Interacting Strings of Characters

Vadim A. Malyshev

INRIA

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Interacting Strings of Characters

Vadim Malyshev

N° 3057

December 1996

THÈME 1

Rapport de recherche

ISSN 0249-6399

Interacting Strings of Characters

Vadim Malyshev

Thème 1 — Réseaux et systèmes
Projet MEVAL

Rapport de recherche n° 3057 — December 1996 — 44 pages

Abstract: We consider several interacting strings of characters. Otherwise speaking, we consider one-dimensional random walks interacting with each other and with their environments (mostly in a one-sided manner). This embraces many applications: queueing networks with different customer types, random walks on some discrete non-commutative groups, random Turing machines and others. We present a scheme for the general theory of such processes, the general theory of random walks in the orthant being a particular case. This scheme is being developed in several papers, a review of which is presented here.

Key-words: String of characters, Markov chains, multi type customer queues, Turing machines.

(Résumé : tsvp)

Unité de recherche INRIA Rocquencourt
Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
Téléphone : (33 1) 39 63 55 11 – Télécopie : (33 1) 39 63 53 30

Chaînes de caractères en interaction

Résumé : Nous analysons plusieurs chaines de caracteères en interaction. Une autre façon de voir les choses est de consideérer des marches aléatoires en dimension 1 en interaction mutuelle, ainsi qu'avec leurs environments respectifs.

Mots-clé : Mots de caractères, Chaines de Markov, files d'attente multi-types, Machines de Turing.

Contents

1 Introduction

1.1 History

Homogeneous random walks in Z^d, a popular reference for which is Spitzer [11], is now a well-established domain of probability, coinciding in several points with summation theory of independent random variables. This theory has many natural generalisations. Amongst them are

- Random walks with boundaries.

  The simplest case is the random walk in Z_+ and its applications in queueing theory. A difficult generalisation are random walks in multidimensional orthants [8]. Many queueing networks are in this class.

- Random walks on discrete groups.

The most popular examples are amenable groups (see [17]) and free non-commutative groups (the first fundamental paper is [18], one of the last references is [20]). Many results for free and modular groups that have been obtained earlier by the generating functions method, are covered by our theory using purely probabilistic arguments.

- Random walks on trees, see [19].
- Random walks in a Random Environment: a frozen environment (see for example [9]), a dynamic environment etc.

Here we present a class of Markov processes that contains or is related to many of the generalisations mentioned above.

1.2 On the Main Result

Here we will present a review of published results and also the scheme of the new general theory of such processes. There are two basic difficulties behind the theory presented below: one is random walks (or diffusion processes) on a manifold with boundary. Contrary to well-known theories for the case of a smooth boundary, we will present here the (much more complicated) situation of a piecewise smooth boundary. The parameter C that is a good measure of complexity for such problems, is the maximum codimension of the intersections of the smooth parts of the boundary. For example, the half line and half space have $C = 1$, and the quarter plane has $C = 2$, because the origin (the intersection of two linear axes) has dimension 0.

The second one is local interaction of a particle in a random homogeneous environment, interacting with it. Here we will consider the simplest situation of an environment that is non-trivial only from one side of the particle and that moves together with it. But this gives us the possibility to obtain deeper results without restricting the assumptions, for example such as small parameters, used in mathematical statistical physics, or locally interacting processes. The case of an environment on both sides of the particle will be considered only in the one dimensional case (it models random Turing machines).

We study our processes by constructing (using scaling) simpler Markov processes or even deterministic dynamical systems (which we call macroprocesses) on a compact space. If such dynamical system is uniquely ergodic, then we can give necessary and sufficient conditions for recurrence and ergodicity of the process in terms of the macro-process (dynamical system). This embraces all results known to me at the moment. It is a long way even to formulate this result in exact terms, so we do it with the simplest possible assumptions that are nevertheless reasonable from the point of view of the models encountered in applications.

2 Homogeneous Walks

Homogeneous random walk in Z^M is the necessary starting point of random walk theory in Z_+^M. In this section we study Markov processes which play similar role with respect to evolution of finite strings we shall study later. This class of Markov processes is more complicated than homogeneous random walks and there are still many open problems.

2.1 Definitions

A finite string $\alpha = x_1 x_2 ... x_n$ is a sequence of symbols x_i from some alphabet $S = \{1, 2, ..., r\}$. We shall always enumerate finite strings from left to the right starting with 1, n is called the length $n(\alpha) =\mid \alpha \mid$ of the string, x_n is its rightmost symbol. The set of all finite strings, including the empty one $\emptyset$, is denoted by $\mathcal{A}$. Concatenation of two strings $\alpha = x_1 ... x_n$ and $\beta = y_1 ... y_m$ is defined by $\alpha\beta = x_1 ... x_{n+m}$ where $x_{n+1} = y_1, ..., x_{n+m} = y_m$.

We shall consider also (left) semiinfinite environments $\xi = ... x_{-1} x_0$, i.e. functions on $Z_- = \{..., -2, -1, 0\} \to S$.

We consider also a particle at the point $n \in Z$ and a pair (n, ξ), which we call a particle in one-sided environment ξ. We call this pair a semiinfinite string (i.e. it is an environment with specified enumeration) and sometimes denote it also by, without abusing the notation, $\alpha = ... y_{n-1} y_n, y_n = x_0, y_{n-1} = x_{-1}, ...$. Thus, semiinfinite string is a function $\alpha : Z \to S$ which is undefined starting from $n + 1$.The set of all semiinfinite strings is denoted by $\mathcal{A}_\infty$, the set of random environments by $\mathcal{E}_\infty$.

One can define the concatenation $\alpha\beta = ...x_{n-1}x_n...x_{n+m}$ of a semiinfinite string $\alpha = ...x_{n-1}x_n$ with finite string $\beta = y_1...y_m$ where again $x_{n+1} = y_1, ..., x_{n+m} = y_m$. Denote θ the shift on Z: $\theta(i) = i+1$. This defines the shift on the set of semiinfinite strings

$$\theta(\alpha) = ...y_{n-2}y_{n-1}, y_k = x_{k+1}$$

For any string $\alpha = ...x_{n-1}x_n$ we define its shift to 0:

$$\theta_0\alpha = \theta^{n(\alpha)}\alpha$$

that is the renumeration of string so that the left-hand symbol had number 0.

To define the evolution one can think about a particle, performing random walk and also changing and creating an environment from the left of itself, with jumps dependent on the environment. Evolution of semiinfinite strings is defined by transition probabilities

$$q(\gamma, \delta) = P(\alpha(t+1) = \rho\delta \mid \alpha(t) = \rho\gamma)$$

where ρ is a semiinfinite string, γ, δ are finite with $n(\gamma) = d, n(\delta) \leq 2d$. Note that the transition probabilities do not depend on ρ, this corresponds to the homogeneity property of random walks in Z^M. The parameter d is fixed and characterizes the "depth" of interaction. We shall assume that $d = 1$, generalization to $d > 1$ mainly does not give new qualitative phenomena but demands a lot of technical work. In this case $q(x, \emptyset)$ is the probability to erase the last symbol x of the environment and to move to the left, $q(x, y)$ is the probability that the particle does not move but substitutes the right-most symbol of the environment by y and $q(x, yz)$ is the probability to jump to the right and to substitute x it by two symbols yz. We shall call this Markov chain an environment interacting random walk (EIRW) or simply, a random string in Z.

The object of our study in this section will be Markov chain $\mathcal{M}$ with the state space $\times_{i=1}^{M}\mathcal{A}_\infty$. We can think about this as a particle at the point $\vec{n} \in Z^M, \vec{n} = (n_1, ..., n_M)$ and the environment vector $\vec{\xi}$ with coordinates

$$\xi_i = ...y_{i,-1}y_{i,0}, i = 1, 2, ..., M$$

This pair gives M-vector A of semiinfinite strings

$$\alpha_i = ...x_{i,(n_i-1)}x_{i,n_i}, i = 1,2,...,M, x_{i,n_i} = y_{i,0}, x_{i,(n_i-1)} = y_{i,-1}$$

If in one dimension one can really think about environment as a function $Z \to S \cup \{\emptyset\}$, for $M > 1$ this is much less natural, because it is a function $Z^M \to S \cup \{\emptyset\}$ and changing this environment by the particle is not local in the topology of Z^M. But the notion of the particle is very useful because we shall always use the analogy with the classical random walk theory ($r = 1$).

Transition probabilities are defined as follows. Let $\vec{x} = \vec{x}(A) = (x_{i,n_i}, i = 1,...,M)$ be the array of all righthand symbols of A and $\vec{\delta} = (\delta_i, i = 1,...,M)$ be an array of strings. Each string δ_i has length $0, 1$ or 2. Let B be the array of strings

$$\beta_i = ...x_{i,(n_i-1)}\delta_i, i = 1,2,...,M$$

Then one step transition probabilities P_{AB} depend only on $\vec{x}$ and $\vec{\delta}$. In other words they are equal to some $q(\vec{x}, \vec{\delta})$ where for all $\vec{x}$

$$\sum_{\vec{\delta}} q(\vec{x}, \vec{\delta}) = 1$$

.

We shall use often the following:

Condition 1 (IIC) *Strong Irreducibility Condition: all $q(\vec{x}, \vec{\delta})$ are nonnegative. Note that in many interesting examples this condition is violated, so we have also to study cases when it does not hold. But we assume it unless otherwise stated.*

2.2 Classification

If $r = 1$ we have classical random walks in Z^M, let us remind the main results for them.

Theorem 1 *1. For $M > 2$ random walk is always transient*

2. For $M = 1, 2$ random walk is recurrent iff the mean drift is zero; otherwise it is transient;

3. *In particular we have that random walk is recurrent only on the set of parameters of Lebesgue measure zero in the parameter space*

We know that law of large numbers and central limit theorem hold for this random walks. For EIRW we need more general definitions. We call the particle process (i.e. the random process $n(t) \in Z^M$) recurrent if for any initial state of the Markov chain particle visits 0 with probability one, and call it null recurrent if it is recurrent and this random recurrence time of the particle to 0 has an infinite expectation for all initial conditions.

Here and farther the statements called theorem-hypothesis mean that it is likely to be true but formal proof does not exist, moreover we always indicate cases where it is proved.

Theorem-Hypothesis 1 1. *For any parameters the particle is either null recurrent or transient;*

2. *For $M > 2$ the particle is always transient (that is non recurrent)*

3. *for $M = 1, 2$ the particle is null recurrent only on the set of parameters of Lebesgue measure zero in the parameter space*

This hypothesis is proved for $M = 1$ (see [3], [4], [7]). Also it can be proved for $M > 1$ independent strings, that is when each string evolves independently of the others.

2.3 Invariant measures

To formulate statements similar to the LLN or CLT we need an important definition. Any measure μ on $(\mathcal{E}_\infty)^M$ is completely characterized by its correlation functions. At the same time the distribution of the Markov chain at time t is completely characterized by two projections: distribution of the position $n(t)$ of the particle and distribution $\mu(t)$ of the environment. For example, for $M = 1$ consider right-end k-point correlation functions p_t^k at time t

$$p_t^1(i) = P(x_{n(t)}(t) = i)$$

$$p_t^2(i, j) = P(x_{n(t)}(t) = i, x_{n(t)-1}(t) = j)$$

etc. They define uniquely the induced distribution on $\mathcal{E}_\infty = \mathcal{A}_\infty^0 = \theta_0 \mathcal{A}_\infty$ which we denote $\mu(t)$.

Definition 1 *Measure* μ *on* $(\mathcal{E}_\infty)^M$ *is called invariant if from* $\mu(0) = \mu$ *follows that* $\mu(t) = \mu$ *for all* t *(initial* $n(0)$ *does not play any role).*

To each invariant measure μ we assign the drift vector $\vec{v} = \vec{v}(\mu)$ with components

$$v_i = \sum_{\vec{\gamma},\vec{\delta}} \mu(\vec{\gamma}) q(\vec{\gamma};\vec{\delta})(n(\delta_i) - d)$$

Note that for classical case $r = 1$ this drift vector is unique (because there is only one environment and point invariant measure) and is a standard drift vector.

Define also the vector of signs

$$sign(\mu) = (sign(v_i), i = 1, ..., M), sign(v_i) = -, 0, +$$

There are 3^M of such vectors , and we shall speak about invariant measures of the corresponding type, for example $(0, ..., 0)$-measure is an invariant measure μ with $sign(\mu) = (0, ..., 0)$.

Theorem-Hypothesis 2

1. *If there exists* $(0, ..., 0)$*-measure then there cannot be invariant measures of other types;*
2. *Stronger hypothesis: if there exists* $(0, ..., 0)$*-measure then it is unique*

This is proved for one-dimensional case (see [7]).

2.4 One-dimensional Case

2.4.1 (+)-measures

Theorem 2 *The following conditions are equivalent:*

1. *For some initial condition the position of the particle* $n(t) \to \infty$ *a.s. as* $t \to \infty$*;*
2. *There exists* $v > 0$ *such that for all initial states* $\frac{n(t)}{t} \to v$ *a.s.;*
3. *There exists an invariant* $(+)$*-measure.*

In this case we have also

- *There exist unique invariant measure μ on $\mathcal{A}^0_\infty$;*
- *Correlation functions p_t^k tend (exponentially fast) to correlation functions of μ.*

For any measure μ on environments define its limiting correlation functions

$$\lim_{k \to -\infty} \mu(x_k = i),$$

$$\lim_{k \to -\infty} \mu(x_k = i, x_{k-1} = j), \ldots$$

etc., if these limits exist. Denote the corresponding measure, defined by the limiting corelation functions, by $Lim(\mu)$.

Theorem-Definition 1 *For the invariant measure μ of theorem 2 the limiting correlation functions exist and define translation invariant measure $Lim(\mu)$ on S^Z.*

This asymptotic measure is a Gibbs measure with exponential mixing properties.

Exponential mixing is proved in [3]. There are two explicit series for the correlation functions of the invariant measure (first see [3], second see [6]).

2.4.2 $(-)$-measures

Theorem 3 *Assume IIC. Then the following conditions are equivalent*

1. *The position of the particle $n(t) = n(\alpha(t)) \to -\infty$ a.s. as $t \to \infty$, for some initial conditions;*
2. *For each initial translation invariant condition (we take the restriction of a stationary process η as the initial condition) there exists $v = v(\eta) < 0$ such that $\frac{n(t)}{t} \to v$ a.s.;*
3. *There exist continuum of extremal invariant measures μ on $\mathcal{A}^0_\infty$;*
4. *For all translation invariant initial states η the correlation functions p_t^k tend to some limit $\mu(\eta)$ and not all of these limits are equal (hopefully all of them are different).*

See [3] , [6]. There are also formulae for the correlation functions of these invariant measures (see [6]).

All invariant measures μ with translation invariant $Lim(\mu)$ can be obtained in this way. A. Zamyatin proved (unpublished) that the limiting measure, which is invariant, also exists if we start from periodic string.

Problem 1 *It is not known whether invariant measures μ with non translation invariant $Lim(\mu)$ exist.*

2.4.3 (0)-measures

Theorem 4 *In all other cases, different from the situations of the previous theorems, there exists a unique invariant measure. The drift vector, corresponding to it, is the zero vector.*

2.5 (+,+,...,+)-measures

Theorem 5 *Assume IIC. Then the following conditions are equivalent*

1. *For all initial conditions the coordinates $n_i(t), i = 1, ..., M$ tend to plus infinity a.s.;*
2. *There exist $v_i > 0, i = 1, ..., M$ such that $\frac{n_i(t)}{t} \to v_i$ a.s. for all initial conditions;*
3. *There exists unique invariant measure on $\mathcal{E}_0$ with $sign(\mu) = (+, ..., +)$*

This is proved in [12]. In the same paper the following facts are also proven.

Lemma 1 *There exist $k > 0, \epsilon > 0$, such that for all environments ξ*

$$E[n_i(k) - n_i(0) \mid n(0) = n_0, \xi(0) = \xi] \geq \epsilon, \forall i$$

Lemma 2 *(Exponential convergence)*

All correlation functions converge exponentially fast to the correlation functions of the invariant measure.

Let us introduce the parameter space $\mathcal{P} = \{q\}$.

Problem 2 *In the generic situation, that is for parameters in $\mathcal{P} \setminus \mathcal{P}_0$ where $\mathcal{P}_0$ is a set of Lebesgue measure zero, the conditions of the theorem are also equivalent to the following: There exists unique invariant measure on $(\mathcal{E}_\infty)^M$. In general it is not true due to the possibility of null-recurrent case (for $M = 1$) when the invariant measure is also unique (for example, for independent strings).*

2.6 Coexistence of measure types

For two semiinfinite strings ($M = 2$) we shall be interested in 4 types of invariant measures in this case: ++, +-, -+, - - (we shall not consider here the situations with 0-type measures: +0, -0, 00, 0+, 0-). We showed already that if ++ measure exists then no other can exist. We shall now consider examples of other situations.

Problem 3 *Do invariant measures μ with translation invariant $Lim(\mu)$ always exist ?*

Only (-,-) measures exist. We shall give examples of such situation.

Example 1. Consider two independent semiinfinite strings, assuming that each has an invariant $(-)$-measure (hence, a continuum of such measures). Denote the corresponding transition probabilities $q_0(.,.)$. In this case there are no invariant measures other than (-,-)-measures.

Small perturbations.

Lemma 3 *If differences $q-q_0$ are sufficiently small then for the system defined by q also only $(-,-)$-measures exist.*

Drift properties.

Problem 4 *Assume that only (-,-) measures exist. Then for all initial environment both particle coordinates tend to $-\infty$ almost surely.*

The intuition behind this problem is the following: all drift vectors $\vec{v}(\mu)$ for the invariant measures have coordinates bounded away from zero due to the closedness of the set $\{\vec{\mu}\}$ (not that we assumed that other sign possibilities cannot occur). Then it seems plausible that Lyapounov function should exist, uniformly in the initial environment. Now we should discuss more about Lyapounov functions. We would like to prove the following

Lyapounov property.

Condition 2 *There exist $k > 0, \epsilon > 0$, such that for all environments ξ*

$$E[n_i(k) - n_i(0) \mid n(0), \xi(0) = \xi] \leq -\epsilon, i = 1, 2$$

Only (+,-) measures exist. Here the same arguments and examples work.

Problem 5 *Assume that only (+,-) measures exist. Then for any initial environment first particle coordinates tends to ∞ and the second tends to $-\infty$ almost surely.*

Lyapounov property. It looks quite similar.

Condition 3 *There exist $k > 0, \epsilon > 0$, such that for all environments ξ*

$$E[n_2(k) - n_2(0) \mid n(0), \xi(0) = \xi] \leq -\epsilon$$

$$E[n_1(k) - n_1(0) \mid n(0), \xi(0) = \xi] \geq \epsilon$$

Note that all this was proved long ago for $r = 1$ (see [1]).

One string controls another one. More examples can be constructed using the following mechanism: assume that one string (say string 1) evolves independently of the string 2 but jumps of the string 2 depend also on the rightmost symbol of the string 1. Choose string 1 to have $(-)$-type invariant measure. Then by appropriate choice of transitions for string 2 one can get the situation when only $(-,-)$-measures exist or only $(-,+)$-type measures exist.

(-,-) and (-,+) measures coexist Let us show that (under IIC condition) there can exist situations when there is a continuum of (-,-) and continuum of (-,+) measures. Take alphabet $S = \{1, 2\}$ and $M = 2$. Assume the following conditions:

- For the string 1 one-step transitions $x \to \emptyset$, that is erasing the rightmost symbol at one step , have probability bigger than $1 - \epsilon$ for all x;
- If the rightmost symbol of string 1 is 1 then transitions $x \to \emptyset$ for string 2 have probability bigger than $1-\epsilon$ for $x = 1, 2$, if the right-hand symbol of string 1 is 2 then transitions $x \to yz$ have probability bigger than $1-\epsilon$ for $x = 1, 2$.

Proposition 1 *There exist in this case (-,-) and (-,+) invariant measures.*

¿From the first condition it follows that string 1 moves always to the left. Assume its initial state was pure 1. Then for $\epsilon > 0$ sufficiently small there exists $\delta = \delta(\epsilon) > 0$ such that for all t the probability that right-hand symbol of string 1 is 1 should be bigger than $1 - \delta$. Then string 2 moves to $-\infty$. This gives invariant measure (-,-). The same argument with initial condition "pure 2" of string 1 gives (-,+)-measure.

Theorem-Hypothesis 3 *Assume that only (-,-) and (-,+) measures exist. Then for all initial environment first particle coordinate tends to $-\infty$ almost surely.*

In this case Lyapounov function can exist only for one coordiate but not for two simultaneously.

Can (-,+) and (+,-) measures coexist ?

Problem 6 *Consider 2-dimensional IIC case. Can (-,+) and (+,-) measures simultaneously exist ?*

Random Turing machine (see below) can be considered as two semiinfinite strings which are both put on Z. Rightmost end of string 1 to the point n, string 2 is reversed (right end becomes left end) and this end is at the point $n+1$. It follows that measure coexistence cannot occur if there is the conservation law $n_1(t) = n_2(t)$, i.e. the sum of coordinates of two particles is constant. Note that condition IIC is violated here.

3-dimensional example We shall show that there can be simultaneously (-,+,-) and (-,-,+) measures. Choose just string 1 as in the previous proposition. Then let the jumps (+,-) of the strings 2 and 3 have probability close to 1 if the right-hand symbol of string 1 is 1, and jumps (-,+) if it is 2.

2.7 Main uniqueness result

For given measure μ on two semiinfinite strings let $P_i\mu, i = 1, 2$, be its projection (marginal distribution) onto the first and second factor correspondingly.

Theorem-Hypothesis 4 *We give two different formulations*

- *Let some point $\{q\} \in \mathcal{P}$ be fixed. Then for any translation invariant measure μ_1 on S^Z there exists not more than one invariant (-,+)-measure μ with $Lim(P_1\mu) = \mu_1$*
- *Assume that starting from μ with $Lim(P_1\mu) = \mu_1$ we have that the second coordinate x_2 of the particle tends to $+\infty$ a.s. Then this holds for all μ with $Lim(P_1\mu) = \mu_1$ and moreover $\mu(t)$ tends to an invariant measure which depends on μ only through $Lim(\mu_1)$.*

Second version is proven (see [12]) for the case when μ_1 is markovian. The same coupling argument should work for μ_1 with exponential mixing.

Intuition. The proof in [12] generalizes the coupling method used in [3] for the transient case. If for given μ_1 there exists invariant (-,+) measure, then starting with this measure second coordinate of the particle goes to $+\infty$. It follows (by coupling) that second coordinate of the particle goes to $+\infty$ for any initial distribution with μ_1 marginal. And the measure at the end of the string 2 depends only on the (-)-infinity tail of the measure μ_1.

There is an obvious generalisation of this theorem for $M > 2$. These uniqueness is crucial for the construction of the scattering operator below.

2.8 About explicit formulae

2.8.1 One dimension

Explicit conditions on the parameters for all cases (ergodic, transient, null-recurrent) are discussed in [4]. Case $d = 1$ is sufficiently explicit, as for explicit conditions for the parameters, when these 3 types of behaviour occur, the case $d > 1$ is technically more complicated and there are some open problems there.

2.8.2 Simple invariant measures.

In some cases invariant measures are sufficiently simple (Bernoulli) and can be explicitly calculated. For example, in LIFO networks without synchronization constraints. For such networks with M nodes we have EIRW with M strings.

Lemma 4 *For the above model in the transient case the only invariant measure is Bernoulli measure.*

Proof. By simple verification. With probability $q(x;\emptyset)$ we erase the last symbol and Bernoulli property does not change. With probability $q(x)$ we append a new symbol to the end, independently of the others. So, invariant probabilities for Bernoulli scheme are $\frac{q(x)}{\sum q(x)}$.

For $M > 1$ we shall prove that there exists Bernoulli measure if the following conditions are satisfied. There can occur only the following events

- erase last symbol x of one of the strings i with probability $p(i;x)$ independently of other symbols;
- put one last symbol from the end of the string i to the end of another string j with probability $p(i,j)$;
- add one symbol x at the end of one string i with probability $q(i,x)$

Proof: Just check that after each event we have still conditional Bernoulli property. We need also that all symbols of all strings were identically distributed. To get this we can find one-particle correlation functions from $M(r-1)$ linear equations, with the same number of unknowns - the probabilities $\pi(i,x)$ that the last symbol of string i is x.

Note that there are many results for product invariant measures for ergodic finite strings, see [14], [15]. It seems that they can be generalized for semiinfinite strings as well.

2.8.3 General case

It is known that even for generic situation with $r = 1, M = 3$ one cannot get explicit ergodicity conditions. In this paper our goal is to subdivide the problem on simpler parts. For each part it is easier to see whether explicit solution is possible or not. One of the possible formulation of SOME seemingly reasonable conditions is the following: just to postulate existence of some Lyapunov functions. The problem is that even this is not possible in the generic situation. But local Lyapunov functions seem to wotk in many cases. To explain their meaning and to give such formulation one needs big machinary: to present one such machinary is the goal of this paper.

Nevertheless, search of parameters for which explicit conditions exist stays a very challenging problem, especially if it is related to important applications.

3 EIRW in Z_+^M

3.1 Definitions

For each EIRW we shall define a class of countable Markov chains. For $M = 1$ the state space of these countable chains (called EIRW on the halfline or one-sided evolution of finite strings) is the set of all finite strings $\mathcal{A}$, including the empty one. We assume that for strings of length $n(\alpha) \geq d$ the transition probabilities are defined by the same formulae as for semiinfinite strings, but we need also to prescribe transitions probabilities for strings of length $n(\alpha) \leq d-1$. We shall define them arbitrarily assuming only "non degeneracy", for example that for all α with $n(\alpha) < d$ we have

$$p_{\alpha\beta} = P(\alpha(t+1) = \beta \mid \alpha(t) = \alpha) > 0$$

for all β with $n(\beta) < 2d$. Further on we shall consider the case $d = 1$, unless otherwise stated. For this case one has to define separately only transition probabilities $p_{\emptyset\emptyset}, p_{\emptyset x}$ from the empty string to strings of lengths zero and one.

The main object of our study will be Markov chains $\mathcal{M}$ with the state space $\times_{i=1}^{M}\mathcal{A}$. Note that for the case $r = 1$ this set is just the orthant Z_+^M and we shall see that for this case we shall get exactly random walks in this orthant.

Some knowledge of the theory of random walks in Z_+^M would be useful in the sequel (see [8] , [1]).

It could be natural, for any $\Lambda \subset \{1, ..., M\}$, to consider the subset

$$Face(\Lambda) = \times_{i=1}^M \mathcal{B}_i \subset \times_{i=1}^M \mathcal{A}$$

where $\mathcal{B}_i$ consists only of the empty string for $i \notin \Lambda$ and equals $\mathcal{A} \setminus \{\emptyset\}$ for $i \in \Lambda$, and call it a face. It is really a face of the orthant when $r = 1$. Note that the whole set $\times_{i=1}^M \mathcal{A}$ is the union of all 2^M faces. Unfortunately, for this definition of faces one cannot define perpendicular to the face and because of this such a useful notion as the induced chain (as it is possible for $r = 1$). We shall introduce below a new definition of induced chains (analog of induced chains, see [1]) in a quite different way.

To define transitions we should, for each Λ including the empty one, introduce a set of transition probabilities, different on different faces.

Condition 4 (Maximal homogeneity condition) .

For each $A \in Face(\Lambda), B \in \times_{i=1}^M \mathcal{A}$, let P_{AB} satisfy the following condition: P_{AB} depend only on Λ, on the last symbols of nonempty strings of A and on the vector $\vec{\delta}$ defined below.

More formally, let A be M-vector of strings

$$\alpha_i = x_{i1}...x_{in_i}, i = 1, 2, ..., M$$

Let $\vec{\gamma}$ be an array of all last symbols $x_i = x_{in_i}$ of these strings; if α_i is empty then x_i also is. Let $\vec{\delta}$ be an array of M strings $\delta_i, i = 1, ..., M$, where δ_i has length zero, one or two if $i \in \Lambda$ and zero or one otherwise. Then the only possible one step transitions are

$$\alpha_i = x_{i1}...x_{in_i} \to x_{i1}...x_{i,n_i-1}\delta_i, i = 1, ..., M$$

Thus, last symbols of nonempty strings are erased and, instead of it, to the right end of each string zero, one or two symbols are appended. To each empty string zero or one symbol is appended.

Condition 5 (Analog of the IIC) . *All such P_{AB} are positive.*

3.2 Induced chains

Related to the countable Markov chain on $\times_{i=1}^{M}\mathcal{A}$ we shall consider Markov chains $\mathcal{M}_\Lambda$, called induced chains, the set of states of which is a product $\times_{i=1}^{M}\mathcal{A}_i$ where $\mathcal{A}_i = \mathcal{A}, i \notin \Lambda$ and $\mathcal{A}_\rangle = \mathcal{A}_\infty, i \in \Lambda$. w We call $\mid \Lambda \mid$ dimension of the face. Transition probabilities on these chains are the same P_{AB} only some of strings in A are semiinfinite, the previous definition holds for this case without any changes. Note that induced chains are uncountable except the case $\Lambda = \emptyset$.

The state of the induced chain is defined by a particle (in $Z_+^{|\Lambda|}$), $\mid \Lambda \mid$ semiinfinite environments and $M- \mid \Lambda \mid$ finite strings.

The idea of the induced chains $\mathcal{M}_\Lambda$ is that they define the behaviour of the system far away from the boundary of Λ, i.e. when the lengths of strings $\alpha_i, i \in \Lambda$ is sufficiently large.

Note that for $r = 1$ the situation was simpler and we defined induced chains (see for example [1]) just by projecting initial transition probabilities onto the perpendicular to the face. So, in this case induced chain is a Markov chain on $Z_+^{M-|\Lambda|}$. We could separate in this case perpendicular and tangential components of the process close to the face. It appears that it is not possible to do it for $r > 1$ and we consider Markov chains on $Z_+^{M-|\Lambda|} \times Z^{|\Lambda|}$. Now, following the new definition, $\Lambda = \{1, ..., M\}$, then $\mathcal{M}_\Lambda$ is just a Markov chain on semiinfinite strings, the homogeneous case studied in the previous section. For the random walks ($r = 1$) this corresponds to the random walks on all Z^M,which is completely homogeneous.

For the random walk $\alpha(t)$ in Z_+^M let us consider the following scaling: we start at $[xN]$ and put $t = [\tau N]$. For $\tau < \frac{1}{d} dist(x, \partial R_+^M)$, if x belongs to the inside part of R_+^M (i.e. the coordinates of the vector x are positive), we have then as $N \to \infty$

$$\frac{1}{N}\alpha([\tau N]) \to v$$

where v is the mean drift inside Z_+^M.

This holds because during this small "macrotime" we cannot reach the boundary of the orthant and our evolution coincides with that for completely homogeneous random walk on all Z^M. Unfortunately, this scaling does not always hold for $r > 1$, but only for special "stationary" initial conditions. To understand this we should start with reviewing results concerning evolution of one string. For example the set of invariant measures will be used below for constructing the fundamental vector bundle.

3.3 One-dimensional Case

Here we consider Markov chains $\mathcal{M}$ on $\mathcal{A}$ and $\mathcal{M}_\infty$ on $\mathcal{A}_\infty$, corresponding to the evolution of one string. Note that Markov chains $\mathcal{M}$ on $\mathcal{A}$ are countable Markov chains and we can use standard terminology concerning countable Markov chains.

3.3.1 Transient case

Denote $\mathcal{P}_d = \mathcal{P}$ the set of all parameters $q(.,.)$ with fixed $d > 0$.

Theorem 6 *The set $\mathcal{P}_0 \subset \mathcal{P}$ of parameters where Markov chain $\mathcal{M}$ is null recurrent has Lebesgue measure 0.*

Theorem 7 *Assume IIC. Then the following conditions are equivalent on $\mathcal{P} \setminus \mathcal{P}_0$*

1. *The Markov chain $\mathcal{M}$ is transient;*
2. *The length of the random string $n(t) = n(\alpha(t)) \to \infty$ a.s. as $t \to \infty$;*
3. *There exists $v > 0$ such that for all initial states $\frac{n(t)}{t} \to v$ a.s.;*
4. *There exists a unique invariant measure μ on $\mathcal{A}^0_\infty$;*
5. *Correlation functions p^k_t tend to a limit which is the same for all initial states.*

Theorem 8 *Under conditions of the previous theorem*

- *The invariant measure μ has translation invariant $Lim(\mu)$;*
- *Moreover $Lim(\mu)$ has the exponential mixing property.*

This coincides with the invariant measure for the corresponding semiinfinite string.

3.3.2 Ergodic case

Theorem 9 *Assume IIC. Then the following conditions are equivalent* $\mathcal{P} \setminus \mathcal{P}_0$

1. *The Markov chain* $\mathcal{M}$ *is ergodic;*

2. *Any of assertions of theorem 3 for the corresponding semiinfinite holds*

In ergodic case also other stabilization laws exist.

Theorem 10 *Let* π *be the stationary distribution and* π_N *be the conditional stationary distribution under the condition that the length of the string is* N*. Then the conditional correlation functions* $\pi_N(\delta)$*, i.e. probabilities that the last substring of length* $n(\delta)$ *of the string (of length* N*) is* δ*, tend as* $N \to \infty$ *to limiting values exponentially fast.*

Another approach to this limiting correlation functions is infinite volume limit. We start from some finite truncation of the countable Makov chain. This means that we consider a sequence $\mathcal{M}_L$ of finite Markov chains. The state space of $\mathcal{M}_L$ is the set all string of the length less or equal to L. If the string has length less than L then its jumps are the same as for the countable chain. If the length is exactly L then we introduce other jumps (boundary conditions, assumed nondegenerate). If $N, L-N$ tend to infinity, then the conditional stationary distribution of the last symbols of a string of length N stabilizes to the same limiting correlation functions independently of boundary conditions. This was proved in [16]. A difficult problem is to see whether it is true for the case $d > 1$.

3.3.3 Null recurrent case

Theorem 11 *Assume IIC. Then the following conditions are equivalent*

1. *The Markov chain* $\mathcal{M}$ *is null recurrent;*

2. *There exists invariant measure* μ *with* $v(\mu) = 0$*;*

3. *Any of assertions of theorem 4 holds*

3.3.4 Non IIC

Simplest case when the condition IIC is violated is the following:

Reducibility in terms of the alphabet. Assume that $R = \cup_i R_i, R_i \cap R_j \neq \emptyset$ if $i \neq j$. Assume that $q(.,.)$ are zero if they contain symbols from different R_i and nonzero otherwise. We shall encounter such situations later on.

3.4 Escape to infinity along the interior.

Now we turn again to multidimensional case.

Theorem 12 *Assume IIC. Then the following conditions are equivalent*

1. *The coordinates $n_i(t), i = 1, ..., M$ of the particle tend to plus infinity a.s. for all initial states;*
2. *There exist $v_i, i = 1, ..., M$, such that $\frac{n_i(t)}{t} \to v_i$ a.s. for all initial conditions;*
3. *There exists unique invariant measure on $\mathcal{M}_0$ with $sign \mu = (+, ..., +)$;*
4. *There are no invariant measures for any of the induced chains with $\Lambda \neq \{1, ..., M\}$*

In the situation of this theorem the chain $\mathcal{M}$ is transient. Proof see in [12]. For random walks ($r = 1$) it corresponds to the case when the mean drift inside the orthant has all its coordinates positive.

3.5 Invariant measures for the induced chains

We should study uncountable induced chains as well as the countable chain $\times_{i=1}^{M} \mathcal{A}$. The study of induced chains gives some inductive (in M) approach to the problem, resembling inductive approaches to the $(M+1)$-body problem in quantum mechanics and also for already better understood case of random walks in Z_+^M. The main role here again will play invariant measures. The first step was the study of invariant measures for the homogeneous case (the simplest induced chain).

Measure on $\times_{i=1}^{M}\mathcal{A}_i^0, \mathcal{A}_i^0 = \mathcal{A}_\infty^0, i \in \Lambda; \mathcal{A}_i^0 = \mathcal{A}, i \notin \Lambda$ is called invariant if it is invariant with respect to dynamics of the induced chain. Note that this definition coincides with the one for the completely homogeneous case ($\Lambda = \{1, ..., M\}$).

We shall give two definitions of ergodicity for a face.

1. The face is called ergodic if there exists an invariant measure on the corresponding $\times_{i=1}^{M}\mathcal{A}_i^0$.

2. The face is called strongly ergodic if the following Lyapounov condition holds: there exist $k > 0, \epsilon > 0$ such that

$$E[n_i(k) - n_i(0) \mid n(0), \xi(0)] \leq -\epsilon, i \notin \Lambda$$

   for all $n_j(0)$ sufficiently large uniformly in the other initial conditions.

By compactness it can be shown that each strongly ergodic face is ergodic.

For $r = 1$ ergodic faces have exactly one invariant measure and the first definition then is exactly the definition of ergodic face for random walks. For general r there can be many invariant measures.

Let us consider one-dimensional chains for the case $M = 2$.

Assume that only (-,-) measures exist for $\Lambda = \{1, 2\}$ and both induced chains are strongly ergodic.

To get ergodicity conditions we shall now consider invariant measures on ergodic one-dimensional chains. Introduce first the drift vector $\vec{v} = \vec{v}(\mu)$ corresponding to a given invariant measure μ on this induced chain. It has only the component along this face i which is equal to

$$v_i = \sum_{\vec{\gamma}, \vec{\delta}} \mu(\vec{\gamma}) q(\vec{\gamma}; \vec{\delta})(n(\delta_i) - 1)$$

Theorem-Hypothesis 5

- *If for some (ergodic) face i there exists invariant measure with $v_i > 0$ then this invariant measure is unique. In this case the (two-dimensional) Markov chain is transient*

- *If for any ergodic face i there exists invariant measure with $v_i < 0$ there are continuum of such invariant measures and all of them have $v_i < 0$. In this case the Markov chain is ergodic.*

We shall give some intuition of the proof. We use two Lyapounov functions: one which forces the second component to go to zero and other one for the induced chain. Lyapounov function for the induced chain can be constructed similarly to the construction of Lyapounov function for one-dimensional EICW (see [4]). Then we glue these Lyaponov functions together as it is standard for random walks in a quarter plane (see [1]).

3.6 Euler limit

We consider first the 2-dimensional case. For classical random walks in Z_+^2 we start from the point $[xN] \in Z_+^2$ with $x \in R_+^2$ fixed and $N \to \infty$. We can prove that for $\tau \in R_+$ small we have in probability

$$\lim_{N\to\infty} \frac{1}{N} n([\tau N]) = x + v\tau$$

where v is the mean drift inside Z_+^2.

For $M > 1$ we should fix not only the initial point but also initial environment for the corresponding induced chain, so let us fix (x, μ), where μ is an invariant measure for homogeneous two-dimensional ECRW. For large N this means that we take both long strings random with almost stationary distribution on them.

Proposition 2 *Assume that μ is (-,-)-measure. Then we have in probability*

$$\lim_{N\to\infty} \frac{1}{N} n([\tau]N) = x + v(\mu)\tau$$

This formula holds for all $\tau \leq \tau_0$, where τ_0 is such that $x + v(\mu)\tau_0$ belongs to one of one-dimensional faces.

The same holds when μ is (+,-)-measure.

Proposition 3 *Assume that only (+,-)-measures exist and μ is one of such measures. Then for $\tau \geq \tau_0$*

$$\lim_{N\to\infty} \frac{1}{N} n([\tau]N) = x + v(\mu)\tau_0 + v_1(\tau - \tau_0)$$

in probability, where the vector v_1 has zero second coordinate (that is, looks along the face 1).

Proof. The proof is similar to that for classical random walk in Z_+^2 (see [2]).

3.7 Escape to infinity along a face

Here we want to find the analog of the proposition 1.2.3 of [8].

Theorem 13 *Assume IIC. Assume that for the induced chain $\mathcal{M}_\Lambda$ there exists an invariant measure μ_Λ such that $v_i(\mu_\Lambda) > 0, i \in \Lambda$. Then*

1. *The chains $\mathcal{M}_\Lambda$ and $\mathcal{M}$ are transient;*
2. *$\frac{n_i(t)}{t} \to v_i$ for all $i \in \Lambda$ with positive probability;*

Proof. It is similar to one of Proposition 1.2.3 in [8] using Birkhof ergodic theorem for functionals n_i of the stationary Markov chain on $\mathcal{A}_0^\Lambda \times \mathcal{A}^{\bar{\Lambda}}$.

4 Macroprocesses on R_+^M

4.1 Scaled dynamics on the measure bundle

It is known that often the dynamical semigroup can be defined only on the tangent bundle but not on the configuration space. Similar situation occurs for evolution of strings for $r > 1$: we have to consider dynamics on the measure bundle over R_+^M instead of R_+^M itself for $r = 1$..

Definition 2 *Measure bundle over R_+^M is the set of pairs (x, μ) where $x \in R_+^M \setminus 0$ and μ is an invariant measure on the induced chain for the face Λ of R_+^M to which x belongs or on some face Λ_1 closure of which contains Λ.*

We assume also that in the latter case μ is an outgoing measure for Λ, that is the corresponding drift vector has positive components perpendicular to Λ.

To understand the intuition behind this definition we explain it in the simplest case. Assume that we start from the string α such that $\mid \alpha \mid = [xN], x \in R_+^M$ has all coordinates positive. Assume that x is fixed but otherwise α is taken random as the restriction of stationary process with distribution μ. Then with Euler scaling for sufficiently small macrotime τ we shall move by the Law of Large Numbers with velocity $v(\mu)$ (in fact, until we reach the boundary of R_+^M).

Now we define the dynamics in the general case. It is a semigroup defined by: if at time 0 we have

$$x(0) = x \in \Lambda, \mu(0) = \mu$$

and μ is a measure for some face Λ_1 closure of which contains Λ (it can be that $\Lambda_1 = \Lambda$).Then

$$x(t) = x + tv(\mu), \mu(t) \equiv \mu$$

until $x(t)$ reaches the boundary of Λ_1. At time t_1 when it reaches the boundary of Λ_1 at the first time at some point $y \in \partial\Lambda_1$ the dynamics becomes discontinuous and defined as follows

$$x(t+0) = y, \mu(t+0) = \nu$$

where (y, ν) is an invariant measure on $\partial\Lambda_1$ or an outgoing invariant measure on some face Λ_2, the closure of which contains $\partial\Lambda_1$. This transition is defined by the COLLISION or SCATTERING operator S which consists of operators S_Λ for all Λ:

$$S_\Lambda : H(in, \Lambda) \to H(out, \Lambda)$$

where $H(in, \Lambda)$ is the set of all ingoing (we already defined outgoing measures, ingoing are defined similarly, see also [8] about similar definitions) measures for Λ, that is measures on the faces of dimension $\mid \Lambda \mid +1$, closure of which contains Λ, and such that their drift is negative in the direction perpendicular to Λ. In $H(out, \Lambda)$ we include all outgoing measures and measures on Λ itself.

In the next section we shall give examples of these collision operators for two-dimensional case. Later we shall give a general definition of collision operators.

However, these dynamical systems help only in two cases: when all trajectories lead to zero or when there exists a face, all trajectories on which lead to infinity. In network examples, that were solved until now by the queueing theory community, only such situations occur. In more complicated situations one should choose different strategy.

4.2 Collision operators in 2-dimensional case

We give now definition and classification of collision operators for the 2-dimensional case.

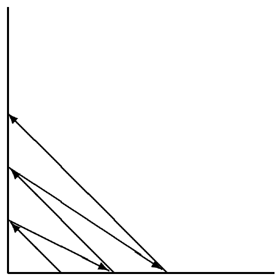

Figure 1: Strange escape

Assume μ to be (+-) invariant measure for the interior and

$$\mu_1 = Lim(P_1(\mu))$$

Denote then by $H(out, \mu_1)$ the set of all invariant (-+) measures ν or invariant measures ν for one-dimensional face 1, with

$$\mu_1 = Lim(P_1(\nu))$$

. Then collision operator S (on face 1) maps the ingoing measure μ to some randomly chosen measure on $H(out, \mu_1)$. If this choice is always unique, we call a collision operator deterministic.

Theorem-Hypothesis 6 *Under IIC condition $H(out, \mu_1)$ consists of only one measure.*

This can be proven if Lyapounov condition for μ holds. If one of the two 1-dimensional faces has no outgoing pairs or both 1-dimensional faces have no outgoing pairs, then the classification of theorem 5 above holds.

In this case for any initial condition after finite macrotime we come to this 1-dimensional face, say face 1. Afterwards, we go along face 1.

In the following subsection we give examples when both 1-dimensional faces have outgoing pairs. This will give us "strange" ways of escape to infinity, see Figure 4.2.

4.3 Deterministic Reflections

Let a partition of the alphabet be given

$$\{1,...,r\}=\bigcup_{i=1}^{s}F_i, F_i\bigcap F_j=\emptyset, i\neq j$$

Evolution of two strings $(\alpha(t),\beta(t))$ is defined by the point $p\in\mathcal{P}$, where $\mathcal{P}$ is the parameter space. We denote symbols of string α at time t by $x_k(t)$,and those of string β by $y_k(t)$.

For given parameters p call a pair of subsets (F_i,F_j) a stable pair if, assuming $x_k(0)\in F_i, y_k(0)\in F_j$ for all k (further on we shall use the following abbreviation instead: $\alpha\in F_i,\beta\in F_j$), we have $x_k(t)\in F_i, y_k(t)\in F_j$ a.s. until at least one of the strings becomes empty.

Definition 3 *We call state (α,β) a pure state if each string is either empty or all its symbols belong to one of F_i.*

We call F_i quasiirreducible classes if the following conditions hold:

- *Each pair (F_i,F_j) is stable;*
- *The set of all pure states is a closed class for our countable Markov chain*

Definition 4 *We call ECIW on Z_+^M deterministic if all collision operators is deterministic*

Definition 5 *Assume that F_i are quasiirreducible. We call pair (i,j) a right admissible pair (and we shall write $i\to j$) if the following conditions hold:*

- *If both strings are nonempty and $\alpha\in F_i,\beta\in F_j$ then α influences on β but not vice vers. This means that α moves independently of β;*
- *String α has an invariant (-)-measure while being in F_i (i.e. restricted to F_i).*

  Moreover, there exists $v>0$ such that for all initial conditions

  $$n(\alpha(t))\leq n(\alpha(0))-vt$$

  until the first moment when $\alpha(t)$ is empty;

- *If $\alpha \in F_i, \beta \in F_j$ then string β tends to infinity and for all initial conditions*
$$n(\beta(t)) \geq n(\beta(0)) + (v+\epsilon)t$$
for some $\epsilon > 0$ until one of the strings becomes empty;

Symmetrically one can define left admissible pairs (i,j) and write $i \leftarrow j$.

Theorem 14 *Let $r = 3, F_i$ are one-point sets, consisting of points i correspondingly, and assume the following pairs to be admissible*

$$1 \to 2, 3 \leftarrow 2, 3 \to 1, 2 \leftarrow 1, 2 \to 3, 1 \leftarrow 3$$

Assume also that

- *If string 2 is in F_2 and string 1 is empty then next moment string 1 is in F_3;*
- *If string 1 is in F_3 and string 2 is empty then next moment string 2 is in F_1;*
- *If string 2 is in F_1 and string 1 is empty then next moment string 1 is in F_2;*
- *If string 1 is in F_2 and string 2 is empty then next moment string 2 is in F_3;*
- *If string 2 is in F_3 and string 1 is empty then next moment string 1 is in F_1;*
- *If string 1 is in F_1 and string 2 is empty then next moment string 2 is in F_2.*

Then we go to infinity as it is shown on Figure 4.2 . More exactly, $n_1(t) + n_2(t) \to \infty$ but each $n_i(t)$ becomes empty infinitely often a.s.

This is a corollary of the theory of random walks in two-dimensional complexes (see [1]).

It was the simplest example of such behaviour but there can occur more complicated phenomena: scattering and asymptotic invariant measures.

Scattering The following case also can be completely immersed into the theory of random walks in two-dimensional complexes as developped in [10]: assume that each F_i consist of only one element i and that for each k only the states when either $n_k = 0$ or when n_k contains only one type of elements. But if, for example, $n_1 > 0, n_2 = 0$ then any type i can appear in the second string with some probability $p(1,k;2,i)$ where k is the type of string 1.

Asymptotic invariant measures. If F_i are not one point but assume the conditions similar to those of the previous theorem. Assume that for admissible pair $i \to j$ there is an invariant (-+)-measure μ_{12}^{ij} with "increasing" marginal μ_2^j (i.e. along which we go to infinity). Then along this measure we "come finally" to axis 2. Then this "increasing" marginal becomes a "decreasing marginal" (we go backwards along the same axis) for some invariant (+-)-measure μ_{21}^{kj}, assuming that $k \leftarrow j$ is an admissible pair. This defines the scattering operator on the axis 2

$$S\mu_{12}^{ij} = \mu_{21}^{kj}$$

Note that in this case the scattering operator is deterministic. The scattering operator on the other axis is defined similarly.

Theorem 15 *Both scattering operators exist here and are deterministic.*

We do not prove the general theorem about the existence of the scattering operator but one can give many examples when it can be easily proven.

One can easily construct examples with more admissible pairs where S is random but takes only finite number of possible values. Also the scattering theory on two-dimensional complexes covers classification results for this case.

Very important is the following continuity property of this operator: S is a continuous mapping (in the weak topology).

4.4 Classification theorem in 2 dimensions

Theorem-Hypothesis 7 *Up to a set of parameters of measure zero we have the following assertion. If there is no scattering then the particle is transient if there exists at least one trajectory of the dynamical system going to infinity. Otherwise it is ergodic.*

Under some additional constraints this theorem is proved in [13]. Intuitive proof is the following. Due to the continuity property, if there exists one trajectory going to infinity then there is an open set of such trajectories. And then one can construct Lyapounov function on this subset.

In the presence of scattering it is necessary to consider more refined techniques, see below.

4.5 Local Lyapounov functions

To prove the existence of Euler limit or finite scattering in the vicinity of nonergodic face one needs Lyapounov functions or some special structure of the random walk. For example FIFO networks, where simpler method to prove convergence to the fluid model exists. For the latter case one considers one node, input and output flows to it. Assuming that there is a scaling limit for the input flows one proves then that it exists also for output flows. Then such local pictures for different nodes are glued together.

As for the first method, we shall formulate exactly definitions of local Lyapounov functions for the case when on each face only one type of invariant measures exists.

Condition 6 *(Local Lyapounov Condition) For ergodic faces we assume the existence of Lyapounov functions in all perpendicular directions, which forces a particle to reach quickly this face. For a nonergodic faces Λ we assume that, uniformly in initial conditions, the perpendicular ingoing (to Λ) coordinates have a Lyapounov function, which forces the particle to reach this face quickly, and there exists unique ergodic outgoing face with Lyapounov function, forcing the particle to go quickly from Λ.*

Under this condition the existence of Euler scaling limit is easily proved, see [13] for the two-dimensional case.

4.6 General Definition of Collision Operators

Let us consider some ingoing measure μ (i.e. a measure with its drift vector looking to the face) and take its projection $\mu_\Lambda = P_\Lambda \mu$ on Λ. If there exists only one outgoing measure ν with $\mu_\Lambda = P_\Lambda \nu$ then we define

$$S\mu = \nu$$

If such situation holds for all Λ and all μ then we call macroprocess deterministic.

In more general cases one should prove the existence of the stochastic scattering kernel. The operator S becomes a stochastic operator: for each Λ and each ingoing measure μ there is a distribution $S(\mu, \nu)$ on outgoing measures ν for the face Λ. There are examples where the stochastis kernel is constructed in the following way. First for each nonergodic Λ a special LOCAL chain is constructed. In terms of this chain ν are the elements of the exit boundary for this local chain. For each μ the kernel $S(\mu, \nu)$ defines a distribution on the exit boundary. So, if for ergodic faces the main instruments are induced chains, defining deterministic evolution along the face, then for non ergodic these are LOCAL chains.

All kernels taken together define Markov process on R_+^M.

A discrete distribution is the simplest particular case (see [1]) but also continuous distributions can exist even for $r = 1$. The simplest example is when Z^2 is subdivided onto 4 quadrants $Z_{++}, Z_{+-}, Z_{-+}, Z_{--}$. In each quadrant we have a homogeneous random walk, the drifts of which are shown on Figure 4.6 . Assume that on the boundaries (half axes) the jumps are the same as inside the outgoing quadrant for the corresponding half axis. Define the vector field $v(x)$ in R^2 constant on each quadrant of the plane. Let C be the isochronous curve: $C = \{x :$ time to reach zero by the reversed process (along the vector field $-v(x)$) is $1\}$.

Theorem-Hypothesis 8 *Euler limit exists in distribution and the limiting distribution has its support on the isochronous curve. The limiting distribution has a density on the isochronous curve.*

First assertion can be proven using a trick similar to one used in the proof of van Hove theorem in statistical mechanics.

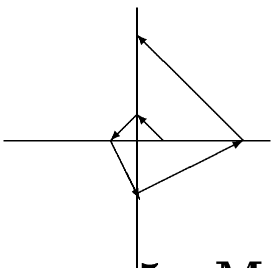

Figure 2: Continuous scattering

5 Macroprocesses on compact measure bundles

Here we can give only very brief exposition, quite parallel to [8]. all notation and definitions can be taken from there.

Let ϕ be the radial projection of R_+^M onto the simplex

$$B^{M-1} = \{(x_1, ..., x_M) : \sum_i x_i = 1\} \subset R_+^M$$

And let U^τ be the radial projection of the dynamical system $T^{\tilde{\tau}}$, accompanied with the time change $\tau = \tau(\tilde{\tau})$, this time change is chosen similarly to [8].

This induces the corresponding map $\phi \times 1$ of the measure bundle over R_+^M onto the measure bundle over B^{M-1}.

The important tool also is dimensional reduction: we can consider only the restriction of U^τ onto the boundary ∂B^{M-1} of B^{M-1} because after some finite time reaches ∂B^{M-1} and stays there forever, except for the trivial case when the drift vector inside the orthant have all its coordinates positive. Note that ∂B^{M-1} is isomorphic to the $(M-2)$-dimensional sphere S^{M-2}. Thus we get a dynamical system on a compact manifold.

It can be that this process of reduction can be continued further. Then one can get dynamical systems on more complicated manifolds $\mathcal{S}$ (with or without boundary) but having smaller dimension.

According to the results for $r = 1$ continuity property of the process could be important. So we should introduce some topology on the measure bundle, using Lebesgue topology on euclidean space and weak topology on the measure spaces.

5.1 Main Theorem

We take definitions of the main constants L and M from [8].

Theorem 16 *Assume that the macroprocess is deterministic and the corresponding dynamical system has finite number of continuous and uniquely (strictly) ergodic components $i = 1, ..., k$. Let ρ_i be the normalized nonnegative invariant (with respect to the macroprocess) measure for i-th component.*

- *If the dynamical system is uniquely ergodic then $M = L$ and the system is ergodic if $L < 0$;*
- *Then under assumption of uniform convergence similar to one in chapter 3 of [8] and at least one constant L_{ρ_i} is larger than zero then the process is transient.*
- *Under the same conditions, if all constants L_{ρ_i} are less than 0 then the process is ergodic.*

Proof. First assertion ca be proved exactly as the theorem 3.1.1 in [8]. Other assertions are also similar to theorems 3.2.1, 3.3.1; 3.3.2 of [8].

6 Applications

6.1 Applications - Noncommutative Groups

Our results for one string cover many problems for free and modular groups, considered earlier (see [18] and many papers in the Annals of Probability).

These groups enjoy the property of boundedness of jumps condition for the corresponding string problem. From our point of view they are characterized by the following property: if at time t the state of the process is the word $\alpha(t)$ then at time $t+1$ the word $\alpha(t+1)$ differs from $\alpha(t)$ by not more than d symbols at the end. Thus, the random walks on such groups produces only local changes at the end of the word. Remind that random walk on G is defined by the set of parameters $\{p_i\}_{i=-r,\dots,r}$, $\sum_i p_i = 1$, $p_i > 0$, defining probabilities of transitions $\rho \to \rho e_i$ where e_i are the generators for $i > 0$, $e_0 = 1, e_i = e_{-i}^{-1}, i < 0$.

This is no longer true for local groups: one step transition can change the string on all its length. In the context of groups one comes very naturally to the new realm of problems for string evolution where boundedness of jumps is no more valid.In [21] locally free groups were introduced as intermediate in complexity between free, modular groups from one side and braid groups from the other side. Braid groups have important applications in quantum field theory.

More exactly, consider the following class of (obviously non-amenable) groups G with generators $S = \{e_i\}, i = 1, \dots, r, -1, \dots, -r$, $e_i^{-1} = e_{-i}$ and the only defining relations:

$$e_i e_j = e_j e_i$$

for some set I of couples (i, j).

Thus, the group is completely defined by the set I. Note that two generators e_i, e_j are either commutative $e_i e_j = e_j e_i$ or generate a free subgroup of the group G.

We call such groups locally free following [21], where a special case of I was considered.

A. Gajrat proved transience, law of large numbers and central limit theorem for this class of groups.

6.2 Other Environments

6.2.1 Double-Sided Evolution

Right-sided evolution of the string was defined by the following one-step transition probabilities:

$$q_r(x, \emptyset), q_r(x, y), q_r(x, yz)$$

The stabilization laws define the following (implicit) functions of $\{q_r\}$

$$v_r > 0, \mu_r, v_{r,\mathrm{erg}}(\mu) > 0$$

where

in the transient case if $t \to \infty$ then

$$\frac{a(t)}{t} \to v_r$$

for the particle coordinate $a(t)$ (right end of the string) and some constant $v_l > 0$;

in the transient case μ_r is the invariant measure and $Lim(\mu)$ is its limiting measure;

in the ergodic case with initial condition defined by the restriction of the stationary process η on Z, we shall denote η also the distribution of this process, we have

$$\frac{a_t}{t} \to -v_{r,\mathrm{erg}}(\mu)$$

In a similar way we can define evolution of the left end of reversed (lefthanded) strings. Denote the corresponding parameters by

$$q_l, v_l > 0, \mu_l, v_{r,\mathrm{erg}}(\eta) > 0$$

where for example in the ergodic case

$$\frac{b_t}{t} \to -v_{r,\mathrm{erg}}(\mu),$$

where b_t is the coordinate of the left end.

Two-sided evolution of finite strings is defined by independent evolution of the left and right ends by the set of parameters q_l, q_r. It appears that double sided evolution can be studied in terms the three parameters for one sided evolution $v_l, \mu_l, v_{l,\mathrm{erg}}(\eta)$ plus their couterparts. The most difficult case is when one string, say the leftsided , is transient and the right string is ergodic. Our main result is that in the two-sided evolution the length of the string tends to infinity a.s. if

$$v_l > v_{r,\mathrm{erg}}(Lim\mu_l)$$

and its mean length stays bounded if

$$v_l < v_{r,\mathrm{erg}}(Lim\mu_l)$$

See complete exposition and proofs in [6].

6.3 Random Turing Machine

Let $\mathcal{L}$ be a discrete time Markov chain on the state space $X = Z_+ \times \{1, \ldots, r\}^{Z_+}$. X is the set of pairs (x, α) with $n \in Z_+$ the position of the particle (or head of Turing machine, Z_+ is considered as the tape of Turing machine) and α the configuration (or environment, or a program on the tape of Turing machine) of spins on Z_+. If $n > 0$ is the position of the particle at time t and $\alpha(n) = a$ the value of the spin in n, then there are two posibilities:

1. With probability q_b^a the particle jumps to $n-1$ at time $t+1$ and the spin in n changes from a to b.

2. With probability $\overline{q}_b^a$ the particle jumps to $n+1$ and the spin in n changes from a to b.

The spins in the other points do not change at time $t+1$.

The Markov chain $\mathcal{L}$ is completely defined by the matrices $Q = \{q_b^a\}$ and $\overline{Q} = \{\overline{q}_b^a\}$, $a, b \in \{1, \ldots, r\}$. It is clear that $Q + \overline{Q}$ is a stochastic $r \times r$ matrix. We assume that $q_b^a > 0, \overline{q}_b^a > 0$ for all a, b.

By (n_t, α_t) we denote the state of the process at time t: n_t is the position of the particle and α_t the configuration on Z_+.

Let $\tau(\alpha)$ be the first time for the particle to visit 0, if the initial state is $(1, \alpha)$:

$$\tau(\alpha) = \min\{t : \quad n_t = 0 \mid n_0 = 1, \alpha_0 = \alpha\}$$

We will use the following definitions.

Definition 6 *$\mathcal{L}$ is ergodic for given α if $P(\tau(\alpha) < \infty) = 1$, i.e. the particle visits point 0 with probability 1, and there exists T, such that $E\tau(\alpha) < T$. We call it uniformly ergodic if the mean time of the first visit to 0 is bounded uniformly in α*

Definition 7 *$\mathcal{L}$ is recurrent for given the initial state $(1, \alpha)$ (uniformly recurrent) if and only if $P(\tau(\alpha) < \infty) = 1$ (for all α)*

Definition 8 *$\mathcal{L}$ is transient for given α (uniformly transient) if and only if there exists $p < 1$, such that $P\{\tau(\alpha) < \infty\} < p$ (for all α).*

Theorem 17 *Let π_0 be the stationary distribution of the stochastic matrix $Q + \overline{Q}$ and*

$$q \stackrel{\text{def}}{=} \pi_0 Q \vec{1},$$

$$\overline{q} \stackrel{\text{def}}{=} \pi_0 \overline{Q} \vec{1}.$$

Then i) $\mathcal{L}$ is uniformly ergodic if and only if

$$q > \overline{q}. \tag{1}$$

ii) $\mathcal{L}$ is uniformly transient if and only if

$$q < \overline{q}. \tag{2}$$

The case $q \neq \overline{q}$ is said to be the *non-critical case.* In the critical case (i.e. $q = \overline{q}$) there is the following result.

Let α be a homogeneous configuration, i.e. $\alpha = ddd\ldots$, or equivalently $\alpha(i) = d$ for all $i \in Z_+$, and some $d \in \{1, \ldots, r\}$.

First of all we introduce some notation:

$$\begin{aligned} a &= \sum_{l=0}^{\infty} (\delta_d - \pi) P^l A \vec{1}, \\ b &= 1 + \sum_{l=0}^{\infty} (\pi A - \pi) P^l A \vec{1}, \end{aligned}$$

where , $P = (I - \overline{Q})^{-1} Q$, π is the stationary distribution associated with the stochastic matrix P, δ_d - δ-measure at the point d on $\{1, ..., r\}$.

Theorem 18 *Let the initial state be* $(1, \alpha)$*. Then:*

- *If* $a < -b$*, then* $\mathcal{L}$ *is ergodic;*
- *If* $a > b$*, then* $\mathcal{L}$ *is transient;*
- *If* $-b < a < b$*, then* $\mathcal{L}$ *is null recurrent*

6.4 Multiclass Queueing Networks

FIFO queue with sinchronization constraints and several customer types in Markovian case is a particular case of the double sided evolution, considered above. So, FIFO networks is a system of M double sided strings. But in FIFO networks another (conventional) language also proved to be useful. The following are the results about behaviour of some classes of FIFO networks in the framework of Euler scaling, which is called fluid limit in these papers:

- First of all deterministic networks (see [24], [25]) are of interest because they can be examples of dynamical systems which can occur. In many cases ergodicity conditions depend ONLY on the nature of the dynamical system (note that dynamical system itself depends essntially on all transition probabilities);
- Fluid models related to some networks are studied in [28];
- Existence of Euler limit (in a somewhat weaker) sense is proved in [32]. The formulation of the result makes no difference between deterministic and scattering cases;
- Ergodicity conditions are studied in the papers [29, 30, 31, 33]
- Different (non fluid) methods for studying ergodicity conditions are proposed in [26], [27].

6.5 Neural Networks

The case $r = 1$ has now many applications to the dynamics of neural networks. The first application was to the less known sandglass model (see [22] and references therein). Recently it was understood that it covers also Figotin-Pastur-Hopfield model with finite number of images (see [23]).

References

[1] G.Fayolle, V.A.Malyshev, M.V.Menshikov. Topics in Constructive Theory of Countable Markov Chains. Cambridge University Press. 1995.

[2] V.A.Malyshev. Stabilization laws for processes with a localized interaction. Rapport de Recherche INRIA, 1992, no.1635.

[3] V.A.Malyshev. Evolution of random string: stabilization laws. Problems of Information Transmission. v.30, 3, 1994, p.79-95

[4] A.S.Gairat, V.A.Malyshev, M.V.Menshikov, K.D.Pelih. Classification of Markov chains describing the evolution of a string of characters. Uspehi Matematicheskih Nauk, v.50,No. 2 (302).1995, p.5-24

[5] A.S.Gairat. Random walk of an active particle in nonhomogeneous environment. Markov Processes and Related Fields v.1.N1,1995,p.91-113

[6] A.S.Gairat, V. Malyshev, A. Zamyatin. Two-sided evolution of a random chain. Markov Processes and Related Fields v.1.No. 2,1995,pp. 281-316.

[7] A.Gairat, R.Iasnogorodski, V.Malyshev. Null Recurrent Random String. *Markov Processes and Related Fields*, 2, 3, 1996, pp. .

[8] V.A.Malyshev. Networks and Dynamical Systems. *Adv.Appl.Prob.*,25, pp. 140-175,1993.

[9] H. Kesten, M. Kozlov, F. Spitzer. A limit law for random walk in a random environment. Compositio Mathematica, 1975, v. 30, 2, pp. 142-168.

[10] G.Fayolle,I.A.Ignatyuk, V.Malyshev, M.Menshikov. Random walks in two-dimensional complexes. Queueing Theory, 9, 3, pp. 269-300, 1991.

[11] F. Spitzer. Principles of Random Walk. Van Nostrand. New York. 1964.

[12] A. Zamyatin, A. Yambartsev. Transient dynamics of two interacting words. Fundamental and Applied Mathematics, v. 2, 1996 (in Russian)

[13] A. Zamyatin, A. Soboleva. Ergodicity and transience conditions for two strings. Preprint. 1996.

[14] F. Kelly. Reversibility and Stochastic Networks. John Wiley. 1979.

[15] F. Baskett, K.M. Chandi, R.R. Muntz, F.G. Palacios. Open, closed and mixed network of queues with different classes of customers. Journ. Assoc. Computing Machinery, vol. 22, no. 2, 1975, pp. 248-260.

[16] N. Andreenko, A. Gajrat, A. Zamyatin. Preprint. Moscow State University. 1996.

[17] R.I. Grigorchuk. Symmetrical Random Walks on Discrete Groups. In "Multicomponent Random Systems. " Adv.Prob. Rel. Topics, v.6" Marcel Dekker. 1980. p. 285-326.

[18] E.B. Dynkin, M.B. Malyutov. Random walks on groups with finite nnumber of generators. Doklady of Ac. Sci. USSR, 1961, v. 2, pp. 399-402.

[19] S. Sawyer, T. Steger. The rate of escape for anisotropic random walk in a tree. Prob. Theory and Rel. Fields. 1987, v. 76, p. 207-230.

[20] Lalley S. Finite range random walk on free groups and homogeneous trees. Annals of Probability, 1993, Vol. 21, No. 4, 2087-2130.

[21] S.K. Nechaev, A.Yu. Grosberg, A.M. Vershik. Random Walks on Braid Groups: Brownian Bridges, Complexity and Statistics. Preprint. 1995.

[22] F. Karpelevich, V. Malyshev, A. Rybko. Stochastic Evolution of Neural Networks. Markov Processes and Related Fields, v.1.No. 1,1995,pp. 141-161.

[23] V. Malyshev, F. Spieksma. Dynamics of Binary Neural Networks. ESI preprint, no. 360, Wien, 1996.

[24] T.I. Seidman. First Come, First Served can be unstable !", IEEE Transactions on Automatic Control, 1994, 39,10.

[25] G.I. Winograd and P.R. Kumar. The FCFS service discipline: stable network topologies, bounds on traffic burstiness and delay, and control by regulators", Preprint. 1994

[26] M. Bramson. Instability of FIFO queueing networks. Annals of Applied Probability, 1994, v. 4, pp. 414-431.

[27] M. Bramson. Instability of FIFO queueing networks with quick service times. Annals of Applied Probability,1994, v. 4, pp. 693-718.

[28] V. Dumas. Instability cycles in fluid Bramson networks. submitted to "IEEE Transactions on Automatic Control" , 1995, May.

[29] H. Chen and H. Zhang. Stability of multiclass queueing networks under FIFO service discipline. Preprint. 1994

[30] A.N. Rybko and A.N. Stolyar. Ergodicity of stochastic processes describing the operation of open queueing networks, Problems of Information Transmission , 1992, 28, 3, 3-26.

[31] M. Bramson. Convergence to equilibria for fluid models of FIFO queueing network. Queueing Theory,1996, 22, 5-45.

[32] A. Stolyar. On the stability of multiclass queueing networks: a relaxed sufficient condition via limiting fluid processes. Markov Processes and Related Field, 1995, v.1, No.4, pp. 491-512.

[33] S.G. Foss and A.N. Rybko, Stability of multiclass Jackson-type networks. Markov Processes and Related Field, 1996, v.2, No.3, pp. 461-486.

Unité de recherche INRIA Lorraine, Technopôle de Nancy-Brabois, Campus scientifique,
615 rue du Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY
Unité de recherche INRIA Rennes, Irisa, Campus universitaire de Beaulieu, 35042 RENNES Cedex
Unité de recherche INRIA Rhône-Alpes, 655, avenue de l'Europe, 38330 MONTBONNOT ST MARTIN
Unité de recherche INRIA Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex
Unité de recherche INRIA Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex

Éditeur
INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
ISSN 0249-6399