of Lie Brackets In closed-loop

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Design of homogeneous time-varying stabilizing control laws for driftless controllable systems via oscillatory approximation of Lie brackets in closed-loop

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Abstract: A constructive method for time-varying stabilization of smooth driftless controllable systems is developed. It provides time-varying homogeneous feedback laws that are continuous and smooth away from the origin. These feedbacks render the closed-loop system globally exponentially asymptotically stable if the driftless controllable system under consideration is homogeneous with respect to a certain family of dilations, and locally exponentially asymptotically stable, using local homogeneous approximation of control systems, in the other case.

The method uses some known algorithms that construct oscillatory control inputs to approximate motion along Lie brackets. These algorithms are modified to adapt them to the closed-loop context.

Key-words: Nonlinear Control, Stabilization, Time-varying stabilization, Controllability, Lie brackets

(Résumé : tsvp)

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Construction de lois de commande homogènes dépendant du temps pour la stabilisation des systèmes commandables sans dérive par approximation en boucle fermée des crochets de Lie à l'aide de commandes oscillantes

Résumé : On développe ici une méthode constructive de stabilisation par retour d'état instationnaire des systèmes commandables sans dérive (infiniment différentiables). On obtient des retours d'état homogènes qui sont continus partout et infiniment différentiables partout sauf à l'origine. Ces lois de commandes rendent le point d'équilibre du système bouclé globalement exponentiellement asymptotiquement stable si le système est naturellement homogène par rapport à une certaine famille de dilatations, et localement exponentiellement asymptotiquement stable sinon, grâce à l'utilisation d'une approximation homogène du système considéré.

On utilise un algorithme connu de construction de commandes oscillantes en boucle ouverte qui permettent d'approcher des mouvements dans la direction des crochets de Lie de champs de vecteurs de commandes. Cet algorithme doit toutefois être sensiblement modifié et adapté pour pouvoir être utilisé dans notre contexte de commande en boucle fermée.

Mots-clé : Automatique non-linéaire, Stabilisation, Stabilisation instationnaire, Commandabilité, Crochets de Lie

1 Introduction

1.1 Related work and contribution

Stabilization by continuous time-varying feedback laws of nonlinear systems that cannot be stabilized by time-invariant continuous feedback laws has been an ongoing subject of research in the past few years.

The fact that for many controllable systems, there exists no continuous stabilizing feedback was first pointed out Sussmann [22] ; a simple necessary condition was given by Brockett [1], since known as "Brockett's condition", and that allows one to identify a wide class of controllable systems for which no continuous stabilizing feedback exists; these include most controllable driftless systems. A weaker necessary condition may be found in [2].

A possible way of stabilizing systems for which these necessary conditions are violated is the use of discontinuous (time-invariant) control laws. This has been explored in the literature, but the present work does not go at all in this direction.

The possibility of stabilizing nonlinear controllable systems via *continuous time-varying feedback control laws* was first noticed in the very detailed study of stabilization of one-dimensional systems by Sontag and Sussmann [20]. More recently, smooth stabilizing control laws for controlled non-holonomic mechanical systems were given by the third author in [18], and this was the starting point of a systematic study of time-varying stabilization. Coron proved in [3] —the paper [15] by the second author deals with a less general class of controllable driftless systems— that all controllable driftless systems may be stabilized by continuous (and even smooth) time-varying feedback, and in [4] that "most" controllable systems (even with drift) can also be stabilized by continuous time-varying feedback.

From here on, only driftless systems are considered in this paper. After the general existence result given in [3], studies on the subject have focused on methods to *construct* continuous time-varying stabilizing feedback laws, and on obtaining feedback laws that provide sufficiently fast convergence.

As far as the constructiveness aspect is concerned, let us, for simplicity, divide the construction methods in two kinds. The first kind of methods apply to rather large classes of controllable driftless systems, like the work of Coron [3] (general controllable driftless systems; the paper is not oriented towards construction of the control, but a method can be extracted from the proofs), by the second autor [15] (controllable driftless systems for which the control Lie algebra is generated by a specific set of vector fields), or by M'Closkey and Murray [12] (same conditions as in [15]). These studies all share the following feature : they use the solution of a certain linear PDE, or the expression of the flow of a certain vector field, to construct the control law. This solution, or this flow, has to be calculated beforehand, either analytically or numerically, and this introduces, especially when no analytical solution is available, a degree of complication which may not be necessary. The second kind of methods found in the literature provides explicit expression of the control laws. Their drawback is that they only apply to specific subclasses of driftless systems, such as models of mobile robots, or systems in the so-called “Chain form” or “Power-form”, like the work by the third author [18], by Teel et al. [25], by Sépulchre et al. [19], among others.

On the other hand, a need to improve the speed of convergence came out of the slow convergence associated with the *smooth* control laws that were first proposed. This concern motivated several studies starting with the work by M'Closkey and Murray [11], yielding the derivation of continuous control laws which are not smooth, or even Lipschitz everywhere, but are homogeneous with respect to some dilation, and thus exponentially stabilizing (not in the standard sense but according to a certain homogeneous norm). See for instance further work by the authors [16, 14], or by M'Closkey and Murray [12], who have also proposed recently in [13] a procedure that transforms a given smooth stabilizing control law into a homogeneous one. Except this last reference, that requires that a smooth stabilizing control laws has been designed beforehand, the construction of homogeneous exponentially stabilizing control laws in the literature is restricted to specific subclasses of driftless systems.

The design method described in the present paper has the advantage of being totally explicit, in the sense that it only requires ordinary differentiation and linear algebraic operations, while it applies to general controllable systems and provides exponential stabilizing homogeneous feedback laws. The fact that it relates controllability with the construction of a stabilizing control law in a more direct way than previous designs also makes it conceptually appealing, all the more so as it may be viewed as converting the open-loop control techniques reported by Liu and Sussmann in [23, 9] into closed-loop ones.

However the generality of the method has also a price. When applied to particular systems for which explicit solutions have long been available, the present method often yields solutions which are significantly more complicated. This is a consequense of the complexity of the approximation algorithm which is proposed in [23, 9], and that we have adapted to our feedback contrl objective.

1.2 Outline of the method

Nonlinear controllability results were first derived for driftless systems, see for instance the work by Lobry [10] where it is shown that such a system is controllable if and only if any direction in the state space can be obtained as a linear combination of iterated Lie brackets of the control vector fields, at least in the real-analytic case. It has was also shown very early on, by Haynes and Hermes [5], that under this same condition, *any* curve in the state-space can be approached by open-loop solutions of the controlled system (note that this property is not shared by all controllable systems, but rather specific to driftless systems). In these studies the key element is that, in addition to the directions of motion corresponding to the control vector fields, motion along other directions corresponding to iterated Lie brackets is also possible by quickly switching motions along the original control vector fields. Take for example a system with two controls

$$\dot{x} \;=\; u_1\, b_1(x) \;+\; u_2\, b_2(x) \tag{1}$$

with state x in $\mathbb{R}^5$, and that assume that at each point x the vectors

$$b_1(x)\,,\; b_2(x)\,,\; [b_1, b_2](x)\,,\; [b_1,[b_1,b_2]](x)\,,\; [b_2,[b_1,b_2]](x) \tag{2}$$

are linearly independent, and thus span $\mathbb{R}^5$. The idea in [5] is the following : on one hand it is clear that *any* (e.g. differentiable) parameterized curve $t \mapsto \gamma(t)$ is a possible solution of the "extended" system with five controls :

$$\dot{x} \;=\; v_1\, b_1(x) + v_2\, b_2(x) + v_3\, [b_1,b_2](x) + v_4\, [b_1,[b_1,b_2]](x) + v_5\, [b_2,[b_1,b_2]](x) \tag{3}$$

(simply decompose $\dot{\gamma}(t)$ on the basis (2) to obtain the controls). Then it is proved in [5] that there exists a sequence of (oscillatory) controls $u_1(\varepsilon, t, v_1, v_2, v_3, v_4, v_5)$ and $u_2(\varepsilon, t, v_1, v_2, v_3, v_4, v_5)$ such that the system (1) "converges to" the system (3) when $\varepsilon \to 0$ in the sense that the solutions of (1) with these controls u_k converge uniformly on finite time intervals to the solutions of (3). The proof in [5] does not give a process to build these sequences of approximating sequence of oscillatory control, and although the case of a simple bracket (approximating $[b_1, b_2]$ by switching between b_1 and b_2) is simple and well known, the above case of brackets of order 3 is already not obvious. The more recent work by Liu, and Liu and Sussmann [23, 9] gives an explicit construction of the approximating sequence. The process of building this sequence is amazingly intricate compared to the simplicity of the existence proof in [5]. Of course, the controls u_k are not defined for $\varepsilon = 0$, and both their frequency and their amplitude tend to infinity when ε goes to zero.

Being aware of these results, and faced with the problem of proving that any controllable driftless system may be stabilized by means of a periodic feedback, the most natural idea is probably the following, which we illustrate for the above case (1) (5 states, 2 controls) :

a- Stabilize the extended system (3) by a control law $v_i(x)$. This is very easy, and $\dot{x}$ may even be assigned to be any desired function, for instance $-x$.

b- Use the approximation results and build the controls $u_k(\varepsilon, t, v_1(x), v_2(x), v_3(x), v_4(x), v_5(x))$, according to the process given in [23, 9] so that when ε tends to zero, the system (1) controlled with these controls "tends to" the extended system (3) controlled with the controls $v_i(x)$.

c- Since the limit system is asymptotically stable (for instance $\dot{x} = -x$), and asymptotic stability is somehow robust, the constructed control laws are hopefully stabilizing for ε nonzero but small enough. For instance, one may take $\|x\|^2$ as a Lyapunov function for the limit system (its time-derivative along the limit system is $-2\|x\|^2$), and it is tempting to believe that its time-derivative along the original system controlled by $u_k(\varepsilon, t, v_1(x), v_2(x), v_3(x), v_4(x), v_5(x))$ is no larger than $-\|x\|^2$ for ε small enough.

Unfortunately, these arguments, which would have been somewhat simpler than those in [3], are not rigorous as they stand. The meaning of the phrase "tends to" in point b is very unprecise. In [5], and in [23, 9], only uniform convergence of the trajectories on finite-time intervals are considered. This is not adequate for asymptotic stabilization. In addition, the fact that feedback controls are considered instead of open-loop controls complicates the proofs because the controls depend on the state and therefore may have a very high derivative with respect to time not only through the high frequencies and amplitudes built in the approximation process but also through their dependence on the state, whose speed is proportionnal to these high amplitudes. The Lyapunov function based argument in point c does not work because, in general, when ε tends to zero, the time-derivative of a given function along the system (1) in feedback with the controls u_k from point b does not tend to the time-derivative of this function along the "limit" system (3).

However, we show in the present paper that the above sketch is basically correct provided that homogeneous controls associated with a homogeneous Lyapunov function, are used. The construction of the approximating sequence has to be modified to take into account the closed-loop nature of the controls. An argument of the type of point c is possible based on a notion of approximation that is not in terms of uniform convergence of trajectories, but in terms of the differential operator defined by derivation along the system.

The paper is organized as follows. After a brief recall of technical material in section 2, we state, in section 3, the control objective, make homogeneity assumptions and explain how they will yield local results for general controllable systems. The design method is developed in section 4; it is done in four distinct steps : choice of the "useful" Lie brackets, construction of stabilizing controls for the extended system (system (3) in the above example), selection of the frequencies by a method that is almost the one exposed in [9], and then construction, using these frequencies, of the controls that stabilize the extended system. The material from these steps is then gathered to give the control law, and the stabilization result is stated. We present in this section all that is needed for the construction of the control law, but the proofs of some properties needed at each steps, and of the theorem, are given separately in section 7. We have presented as a separate theorem, in section 6, a convergence result needed in the proof of the stability theorem ; it is a translation in terms of differential operators (instead of trajectories) of the averaging results presented in [23, 9, 24], and also in [8]. An illustrative example is given in section 5.

2 Background on homogeneous vector fields

For any $\lambda > 0$, the "dilation operator" δ_λ associated with a "weight vector" $r = (r_1, \dots, r_n)$, $(r_i > 0)$ is defined on $\mathbb{R}^n$ by:

$$\delta_\lambda(x_1, \dots, x_n) = (\lambda^{r_1} x_1, \dots, \lambda^{r_n} x_n).$$

A function $f \in C^o(\mathbb{R}^n; \mathbb{R})$ is said to be homogeneous of degree τ with respect to the family of dilations (δ_λ) if :

$$\forall \lambda > 0, \quad f(\delta_\lambda(x)) = \lambda^\tau f(x)$$

A continuous vector field X on $\mathbb{R}^n$ is said to be homogeneous of degree σ with respect to the family of dilations (δ_λ) if one of the following equivalent properties is satisfied :

1. For any $i = 1, \dots, n$, its ith component, i.e. the function $x \mapsto X_i(x)$, is homogeneous of degree $r_i + \sigma$.

2. For any function h homogeneous of degree $\tau > 0$ with respect to the same dilation, the function $L_X h$ (its Lie derivative along X) is homogeneous with degree $\sigma + \tau$.

3. For all positive constant λ, the vector field $((\delta_\lambda)_* X)$, conjugate of X by the diffeomorphism δ_λ —away from the origin— satisfies $((\delta_\lambda)_* X)(x) = \lambda^{-\sigma} X(x)$ for $x \neq 0$.

The previous definitions of homogeneity can be extended to time varying functions and vector fields, by considering an "extended dilation":

$$\delta_\lambda(x_1, \dots, x_n, t) = (\lambda^{r_1} x_1, \dots, \lambda^{r_n} x_n, t).$$

Finally, let $f \in C^o(\mathbb{R}^n \times \mathbb{R}; \mathbb{R}^n)$, with $f(x,.)$ T periodic, defining an homogeneous vector field of degree 0 with respect to a family of dilations (δ_λ). Then, the two following properties are equivalent:

i) the origin $x = 0$ of the system $\dot{x} = f(x,t)$ is locally asymptotically stable,

ii) $x = 0$ is globally exponentially asymptotically stable.

3 Problem Statement

Consider a general smooth driftless controllable system

$$\dot{x} \ = \ \sum_{i=1}^{m} u_i f_i(x) \ . \tag{4}$$

It is sometimes the case that the control vector fields are homogeneous along a certain "natural" dilation. In general however, there does not exist such a dilation, or at least it is not easy to find one. Yet, controllability is sufficient to construct a dilation, and a *homogeneous approximation* [6, 7] of the system (4) around the origin. It is a driftless control system whose control vector fields are all homogeneous with degree -1, and has the property that if some homogeneous feedback law asymptotically stabilizes it globally, then it locally asymptotically stabilizes the original system.

The present work constructs an homogeneous feedback that ensures global exponential stabilization for homogeneous systems, hence in particular for homogeneous approximations of general systems. It will therefore provide local exponential stabilization of general systems (4).

In the sequel, we always consider a system

$$\dot{x} \;=\; \sum_{i=1}^{m} u_i b_i(x) \tag{5}$$

where the b_i's are smooth vector fields and the system of coordinates is such that there exists some integers $(r_1, \ldots, r_n)$ such that,

1. each vector field b_i is homogeneous of degree -1 with respect to the family of dilations δ_λ with weights $(r_1, \ldots, r_n)$,

2. the rank at the origin of the Lie algebra generated by the b_i's is n :

$$Rank(\; Lie\{b_1, \ldots, b_m\}(0)\;) \;=\; n \;. \tag{6}$$

The integer valued weights $r_1, \ldots, r_n$ are now fixed, and we denote

$$P \;=\; \mathrm{Max}\,\{r_i;\ i = 1, \ldots, n\} \;. \tag{7}$$

Our objective is to design feedback laws $u = (u_1, \ldots, u_m) \in C^o(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^m)$ such that the origin $x = 0$ of the closed loop system (5) is exponentially asymptotically stable.

Remark 1 *We only require full rank Control Lie Algebra at the origin, but controllability follows, because homogeneity allows to deduce the same rank condition everywhere.*

Remark 2 *We assume that the degrees are all equal to -1. It is the value given by the contructio of a homogeneous approximation in [21]. If a system is naturallt homogenous, and the degrees are not all equal (if they are equal, a simple scaling makes them all equal to -1), it is of course better to use this natural homogeneity than to construct a different homogeneity approximation that will have all the degrees equal to -1 : one can adapt the present method to the case when the degrees of homogeneity are not all equal, this requires only a modification of the first step. See remark 3. More details are available from the authors.*

4 Controller design

The control design consists in five steps which we describe hereafter. The proofs needed to justify this construction are given in Section 7.

Step 1 (Selection of Lie brackets)

In this first step, we select Lie brackets $\tilde{b}_j$ $(j = 1, \ldots, N)$ from the Lie algebra generated by the b_i's. If the vector fields $b_1, \ldots, b_m$ are independent at the origin and the rank of all the Lie brackets of length less than k is constant around the origin for all k, then we just select enough brackets, as short as possible to make a basis of $\mathbb{R}^n$ (and $N = n$ in this case). In order to be able to deal with singular cases, the selected brackets $\tilde{b}_j$ are, in general, selected recursively as follows.

First, select vector fields $\tilde{b}_1, \ldots, \tilde{b}_{m_1}$ taken among $b_1, \ldots, b_m$ in the following way.

- Pick the maximal number of vector fields $\tilde{b}_1, \ldots, \tilde{b}_{m'_1}$ among $b_1, \ldots, b_m$ such that the vectors $\tilde{b}_1(0), \ldots, \tilde{b}_{m'_1}(0)$ are linearly independent.

- Add to these vector fields the maximum number of vector fields $\tilde{b}_{m'_1+1}, \ldots, \tilde{b}_{m_1}$ taken among $b_1, \ldots, b_m$ such that $\tilde{b}_1, \ldots \tilde{b}_{m_1}$ are linearly independent on $\mathbb{R}$ — i.e., such that for any vector $(\lambda_1, \ldots, \lambda_{m_1})$ in $\mathbb{R}^{m_1}$, the vector field $\lambda_1 \tilde{b}_1 + \ldots + \lambda_{m_1} \tilde{b}_{m_1}$ is identically zero on $\mathbb{R}^n$ only if $\lambda_1 = \ldots = \lambda_{m_1} = 0$.

Assume now that a family $\tilde{b}_1, \ldots, \tilde{b}_{m_1}, \ldots, \tilde{b}_{m_{p-1}+1}, \ldots, \tilde{b}_{m_p}$ $(1 \leq p < P)$, made of brackets of length not larger than p, has been computed —for $p = 1$, $m_{p-1} = m_0 \stackrel{\Delta}{=} 0$. Then, compute all brackets of length $p + 1$ and,

- Pick, among these brackets, the maximal number of brackets $\tilde{b}_{m_p+1}, \ldots, \tilde{b}_{m'_{p+1}}$ such that the vectors $\tilde{b}_{m_p+1}(0), \ldots, \tilde{b}_{m'_{p+1}}(0)$ are linearly independent.

- Add the maximum number of brackets $\tilde{b}_{m'_{p+1}+1}, \ldots, \tilde{b}_{m_{p+1}}$ taken among the brackets of order $p + 1$ such that $\tilde{b}_{m_p+1}, \ldots \tilde{b}_{m_{p+1}}$ are linearly independent on $\mathbb{R}$.

Stop at length P (let us recall that P has been defined in Section 3), and denote N the integer m_P. The family $\tilde{b}_j$ $(j = 1, \ldots, N)$ is then well defined.

Example : Let us illustrate this step, on the following academic example :

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= x_3^2(u_1 + u_2) \\ \dot{x}_3 &= u_3 \end{aligned}$$

which is of the form (5) with $m = 3$ and

$$b_1 = \frac{\partial}{\partial x_1} + x_3^2 \frac{\partial}{\partial x_2} , \quad b_2 = x_3^2 \frac{\partial}{\partial x_2} , \quad b_3 = \frac{\partial}{\partial x_3} .$$

The assumptions are met with $r_1 = 1$, $r_2 = 3$ and $r_3 = 1$. For the brackets of length 1, i.e. the control vector fields, b_1 and b_3 are independent at the origin while b_2 is zero at the origin, but independent from b_1 and b_3 away from $x_3 = 0$; hence one takes $m'_1 = 2$, with $\tilde{b}_1 = b_1$ and $\tilde{b}_2 = b_3$, and $m_1 = 3$ with $\tilde{b}_3 = b_2$. At length 2, all the brackets vanish at the origin, so $m'_2 = m_1 = 3$, but they are not identically zero : $[b_2, b_3] = -2x_3 \frac{\partial}{\partial x_2}$, and $[b_3, b_1] = -[b_2, b_3]$, $[b_1, b_2] = 0$, hence $m_2 = 4$, with for instance $\tilde{b}_4 = [b_2, b_3]$; finally, since $[b_3, [b_2, b_3]] = -2\frac{\partial}{\partial x_2}$, $m'_3 = m_3 = 5$ with for instance $\tilde{b}_5 = [b_3, [b_2, b_3]]$. Note that here, due to the origin being a singular point for the distributions spanned by the control vector fields, and by the brackets of order at most 2, N is strictly larger than n.

In the sequel, $\ell(j)$ denotes the length of the bracket $\tilde{b}_j$, i.e.

$$\ell(j) = p \quad \Leftrightarrow \quad m_{p-1} + 1 \leq j \leq m_p \ .$$

The following properties of this family of vector fields will be important.

Property 1 *For any family $(\tilde{b}_j)_{j=1,\ldots,N}$ defined as above we have:*

a) For all x,

$$\{\tilde{b}_1(x), \ldots, \tilde{b}_{m'_1}(x), \tilde{b}_{m_1+1}(x), \ldots, \tilde{b}_{m'_2}(x), \ldots, \tilde{b}_{m_{P-1}+1}(x), \ldots, \tilde{b}_{m'_P}(x)\}$$

is a basis of $I\!\!R^n$.

b) Any Lie bracket b made with the vector fields b_i $(i = 1, \ldots, m)$ and of length $p \leq P = \max_j \ell(j)$ can be decomposed as:

$$b = \sum_{j=m_{p-1}+1}^{m_p} \lambda_j \tilde{b}_j = \sum_{\ell(j)=p} \lambda_j \tilde{b}_j$$

for a constant $\lambda_j \in I\!\!R$.

c) The family $(\tilde{b}_j)_{j=1,\ldots,N}$ is linearly independent on $I\!\!R$, i.e.:

$$(\tilde{b}_j = \sum_{\ell(k)=\ell(j)} \alpha_k \tilde{b}_k \quad \textit{with} \ \ \alpha_k \in I\!\!R) \Longrightarrow (\alpha_j = 1 \ \textit{and} \ \alpha_k = 0 \ \ \forall k \neq j)$$

(Proof in Section 7.1)

Remark 3 *1. Since the vector fields b_i $(i = 1, \ldots, m)$ are smooth homogeneous vector fields (and the degrees and weights are integers), they are in fact polynomials. Using a (finite) basis of the polynomials homogeneous of degree k ($k \in \{0, \ldots, P-1\}$), the computation of the vector fields $\tilde{b}_{m'_p+1}, \ldots, \tilde{b}_{m_p}$ ($p = 1, \ldots, P$) consists in computing a basis of a finite dimensional vector space.*

2. *In [23, 9, 24], instead of the brackets $\tilde{b}_j$, Liu and Sussmann use a basis of the* free *Lie algebra generated by $b_1, \ldots, b_m$ up to a certain order, namely a P. Hall basis. We only choose here to retain the Lie brackets that are important in the Lie algebra corresponding to the particular values of the b_i for the control system. Note that at each step above, instead of considering* all *the Lie bracket of a certain length, one may of course consider only these corresponding to a basis of the free Lie algebra.*

3. *If the degrees of the vector fields b_i are not all equal, the above construction has to be modified. More precisely, in the recursive construction of the family $(\tilde{b}_j)_{j=1,\ldots,N}$, we have to consider an induction on the degree of homogeneity, instead of an induction on the length of the Lie brackets —remark that this is just a generalization of the above construction since for vector fields of the same degree -1, the set of Lie brackets of length p is the same as the set of Lie brackets of degree $-p$. This means that at each step, we have to compute the set of Lie brackets of a certain degree and select, among them, a finite number of vector fields which form a basis of this set.*

Step 2 (Stabilization of the extended system)

- Take for a be any smooth vector field homogeneous of degree 0 with respect to the family of dilations (δ_λ), and such that the origin $x = 0$ of the system $\dot{x} = a(x)$ is asymptotically stable (one may take for instance $a(x) = -x$).

- Let $j_1, \ldots, j_n \in \{1, \ldots, N\}$ be the integers that (see step 1) are between $m'_{k-1}+1$ and m_k for a certain k, rather than between $m_k + 1$ and m'_{k+1} :

$$(j_1, \ldots, j_n) \;=\; (1, \ldots, m'_1, m_1 + 1, \ldots, m'_2, \ldots, m_{P-1} + 1, \ldots, m'_P) \,.$$

  In view of Property 1-a, the $n \times n$ matrix whose columns are $\tilde{b}_{j_1}(x), \ldots, \tilde{b}_{j_n}(x)$ is invertible for all x.

- Define the functions $\tilde{u}_j$ $(j = 1, \ldots, N)$ by :

$$\bullet \begin{pmatrix} \tilde{u}_{j_1}(x) \\ \vdots \\ \tilde{u}_{j_n}(x) \end{pmatrix} = \left(\tilde{b}_{j_1}(x), \ldots, \tilde{b}_{j_n}(x)\right)^{-1} a(x) \quad , \tag{8}$$

$$\bullet\ \tilde{u}_j = 0 \quad \forall j \notin \{j_1, \ldots, j_n\} \quad .$$

These functions are obviously such that

$$a = \sum_{j=1}^{N} \tilde{u}_j \, \tilde{b}_j \ , \tag{9}$$

and furthermore, one has :

Property 2 *For any $j = 1, \ldots, N$, the above constructed function $\tilde{u}_j$ is in $C^\infty(\mathbb{R}^n - \{0\}; \mathbb{R}) \cap C^0(\mathbb{R}^n; \mathbb{R})$, and is homogeneous of degree $\ell(j)$.*

Proof : Continuity and smoothness away from the origin are inherited from the vector fields $\tilde{b}_j$ and the vector field a. Each $\tilde{u}_{j_k}$ is homogeneous of degree $\ell(j_k)$ because the lth component of the vector field a is homogeneous of degree r_l and the element (k, l) of the matrix $\left(\tilde{b}_{j_1}(x), \ldots, \tilde{b}_{j_n}(x)\right)^{-1}$ is homogeneous of degree $\ell(j_k) - r_l$. This last statement is true because the element (k, l) of the matrix $\left(\tilde{b}_{j_1}(x), \ldots, \tilde{b}_{j_n}(x)\right)$ is homogeneous of degree $\ell(j_k) - r_l$ for the vector field $\tilde{b}_{j_k}$ is an iterated Lie bracket of $\ell(j_k)$ homogeneous vector fields of degree -1, and hence is homogeneous of degree $-\ell(j_k)$. ∎

Step 3 (Selection of the frequencies)

This step borrows a lot of material from the work of Liu [9] and Sussmann and Liu [23, 24]. Since our purpose is rather different, we have to adapt their construction, and cannot simply refer to it. The main difference is that we do not work with the *free* Lie algebra generated by the control vector fields $b_1, \ldots, b_m$, but with the Lie algebra *of the system* —i.e., the one corresponding to the particular expression of the vector fields defining the control system, and not the one where the b_i's are considered as independent symbols. Of course this has the drawback that the construction depends on the system, whereas the construction in [9, 23, 24] does not —i.e. the controls needed to produce the desired trajectory for the extended system depend on the expressions of the vector fields, but the construction of highly oscillatory controls (for the real systems) which approximate the extended system does not.

To expose this third step, we first need two definitions from [23, 9].

Definition 1 *[23, 9] Let Ω be a finite set of $\mathbb{R}$ and $|\Omega|$ denote the number of elements of Ω. The set Ω is said to be "Minimally Canceling" (in short, MC) if:*

i) $\sum_{\omega\in\Omega} \omega = 0$

ii) this is the only zero sum with at most $|\Omega|$ terms taken in Ω with possible repetitions:

$$\left.\begin{array}{c} \sum_{\omega\in\Omega} \lambda_\omega \omega = 0 \\ (\lambda_\omega)_{\omega\in\Omega} \in \mathbb{Z}^{|\Omega|} \\ \sum_{\omega\in\Omega} |\lambda_\omega| \leq |\Omega| \end{array}\right\} \Longrightarrow \left\{\begin{array}{ccc} (\lambda_\omega)_{\omega\in\Omega} & = & (0,\dots,0) \\ & or & (1,\dots,1) \\ & or & (-1,\dots,-1) \end{array}\right. \tag{10}$$

Definition 2 *[23, 9] Let $(\Omega_\alpha)_{\alpha\in I}$ be a finite family of finite sets Ω_α of $\mathbb{R}$. The family $(\Omega_\alpha)_{\alpha\in I}$ is said "Independent with respect to p" if:*

$$\left.\begin{array}{l} \bullet \sum_{\alpha\in I}\sum_{\omega\in\Omega_\alpha} \lambda_\omega \omega = 0 \\ \bullet\ (\lambda_\omega)_{\omega\in\Omega_\alpha,\alpha\in I} \in \mathbb{Z}^{\Sigma|\Omega_\alpha|} \\ \bullet \sum_{\alpha\in I}\sum_{\omega\in\Omega} |\lambda_\omega| \leq p \end{array}\right\} \Longrightarrow \sum_{\omega\in\Omega_\alpha} \lambda_\omega\omega = 0 \quad \forall\alpha\in I \tag{11}$$

We also introduce the following notations :

- We denote by $\tau_j^1,\dots,\tau_j^{\ell(j)}$ the indices i of the control vector fields b_i from which the bracket $\tilde{b}_j$ is made. For instance of $\ell(j) = 4$ and $\tilde{b}_j = [b_1,[[b_3,b_4],b_1]]$, we have $(\tau_j^1,\tau_j^2,\tau_j^3,\tau_j^4) = (1,3,4,1)$.
- For any $p\in\{2,\dots,P\}$, any MC set $\Omega = \{\omega^1,\dots,\omega^p\}$ and any $q\in\{m_{p-1}+1,\dots,m_p\}$, we denote:

$$I_q(\Omega) \triangleq \sum_{\sigma\in\mathfrak{S}(p)} \frac{[b_{\tau_q^{\sigma(1)}},[b_{\tau_q^{\sigma(2)}},[\dots,b_{\tau_q^{\sigma(p)}}]\dots]]}{\omega^{\sigma(1)}(\omega^{\sigma(1)}+\omega^{\sigma(2)})\dots(\omega^{\sigma(1)}+\dots+\omega^{\sigma(p-1)})} \tag{12}$$

  with $\mathfrak{S}(p)$ the group of permutations of order p. By an abuse of notation, Ω will sometimes be identified to a vector $(\omega_1,\dots,\omega_p)$ in $\mathbb{R}^p$, and $I_q(\Omega)$ to the value of a function I_q of the p variables $(\omega_1,\dots,\omega_p)$.

We start this third step by the following computations.

i) For any $p \in \{2, \ldots, P\}$, any $q \in \{m_{p-1+1}, \ldots, m_p\}$, and any MC set Ω, it follows from Property 1-b that $I_q(\Omega)$ can be decomposed as a linear combination of $\tilde{b}_{m_{p-1}+1}, \ldots, \tilde{b}_{m_p}$. The coefficients obviously depend on Ω, but from Property 1-b, they do not depend on x. Let us define g_q^k $(k = m_{p-1+1}, \ldots, m_p)$ by:

$$I_q(\Omega) \;=\; g_q^{m_{p-1}+1}(\Omega)\,\tilde{b}_{m_{p-1}+1} + \ldots + g_q^{m_p}(\Omega)\,\tilde{b}_{m_p} \tag{13}$$

Note that each g_q^k is a rational function of $\omega^1, \ldots, \omega^p$.

ii) Let g_q be defined by $g_q(\Omega) = (g_q^{m_{p-1}+1}(\Omega), \ldots, g_q^{m_p}(\Omega))$ and let M_p $(p \in \{2, \ldots, P\}$ denote the $(m_p - m_{p-1})^2 \times (m_p - m_{p-1})$ matrix defined by:

$$M_p(\Omega_{i,j},\; i,j = m_{p-1}+1, \ldots, m_p) \;=\; \begin{pmatrix} g_{m_{p-1}+1}(\Omega_{m_{p-1}+1,m_{p-1}+1}) \\ \vdots \\ g_{m_{p-1}+1}(\Omega_{m_{p-1}+1,m_p}) \\ \vdots \\ g_{m_p}(\Omega_{m_p,m_{p-1}+1}) \\ \vdots \\ g_{m_p}(\Omega_{m_p,m_p}) \end{pmatrix} \tag{14}$$

where each $\Omega_{i,j}$ $(i,j = m_{p-1}+1, \ldots, m_p)$ is a set $\{\omega_{i,j}^1, \ldots, \omega_{i,j}^p\}$, identified with a vector in $I\!R^p$. Each M_p can be identified to a matrix-valued function of $p \times (m_p - m_{p-1})^2$ variables (the $\omega_{i,j}^k$), whose components are rational functions. Since each M_p is only defined for MC sets of frequencies $\Omega_{i,j}$, only the linear subspace Π, of dimension $(p-1) \times (m_p - m_{p-1})^2$, given by:

$$\sum_{k=1}^{p} \omega_{i,j}^k = 0 \qquad (i,j = m_{p-1}+1, \ldots, m_p)$$

is of interest to us.

Then we have:

Lemma 1 *Let us consider, for any $p = 2, \ldots, P$ the minors of order $m_p - m_{p-1}$ of M_p. Then, the determinants of these minors are rational functions from $I\!R^{p(m_p - m_{p-1})^2}$ to $I\!R$, and the restriction to Π of these functions (which may be viewed as rational functions of $(p-1)(m_p - m_{p-1})^2$ are not all identically zero.*

(Proof in Section 7.2)

We continue this thrid step by :

iii) For any $p = 2, \ldots, P$, choose integers $\gamma_{m_{p-1}+1}, \ldots, \gamma_{m_p} \in \{m_{p-1}+1, \ldots, m_p\}$ such that the determinant of the minor

$$S_p = \begin{pmatrix} g_{\gamma_{m_{p-1}+1}}(\Omega_{m_{p-1}+1}) \\ \vdots \\ g_{\gamma_{m_p}}(\Omega_{m_p}) \end{pmatrix} \tag{15}$$

is not identically zero —and thus different from zero on an open and dense subset of $I\!R^{(p-1)(m_p-m_{p-1})}$: the complementary of the zeros of a finite number of polynomials, intersected with the set on which S_p is defined (which is also open and dense, cf. Property 1 in Section 7.2).

iv) Find some sets Ω_j ($j = m_1+1, \ldots, N$) such that:

1. Each Ω_j ($j \in \{m_{p-1}+1, \ldots, m_p\}$) is an MC set of $p-$uples,
2. The family $(\Omega_j)_{j=m_1+1,\ldots,N}$ is independent with respect to P,
3. Each matrix S_p, *evaluated at these specific values of* $\Omega_{m_{p-1}+1}, \ldots, \Omega_{m_p}$, is invertible.

The justification of part *iii)* relies on Lemma 1 proved in Section 7.2. The justification of part *iv)* is also given in that section.

Remark 4 *The integers γ_j and the sets Ω_j can be computed recursively in the sense that if a family of integers γ_j and MC sets Ω_j ($j = m_1+1, \ldots, m_p, 2 \le p \le P$) have been found such that the family $(\Omega_j)_{j=m_1+1,\ldots,m_p}$ is independent with respect to P and each matrix S_r ($r = 2, \ldots, p$) is of full rank $m_r - m_{r-1}$ then, one can find a family of integers $\gamma_{m_p+1}, \ldots, \gamma_{m_{p+1}} \in \{m_p+1, \ldots, m_{p+1}\}$ and a family of MC sets $\Omega_{m_p+1}, \ldots, \Omega_{m_{p+1}}$ such that the family $(\Omega_j)_{j=m_1+1,\ldots,m_{p+1}}$ is independent with respect to P and the matrix S_{p+1} is of full rank $m_{p+1} - m_p$.*

Finally, one directly infers from (13) and (15) the following property.

Property 3 *With the integers $\gamma_{m_{p-1}+1}, \ldots, \gamma_{m_p}$ and the sets $\Omega_{m_{p-1}+1}, \ldots, \Omega_{m_p}$ defined above we have:*

$$\begin{pmatrix} \tilde{b}^T_{m_{p-1}+1} \\ \vdots \\ \tilde{b}^T_{m_p} \end{pmatrix} = S_p^{-1} \begin{pmatrix} I^T_{\gamma_{m_{p-1}+1}}(\Omega_{m_{p-1}+1}) \\ \vdots \\ I^T_{\gamma_{m_p}}(\Omega_{m_p}) \end{pmatrix} \tag{16}$$

Step 4 (Construction of the state-dependent amplitudes)

Let ν_j^k $j = 1, \ldots, N$, $k = 1, \ldots, \ell(j)$ be integers defined by:

$$\begin{array}{lllll} \bullet\ \nu_j^1 & = & \tau_j^1 & \text{for } j = 1, \ldots, m_1 \\ \bullet\ \nu_j^k & = & \tau_{\gamma_j}^k & \text{for } j = m_1 + 1, \ldots, N \end{array} \tag{17}$$

where $(\tau_j^1, \ldots, \tau_j^{\ell(j)})$ are, as in step 3, the indices i of the control vector fields b_i from which the bracket $\tilde{b}_j$ is made, and the γ_j's have been defined by step 3.

The aim of this step is to compute a family of functions v_j^k ($j = 1, \ldots, N$, $k = 1, \ldots, \ell(j)$) such that:

$$\sum_{j=1}^{N} \sum_{\sigma \in \mathfrak{S}(\ell(j))} \frac{[b_{\nu_j^{\sigma(1)}} v_j^{\sigma(1)}, [b_{\nu_j^{\sigma(2)}} v_j^{\sigma(2)}, [\ldots, b_{\nu_j^{\sigma(\ell(j))}} v_j^{\sigma(\ell(j))}] \ldots]]}{\omega_j^{\sigma(1)} (\omega_j^{\sigma(1)} + \omega_j^{\sigma(2)}) \ldots (\omega_j^{\sigma(1)} + \ldots + \omega_j^{\sigma(\ell(j)-1)})} = \sum_{j=1}^{N} \tilde{u}_j \tilde{b}_j \tag{18}$$

If one could simply pull the functions v_j^k out of the brackets, the left-hand side in (18) would be equal[1], in view of (12) and (17), to

$$\sum_{j=1}^{N} v_j^1 \ldots v_j^{\ell(j)} I_{\gamma_j}(\Omega_j)$$

Then, using (16), equality (18) would be satisfied by taking[2], for each $p \in \{1, \ldots, P\}$,

$$\begin{pmatrix} v_{m_{p-1}+1}^1 \ldots v_{m_{p-1}+1}^p \\ \vdots \\ v_{m_p}^1 \ldots v_{m_p}^p \end{pmatrix} = (S_p^{-1})^T \begin{pmatrix} \tilde{u}_{m_{p-1}+1} \\ \vdots \\ \tilde{u}_{m_p} \end{pmatrix} \tag{19}$$

Unfortunately, when v_1 and v_2 are two functions, and b_{i_1} and b_{i_2} two vector fields, $[v_1 b_{i_1}, v_2 b_{i_2}]$ is not equal to $v_1 v_2 [b_{i_1}, b_{i_2}]$ but to $v_1 v_2 [b_{i_1}, b_{i_2}] - v_2 (L_{b_{i_2}} v_1) b_{i_1} + v_1 (L_{b_{i_1}} v_2) b_{i_2}$. More generally we have:

---

[1] As a (natural) convention, for $\ell(j) = 1$, the partial sum

$$\sum_{\sigma \in \mathfrak{S}(\ell(j))} \frac{[b_{\nu_j^{\sigma(1)}} v_j^{\sigma(1)}, [b_{\nu_j^{\sigma(2)}} v_j^{\sigma(2)}, [\ldots, b_{\nu_j^{\sigma(\ell(j))}} v_j^{\sigma(\ell(j))}] \ldots]]}{\omega_j^{\sigma(1)} (\omega_j^{\sigma(1)} + \omega_j^{\sigma(2)}) \ldots (\omega_j^{\sigma(1)} + \ldots + \omega_j^{\sigma(\ell(j)-1)})}$$

in (18) should simply be understood as $b_{\nu_j^1} v_j^1$.

[2] For $p = 1$, S_p should be understood as the identity matrix.

Lemma 2 *Let b be a Lie bracket of length p ($p \in \{2, \dots, P\}$) made with the vector fields $b_{i_1}v_1, \dots, b_{i_p}v_p$, with $i_k \in \{1, \dots, m\}$ for $k = 1, \dots, p$, and with $v_k \in C^\infty(I\!R^n - \{0\}; I\!R) \cap C^0(I\!R^n; I\!R)$ some functions homogeneous of degree 1. We denote $\mathcal{C}(b_{i_1}v_1, \dots, b_{i_p}v_p)$ the symbolic expression of this bracket. Then,*

i) $b = v_1 \dots v_p \, \mathcal{C}(b_{i_1}, \dots, b_{i_p}) - \sum_{j=1}^{m_{p-1}} h_j \, \tilde{b}_j$

ii) for any $j = 1, \dots, m_{p-1}$, $h_j \in C^\infty(I\!R^n - \{0\}; I\!R) \cap C^0(I\!R^n; I\!R)$ and is homogeneous of degree $\ell(j)$

The proof of this lemma follows, from Property 1-b, by a direct induction on the length p of the bracket b. It is left to the reader.
In any case, the functions h_j can be explicitly computed by expressing brackets of order not larger than $p-1$ as linear combinations of $\tilde{b}_1, \dots, \tilde{b}_{m_{p-1}}$.

A family of functions v_j^k for which (18) holds can now be computed as follows. We proceed recursively by a decreasing induction on p and compute, at each step, a family of functions v_j^k and h_j^k.

Step P:

For brackets of highest order, the functions v_j^k are computed according to (19). More precisely, let $N_P = (n_P^{rs})_{r,s=m_{P-1}+1,\dots,m_P}$ denote the inverse of the matrix S_P computed in Step 3 (cf. (15)) . Then, for any $j = m_{P-1}+1, \dots, m_P$, we define:

$$
\begin{aligned}
v_j^r &= \rho \,, \qquad \forall r = 1, \dots, P-1 \,, \\
v_j^P &= \frac{1}{\rho^{P-1}} \sum_{r=m_{P-1}+1}^{m_P} n_P^{rj} \tilde{u}_r
\end{aligned}
\tag{20}
$$

with $\rho \in C^\infty(I\!R^n - \{0\}; I\!R) \cap C^0(I\!R^n; I\!R)$ an homogeneous norm (for instance one may take $\rho(x) = (\sum |x_i|^{\frac{q}{r_i}})^{\frac{1}{q}}$ with $q = 2\prod_{i=1}^n r_i$).
We also define the functions h_j^P ($j = 1, \dots, m_{P-1}$), which keep track of the corrective terms involving brackets of order not larger than $P-1$, by:

$$
\begin{aligned}
&\sum_{j=m_{P-1}+1}^{m_P} \sum_{\sigma \in \mathfrak{S}(P)} \frac{[b_{\nu_j^{\sigma(1)}} v_j^{\sigma(1)}, [b_{\nu_j^{\sigma(2)}} v_j^{\sigma(2)}, [\dots, b_{\nu_j^{\sigma(P)}} v_j^{\sigma(P)}] \dots]]}{\omega_j^{\sigma(1)}(\omega_j^{\sigma(1)} + \omega_j^{\sigma(2)}) \dots (\omega_j^{\sigma(1)} + \dots + \omega_j^{\sigma(P-1)})} \\
&= \sum_{j=m_{P-1}+1}^{m_P} \tilde{u}_j \tilde{b}_j - \sum_{j=1}^{m_{P-1}} h_j^P \tilde{b}_j
\end{aligned}
\tag{21}
$$

These functions are obtained by expanding the brackets in the left-hand side of (21) with respect to the variables v_j^k and their derivatives (cf. Lemma 2). The ω_j^k are those defined in Step 3 by $\Omega_j = \{\omega_j^1, \dots, \omega_j^{\ell(j)}\}$.

Step $1 \leq p < P$:

First, the functions v_j^k and h_j^k computed in Steps P to $p+1$ satisfy:

$$\begin{aligned} &\sum_{j=m_p+1}^{N} \sum_{\sigma \in \mathfrak{S}(\ell(j))} \frac{[b_{\nu_j^{\sigma(1)}} v_j^{\sigma(1)}, [b_{\nu_j^{\sigma(2)}} v_j^{\sigma(2)}, [\dots, b_{\nu_j^{\sigma(\ell(j))}} v_j^{\sigma(\ell(j))}] \dots]]}{\omega_j^{\sigma(1)}(\omega_j^{\sigma(1)} + \omega_j^{\sigma(2)}) \dots (\omega_j^{\sigma(1)} + \dots + \omega_j^{\sigma(\ell(j)-1)})} \\ &= \sum_{j=m_p+1}^{N} \tilde{u}_j \tilde{b}_j - \sum_{j=1}^{m_p} h_j^{p+1} \tilde{b}_j \end{aligned} \tag{22}$$

Let us now denote $N_p = (n_p^{rs})_{r,s=m_{p-1}+1,\dots,m_p}$ the inverse of the matrix S_p. Then, for any $j = m_{p-1}+1, \dots, m_p$, the functions $v_j^1, \dots, v_j^p$ are defined by:

$$\begin{aligned} v_j^r &= \rho\,, \qquad \forall r = 1, \dots, p-1\,, \\ v_j^p &= \frac{1}{\rho^{p-1}} \sum_{r=m_{p-1}+1}^{m_p} n_p^{rj} (\tilde{u}_r + h_r^{p+1}) \end{aligned} \tag{23}$$

and the functions h_j^p $(j = 1, \dots, m_{p-1})$ are defined by:

$$\begin{aligned} &\sum_{j=m_{p-1}+1}^{m_p} \sum_{\sigma \in \mathfrak{S}(p)} \frac{[b_{\nu_j^{\sigma(1)}} v_j^{\sigma(1)}, [b_{\nu_j^{\sigma(2)}} v_j^{\sigma(2)}, [\dots, b_{\nu_j^{\sigma(p)}} v_j^{\sigma(p)}] \dots]]}{\omega_j^{\sigma(1)}(\omega_j^{\sigma(1)} + \omega_j^{\sigma(2)}) \dots (\omega_j^{\sigma(1)} + \dots + \omega_j^{\sigma(p-1)})} \\ &= \sum_{j=m_{p-1}+1}^{m_p} (\tilde{u}_j + h_j^{p+1}) \tilde{b}_j - \sum_{j=1}^{m_{p-1}} (h_j^p - h_j^{p+1}) \tilde{b}_j \end{aligned} \tag{24}$$

and are also obtained by expanding the brackets in the left-hand side of (24) with respect to the variables v_j^k and their derivatives.
The computation of the functions v_j^k and h_j^k ends after Step $p = 1$ has been performed. Let us remark that in the last step ($p = 1$), there is no function h_j^k to compute.

With this construction, we have:

Property 4 *Let us consider the functions v_j^k and h_j^k defined above. Then,* *a)* *The functions* v_j^k $(j = 1, \ldots, N\,,\ k = 1, \ldots, \ell(j))$ *belong to* $C^\infty(I\!R^n - \{0\}; I\!R) \cap C^0(I\!R^n; I\!R)$ *and are homogeneous of degree* 1.

b) *The functions* h_j^k $(k = 2, \ldots, P,\ j = 1, \ldots, m_{k-1})$ *belong to* $C^\infty(I\!R^n - \{0\}; I\!R) \cap C^0(I\!R^n; I\!R)$ *and are homogeneous of degree* $\ell(j)$.

c) *Equation (18) is satisfied.*

(Proof in Section 7.3)

Final step

The time varying stabilizing feedback laws are computed in the following way. Define some complex numbers

$$\eta_j^s\,, \qquad 0 \leq j \leq N\,,\ 1 \leq s \leq \ell(j)$$

such that

$$\Re\left(\frac{\eta_j^1 \eta_j^2 \cdots \eta_j^{\ell(j)}}{i^{\ell(j)-1}}\right) \;=\; \frac{\ell(j)}{2}\,. \tag{25}$$

For example

$$\eta_j^1 \;=\; \frac{\ell(j)}{2}\, i^{\ell(j)-1}\,,\ \eta_j^2 \;=\cdots=\; \eta_l^{\ell(j)} \;=\; 1\,. \tag{26}$$

The time-varying controls are then given by :

$$u_i^\varepsilon(x,t) \;=\; w_i \;+\; 2 \sum_{(j,s),\, \nu_j^s = i} \varepsilon^{-\frac{\ell(j)-1}{\ell(j)}}\, \Re\left(\eta_j^s\, e^{i\omega_j^s t/\varepsilon}\right)\, v_j^s(x) \tag{27}$$

with

$$w_i \;=\; \begin{cases} v_j^1 & \text{if there exist a (unique) integer } j \text{ such that} \\ & \ell(j) = 1 \text{ and } \nu_j^1 = i \\ 0 & \text{if there is no } j \text{ such that } \ell(j) = 1 \text{ and } \nu_j^1 = i \end{cases} \tag{28}$$

The second case above can occur only if the vector fields b_i are not independent at the origin. Let us also remark that in view of Property 4-a, each u_i^ε belongs to $C^\infty((I\!R^n - \{0\}) \times I\!R; I\!R) \cap C^0(I\!R^n \times I\!R; I\!R)$.

If the complex numbers η_j^s are chosen according to (26), the term in the sum in (27) is

$$\begin{array}{ll} \varepsilon^{-\frac{\ell(j)-1}{\ell(j)}}\cos(\omega_j^s t/\varepsilon)\, v_j^s(x) & \text{if } s \neq 1, \\ \frac{\ell(j)}{2}(-1)^{\frac{\ell(j)-1}{2}}\varepsilon^{-\frac{\ell(j)-1}{\ell(j)}}\cos(\omega_j^1 t/\varepsilon)\, v_j^s(x) & \text{if } s = 1 \text{ and } \ell(j) \text{ is odd,} \\ \frac{\ell(j)}{2}(-1)^{\frac{\ell(j)}{2}}\varepsilon^{-\frac{\ell(j)-1}{\ell(j)}}\sin(\omega_j^1 t/\varepsilon)\, v_j^s(x) & \text{if } s = 1 \text{ and } \ell(j) \text{ is even.} \end{array}$$

To sum up this control design method, we have the following theorem.

Theorem 1 *Let the controls u_i^ε be these described above. Then the vector field in the right-hand side of the time-varying closed-loop system*

$$\dot{x} = \sum_{i=1}^{m} u_i^\epsilon(x,t)\, b_i(x) \tag{29}$$

is homogeneous of degree zero, and for $\varepsilon > 0$ sufficiently small, the origin is exponentially uniformly asymptotically stable.

It remains to show that each of the five steps exposed above can be performed, and to prove Theorem 1. These proofs are given in Section 7.

5 An illustrative example

We now illustrate the control design method exposed in Section 4. Let us consider the following system in $I\!R^4$:

$$\dot{x} = b_1 u_1 + b_2 u_2 \tag{30}$$

with $b_1 = \frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + x_4\frac{\partial}{\partial x_3}$ and $b_2 = \frac{\partial}{\partial x_4}$, which can be used to model the kinematic equations of a car like mobile robot. One easily verifies that the vector fields b_1 and b_2 are homogeneous of degree -1 with respect to the family of dilations of weight $r = (1,3,2,1)$, and that this system is controllable. We follow on this example the five steps of our control design procedure.

Step 1

Since $[b_1, b_2] = -\frac{\partial}{\partial x_3}$, $[b_1,[b_1,b_2]] = \frac{\partial}{\partial x_2}$ and $[b_2,[b_2,b_1]] = 0$, the family $(\tilde{b}_j)$ is directly given by:

$$(\tilde{b}_j) \quad = (\tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \tilde{b}_4) = (b_1, b_2, [b_1,b_2], [b_1,[b_1,b_2]]) \tag{31}$$

This implies that $\tau_1^1 = 1, \tau_2^1 = 2, \tau_3^1 = 1, \tau_3^2 = 2, \tau_4^1 = \tau_4^2 = 1, \tau_4^3 = 2$, and that $m_1 = m_1' = 2$, $m_2 = m_2' = 3$, and $m_3 = m_3' = N = 4$.

Step 2

Let us for instance define the vector field a by $a(x) = -x$ (the origin $x = 0$ of $\dot{x} = a(x)$ is obviously asymptotically stable). Then the integers j_k are simply defined by $j_k = k$ $(k = 1, \dots, 4)$. By a direct computation, one obtains the following expression for the functions $\tilde{u}_j$:

$$\begin{aligned} (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{u}_4)^T(x) &= (\tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \tilde{b}_4)^{-1}(x)\, a(x) \\ &= (-x_1, -x_4, -x_1x_4 + x_3, x_1x_3 - x_2)^T \end{aligned} \tag{32}$$

Step 3

We first determine the expression of I_3 and I_4.
In view of (12) and (13) we have:

$$I_3(\Omega) = \sum_{\sigma \in \mathfrak{S}(2)} \frac{[b_{\tau_3^{\sigma(1)}}, b_{\tau_3^{\sigma(2)}}]}{\omega^{\sigma(1)}} = (\frac{1}{\omega^1} - \frac{1}{\omega^2})[b_1, b_2] = (\frac{1}{\omega^1} - \frac{1}{\omega^2})\tilde{b}_3 \tag{33}$$

As a consequence, $g_3^3(\Omega) = \dfrac{1}{\omega^1} - \dfrac{1}{\omega^2}$. Similarly,

$$\begin{aligned} I_4(\Omega) &= \sum_{\sigma \in \mathfrak{S}(3)} \frac{[b_{\tau_4^{\sigma(1)}}, [b_{\tau_4^{\sigma(2)}}, b_{\tau_4^{\sigma(3)}}]}{\omega^{\sigma(1)}(\omega^{\sigma(1)} + \omega^{\sigma(2)})} \\ &= (\tfrac{1}{\omega^1(\omega^1+\omega^2)} - \tfrac{1}{\omega^1(\omega^1+\omega^3)} + \tfrac{1}{\omega^2(\omega^2+\omega^1)} - \tfrac{1}{\omega^2(\omega^2+\omega^3)})[b_1, [b_1, b_2]] \\ &= (\tfrac{1}{\omega^1(\omega^1+\omega^2)} - \tfrac{1}{\omega^1(\omega^1+\omega^3)} + \tfrac{1}{\omega^2(\omega^2+\omega^1)} - \tfrac{1}{\omega^2(\omega^2+\omega^3)})\tilde{b}_4 \end{aligned} \tag{34}$$

and $g_4^4(\Omega) = \dfrac{1}{\omega^1(\omega^1+\omega^2)} - \dfrac{1}{\omega^1(\omega^1+\omega^3)} + \dfrac{1}{\omega^2(\omega^2+\omega^1)} - \dfrac{1}{\omega^2(\omega^2+\omega^3)}$.

Since for $p = 2, 3$, $\{m_{p-1} + 1, \dots, m_p\} = \{m_p\}$, S_2 and S_3 are simply defined by $S_2 = g_{\gamma_3}(\Omega_3)$, $S_3 = g_{\gamma_4}(\Omega_4)$ with $\gamma_3 = m_2 = 3$, $\gamma_4 = m_3 = 4$. It remains to find two sets $\Omega_3 = \{\omega_3^1, \omega_3^2\}$ and $\Omega_4 = \{\omega_4^1, \omega_4^2, \omega_4^3\}$ such that:

1. Ω_3 and Ω_4 are MC,

2. the family (Ω_3, Ω_4) is independent with respect to $P = 3$,

3. $g_{\gamma_3}(\Omega_3) = g_3^3(\Omega_3) \neq 0$ and $g_{\gamma_4}(\Omega_4) = g_4^4(\Omega_4) \neq 0$.

Using the expression of g_3^3 and g_4^4 derived above, one easily shows that for any sets Ω_3 and Ω_4 M.C, g_3^3 and g_4^4 are in fact given by:

$$g_3^3(\Omega_3) = \frac{2}{\omega_3^1}, \qquad g_4^4(\Omega_4) = \frac{3}{\omega_4^1\omega_4^2} \tag{35}$$

As a consequence, these functions are different from zero for any MC sets Ω_3 and Ω_4, and a choice of these sets ensuring that the family (Ω_3, Ω_4) is independent with respect to P is for instance given by $\Omega_3 = \{\frac{7}{2}, -\frac{7}{2}\}$ and $\Omega_4 = \{2, 3, -5\}$.

Step 4

In view of (17), and since $\gamma_j = j\,(j = 3, 4)$, the integers ν_j^k are equal to the integers τ_j^k defined in Step 1. We now follow the procedure specified in Section 4. We proceed recursively.

Step $P = 3$.
The functions v_4^1, v_4^2 and v_4^3 are given, in view of (20), by:

$$v_4^1 = v_4^2 = \rho, \quad v_4^3 = 2\frac{\tilde{u}_4}{\rho^2} \tag{36}$$

where $\rho \in C^\infty(\mathbb{R}^4 - \{0\}; \mathbb{R}) \cap C^0(\mathbb{R}^4; \mathbb{R})$ is any function homogeneous of degree 1 with respect to the family of dilations (δ_λ) (for instance, one may take $\rho(x) = (x_1^{12} + x_2^4 + x_3^6 + x_4^{12})^{\frac{1}{12}})$.
We now want to compute the functions h_j^3 $(j = 1, \ldots, 3)$ such that:

$$\sum_{\sigma \in \mathfrak{S}(3)} \frac{[b_{\nu_4^{\sigma(1)}} v_4^{\sigma(1)}, [b_{\nu_4^{\sigma(2)}} v_4^{\sigma(2)}, b_{\nu_4^{\sigma(3)}} v_4^{\sigma(3)}]]}{\omega_4^{\sigma(1)}(\omega_4^{\sigma(1)} + \omega_4^{\sigma(2)})} = \tilde{u}_4 \tilde{b}_4 - \sum_{j=1}^{3} h_j^3 \tilde{b}_j \tag{37}$$

A tedious but simple calculation shows that the equality (37) is satisfied with h_1^3, h_2^3, and h_3^3 defined by:

$$\begin{aligned} h_1^3 &= -\tfrac{1}{2}(-L_{[b_1,b_2]} v_4^2 v_4^3 - L_{b_2 L_{b_1 v_4^2} v_4^3} v_4^1 - v_4^1 L_{b_1 v_4^1} L_{b_2 v_4^3} v_4^2 + L_{b_2 v_4^3} v_4^2 L_{b_1} v_4^1) \\ h_2^3 &= -\tfrac{1}{2}(v_4^1 L_{b_1} L_{b_1 v_4^2} v_4^3) \\ h_3^3 &= -\tfrac{1}{2}(L_{b_1 v_4^1} v_4^2 v_4^3 + v_4^1 L_{b_1 v_4^2} v_4^3) \end{aligned} \tag{38}$$

Step $p = 2$.
The functions v_3^1 and v_3^2 are given, in view of (23), by:

$$v_3^1 = \rho, \quad v_3^2 = \frac{7}{4}\frac{(\tilde{u}_3 + h_3^3)}{\rho} \tag{39}$$

Let us compute two functions h_1^2 and h_2^2 such that:

$$\sum_{\sigma\in\mathfrak{S}(2)} \frac{[b_{\nu_3^{\sigma(1)}} v_3^{\sigma(1)}, b_{\nu_3^{\sigma(2)}} v_3^{\sigma(2)}]}{\omega_3^{\sigma(1)}} = (\tilde{u}_3 + h_3^3)\tilde{b}_3 - \sum_{j=1}^{2}(h_j^2 - h_j^3)\tilde{b}_j \tag{40}$$

One easily verifies that equality (40) is satisfied with h_1^1 and h_2^1 defined by:

$$\begin{array}{l} h_1^2 = \frac{4}{7} v_3^2 L_{b_2} v_3^1 + h_1^3 \\ h_2^2 = -\frac{4}{7} v_3^1 L_{b_1} v_3^2 + h_2^3 \end{array} \tag{41}$$

Step $p = 1$.
Finally, the functions v_1^1 and v_2^1 are given by

$$v_1^1 = \tilde{u}_1 + h_1^2, \quad v_2^1 = \tilde{u}_2 + h_2^2 \tag{42}$$

To sum up, the functions v_j^k are defined by:

$$\begin{array}{l} v_1^1 = \tilde{u}_1 + h_1^2, \quad v_2^1 = \tilde{u}_2 + h_2^2 \\ v_3^1 = \rho, \quad v_3^2 = \dfrac{7}{4}\dfrac{(\tilde{u}_3 + h_3^2)}{\rho} \\ v_4^1 = v_4^2 = \rho, \quad v_4^3 = 2\dfrac{\tilde{u}_4}{\rho^2} \end{array} \tag{43}$$

where the functions h_j^k are defined by (38) and (41), and the functions $\tilde{u}_j$ are given by (32).

Final Step

Time-varying feedback laws which, for ϵ small enough, exponentially stabilize the origin of the system (30) are finally given, using (27) and (26), by:

$$\left\{ \begin{array}{rcl} u_1^\epsilon(x,t) &=& v_1^1(x) - 2\epsilon^{-\frac{1}{2}} \sin(\frac{7t}{2\epsilon}) v_3^1(x) - 3\epsilon^{-\frac{2}{3}} \cos(\frac{2t}{\epsilon}) v_4^1(x) + 2\epsilon^{-\frac{2}{3}} \cos(\frac{3t}{\epsilon}) v_4^2(x) \\ u_2^\epsilon(x,t) &=& v_2^1(x) + 2\epsilon^{-\frac{1}{2}} \cos(\frac{7t}{2\epsilon}) v_3^2(x) + 2\epsilon^{-\frac{2}{3}} \cos(\frac{5t}{\epsilon}) v_4^3(x) \end{array} \right. \tag{44}$$

6 Convergence of highly oscillatory vector fields as differential operators

As explained in the introduction (section 1.2), the convergence results which are implicitly contained in [5], and explicitly in [23, 9] or [8], in terms of uniform convergence of solutions on finite time intervals, are not sufficient here.

In this section, we state separately the "convergence" result which is used to prove Theorem 1. The word convergence is maybe a bit farfetched here since there is no notion of limit in the topological sense, the convergence is more of an algebraic nature : we simply decompose the operator as the sum of a non-oscillating term (the "limit") and a term which is a differential operator whose coefficients remain bounded when ε goes to zero multiplied by ε at a positive power. We do not want to give a meaning to a sentence like "these terms tend to zero when ε does". However this result will prove to be sufficient for our needs. It is also sufficient to recover the uniform convergence stated in [5, 8, 23, 9] ; we briefly state this corollary at the end of the section.

Since we want to state a result that can also be used for open-loop controls (the added complexity is very minor), we leave the time-interval $\mathcal{T}$, the vector fields X_j^s and the functions of time η_j^s free : for stabilization, the time interval $\mathcal{T}$ is all $\mathbb{R}$, the vector fields X_j^s are given by (67) and the functions η_j^s are constant complex numbers.

Theorem 2 *Let N be a positive integer. For all j, $1 \leq j \leq N$, let $\ell(j)$ be an integer no smaller than one. Let P be the largest $\ell(j)$. Let a family of real numbers ω_j^s be defined for $1 \leq j \leq N$ and $1 \leq s \leq \ell(j)$, such that $\omega_j^s = 0$ if $\ell(j) = 1$ and the family of sequences $\Omega_j = \{\omega_j^1, \ldots, \omega_j^{\ell(j)}\}$, for all the integers j such that $\ell(j) \geq 2$ are minimally canceling (MC), and are independent with respect to P. Let :*

- *$X_j^s \in C^\infty(\mathbb{R}^n - \{0\}; \mathbb{R}^n) \cap C^0(\mathbb{R}^n; \mathbb{R}^n)$, for all (j,s) such that $1 \leq j \leq N$, $1 \leq s \leq \ell(j)$, be homogeneous vector fields of degree zero,*
- *η_j^s, for all (j,s) such that $1 \leq j \leq N$, $1 \leq s \leq \ell(j)$, be a family of smooth complex valued functions of time such that there is a uniform bound $M < +\infty$ on the considered period of time $\mathcal{T}$, on all these functions and their time-derivatives :*

$$\left|\eta_j^s(t)\right| \leq M \text{ and } \left|\dot{\eta}_j^s(t)\right| \leq M \quad \forall (j,s),\, 1 \leq j \leq N,\, 1 \leq s \leq \ell(j),\ \forall t \in \mathcal{T} \quad (45)$$

Let the functions $\alpha^s_{j,\varepsilon}(t)$ *be given by*

$$\alpha^s_{j,\varepsilon}(t) \;=\; 2\varepsilon^{-\frac{\ell(j)-1}{\ell(j)}}\,\Re\left(\eta^s_j(t)\,e^{i\omega^s_j t/\varepsilon}\right) \tag{46}$$

(recall that for j *such that* $\ell(j) = 1$ *(then* $s = 1$*),* $\omega^s_j = 0$ *and the corresponding exponentials are then simply equal to 1). With these "highly oscillatory controls", the differential equation*

$$\dot x \;=\; \sum_{j=1}^{N}\sum_{s=1}^{\ell(j)} \alpha^s_{j,\varepsilon} X^s_j$$

"converges", when $\varepsilon \to 0$*, to the equation*

$$\dot x \;=\; \sum_{j=1}^{N}\frac{2}{\ell(j)}\Re\left(\frac{\eta^1_j(t)\cdots\eta^{\ell(j)}_j(t)}{i^{\ell(j)-1}}\right)\sum_{\sigma\in\mathfrak{S}(\ell(j))}\frac{[X^{\sigma(1)}_j,[X^{\sigma(2)}_j,[\ldots,X^{\sigma(\ell(j))}_j]\ldots]](x)}{\omega^{\sigma(1)}_j(\omega^{\sigma(1)}_j+\omega^{\sigma(2)}_j)\ldots(\omega^{\sigma(1)}_j+\ldots+\omega^{\sigma(\ell(j)-1)}_j)}$$

in the following sense. The closed-loop vector field F^ε *in* $I\!R^{1+n}$ *given by :*

$$F^\varepsilon \;=\; \frac{\partial}{\partial t} + \sum_{j=1}^{N}\sum_{s=1}^{\ell(j)} \alpha^s_{j,\varepsilon} X^s_j \tag{47}$$

considered as a differential operator of order 1 on functions of t *and* x *may be expressed as :*

$$\begin{aligned} F^\varepsilon \;&=\; \frac{\partial}{\partial t} \\ &+\; \sum_{j=1}^{N}\frac{2}{\ell(j)}\Re\left(\frac{\eta_1\cdots\eta_{\ell(j)}}{i^{\ell(j)-1}}\right)\sum_{\sigma\in\mathfrak{S}(\ell(j))}\frac{[X^{\sigma(1)}_j,[X^{\sigma(2)}_j,[\ldots,X^{\sigma(\ell(j))}_j]\ldots]]}{\omega^{\sigma(1)}_j(\omega^{\sigma(1)}_j+\omega^{\sigma(2)}_j)\ldots(\omega^{\sigma(1)}_j+\ldots+\omega^{\sigma(\ell(j)-1)}_j)} \\ &+\; \varepsilon^{\gamma_1}\left(F^\varepsilon D^\varepsilon_1 - D^\varepsilon_1\frac{\partial}{\partial t}\right) \;+\; \varepsilon^{\gamma_2}D^\varepsilon_2 \end{aligned} \tag{48}$$

with $\gamma_1 > 0$*,* $\gamma_2 > 0$*, and* D^ε_1 *and* D^ε_2 *two higher order differential operators whose coefficients are continuous, smooth outside the origin, and locally uniformly bounded when* $\varepsilon \to 0$*, in the following sense : there exists* $\varepsilon_0 > 0$ *such that for all compact subset* K *of* $I\!R^n$*, all these coefficients are bounded for* $(\varepsilon, t, x) \in (0, \varepsilon_0] \times \mathcal{T} \times K$*. This equality is in the sense of differential operators acting on functions of* (t, x)*.*

If all the vector fields X^s_j *are homogeneous of degree zero, then all the terms in the sums in (48) are differential operators homogeneous of degree zero.*

(Proof in Section 7) (In [5, 8, 23, 9], the main ingredient of the proof was iterated integrations by parts ; here we mimic these integrations by parts, but at the level of products of differential operators instead of integrals along the solutions.)

The following convergence result, sufficient to recover the results in [23, 9]. or in [8], is a direct consequence :

Corollary *Let $b_1, \ldots, b_m$ be smooth vector fields. Let N be a positive integer, $\ell(j)$ be a positive integer for all j between 1 and N, and $\tau(j,s)$ be an integer between 1 and m for all (j,s) such that $1 \leq j \leq N$ and $1 \leq s \leq \ell(j)$. Let the real numbers ω_j^s be defined for $1 \leq j \leq N$ and $1 \leq s \leq \ell(j)$, such that $\omega_j^s = 0$ if $\ell(j) = 1$ and the family of sequences $\Omega_j = \{\omega_j^1, \ldots, \omega_j^{\ell(j)}\}$, for all the integers j such that $\ell(j) \geq 2$ are minimally canceling (MC), and are independent with respect to P.*

Let $v_1, \ldots, v_N$ be N functions of time defined on $[0,T]$ and x_0 be a point in $\mathbb{R}^n$. Suppose that the (unique) solution of

$$\begin{aligned} \dot{x} &= \sum_{j=1}^{N} v_j(t) \sum_{\sigma \in \mathfrak{S}(\ell(j))} \frac{[b_{\tau(j,\sigma(1))}, [b_{\tau(j,\sigma(2))}, [\ldots, b_{\tau(j,\sigma(\ell(j)))}] \ldots]](x)}{\omega_j^{\sigma(1)} (\omega_j^{\sigma(1)} + \omega_j^{\sigma(2)}) \ldots (\omega_j^{\sigma(1)} + \ldots + \omega_j^{\sigma(\ell(j)-1)})} \\ x(0) &= x_0 \ , \end{aligned} \tag{49}$$

is defined on all the time-interval $[0,T]$. Call this solution $x^0(t)$.

Let the functions $\eta_j^s(t)$ ($1 \leq s \leq \ell(j)$) be chosen such that, for all j,

$$\eta_j^1(t) \cdots \eta_j^{\ell(j)}(t) \;=\; \frac{\ell(j)}{2}\, i^{\ell(j)-1}\, v_j(t) \ , \tag{50}$$

and the functions $\alpha_{j,\varepsilon}^s$ be defined according to (46), and $w_{k,\varepsilon}(t,x)$ be given by

$$w_{k,\varepsilon}(t,x) \;=\; \sum_{\substack{(j,s) \text{ such that} \\ 1 \leq j \leq N,\ 1 \leq s \leq \ell(j), \\ \tau(j,s) = k}} \alpha_{j,\varepsilon}^s(t,x) \ . \tag{51}$$

Then, for ε small enough, the unique solution x^ε of

$$\begin{aligned} \dot{x} &= \sum_{k=1}^{m} w_{k,\varepsilon}(t,x)\, b_k(x) \\ x(0) &= x_0 \end{aligned} \tag{52}$$

is defined on $[0,T]$, and $x^\varepsilon(t)$ converges to $x^0(t)$ uniformly on $[0,T]$.

Proof : Apply Theorem 2 with $\mathcal{T} = I\!\!R$ and the vector fields X_j^s given by :

$$X_j^s \;=\; b_{\tau(j,s)} \;. \tag{53}$$

Since

$$\sum_{k=1}^{m} w_k(\varepsilon,t,x)\, b_k(x) \;=\; \sum_{j=1}^{N}\sum_{s=1}^{\ell(j)} \alpha_{j,\varepsilon}^s(t,x)\, X_j^s(x) \;,$$

(47)-(48) yields

$$\begin{aligned} &\left(\frac{\partial}{\partial t} \;+\; \sum_{k=1}^{m} w_{k,\varepsilon} b_k\right)(I \;-\; \varepsilon^{\gamma_1}\, D_1^\varepsilon) \\ &\quad = \;\frac{\partial}{\partial t} \;+\; \sum_{j=1}^{N} v_j \sum_{\sigma\in\mathfrak{S}(\ell(j))} \frac{[b_{\tau(j,\sigma(1))},[b_{\tau(j,\sigma(2))},[\ldots,b_{\tau(j,\sigma(\ell(j)))}]\ldots]]}{\omega_j^{\sigma(1)}(\omega_j^{\sigma(1)}+\omega_j^{\sigma(2)})\ldots(\omega_j^{\sigma(1)}+\ldots+\omega_j^{\sigma(\ell(j)-1)})} \\ &\qquad -\,\varepsilon^{\gamma_1} D_1^\varepsilon \frac{\partial}{\partial t} \;+\; \varepsilon^{\gamma_2} D_2^\varepsilon \;. \end{aligned} \tag{54}$$

This is an equality between differential operators; apply each side to the coordinate functions x_i. $D_1^\varepsilon x_i$ and $D_2^\varepsilon x_i$ are simply the coefficient in front of $\frac{\partial}{\partial x_i}$ in the expression of the differential operator $D_1^\varepsilon x_i$ or $D_2^\varepsilon x_i$. This implies (coordinate by coordinate) that the differential equation (52) may be rewritten

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(x \;-\; \varepsilon^{\gamma_1} d_1(\varepsilon,t,x)) \;=\; F(t,x) \;+\; \varepsilon^{\gamma_2} d_2(\varepsilon,t,x) \;, \tag{55}$$

where $F(t,x)$ stands for the right-hand side of (49), and $d_i(\varepsilon,t,x)$ $(i\in\{1,2\})$ is the vector whose jth component is the coefficient of $\frac{\partial}{\partial x_j}$ in D_i^ε. This implies that the difference between $x^0(t) - x^\varepsilon(t)$ satisfies

$$\begin{aligned} \|x^\varepsilon(t) - x^0(t)\| \;\le\;& \varepsilon^{\gamma_1}\|d_1(\varepsilon,t,x^0(t))\| \;+\; \varepsilon^{\gamma_1}\|d_1(\varepsilon,t,x_0)\| \\ &+\int_0^t \|F(\tau,x^\varepsilon(\tau)) - F(\tau,x^0(\tau))\|\mathrm{d}\tau \;+\; \varepsilon^{\gamma_2}\int_0^t \|d_2(\varepsilon,\tau,x^\varepsilon(\tau)))\|\mathrm{d}\tau \;. \end{aligned} \tag{56}$$

The standard Gronwall lemma then yields, for all $\varepsilon\in(0,\varepsilon_0]$ (ε_0 defined by theorem 2), and all $t\in[0,T]$ such that x^ε remains in the interior of a certain compact neighborhood K of the trajectory x^0, the estimate $\|x^\varepsilon(t) - x^0(t)\| \le (2\varepsilon^{\gamma_1} + T\varepsilon^{\gamma_2})Me^{\lambda t}$, where λ is a Lipschitz constant (with respect to x) of F on $[0,T]\times K$ and M is an upperbound on $(0,\varepsilon_0]\times[0,T]\times K$ for both $\|d_1\|$ and $\|d_2\|$, given by theorem 2. This proves the Corollary. ∎

7 Proofs

7.1 Proofs related to Step 1

Let us show that any family $(\tilde{b}_j)_{j=1,\dots,N}$ computed following the design procedure satisfies the Properties 1-a, 1-b, and 1-c.
In view of the construction, Property 1-b easily follows since otherwise, there would exist a Lie bracket b of length p which is not a linear combination of the vector fields $\tilde{b}_j$ $(\ell(j)=p)$. By construction, it should have been added to the family $(\tilde{b}_j)$.
Property 1-c is a direct consequence of the definition of the family $(\tilde{b}_j)$.
There only remains to prove Property 1-a.
Let us denote $\tilde{b}$ the matrix-valued function defined by:

$$\tilde{b}(x) \;=\; (\tilde{b}_1,\dots,\tilde{b}_{m'_1},\tilde{b}_{m_1+1},\dots,\tilde{b}_{m'_2},\dots,\tilde{b}_{m_{P-1}+1},\dots,\tilde{b}_{m'_P})(x)$$

We first show that $\text{Rank}\,\tilde{b}(0)=n$. In order to show that, we first remark that the Lie algebra generated by the b_i $(i=1,\dots,m)$ is nilpotent in the sense that any Lie bracket of length $p>P$ of this Lie algebra is identically zero. Indeed, since the vector fields b_i are homogeneous of degree -1 , any Lie bracket b of length p made with the b_i is homogeneous of degree $-p$ and each kth component b^k of b is a homogeneous function of degree $-p+r_k$. Since $P=\ \text{Max}\,\{r_k;\ k=1,\dots,n\}$, $-p+r_k$ is strictly negative if $p>P$ so that, if b^k where not identically zero, it would tend to infinity at the origin, which would contradict the fact that the vector fields are smooth. Using this property and the Lie Algebra Rank Condition (6), the construction of the family $(\tilde{b}_j)$ implies that $\text{Rank}\,\tilde{b}(0)=n$.
Let us now show that $Rank\,\tilde{b}(x)=n$ for all $x\in I\!R^n$.
Since $Rank\,\tilde{b}(0)=n$, there exist a neighborhood W of $x=0$ on which $Rank\,\tilde{b}(x)=n$. There remains to prove that this property holds everywhere. Let x be outside W. There exist $\lambda>0$ such that $\bar{x}=\delta_\lambda(x)$ is in W and hence, $Rank\,\tilde{b}(\bar{x}) \;=\; n$. As a consequence, since δ_λ is a local diffeomorphism from a neighborhood of x to a neighborhood of $\bar{x}$,

$$Rank\,\{\left((\delta_\lambda^{-1})_*\tilde{b}_{j_1}\right)(x),\dots,\left((\delta_\lambda^{-1})_*\tilde{b}_{j_n}\right)(x)\} \;=\; n\ .$$

Now, from the homogeneity, $(\delta_\lambda^{-1})_*\tilde{b}_{j_k}=\lambda^{-\ell(j_k)}\tilde{b}_{j_k}$, which implies that $Rank\,\tilde{b}(x)=n$.
Finally, since the column vectors of $\tilde{b}(0)$ are independent, $\tilde{b}$ is necessarily a square $n\times n$ matrix and the family associated to the matrix $\tilde{b}(x)$ is both free and generating.
∎

7.2 Proofs related to Step 3

Proof of Lemma 1. This is a direct consequence of the results by Liu and Sussmann. More precisely, it is shown in [9, section 7] that for any $q \in \{m_{p-1}+1,\ldots,m_p\}$, the set $\{I_q(\Omega);\ \Omega = \{\omega^1,\ldots,\omega^p\}\,MC\}$ is a generating family of the set of Lie brackets of length p made with the vector fields $b_{\tau_q^1},\ldots,b_{\tau_q^p}$. This is proved in the *free* Lie algebra generated by $b_{\tau_q^1},\ldots,b_{\tau_q^p}$, but is a fortiori true in the Lie algebra of the system. As a consequence, for each $j \in \{m_{p-1}+1,\ldots,m_p\}$, the vector field $\tilde{b}_j$ can be expressed as a linear combination of a finite number of vectors $I_j(\Omega)$, with Ω MC. Since, in view of Step 1 (Property 1-b), each of these $I_j(\Omega)$ can in turn be expressed as a linear combination of the $\tilde{b}_l$ $(l = m_{p-1}+1,\ldots,m_p)$, there can be at most $m_p - m_{p-1}$ $I_j(\Omega)$ independent. As a consequence, at most $m_p - m_{p-1}$ $I_j(\Omega)$ — which will be our choice for $\Omega_{j,m_{p-1}+1},\ldots,\Omega_{j,m_p}$— are needed to express $\tilde{b}_j$. Hence with these values of the $\Omega_{j,k}$, and using the fact that the $\tilde{b}_j$ are independent (Property 1-b), the matrix M has full rank $m_p - m_{p-1}$. By extracting $m_p - m_{p-1}$ independent lines from the matrix M, the existence of exactly $m_p - m_{p-1}$ integers $\gamma_{m_{p-1}+1},\ldots,\gamma_{m_p} \in \{m_{p-1}+1,\ldots,m_p\}$ and $m_p - m_{p-1}$ MC sets $\Omega_{m_{p-1}+1},\ldots,\Omega_{m_p}$ such that the matrix S_p is invertible readily follows. ∎

Finally let us prove that part *iv)* can also be achieved. The proof relies on the following properties of MC sets.

Properties

1. Let $p \in \mathbb{N}$, then, if $\Omega = \{\omega^1,\ldots,\omega^p\}$ is minimally canceling, Ω can be identified with a point in the hyperplane: $\{\omega \in \mathbb{R}^p : \omega^1 + \ldots + \omega^p = 0\}$, which itself is diffeomorphic to $\mathbb{R}^{p-1}$. Using the fact that the left-hand side of the implication (10) defines only a finite set of linear equations in the variables $\omega^1,\ldots\omega^p$, the set of all MC sets Ω with $|\Omega| = p$ can be identified to $\mathbb{R}^{p-1} - \{E_1,\ldots,E_R\}$ where each E_r $(r = 1,\ldots,R)$ is a linear subspace (of dimension strictly less than $p-1$) of $\mathbb{R}^{p-1}$. Hence, the set of all MC sets Ω with $|\Omega| = p$ can be identified to an open and dense subset of $\mathbb{R}^{p-1}$.

2. Similarly, let us consider a family $(\Omega_\alpha)_{\alpha\in I} = (\Omega_1,\Omega_2,\ldots,\Omega_Q)$ where each Ω_α is a MC set of cardinality $|\Omega_\alpha| \geq 2$. Then, the family $(\Omega_\alpha)_{\alpha\in I}$ can be identified to a point in $\mathbb{R}^{|\Omega_1|-1} \times \mathbb{R}^{|\Omega_2|-1} \times \ldots \times \mathbb{R}^{|\Omega_Q|-1}$ and, in view of the previous property, the set of families $(\Omega'_\alpha)_{\alpha\in I}$ —where each Ω'_α is MC, and $|\Omega'_\alpha| = |\Omega_\alpha|$— can be identified to an open and dense subset $\mathcal{O}$ of $\mathbb{R}^{|\Omega_1|-1} \times \mathbb{R}^{|\Omega_2|-1} \times \ldots \times \mathbb{R}^{|\Omega_Q|-1}$. Moreover, by an argument similar to the one used in the previous property, the set of families $(\Omega'_\alpha)_{\alpha\in I}$ —with each Ω'_α MC and of cardinality

$|\Omega_\alpha|$— independent with respect to a given integer p, can be identified to $\mathcal{O} - \{D_1, \ldots, D_M\}$ where each D_m $(m = 1, \ldots, M)$ is a linear subspace — of dimension strictly less than $(|\Omega_1| - 1) + (|\Omega_2| - 1) + \ldots + (|\Omega_Q| - 1)$— of $\mathbb{R}^{|\Omega_1|-1} \times \mathbb{R}^{|\Omega_2|-1} \times \ldots \times \mathbb{R}^{|\Omega_Q|-1}$. Hence, this set can be identified to an open and dense subset of $\mathbb{R}^{|\Omega_1|-1} \times \mathbb{R}^{|\Omega_2|-1} \times \ldots \times \mathbb{R}^{|\Omega_Q|-1}$.

In view of *iii)*, each matrix S_p is invertible on an open and dense subset of $\mathbb{R}^{(p-1)(m_p - m_{p-1})} = \mathbb{R}^{|\Omega_{m_{p-1}+1}|-1} \times \ldots \times \mathbb{R}^{|\Omega_{m_p}|-1}$. This directly implies, from Property 2 above, that part *iv)* can indeed be achieved. ∎

7.3 Proofs related to Step 4

We first remark that Equation (18) is a particular case of (22) with $p = 0$. As a consequence, Property 4 will be proved if we can show, for any $p = P, P-1, \ldots, 1$ the following

Facts:

1. The functions $v_j^1, \ldots, v_j^p$ $(j = m_{p-1}+1, \ldots, m_p)$ belong to $C^\infty(\mathbb{R}^n - \{0\}; \mathbb{R}) \cap C^0(\mathbb{R}^n; \mathbb{R})$ and are homogeneous of degree 1.

2. The functions h_j^p $(p > 1,\ j = 1, \ldots, m_{p-1})$, for which (21) and (24) hold, exist, belong to $C^\infty(\mathbb{R}^n - \{0\}; \mathbb{R}) \cap C^0(\mathbb{R}^n; \mathbb{R})$, and are homogeneous of degree $\ell(j)$.

3. $$\begin{aligned} &\sum_{j=m_{p-1}+1}^{N} \sum_{\sigma \in \mathfrak{S}(\ell(j))} \frac{[b_{\nu_j^{\sigma(1)}} v_j^{\sigma(1)}, [b_{\nu_j^{\sigma(2)}} v_j^{\sigma(2)}, [\ldots, b_{\nu_j^{\sigma(\ell(j))}} v_j^{\sigma(\ell(j))}] \ldots]]}{\omega_j^{\sigma(1)}(\omega_j^{\sigma(1)} + \omega_j^{\sigma(2)}) \ldots (\omega_j^{\sigma(1)} + \ldots + \omega_j^{\sigma(\ell(j)-1)})} \\ &= \sum_{j=m_{p-1}+1}^{N} \tilde{u}_j \tilde{b}_j - \sum_{j=1}^{m_{p-1}} h_j^p \tilde{b}_j \end{aligned} \tag{57}$$

Proof:
We proceed by a decreasing induction.

<u>$p = P$</u>:
We first prove Fact 1. It follows from Property 2-a, and from the assumptions on the homogeneous norm ρ, that the functions $v_j^1, \ldots, v_j^P$ defined by (20) are homogeneous of degree 1 and belong to $C^\infty(\mathbb{R}^n - \{0\}; \mathbb{R}) \cap C^0(\mathbb{R}^n; \mathbb{R})$. The continuity at the origin directly follows from the fact that these functions are homogeneous of strictly

positive degree.
We now prove Fact 2. Using Fact 1 and Lemma 2 we deduce the existence of m_{P-1} functions h_j^P ($j = 1, \dots, m_{P-1}$) belonging to $C^\infty(I\!R^n - \{0\}; I\!R) \cap C^0(I\!R^n; I\!R)$ and homogeneous of degree $\ell(j)$ such that:

$$\begin{aligned}
&\sum_{j=m_{P-1}+1}^{m_P} \sum_{\sigma\in\mathfrak{S}(P)} \frac{[b_{\nu_j^{\sigma(1)}} v_j^{\sigma(1)}, [b_{\nu_j^{\sigma(2)}} v_j^{\sigma(2)}, [\dots, b_{\nu_j^{\sigma(P)}} v_j^{\sigma(P)}] \dots]]}{\omega_j^{\sigma(1)}(\omega_j^{\sigma(1)} + \omega_j^{\sigma(2)}) \dots (\omega_j^{\sigma(1)} + \dots + \omega_j^{\sigma(P-1)})} \\
&= \sum_{j=m_{P-1}+1}^{m_P} \sum_{\sigma\in\mathfrak{S}(P)} v_j^{\sigma(1)} \dots v_j^{\sigma(p)} \frac{[b_{\nu_j^{\sigma(1)}}, [b_{\nu_j^{\sigma(2)}}, [\dots, b_{\nu_j^{\sigma(P)}}] \dots]]}{\omega_j^{\sigma(1)}(\omega_j^{\sigma(1)} + \omega_j^{\sigma(2)}) \dots (\omega_j^{\sigma(1)} + \dots + \omega_j^{\sigma(P-1)})} \\
&- \sum_{j=1}^{m_{P-1}} h_j^P \tilde{b}_j
\end{aligned} \tag{58}$$

In view of (20) this implies that

$$\begin{aligned}
&\sum_{j=m_{P-1}+1}^{m_P} \sum_{\sigma\in\mathfrak{S}(P)} \frac{[b_{\nu_j^{\sigma(1)}} v_j^{\sigma(1)}, [b_{\nu_j^{\sigma(2)}} v_j^{\sigma(2)}, [\dots, b_{\nu_j^{\sigma(P)}} v_j^{\sigma(P)}] \dots]]}{\omega_j^{\sigma(1)}(\omega_j^{\sigma(1)} + \omega_j^{\sigma(2)}) \dots (\omega_j^{\sigma(1)} + \dots + \omega_j^{\sigma(P-1)})} \\
&= \sum_{j=m_{P-1}+1}^{m_P} \sum_{r=m_{P-1}+1}^{m_P} n_P^{rj} \, \tilde{u}_r \sum_{\sigma\in\mathfrak{S}(P)} \frac{[b_{\nu_j^{\sigma(1)}}, [b_{\nu_j^{\sigma(2)}}, [\dots, b_{\nu_j^{\sigma(P)}}] \dots]]}{\omega_j^{\sigma(1)}(\omega_j^{\sigma(1)} + \omega_j^{\sigma(2)}) \dots (\omega_j^{\sigma(1)} + \dots + \omega_j^{\sigma(P-1)})} \\
&- \sum_{j=1}^{m_{P-1}} h_j^P \tilde{b}_j \\
&= \sum_{r=m_{P-1}+1}^{m_P} \tilde{u}_r \sum_{j=m_{P-1}+1}^{m_P} n_P^{rj} \sum_{\sigma\in\mathfrak{S}(P)} \frac{[b_{\nu_j^{\sigma(1)}}, [b_{\nu_j^{\sigma(2)}}, [\dots, b_{\nu_j^{\sigma(P)}}] \dots]]}{\omega_j^{\sigma(1)}(\omega_j^{\sigma(1)} + \omega_j^{\sigma(2)}) \dots (\omega_j^{\sigma(1)} + \dots + \omega_j^{\sigma(P-1)})} \\
&- \sum_{j=1}^{m_{P-1}} h_j^P \tilde{b}_j
\end{aligned} \tag{59}$$

By definition of the matrix N_P (let us recall that $N_P = S_P^{-1}$), and in view of (12), (16) and (17) it follows from (59) that:

$$
\begin{aligned}
&\sum_{j=m_{P-1}+1}^{m_P} \sum_{\sigma\in\mathfrak{S}(P)} \frac{[b_{\nu_j^{\sigma(1)}} v_j^{\sigma(1)}, [b_{\nu_j^{\sigma(2)}} v_j^{\sigma(2)}, [\ldots, b_{\nu_j^{\sigma(P)}} v_j^{\sigma(P)}]\ldots]]}{\omega_j^{\sigma(1)}(\omega_j^{\sigma(1)}+\omega_j^{\sigma(2)})\ldots(\omega_j^{\sigma(1)}+\ldots+\omega_j^{\sigma(P-1)})} \\
&= \sum_{r=m_{P-1}+1}^{m_P} \tilde{u}_r \sum_{j=m_{P-1}+1}^{m_P} n_P^{rj} I_{\gamma_j}(\Omega_j) - \sum_{j=1}^{m_{P-1}} h_j^P \tilde{b}_j \\
&= \sum_{r=m_{P-1}+1}^{m_P} \tilde{u}_r \tilde{b}_r - \sum_{j=1}^{m_{P-1}} h_j^P \tilde{b}_j
\end{aligned}
\tag{60}
$$

Therefore, relation (21) is satisfied and this concludes the proof of Fact 2.
Finally, since $m_P = N$, Fact 3 follows from (21).

From step $p+1$ to step p:
We now assume that Facts 1, 2, and 3 are true from P down to $p+1$ and show that they are also true at rank p.
The proof of Fact 1 is similar to the proof given in the case $p = P$ by using the expression (23) of the functions v_j^r and the fact that the functions $\tilde{u}_r$ and h_r^{p+1} ($r = m_{p-1}+1,\ldots,m_p$) are homogeneous of degree p (in view of Property 2.b and the induction hypothesis).
The proof of Fact 2 is also similar to the proof of Fact 2 for the case $p = P$.
Let us finally proceed with the proof of Fact 3. Using the induction hypothesis (Fact 3 for $p+1$), we have:

$$
\begin{aligned}
&\sum_{j=m_p+1}^{N} \sum_{\sigma\in\mathfrak{S}(\ell(j))} \frac{[b_{\nu_j^{\sigma(1)}} v_j^{\sigma(1)}, [b_{\nu_j^{\sigma(2)}} v_j^{\sigma(2)}, [\ldots, b_{\nu_j^{\sigma(\ell(j))}} v_j^{\sigma(\ell(j))}]\ldots]]}{\omega_j^{\sigma(1)}(\omega_j^{\sigma(1)}+\omega_j^{\sigma(2)})\ldots(\omega_j^{\sigma(1)}+\ldots+\omega_j^{\sigma(\ell(j)-1)})} \\
&= \sum_{j=m_p+1}^{N} \tilde{u}_j \tilde{b}_j - \sum_{j=1}^{m_p} h_j^{p+1} \tilde{b}_j
\end{aligned}
\tag{61}
$$

Adding (24) and (61) term to term, relation (57) directly follows. ∎

7.4 Proof of Theorem 1

In order to prove Theorem 1 it is now sufficient, in view of Properties 2-b and 4-c, to show that when the origin is asymptotically stable for the (time-invariant) system

$$\dot{x} = \sum_{j=1}^{N} \sum_{\sigma\in\mathfrak{S}(\ell(j))} \frac{[b_{\nu_j^{\sigma(1)}} v_j^{\sigma(1)}, [b_{\nu_j^{\sigma(2)}} v_j^{\sigma(2)}, [\ldots, b_{\nu_j^{\sigma(\ell(j))}} v_j^{\sigma(\ell(j))}] \ldots]]}{\omega_j^{\sigma(1)}(\omega_j^{\sigma(1)} + \omega_j^{\sigma(2)}) \ldots (\omega_j^{\sigma(1)} + \ldots + \omega_j^{\sigma(\ell(j)-1)})} = \sum_{j=1}^{N} \tilde{u}_j \tilde{b}_j \quad (62)$$

then, for $\varepsilon > 0$ sufficiently small, the origin is uniformly (exponentially) asymptotically stable for the time varying system (29).

Let us therefore assume that the origin is asymptotically stable for (62). Since the right-hand side is homogeneous (of degree zero), there exist, from [17], a *homogeneous* Lyapunov function V, positive definite, and whose derivative along (62) may be written

$$\dot{V}_{(62)} = \sum_{j=1}^{N} \sum_{\sigma\in\mathfrak{S}(\ell(j))} \frac{\Big([b_{\nu_j^{\sigma(1)}} v_j^{\sigma(1)}, [b_{\nu_j^{\sigma(2)}} v_j^{\sigma(2)}, [\ldots, b_{\nu_j^{\sigma(\ell(j))}} v_j^{\sigma(\ell(j))}] \ldots]]\Big)\, V}{\omega_j^{\sigma(1)}(\omega_j^{\sigma(1)} + \omega_j^{\sigma(2)}) \ldots (\omega_j^{\sigma(1)} + \ldots + \omega_j^{\sigma(l(j)-1)})} = -W \quad (63)$$

—here $X\,V$, for X a vector field, denotes the Lie derivative of V along X— with W homogeneous positive definite, of the same degree as V, i.e.

$$W(x) \;\geq\; c\,V(x)\,. \quad (64)$$

Let us now compute the derivative of V along the "real" closed-loop equation (29) :

$$\dot{V}_{(29)} \;=\; F^{\varepsilon}\,V \quad (65)$$

with F^{ε} the closed-loop vector field, in the extended space where time is one coordinate :

$$F^{\varepsilon} \;=\; \frac{\partial}{\partial t} + \sum_{i=1}^{m} u_i^{\varepsilon}\, b_i \quad (66)$$

(of course the term $\frac{\partial}{\partial t}$ does not act on V because V does not depend on t).

Now use Theorem 2, with $\mathcal{T} = \mathbb{R}$, the —obviously not linearly independent— vector fields X_j^s given by :

$$X_j^s \;=\; v_j^s\, b_{\nu_j^s}\;, \quad (67)$$

and the complex valued functions of time η_j^s taken equal to the complex *constants* decided above (for instance chosen according to (26), and in any case in such a way that (25) is satisfied). From (48), (25), and (63), one has

$$F^\varepsilon V \;=\; -W \;+\; \varepsilon^{\gamma_1} F^\varepsilon D_1^\varepsilon V \;-\; \varepsilon^{\gamma_1} D_1^\varepsilon \frac{\partial V}{\partial t} \;+\; \varepsilon^{\gamma_2} D_2^\varepsilon V$$

which may be rewritten, since V does not depend on t,

$$F^\varepsilon V_\varepsilon \;=\; -W \;+\; \varepsilon^{\gamma_2} D_2^\varepsilon V \tag{68}$$

with

$$V_\varepsilon \;=\; V \;-\; \varepsilon^{\gamma_1} D_1^\varepsilon V \;. \tag{69}$$

Since, from Theorem 2, the operators D_1^ε and D_2^ε are homogeneous of degree zero, and locally uniformly bounded with respect to $\varepsilon > 0$, one has, since V is positive definite,

$$|D_1^\varepsilon V| \;\leq\; k\,V \;, \quad |D_2^\varepsilon V| \;\leq\; k\,V$$

for all $\varepsilon > 0$. Hence for ε sufficiently small, V_ε is arbitrarily close to V and hence positive definite, and also

$$\dot{V}_\varepsilon \;=\; F^\varepsilon V_\varepsilon \;\leq\; -\frac{c}{2} V \tag{70}$$

Hence for ε small enough, V_ε is a strict Lyapunov function for the closed-loop system (29). This ends the proof of Theorem 1 via Lyapunov's first method. ∎

7.5 Proof of Theorem 2

The closed-loop vector field F^ε can be re-written as

$$\begin{aligned} F^\varepsilon \;=\; & \frac{\partial}{\partial t} \;+\; \sum_{\substack{1 \leq j \leq N \\ \ell(j) = 1}} 2\eta_j^1 X_j^1 \\ & +\; \sum_{\substack{1 \leq j \leq N \\ \ell(j) \geq 2}} \sum_{s=1}^{\ell(j)} \varepsilon^{-\frac{\ell(j)-1}{\ell(j)}} \left(\eta_j^s e^{i\omega_j^s t/\varepsilon} \;+\; \overline{\eta_j^s} e^{-i\omega_j^s t/\varepsilon} \right) X_j^s \end{aligned} \tag{71}$$

Let us make some conventions and definitions, used only in the present proof. We define the following sets of indices :

$$J \;=\; \{\, j\in\{1,\ldots,N\}\,,\ \ell(j)\geq 2\,\} \;=\; \{m_1+1,\ldots,N\} \tag{72}$$

$$J_l \;=\; \{\, j\in\{1,\ldots,N\}\,,\ \ell(j)=l\,\} \;=\; \{m_{l-1}+1,\ldots,m_l\} \tag{73}$$

$$K_j \;=\; \{\,-\ell(j)\,,\ -\ell(j)-1\,,\ \ldots\,,\ -1\,,\,1\,,\,2\,,\ \ldots\ell(j)\,\}\ . \tag{74}$$

and the following sets of pairs of indices :

$$I \;=\; \{\,(j,s),\ j\in J\,,\ s\in K_j\,\} \;=\; \bigcup_{j\in J}\{j\}\times K_j \tag{75}$$

$$I_l \;=\; \{\,(j,s)\in I\,,\ \ell(j)=l\,\} \;=\; \bigcup_{j\in J_l}\{j\}\times K_j \tag{76}$$

We call F_1 the vector field

$$F_1 \;=\; \sum_{\substack{1\leq j\leq N\\ \ell(j)=1}} 2\eta_j^1 X_j^1 \;=\; \sum_{(j,s)\in I_1} 2\eta_j^s X_j^s\ . \tag{77}$$

Clearly, if we define, for $s<0$, the real numbers ω_j^s, the complex numbers η_j^s and the vector fields X_j^s by :

$$\left.\begin{array}{rcl} \omega_j^{-s} &=& -\,\omega_j^s\\ \eta_j^{-s} &=& \overline{\eta_j^s}\\ X_j^{-s} &=& X_j^s \end{array}\right\}\ \text{for } j\in J\,,\ s\in K_j\,,\ s>0\ , \tag{78}$$

the vector field F^ε from (71) may be rewritten as

$$F^\varepsilon \;=\; \frac{\partial}{\partial t} + F_1 + \sum_{(j,s)\in I} \varepsilon^{-\frac{\ell(j)-1}{\ell(j)}}\,\eta_j^s e^{i\omega_j^s t/\varepsilon} X_j^s \tag{79}$$

$$\;=\; \frac{\partial}{\partial t} + F_1 + \varepsilon^{-\frac{1}{2}}F_2^\varepsilon + \varepsilon^{-\frac{2}{3}}F_3^\varepsilon + \cdots + \varepsilon^{-\frac{P-1}{P}}F_P^\varepsilon \tag{80}$$

where

$$F_l^\varepsilon \;=\; \sum_{(j,s)\in I_l} \eta_j^s e^{i\omega_j^s t/\varepsilon} X_j^s\ . \tag{81}$$

Note that the interest of the last sum is that the negative powers of ε are written apart, and the vector fields F_j^ε have the “boundedness” property that their coefficients are continuous functions of x and t, smooth outside $x = 0$, indexed by $\varepsilon > 0$, and locally uniformly bounded with respect to $\varepsilon > 0$ (it is not the case of F^ε itself because of the negative powers of ε). We shall in the remainder of the proof always write the negative powers of ε apart, so that all the differential operators written as capital letters never contain coefficients that are unbounded when ε goes to zero.

We now define a certain number of differential operators $F^\varepsilon_{p_1,p_2,\ldots,p_d}$ of order d for d between 1 and P, and for all d-uple $(p_1, p_2, \ldots, p_d)$ of integers such that :

$$\left.\begin{array}{l} 1 \leq p_k \leq P \quad \text{for } 1 \leq k \leq d \ , \\ \frac{1}{p_1} + \cdots + \frac{1}{p_{d-1}} \leq 1 \ , \\ \qquad (p_1, p_2) \neq (2, 2) \ , \\ \quad (p_1, p_2, p_3) \neq (3, 3, 3) \ , \\ \qquad\qquad \vdots \\ (p_1, \ldots, p_{d-1}) \neq (d-1, \ldots, d-1) \ . \end{array}\right\} \tag{82}$$

$F^\varepsilon_{p_1,p_2,\ldots,p_d}$ is defined by :

$$F^\varepsilon_{p_1,p_2,\ldots,p_d} = \sum_{(\,(j_1,s_1)\,,\ldots,\,(j_d,s_d)\,)\,\in\, I^d(p_1,\ldots,p_d)} \frac{\eta_{j_1}^{s_1}\eta_{j_2}^{s_2}\cdots\eta_{j_d}^{s_d} e^{i(\omega_{j_1}^{s_1}+\cdots+\omega_{j_d}^{s_d})\frac{t}{\varepsilon}}\, X_{j_d}^{s_d} X_{j_{d-1}}^{s_{d-1}} \cdots X_{j_1}^{s_1}}{i^{(d-1)}\omega_{j_1}^{s_1}(\omega_{j_1}^{s_1}+\omega_{j_2}^{s_2})\cdots(\omega_{j_1}^{s_1}+\cdots+\omega_{j_{d-1}}^{s_{d-1}})} \tag{83}$$

where $I^d(p_1, \ldots, p_d)$ is the set of d-uples of indices $(\,(j_1, s_1)\,, \ldots\,, (j_d, s_d)\,)$ such that $\ell(j_k) = p_k$, and which are neither a collection of $\frac{d}{2}$ pairs of the form (j, s), $(j, -s)$ nor such that, for some (even) k, $2 \leq k \leq d$, $(\,(j_1, s_1)\,, \ldots\,, (j_k, s_k)\,)$ would be a collection of $\frac{k}{2}$ pairs of the form (j, s), $(j, -s)$. More precisely, $I^d(p_1, \ldots, p_d)$ may be defined recursively by $I^1(p) = I_1$ and :

$$((j_1, s_1), \ldots, (j_d, s_d)) \in I^d(p_1, \ldots, p_d)$$
$$\Leftrightarrow \left\{\begin{array}{l} \bullet\ (j_k, s_k) \in I_{p_k} \text{ for all } k, \\ \bullet\ ((j_1, s_1), \ldots, (j_{d-1}, s_{d-1})) \in I^{d-1}(p_1, \ldots, p_{d-1}), \\ \bullet \text{ there exist no permutation } \tau \in \mathfrak{S}(d) \\ \quad \text{such that } (j_{\tau(k)}, s_{\tau(k)}) = (j_k, -s_k) \ . \end{array}\right. \tag{84}$$

With the above definition of the sets of indices $I^d(p_1, \ldots, p_d)$, the denominators in (83) cannot be zero because of the following lemma.

Lemma 3 *Let* $((j_1, s_1), \ldots, (j_d, s_d)) \in I^d$ *—see the definition of* I *in (75)— be such that*
$\omega_{j_1}^{s_1} + \cdots + \omega_{j_d}^{s_d} = 0$ *. Then :*

- *either* $(\ell(j_1),\ldots,\ell(j_d)) = (d,\ldots,d)$ *,*
- *or* $\dfrac{1}{\ell(j_1)}+\cdots+\dfrac{1}{\ell(j_d)} \;>\; 1$ *,*
- *or there exist a permutation* $\tau\in\mathfrak{S}(d)$ *such that* $(j_{\tau(k)},s_{\tau(k)}) = (j_k,-s_k)$ *for all* k*.*

Proof of lemma 3 : The equality $\omega_{j_1}^{s_1}+\cdots+\omega_{j_d}^{s_d}=0$ may be rewritten

$$\sum_{\substack{j\in\{1,\ldots,N\}\\ \ell(j)\geq 2}} \sum_{s=1}^{\ell(j)} \lambda_j^s\omega_j^s \;=\; 0 \tag{85}$$

where the integer λ_j^s is equal to the number of times that (j,s) appears in $((j_1,s_1),\ldots,(j_d,s_d))$ minus the number of times $(j,-s)$ appears. Of course, (85) may be rewritten as

$$\sum_{j\in J}\sum_{\omega\in\Omega_j}\lambda_\omega\omega = 0$$

with $\lambda_{\omega_j^s}=\lambda_j^s$. Note that

$$\sum_\omega |\lambda_w| \;=\; \sum_{j,s}|\lambda_j^s| \;\leq\; d \;\leq\; P\;.$$

Hence, from the assumption that the sequences of frequencies are mutually "independent with respect to P", and are all "minimally canceling" (see (11)-(10)), each $(\lambda_j^1,\ldots,\lambda_j^{\ell(j)})$ is equal to either $(0,\ldots,0)$ or $(1,\ldots,1)$ or $(-1,\ldots,-1)$. If it is not $(0,\ldots,0)$ for at least one j, then all the couples $(j,1),\ldots,(j,\ell(j))$, or all the couples $(j,-1),\ldots,(j,-\ell(j))$, appear in $((j_1,s_1),\ldots,(j_d,s_d))$. If $d=\ell(j)$ for this j, i.e. if $((j_1,s_1),\ldots,(j_d,s_d))$ is a re-ordering of $((j,1),\ldots,(j,\ell(j)))$, or of $((j,-1),\ldots,(j,-\ell(j)))$, then we are in the first case of the lemma ; if $d>\ell(j)$, then there is at least another couple (j',s') in $((j_1,s_1),\ldots,(j_d,s_d))$ and hence the sum $\frac{1}{\ell(j_1)}+\cdots+\frac{1}{\ell(j_d)}$ can be no less than $1+\frac{1}{\ell(j')}$ and hence we are in the second case of the lemma. Let us now examine the case where all the $(\lambda_j^1,\ldots,\lambda_j^{\ell(s)})$'s are equal to $(0,\ldots,0)$. This means that for all j,s, the couple (j,s) and the couple $(j,-s)$ appears the same numbers of time in $((j_1,s_1),\ldots,(j_d,s_d))$. This allows to build the permutation having the property required in the third point of the lemma : it is the one that exchanges 1 with the first k_1 such that $(j_{k_1}, s_{k_1}) = (j_1, -s_1)$, 2 (3 if $k_1 = 2$) with the first $k_2 \neq k_1$ such that $(j_{k_2}, s_{k_2}) = (j_2, -s_2)$, and so on. ∎

We shall now prove the following two facts.

Fact 1 : For all q, $1 \leq q \leq P$, there exist $\gamma_{1,q}$ and $\gamma_{2,q}$ strictly positive such that:

$$\begin{aligned} F^\varepsilon \;=\; & \frac{\partial}{\partial t} + F_1 + \sum_{p=2}^{q} (-1)^{p-1} F^\varepsilon_{\underbrace{p,p,\ldots,p}_{p \text{ times}}} \\ & + \varepsilon^{\gamma_{1,q}} \left(F^\varepsilon D^\varepsilon_{1,q} - D^\varepsilon_1 \frac{\partial}{\partial t} \right) + \varepsilon^{\gamma_{2,q}} D^\varepsilon_{2,q} \\ & + \sum_{\substack{(p_1,\ldots,p_q) \in \{2,\ldots,P\}^q, \\ \frac{1}{p_1} + \cdots + \frac{1}{p_q} \leq 1 \\ (p_1,\ldots,p_q) \neq (q,\ldots,q)}} (-1)^{q-1} \varepsilon^{-\left(1 - \frac{1}{p_1} - \cdots - \frac{1}{p_q}\right)} F^\varepsilon_{p_1,\ldots,p_q} \end{aligned} \tag{86}$$

Fact 2 : For all p, $1 \leq p \leq P$, there exist $\gamma'_{1,p}$ and $\gamma'_{2,p}$ strictly positive such that:

$$\begin{aligned} F^\varepsilon_{\underbrace{p,p,\ldots,p}_{p \text{ times}}} \;=\; & 2 \frac{(-1)^{p-1}}{p} \sum_{j \in J_p} \Re \left(\frac{\eta^1_j \cdots \eta^p_j}{i^{(p-1)}} \right) B_j \\ & + \varepsilon^{\gamma'_{1,p}} \left(F^\varepsilon D'^\varepsilon_{1,p} - D'^\varepsilon_{1,p} \frac{\partial}{\partial t} \right) + \varepsilon^{\gamma'_{2,p}} D'^\varepsilon_{2,p} \end{aligned} \tag{87}$$

with

$$B_j = \sum_{\sigma \in \mathfrak{S}(\ell(j))} \frac{[X^{\sigma(1)}_j, [X^{\sigma(2)}_j, [\ldots, X^{\sigma(\ell(j))}_j] \ldots]]}{\omega^{\sigma(1)}_j (\omega^{\sigma(1)}_j + \omega^{\sigma(2)}_j) \ldots (\omega^{\sigma(1)}_j + \ldots + \omega^{\sigma(\ell(j)-1)}_j)} \tag{88}$$

These two facts imply Theorem 2. Indeed, for $q = P$, the last sum in (86) is empty since $\frac{1}{p_1} + \cdots + \frac{1}{p_P} \leq 1$ with all the integers p_j no larger that P implies $(p_1, \ldots, p_P) = (P, \ldots, P)$. Hence for $q = P$, (86) reads

$$\begin{aligned} F^\varepsilon \;=\; & \frac{\partial}{\partial t} + F_1 + \sum_{p=2}^{P} (-1)^{p-1} F^\varepsilon_{\underbrace{p,p,\ldots,p}_{p \text{ times}}} \\ & + \varepsilon^{\gamma_{1,P}} \left(F^\varepsilon D^\varepsilon_{1,P} - D^\varepsilon_1 \frac{\partial}{\partial t} \right) + \varepsilon^{\gamma_{2,P}} D^\varepsilon_{2,P} \,. \end{aligned} \tag{89}$$

Substituting in the above the expression of $F^\varepsilon_{p,\dots,p}$ given by (87), one clearly gets (48) with the appropriate differential operators D^ε_1 and D^ε_2 and the appropriate positive real numbers γ_1 and γ_2.

Proof of fact 1. We prove (86) by induction on q, from $q=1$ to $q=P$.

For $q=1$, the sum on the first line of (86) is empty, one may take $D^\varepsilon_{1,1}, D^\varepsilon_1$ and $D^\varepsilon_{2,1}$ to be zero, and (86) is simply (80).

Let us now suppose that (86) holds for a certain $q \geq 1$ and let us prove it for $q+1$. This is done through a manipulation on differential operator that more or less mimics an integration by parts. Since we shall use it elsewhere, let us explain it on a "general" differential operator Y before applying it.

Consider a differential operator of order d on functions of t and x that does not contain derivations with respect to t :

$$Y \;=\; \sum_{\text{multi-indices } I \text{ of length } d} \eta_I(t) a_I(t,x) \frac{\partial^{|I|}}{\partial x_I} \;. \tag{90}$$

Define $Y^{[1]}$ and $Y^{[-1]}$ to be

$$Y^{[-1]} \;=\; \sum_{\text{multi-indices } I \text{ of length } d} \eta_I(t) \left(\int_*^t a_I(\tau,x)\mathrm{d}\tau \right) \frac{\partial^{|I|}}{\partial x_I} \tag{91}$$

$$Y^{[1]} \;=\; \sum_{\text{multi-indices } I \text{ of length } d} \frac{\mathrm{d}\eta_I}{\mathrm{d}t}(t) \left(\int_*^t a_I(\tau,x)\mathrm{d}\tau \right) \frac{\partial^{|I|}}{\partial x_I} \tag{92}$$

Note that these are defined up to a function of x (through the initial time in the integrals) and that $Y^{[1]}$ is zero if the η's are constants. The "derivative with respect to t" of $Y^{[-1]}$ is $Y+Y^{[1]}$ in the following sense :

$$Y + Y^{[1]} \;=\; [\,\frac{\partial}{\partial t}\,,\,Y^{[-1]}\,] \;=\; \frac{\partial}{\partial t} Y^{[-1]} \,-\, Y^{[-1]} \frac{\partial}{\partial t} \;. \tag{93}$$

indeed it is obvious that for any smooth function h of x and t, one has

$$Y.h\,(t,x) \;+\; Y^{[1]}.h\,(t,x) \;=\; \frac{\partial}{\partial t} \left(Y^{[-1]}.h\,(t,x) \right) \,-\, Y^{[-1]}.\frac{\partial h}{\partial t}\,(t,x) \;, \tag{94}$$

simply because $\frac{\partial}{\partial t}$ commutes with $\frac{\partial}{\partial^{|I|} x_I}$. Then we re-write (93) in the following way :

$$\begin{aligned} Y \;&=\; [\,\frac{\partial}{\partial t}\,,\,Y^{[-1]}\,] \,-\, Y^{[1]} \\ &=\; F^\varepsilon\, Y^{[-1]} \,-\, \left(\sum_{r=1}^{P} \varepsilon^{-\frac{r-1}{r}} F^\varepsilon_r \right) Y^{[-1]} \,-\, Y^{[-1]} \frac{\partial}{\partial t} \,-\, Y^{[1]} \;. \end{aligned} \tag{95}$$

In order to prove that if (86) holds for q, it also holds for $q+1$, we apply the identity (95) with

$$\begin{array}{rcl} Y & = & F^{\varepsilon}_{p_1,\dots,p_q} \\ Y^{[-1]} & = & \varepsilon\, G^{\varepsilon}_{p_1,\dots,p_q}\ , \\ Y^{[1]} & = & \varepsilon\, H^{\varepsilon}_{p_1,\dots,p_q}\ , \end{array}$$

for

$$(p_1,\dots,p_q) \neq (q,\dots,q) \quad \text{and} \quad \frac{1}{p_1}+\cdots+\frac{1}{p_q} \leq 1\ , \tag{96}$$

where $G^{\varepsilon}_{p_1,\dots,p_q}$ and $H^{\varepsilon}_{p_1,\dots,p_q}$ are given by :

$$G^{\varepsilon}_{p_1,\dots,p_q} \;=\; \sum_{((j_1,s_1),\dots,(j_q,s_q))\,\in\, I^q(p_1,\dots,p_q)} \frac{\eta^{s_1}_{j_1}\cdots\eta^{s_q}_{j_q} e^{i(\omega^{s_1}_{j_1}+\cdots+\omega^{s_q}_{j_q})t/\varepsilon}\, X^{s_q}_{j_q} X^{s_{q-1}}_{j_{q-1}}\dots X^{s_1}_{j_1}}{i^q\omega^{s_1}_{j_1}(\omega^{s_1}_{j_1}+\omega^{s_2}_{j_2})\dots(\omega^{s_1}_{j_1}+\cdots+\omega^{s_q}_{j_q})} \tag{97}$$

$$H^{\varepsilon}_{p_1,\dots,p_q} \;=\; \sum_{((j_1,s_1),\dots,(j_q,s_q))\in I^q(p_1,\dots,p_q)} \frac{\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\eta^{s_1}_{j_1}\cdots\eta^{s_q}_{j_q}\right)\right) e^{i(\omega^{s_1}_{j_1}+\cdots+\omega^{s_q}_{j_q})t/\varepsilon}\, X^{s_q}_{j_q} X^{s_{q-1}}_{j_{q-1}}\dots X^{s_1}_{j_1}}{i^q\omega^{s_1}_{j_1}(\omega^{s_1}_{j_1}+\omega^{s_2}_{j_2})\dots(\omega^{s_1}_{j_1}+\cdots+\omega^{s_q}_{j_q})} \tag{98}$$

Note that the denominators are nonzero because, from lemma 3, the definition (84) of the set of indices $I^q(p_1,\dots,p_q)$ precisely removes the terms where the denominators would be zero.

Then (95) with the above expressions for Y, $Y^{[1]}$ and $Y^{[-1]}$ yields :

$$F^{\varepsilon}_{p_1,\dots,p_q} \;=\; -\sum_{r=1}^{P} \varepsilon^{\frac{1}{r}} F^{\varepsilon}_r G^{\varepsilon}_{p_1,\dots,p_q} + \varepsilon\, F^{\varepsilon} G^{\varepsilon}_{p_1,\dots,p_q} - \varepsilon\, G^{\varepsilon}_{p_1,\dots,p_q}\frac{\partial}{\partial t} - \varepsilon\, H^{\varepsilon}_{p_1,\dots,p_q}\ . \tag{99}$$

From (97) and (81) we have :

$$F^{\varepsilon}_r G^{\varepsilon}_{p_1,\dots,p_q} =$$

$$\sum_{\substack{((j_1,s_1),\dots,(j_q,s_q))\,\in\, I^q(p_1,\dots,p_q) \\ (j_{q+1},s_{q+1})\in I_r}} \frac{\eta^{s_1}_{j_1}\cdots\eta^{s_{q+1}}_{j_{q+1}} e^{i(\omega^{s_1}_{j_1}+\cdots+\omega^{s_{q+1}}_{j_{q+1}})t/\varepsilon}\, X^{s_{q+1}}_{j_{q+1}} X^{s_q}_{j_q}\dots X^{s_1}_{j_1}}{i^q\omega^{s_1}_{j_1}(\omega^{s_1}_{j_1}+\omega^{s_2}_{j_2})\dots(\omega^{s_1}_{j_1}+\cdots+\omega^{s_q}_{j_q})}$$

The right-hand side of the above equation is equal to $F^{\varepsilon}_{p_1,\dots,p_q,r}$ given by (83) because the $(q+1)$-tuples $((j_1,s_1),\dots,(j_{q+1},s_{q+1}))$ which are in $I^q(p_1,\dots,p_q)\times I_r$ but not in $I^{q+1}(p_1,\dots,p_q,r)$ are —compare (84)— these such that —this is possible only if q is odd— there exist a permutation τ of the set of integers $\{1,\dots,q+1\}$ for which

$$((j_{\tau(1)},s_{\tau(1)}),(j_{\tau(2)},s_{\tau(2)}),\dots,(j_{\tau(q+1)},s_{\tau(q+1)})) = ((j_1,-s_1),(j_2,-s_2),\dots,(j_{q+1},-s_{q+1}))\ . \tag{100}$$

But these terms sum to zero in sum above which is equal to $F_r^\varepsilon G_{p_1,\dots,p_q}$ because, for $((j_1,s_1),$
$(j_2,s_2),\dots,(j_{q+1},s_{q+1}))$ such that a permutation τ satisfying (100) exists, the term corresponding to $((j_1,-s_1),(j_2,-s_2),\dots,(j_{q+1},-s_{q+1}))$ is opposite to the term corresponding to $((j_1,s_1),(j_2,s_2),\dots,(j_{q+1},s_{q+1}))$. Indeed, (78) —for X and ω, not for η— and (100) imply that the term corresponding to $((j_1,-s_1),(j_2,-s_2),\dots,(j_{q+1},-s_{q+1}))$ is equal to :

$$\frac{\eta_{j_{\tau(1)}}^{s_{\tau(1)}}\cdots\eta_{j_{\tau(q+1)}}^{s_{\tau(q+1)}}e^{i(\omega_{j_{\tau(1)}}^{s_{\tau(1)}}+\cdots+\omega_{j_{\tau(q+1)}}^{s_{\tau(q+1)}})t/\varepsilon}\,X_{j_{q+1}}^{s_{q+1}}X_{j_q}^{s_q}\dots X_{j_1}^{s_1}}{i^q(-\omega_{j_1}^{s_1})(-\omega_{j_1}^{s_1}-\omega_{j_2}^{s_2})\dots(-\omega_{j_1}^{s_1}-\cdots-\omega_{j_q}^{s_q})}$$

which, since q must be odd (if not, there is no such terms anyway), is equal to

$$-\frac{\left(\prod_{k=1}^{q+1}\eta_{j_{\tau(k)}}^{s_{\tau(k)}}\right)e^{i(\omega_{j_1}^{s_1}+\cdots+\omega_{j_{q+1}}^{s_{q+1}})t/\varepsilon}\,X_{j_{q+1}}^{s_{q+1}}X_{j_q}^{s_q}\dots X_{j_1}^{s_1}}{i^q\omega_{j_1}^{s_1}(\omega_{j_1}^{s_1}+\omega_{j_2}^{s_2})\dots(\omega_{j_1}^{s_1}+\cdots+\omega_{j_q}^{s_q})}$$

and τ gives the change of index in the product allows to say that this is the opposite of the term corresponding to $((j_1,s_1),(j_2,s_2),\dots,(j_{q+1},s_{q+1}))$. Hence $F_r^\varepsilon G_{p_1,\dots,p_q}^\varepsilon = F_{p_1,\dots,p_q,r}^\varepsilon$. Substituting this in (99) yields (we rename r as p_{q+1}) :

$$F_{p_1,\dots,p_q}^\varepsilon \;=\; -\sum_{p_{q+1}=1}^{P}\varepsilon^{\frac{1}{p_{q+1}}}F_{p_1,\dots,p_q,p_{q+1}}^\varepsilon+\varepsilon\,F^\varepsilon G_{p_1,\dots,p_q}^\varepsilon-\varepsilon\,G_{p_1,\dots,p_q}^\varepsilon\frac{\partial}{\partial t}-\varepsilon\,H_{p_1,\dots,p_q}^\varepsilon\,. \tag{101}$$

Hence (86) yields :

$$\begin{aligned}F^\varepsilon \;=\;& \frac{\partial}{\partial t}+F_1+\sum_{p=2}^{q}(-1)^{p-1}F^\varepsilon_{\underbrace{p,p,\dots,p}_{p\text{ times}}}\\ &+\varepsilon^{\gamma_{1,q}}\left(F^\varepsilon D_{1,q}^\varepsilon-D_1^\varepsilon\frac{\partial}{\partial t}\right)+\varepsilon^{\gamma_{2,q}}D_{2,q}^\varepsilon\\ +(-1)^q&\sum_{\substack{(p_1,\dots,p_q)\in\{2,\dots,P\}^q\\ \frac{1}{p_1}+\cdots+\frac{1}{p_q}\le 1\\ (p_1,\dots,p_q)\ne(q,\dots,q)\\ p_{q+1}\in\{1,\dots,P\}}}\varepsilon^{-\left(1-\frac{1}{p_1}-\cdots-\frac{1}{p_{q+1}}\right)}F_{p_1,\dots,p_q,p_{q+1}}^\varepsilon\end{aligned}$$

$$+ (-1)^{q-1} \sum_{\substack{(p_1,\ldots,p_q) \in \{2,\ldots,P\}^q, \\ \frac{1}{p_1}+\cdots+\frac{1}{p_q} \leq 1 \\ (p_1,\ldots,p_q) \neq (q,\ldots,q)}} \varepsilon^{\frac{1}{p_1}+\cdots+\frac{1}{p_q}} \left(F^\varepsilon G^\varepsilon_{p_1,\ldots,p_q} - G^\varepsilon_{p_1,\ldots,p_q} \frac{\partial}{\partial t} - H^\varepsilon_{p_1,\ldots,p_q} \right) \tag{102}$$

This yields (86) for $q+1$ because the term corresponding to $(p_1,\ldots,p_{q+1}) = (q+1,\ldots,q+1)$ in the sum on the third line is $(-1)^q F^\varepsilon_{q+1,\ldots,q+1}$, it adds to the sum on the first lie and this yields the first line of (86) for $q+1$, the other terms in this sum such that $\frac{1}{p_1}+\cdots+\frac{1}{p_{q+1}} \leq 1$ yield exactly the third line of (86) for $q+1$, and the terms in this sum such that $\frac{1}{p_1}+\cdots+\frac{1}{p_{q+1}} > 1$, as well as all the last sum add up with the second line to give the second line (the "small" terms) of (86) for $q+1$. This ends the proof by induction of fact 1.

Proof of fact 2. From the definition (83), we have :

$$F^\varepsilon_{\underbrace{p,\ldots,p}_{p \text{ times}}} =$$

$$\sum_{\substack{((j_1,s_1),\ldots,(j_p,s_p)) \in I^p(p,\ldots,p) \\ \omega^{s_1}_{j_1}+\cdots+\omega^{s_p}_{j_p} = 0}} \frac{\eta^{s_1}_{j_1}\eta^{s_2}_{j_2}\cdots\eta^{s_p}_{j_p} X^{s_p}_{j_p} X^{s_{p-1}}_{j_{p-1}} \ldots X^{s_1}_{j_1}}{i^{(p-1)}\omega^{s_1}_{j_1}(\omega^{s_1}_{j_1}+\omega^{s_2}_{j_2})\ldots(\omega^{s_1}_{j_1}+\cdots+\omega^{s_{p-1}}_{j_{p-1}})}$$

$$+ \sum_{\substack{((j_1,s_1),\ldots,(j_p,s_p)) \in I^p(p,\ldots,p) \\ \omega^{s_1}_{j_1}+\cdots+\omega^{s_p}_{j_p} \neq 0}} \frac{\eta^{s_1}_{j_1}\eta^{s_2}_{j_2}\cdots\eta^{s_p}_{j_p} e^{i(\omega^{s_1}_{j_1}+\cdots+\omega^{s_p}_{j_p})t/\varepsilon} X^{s_p}_{j_p} X^{s_{p-1}}_{j_{p-1}} \ldots X^{s_1}_{j_1}}{i^{(p-1)}\omega^{s_1}_{j_1}(\omega^{s_1}_{j_1}+\omega^{s_2}_{j_2})\ldots(\omega^{s_1}_{j_1}+\cdots+\omega^{s_{p-1}}_{j_{p-1}})} \tag{103}$$

Now, apply (90)-(91)-(92)-(95) with Y equal to the second sum, and therefore

$$Y^{[-1]} = \varepsilon\, G^\varepsilon_{p,\ldots,p} \,, \quad Y^{[1]} = \varepsilon\, H^\varepsilon_{p,\ldots,p} \,,$$

with

$$G^\varepsilon_{\underbrace{p,\ldots,p}_{p \text{ times}}} = \tag{104}$$

$$\sum_{\substack{((j_1,s_1),\ldots,(j_p,s_p)) \in I^p(p,\ldots,p) \\ \omega^{s_1}_{j_1}+\cdots+\omega^{s_p}_{j_p} \neq 0}} \frac{\eta^{s_1}_{j_1}\cdots\eta^{s_p}_{j_p} e^{i(\omega^{s_1}_{j_1}+\cdots+\omega^{s_p}_{j_p})t/\varepsilon} X^{s_p}_{j_p} X^{s_{p-1}}_{j_{p-1}} \ldots X^{s_1}_{j_1}}{i^p\omega^{s_1}_{j_1}(\omega^{s_1}_{j_1}+\omega^{s_2}_{j_2})\ldots(\omega^{s_1}_{j_1}+\cdots+\omega^{s_p}_{j_p})}$$

$$H^{\varepsilon}_{\underbrace{p,\ldots,p}_{p \text{ times}}} = \tag{105}$$

$$\sum_{\substack{((j_1,s_1),\ldots,(j_p,s_p)) \in I^p(p\ldots p) \\ \omega^{s_1}_{j_1}+\cdots+\omega^{s_p}_{j_p} \neq 0}} \frac{\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\eta^{s_1}_{j_1}\cdots\eta^{s_p}_{j_p}\right)\right) e^{i(\omega^{s_1}_{j_1}+\cdots+\omega^{s_p}_{j_p})t/\varepsilon} X^{s_p}_{j_p} X^{s_{p-1}}_{j_{p-1}} \ldots X^{s_1}_{j_1}}{i^p \omega^{s_1}_{j_1}(\omega^{s_1}_{j_1}+\omega^{s_2}_{j_2})\ldots(\omega^{s_1}_{j_1}+\cdots+\omega^{s_p}_{j_p})}$$

This allows to rewrite the second sum in (103) as

$$\varepsilon\, F^{\varepsilon}\, G^{\varepsilon}_{p,\ldots,p} \; - \; \varepsilon\, G^{\varepsilon}_{p,\ldots,p}\,\frac{\partial}{\partial t} \; - \; \varepsilon\, H^{\varepsilon}_{p,\ldots,p} \; - \; \sum_{r=1}^{P} \varepsilon^{\frac{1}{r}} F^{\varepsilon}_{p,\ldots,p,r}$$

with

$$F^{\varepsilon}_{\underbrace{p,\ldots,p}_{p \text{ times}},r} = \tag{106}$$

$$\sum_{\substack{((j_1,s_1),\ldots,(j_p,s_p)) \;\in\; I^p(p,\ldots,p) \\ \omega^{s_1}_{j_1}+\cdots+\omega^{s_p}_{j_p} \neq 0 \\ (j_{p+1},s_{p+1}) \in I_r}} \frac{\eta^{s_1}_{j_1}\cdots\eta^{s_{p+1}}_{j_{p+1}} e^{i(\omega^{s_1}_{j_1}+\cdots+\omega^{s_{p+1}}_{j_{p+1}})t/\varepsilon} X^{s_{p+1}}_{j_{p+1}} X^{s_p}_{j_p} \ldots X^{s_1}_{j_1}}{i^p \omega^{s_1}_{j_1}(\omega^{s_1}_{j_1}+\omega^{s_2}_{j_2})\ldots(\omega^{s_1}_{j_1}+\cdots+\omega^{s_p}_{j_p})}$$

Let us now consider the *first* sum in (103). Since all sequences are minimally canceling, and independent with respect to p, and the sequences in $I^p(p,\ldots,p)$ never contain a couple with the same j and an opposite s (see (84)), the p-uples $((j_1,s_1),\ldots,(j_p,s_p))$ such that $\omega^{s_1}_{j_1}+\cdots+\omega^{s_p}_{j_p}=0$ are exactly of the form $((j,\sigma(1)),\ldots,(j,\sigma(p)))$ or $((j,-\sigma(1)),\ldots,(j,-\sigma(p)))$ with $\ell(j)=p$ and σ a permutation of $\{1,\ldots,p\}$. Hence the first sum may be rewritten (recall that $X^{-s}_j = X^s_j$) as :

$$2\sum_{j\in J_p} \Re\left(\frac{\eta^1_j\cdots\eta^p_j}{i^{p-1}}\right) C_j$$

with

$$C_j \;=\; \sum_{\sigma\in\mathfrak{S}(p)} \frac{X^{\sigma(p)}_j X^{\sigma(p-1)}_j \ldots X^{\sigma(1)}_j}{\omega^{\sigma(1)}_j(\omega^{\sigma(1)}_j+\omega^{\sigma(2)}_j)\cdots(\omega^{\sigma(1)}_j+\cdots+\omega^{\sigma(p-1)}_j)} \tag{107}$$

If one replaces in the above sum σ by $\sigma \circ \tau$ where τ is the permutation that sends $(1, 2, \ldots, p)$ on $(p, p-1, \ldots, 1)$ (change of indices in the summation), one gets

$$C_j = \sum_{\sigma \in \mathfrak{S}(p)} \frac{X_j^{\sigma(1)} X_j^{\sigma(2)} \ldots X_j^{\sigma(p)}}{(\omega_j^{\sigma(p)} + \cdots + \omega_j^{\sigma(2)})(\omega_j^{\sigma(p)} + \cdots + \omega_j^{\sigma(3)}) \cdots (\omega_j^{\sigma(p)} + \omega_j^{\sigma(p-1)})\omega_j^{\sigma(p)}}$$

since $\omega_j^1 + \cdots + \omega_j^p = 0$, the denominator may be transformed :

$$C_j = (-1)^{p-1} \sum_{\sigma \in \mathfrak{S}(p)} \frac{X_j^{\sigma(1)} X_j^{\sigma(2)} \ldots X_j^{\sigma(p)}}{\omega_j^{\sigma(1)}(\omega_j^{\sigma(1)} + \omega_j^{\sigma(2)}) \cdots (\omega_j^{\sigma(1)} + \cdots + \omega_j^{\sigma(p-1)})}$$

Finally, a combinatorial computation in the free Lie algebra (see [8], or [9] in which this identity is also obtained but in a less computational way) gives:

$$\begin{aligned} &\sum_{\sigma \in \mathfrak{S}(p)} \frac{X_j^{\sigma(1)} X_j^{\sigma(2)} \ldots X_j^{\sigma(p)}}{\omega_j^{\sigma(1)}(\omega_j^{\sigma(1)} + \omega_j^{\sigma(2)}) \cdots (\omega_j^{\sigma(1)} + \cdots + \omega_j^{\sigma(p-1)})} \\ &\qquad = \frac{1}{p} \sum_{\sigma \in \mathfrak{S}(p)} \frac{[X_j^{\sigma(1)}, [X_j^{\sigma(2)}, [\cdots, X_j^{\sigma(p)}] \cdots]]}{\omega_j^{\sigma(1)}(\omega_j^{\sigma(1)} + \omega_j^{\sigma(2)}) \cdots (\omega_j^{\sigma(1)} + \cdots + \omega_j^{\sigma(p-1)})} \end{aligned}$$

Hence $C_j = \frac{(-1)^{p-1}}{p} B_j$ with B_j given by (88). Substituting the above in (103) yields

$$\begin{aligned} F^{\varepsilon}_{\underbrace{p, \ldots, p}_{p \text{ times}}} &= \frac{2\,(-1)^{p-1}}{p} \sum_{j \in J_p} \Re\left(\frac{\eta_j^1 \cdots \eta_j^p}{i^{p-1}}\right) B_j \qquad (108) \\ &\quad + \varepsilon\,(F^{\varepsilon}\, G^{\varepsilon}_{\underbrace{p, \ldots, p}_{p \text{ times}}} - G^{\varepsilon}_{\underbrace{p, \ldots, p}_{p \text{ times}}} \frac{\partial}{\partial t}) - \varepsilon\, H^{\varepsilon}_{p, \ldots, p} - \sum_{r=1}^{P} \varepsilon^{\frac{1}{r}} F^{\varepsilon}_{\underbrace{p, \ldots, p}_{p \text{ times}}, r} \end{aligned}$$

This clearly yields (87), and ends the proof of fact 2, and hence the proof of Theorem 2.

References

[1] R. W. BROCKETT, *Asymptotic stability and feedback stabilization*, in Differential Geometric Control Theory, R. W. Brockett, R. S. Millman, and H. J. Sussmann, eds., Basel-Boston, 1983, Birkäuser.

[2] J.-M. Coron, *A necessary condition for feedback stabilization*, Systems & Control Lett., 14 (1990), pp. 227–232.

[3] ——, *Global asymptotic stabilization of for controllable systems without drift*, Mathematics of Cont. Sign. and Systems, 5 (1992), pp. 295–312.

[4] ——, *Stabilization in finite time of locally controllable systems by means ofcontinuous stabilizing feedback laws*, SIAM J. on Control and Optim., 33 (1995), pp. 804–833.

[5] G. W. Haynes and H. Hermes, *Nonlinear controllability via Lie theory*, SIAM J. on Control, 8 (1970), pp. 450–460.

[6] H. Hermes, *Nilpotent and high-order approximations of vector fields systems*, SIAM Reviews, 33 (1991), pp. 238–264.

[7] M. Kawski, *Homogeneous stabilizing feedback laws*, Control Theory and Advanced Technology, 6 (1990), pp. 497–516.

[8] J. Kurzweil and J. Jarnik, *Iterated Lie brackets in limit processes in ordinary differential equations*, Results in Mathematics, 14 (1988), pp. 125–137.

[9] W. Liu, *An approximation algorithm for nonholonomic systems*, Tech. Report SYCON-93-11, Rutgers Univ., New Brunswick, U.S.A., 1993. To appear in SIAM J. on Control & Optim., 1997.

[10] C. Lobry, *Contrôlabilité des systèmes non linéaires*, SIAM J. on Control, 8 (1970), pp. 573–605.

[11] R. T. M'Closkey and R. M. Murray, *Nonholonomic systems and exponential convergence : Some analysis tools*, in 32th IEEE Conf. on Decision and Control, San Antonio, USA, Dec. 1993, IEEE.

[12] ——, *Exponential stabilization of driftless nonlinear control systems via time-varying homogeneous feedback*, in 33rd IEEE Conf. on Decision and Control, Lake Buena Vista, USA, Dec. 1994, IEEE.

[13] ——, *Exponential stabilization of driftless nonlinear control systems using homogeneous feedback*, Tech. Report CDS95-012, California Inst. of Tech., 1995. To appear in IEEE Trans. on Automatic Control.

[14] P. Morin and C. Samson, *Applications of backstepping techniques to the time-varying exponential stabilization of chained-form systems*, European J. of Control, (1997 (to appear)).

[15] J.-B. Pomet, *Explicit design of time-varying stabilizing control laws for a class of controllable systems without drift*, Systems & Control Lett., 18 (1992), pp. 147–158.

[16] J.-B. Pomet and C. Samson, *Exponential stabilization of nonholonomic systems in power form*, in IFAC Symposium on Robust Control Design, Rio de Janeiro (Brasil), Sept. 1994, pp. 447–452.

[17] L. Rosier, *Homogeneous lyapunov function for homogeneous continuous vector field*, Systems & Control Lett., 19 (1992), pp. 467–473.

[18] C. Samson, *Velocity and torque feedback control of a nonholonomic cart*, in Int. Workshop in Adaptive and Nonlinear Control : Issues in Robotics, vol. 162, Springer-Verlag, New-York, 1991. Proceedings of a Conf. held in Grenoble, France, 1990.

[19] R. Sépulchre, G. Campion, and V. Wertz, *Some remarks on periodic feedback stabilization*, in 2nd NOLCOS, Bordeaux, France, June 1992, IFAC, pp. 418–423.

[20] E. D. Sontag and H. J. Sussmann, *Remarks on continuous feedback*, in 19th IEEE Conf. on Decision and Control, Albuquerque, NM, USA, Dec. 1980, IEEE.

[21] G. Stefani, *Polynomial approximations to control systems and local controllability*, in 24th IEEE Conf. on Dec. and Control, Dec. 1985.

[22] H. J. Sussmann, *Subanalytic sets and feedback control*, J. Diff. Equations, 31 (1979), pp. 31–52.

[23] H. J. Sussmann and W. Liu, *Limits of highly oscillatory controls and approximation of general paths by admissible trajectories*, in 30th IEEE Conf. on Dec. and Control, Dec. 1991.

[24] ——, *Lie bracket extensions and averaging : the single bracket case*, in Nonholonomic Motion Planning, Z. Li and J. Canny, eds., Boston, 1993, KAP, pp. 107–147.

[25] A. R. Teel, R. M. Murray, and G. Walsh, *Nonholonomic control systems : from steering to stabilization with sinusoids*, in 31th IEEE Conf. on Decision and Control, Tucson, AZ, USA, Dec. 1992, IEEE.

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