An Algebraic Design for the Simultaneous Stabilization of Two Systems

Christophe Fonte, Christine Bernier-Kazantsev, Michel Zasadzinski

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

An algebraic design for the simultaneous stabilization of two systems.

Christophe Fonte , Christine Bernier-Kazantsev , Michel Zasadzinski

N° 3043

Novembre 1996

THÈME 4b

Rapport de recherche

ISSN 0249-6399

An algebraic design for the simultaneous stabilization of two systems.

Christophe Fonte [*], Christine Bernier-Kazantsev [†], Michel Zasadzinski [‡]

Thème 4b — Simulation et optimisation
de systèmes complexes
Projet NUMATH

Rapport de recherche n° 3043 — Novembre 1996 — 36 pages

Abstract: In this report we present a unified description for studying the problem of the simultaneous stabilization of two plants. Three approaches for the simultaneous stabilizability are defined. The first one corresponds to the definition commonly used in the literature. For the third one, we show that, like for the first one, the design of a simultaneous compensator leads to a divisibility condition in the ring of RH_∞. A simple formulation of the existence condition for simultaneous stabilization is proposed. Moreover, the equivalence between the existence conditions for the first and third approaches is shown. Finally an explicit method is given to compute simultaneous compensators.

Key-words: Simultaneous stabilization, strong stabilization, unit interpolation, Youla-parametrization.

(Résumé : tsvp)

[*] CRAN, CNRS URA 821, Université H. Poincaré-Nancy 1, 186 rue de Lorraine , 54400 Cosnes et Romain, France.
[†] Institut Elie Cartan, CNRS UMR 9973, INRIA-Lorraine, Université H. Poincaré-Nancy 1, 54506 Vandoeuvre-les-Nancy.
[‡] CRAN, CNRS URA 821, Université H. Poincaré-Nancy 1, 186 rue de Lorraine, 54400 Cosnes et Romain, France.

Unité de recherche INRIA Lorraine
Technopôle de Nancy-Brabois, Campus scientifique,
615 rue de Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY (France)
Téléphone : (33) 83 59 30 30 – Télécopie : (33) 83 27 83 19
Antenne de Metz, technopôle de Metz 2000, 4 rue Marconi, 55070 METZ
Téléphone : (33) 87 20 35 00 – Télécopie : (33) 87 76 39 77

Approche algébrique pour la stabilisation simultanée de deux systèmes.

Résumé : Dans ce rapport de recherche nous présentons le problème de la stabilisation simultanée de deux systèmes. Nous définissons ce problème de trois manières. La première correspond à la définition habituelle présentée dans la littérature. Nous montrons que pour la troisième méthode ainsi que pour la première, l'étude du compensateur simultané conduit à une condition de divisibilité dans RH_∞ et à ce propos, nous proposons une nouvelle formulation de la condition d'existence. De plus nous montrons l'équivalence des conditions d'existence pour la première et la troisième approche. Finalement une méthode de synthèse explicite est donnée afin de construire des correcteurs simultanés pour deux systèmes.

Mots-clé : Stabilisation simultanée, stabilisation forte, interpolation de fonctions rationnelles unimodulaires dans RH_∞, paramétrage de Youla.

Notations.

$R^p[s]$	:	The set of proper rational functions with real coefficients.
$R[s]$	:	The set of rational functions with real coefficients.
RH_∞	:	The set of proper and stable rational functions with real coefficients.
H	:	The set of Hurwitz's polynomials.
$\mathbb{R}[s]$	:	The set of polynomials with real coefficients.
R_{+e}	:	Positive reals including *0* and ∞.
U	:	The set of units in RH_∞.
C_{+e}	:	Re(s) $\geq$*0* including ∞.
C_+	:	Re(s) $\geq$ *0*.
c.f.	:	Coprime factorization over RH_∞.

1 Introduction.

In automatic control, the simultaneous stabilization problem may arise in many situations. One is the control with integrity: the same compensator must guarantee stability and performance requirements even if some sensors, transducers or actuators fail. A second example is the control of a nonlinear plant with operating point changes: all the linearized models obtained at each operating point must be stabilized with only one compensator. Then the simultaneous stabilization is an important subject with rising interest of the control community.
One of the first to study this question in an intensive way was Vidyasagar et al. [5]. The main contribution of [5] was to connect the simultaneous stabilization to the strong stabilization (i.e. the stabilization of a plant by a stable compensator). The strong stabilization was solved by Youla et al. in [8] where necessary and sufficient conditions for existence of a stable compensator in the scalar case are provided: the only condition to check is the relative location of the unstable real poles and zeroes. This condition is known as the parity interlacing property denoted p.i.p. An efficient extension of this result to the simultaneous stabilization of two plants in the multivariable case was completed by Vidyasagar et al. [5] by replacing the zeroes in the scalar case by the blocking zeroes. Another important point proven in [5] is that the strong stabilization is equivalent to an euclidean division in the RH_∞ ring under the constraint that the remainder is a unit in RH_∞. The contribution of [5] was also the use of the strong stabilization in the simultaneous stabilization of N plants ($N \geq 2$): to stabilize simultaneously N plants is equivalent to stabilize $(N-1)$ associated plants with a stable compensator. The reader can find an overview of these results in [6]. After these introductive contributions of Youla et al. [8] and Vidyasagar et al. [5], the strong stabilization and simultaneous stabilization problems became an active field of research. In [1], Blondel showed that, unlike for two plants, there does not exist necessary and sufficient conditions for the simultaneous stabilization when N is strictly greater than two. In this case, the simultaneous stabilization is a rationally undecidable question, and thus only sufficient or necessary conditions may be found.

In addition, Blondel [1] and Blondel et al. [2] introduced the concept of avoidance which can be used instead of the strong stabilization to check the existence of a simultaneous compensator. Wang et al. [10] and Gündes et al. [4] also developed robustness considerations in the simultaneous stabilization problem for uncertain systems. A common feature of these different approaches [1], [5], [8] and [10] is an intensive use of the fractional representation, especially the coprime factorization over RH_∞. This interest is due to stability notions easier to use with rational functions in RH_∞ than with a polynomial formulation, particularly in the case of poles zeroes cancellations at infinity. Note that a polynomial version of the simultaneous stabilization problem has been treated by Wei in [7] as a transcendental problem. Several numerical methods exist to compute simultaneous compensators. They require the manipulation of some complex mathematical concepts and especially algebraic tools such as unit interpolation in RH_∞.

The main contribution of our work is to study different notions of simultaneous stabilizability of two plants. The simultaneous stabilizability is introduced by using the Youla-parametrization also called the Q-parametrization. We consider a controller stabilizing a given plant and we express a constraint on the Q-parametrization such that an another given plant can be also stabilized by the modified compensator obtained from the Q-parametrization. The motivation to use the Youla-parametrization is to provide a systematic way to generate simultaneous compensators for a pair of plants if we have already stabilized a first one, i.e. if we claim a unit constraint on the first. Then, from the stabilization of the first system, the free parameter Q can be selected in the linear space corresponding to this unit constraint.

In our approach, only one parameter in RH_∞ must be determined by solving an appropriate euclidean division in RH_∞ with remainder in U, in contrast to [5] and [6] where the computation of two parameters in RH_∞ is required, namely one for each plant to stabilize (see equation (5.4.6) in [6]). The proposed approach allows us to give necessary and sufficient conditions based on evaluations of one of the plant coprime factors at the unstable real zeroes of the difference of an associated plant described by the coprime factors of the two plants. Unlike in the p.i.p. condition given in [5] and [8], we do not consider unstable real poles of an associated plant.

This report is organized as follows. Section 2 is devoted to the mathematical background on coprime factorization over RH_∞ and its use to the parametrization of stabilizing controllers. Two new definitions to characterize the set of all stabilizing controllers are introduced. In section 3, these definitions allow us to describe three ways for stabilizing two plants with the same compensator simultaneously. Section 4 considers only the third approach and show that in this case the simultaneous stabilization problem can be reduced to search one parameter in RH_∞ by solving an euclidean division in RH_∞ with remainder in the set of units in RH_∞. In this section, an another formulation than this given in [5] of the necessary and sufficient conditions for stabilizing simultaneously two plants is presented. This formulation does not use the Bezout's identity. Finally, in section 5, an algorithm summarizes this approach to design simultaneous compensators, and an illustrative example is given.

For readability we use these simplified notations: when we write the system $P = ND^{-1}$, we consider the pair (N, D) and not the ratio ND^{-1}. Similarly, when we write the compensator $C = RS^{-1}$, we consider the pair (R, S) and not the ratio RS^{-1}.

2 Preliminaries and definitions.

Consider an element M of $R[s]$, which does not admit an unstable pole-zero cancellation. Then there exist X and Y belonging to RH_∞ such that $M = XY^{-1}$ where X and Y have no common unstable zero [6]. The pair (X, Y) is called a coprime factorization (c.f.) of M over RH_∞. To each plant $P_i \in R^p[s]$, we associate a c.f. denoted by (N_i, D_i) with $P_i = N_i D_i^{-1}$. Similarly, to each compensator $C_i \in R[s]$ corresponds a c.f. denoted by (R_i, S_i) with $C_i = R_i S_i^{-1}$. As in [6], the degree of $X \in RH_\infty$ denoted by $\delta(X)$ is defined as

$$\delta(X) := \{\textit{number of zeroes of}\, X \text{ in } C_+\} + \{\textit{relative degree of}\, X\}.$$

Let (N_i, D_i) and (R_i, S_i) be c.f. of the plant $P_i \in R^p[s]$ and of the compensator $C_i \in R[s]$ respectively. According to [6], we say that C_i stabilizes P_i if and only if $\Phi(N_i, D_i, R_i, S_i) \in U$ where

$$\Phi(N_i, D_i, R_i, S_i) = N_i R_i + D_i S_i. \tag{1}$$

From now on, $\Phi(N_i, D_i, R_i, S_i)$ is denoted by $\Phi(P_i, C_i)$ if no confusion may arise.

By considering the unit $\Phi(P_i, C_i)$ in (1), two definitions are introduced to characterize the set of stabilizing compensators.

Definition 2.1 The set of stabilizing compensators with the property to preserve the initial unit $\Phi(P_i, C_i)$.
Select (N_i, D_i) a c.f. of a plant $P_i \in R^p[s]$ and (R_i, S_i) a c.f. of a compensator $C_i \in R[s]$ such that $\Phi(P_i, C_i) \in U$. The set of all compensators that stabilize $P_i \in R^p[s]$ with the property of preserving the initial unit $\Phi(P_i, C_i)$ is defined by

$$S_1(P_i, C_i) := \{C_k \in R[s] \textit{ such that } \Phi(P_i, C_k) = \Phi(P_i, C_i)\}.$$

Definition 2.2 The set of all stabilizing compensators.
The set of all compensators that stabilize $P_i \in R^p[s]$ is defined by

$$S_2(P_i) := \{C_g \in R[s] \textit{ such that } \Phi(P_i, C_g) \in U\}.$$

In Definition 2.1 and in Definition 2.2, $S_1(P_i, C_i)$ and $S_2(P_i)$ are sets of c.f. (R, S) of stabilizing compensators C_g. Unlike the set $S_2(P_i)$, the set $S_1(P_i, C_i)$ makes explicitely reference to the unit $\Phi(P_i, C_i)$.

These two definitions will be used in section 3 for the statement of the simultaneous stabilization problem. Now, the parametrization of the stabilizing compensators is given by two lemmas which are related to Definition 2.1 and Definition 2.2 respectively.

Lemma 2.1 [6] Parametrization of the stabilizing compensators belonging to $S_1(P_i, C_i)$.
Let (N_i, D_i) be a c.f. of a plant P_i and (R_i, S_i) a c.f. of a compensator $C_i \in S_2(P_i)$. All elements C_k in the set $S_1(P_i, C_i)$ may be parametrized as a pair (R_k, S_k) where (R_k, S_k) is a c.f. of C_k given by

$$(R_k, S_k) = (R_i + Q_i D_i, S_i - Q_i N_i) \tag{2}$$

with $Q_i \in RH_\infty$ such that $S_i \neq Q_i N_i$.

Lemma 2.2 Parametrization of all stabilizing compensators.
Let (N_i, D_i) be a c.f. of a plant P_i and (R_i, S_i) a c.f. of a compensator $C_i \in S_2(P_i)$. Then for all $C_g \in S_2(P_i)$ there exist $U_i \in U$ and $\widetilde{Q}_i \in RH_\infty$ fulfilling

$$(R_g, S_g) = (U_i R_i + \widetilde{Q}_i D_i, U_i S_i - \widetilde{Q}_i N_i) \tag{3}$$

under the condition $S_i \neq U_i^{-1} \widetilde{Q}_i N_i$. Note that (R_g, S_g) is any c.f. of C_g.

Proof. Let (N_i, D_i), (R_i, S_i) and (R_g, S_g) be c.f. of P_i, C_i and C_g respectively. Recall that

$$\begin{cases} \Phi(P_i, C_i) &= R_i N_i + S_i D_i \\ \Phi(P_i, C_g) &= R_g N_i + S_g D_i. \end{cases}$$

So we have

$$\Phi^{-1}(P_i, C_i)(R_i N_i + S_i D_i) = 1 \tag{4}$$
$$\Phi^{-1}(P_i, C_g)(R_g N_i + S_g D_i) = 1. \tag{5}$$

Adding (4) and (5) yields

$$\Phi^{-1}(P_i, C_i) R_i N_i + \Phi^{-1}(P_i, C_i) S_i D_i = \Phi^{-1}(P_i, C_g) R_g N_i + \Phi^{-1}(P_i, C_g) S_g D_i$$

or equivalently

$$(\Phi^{-1}(P_i, C_i) R_i - \Phi^{-1}(P_i, C_g) R_g) N_i + (\Phi^{-1}(P_i, C_i) S_i - \Phi^{-1}(P_i, C_g) S_g) D_i = 0.$$

Since N_i and D_i are coprime over RH_∞, there exists Q_i belonging to RH_∞ such that

$$\begin{cases} \Phi^{-1}(P_i, C_g) R_g &= \Phi^{-1}(P_i, C_i) R_i + Q_i D_i \\ \Phi^{-1}(P_i, C_g) S_g &= \Phi^{-1}(P_i, C_i) S_i - Q_i N_i \end{cases}$$

under the condition $\Phi^{-1}(P_i, C_i)\ S_i \neq Q_i\ N_i$.
This relation is rewrite as

$$\left\{ \begin{array}{lll} R_g & = & \Phi(P_i, C_g)\Phi^{-1}(P_i, C_i)R_i + \Phi(P_i, C_g)Q_iD_i \\ S_g & = & \Phi(P_i, C_g)\Phi^{-1}(P_i, C_i)S_i - \Phi(P_i, C_g)Q_iN_i. \end{array} \right.$$

Then the pair (R_g, S_g) is given by

$$(R_g, S_g) = (U_iR_i + \widetilde{Q}_iD_i, U_iS_i - \widetilde{Q}_iN_i) \tag{6}$$

with

$$\left\{ \begin{array}{lll} U_i & = & \Phi(P_i, C_g)\Phi^{-1}(P_i, C_i) \\ \widetilde{Q}_i & = & \Phi(P_i, C_g)Q_i. \end{array} \right.$$

Since $C_i \in S_2(P_i)$, $C_g \in S_2(P_i)$ so $\Phi(P_i, C_i) \in U$ and $\Phi(P_i, C_g) \in U$ then $U_i \in RH_\infty$.

End of the proof.

Remark 2.1 The connection between $S_1(P_i, C_i)$ and $S_2(P_i)$ is given for any $C_i \in S_2(P_i)$, by

$$S_1(P_i, C_i) \subset S_2(P_i). \tag{7}$$

Relation (7) is obvious. Indeed, from Lemma 2.1 and Lemma 2.2, if (R_k, S_k) is a c.f. of a compensator $C_k \in S_1(P_i, C_i)$, then for any $C_g \in S_2(P_i)$ there exists a unit U_k such that

$$(R_g, S_g) = (U_kR_k, U_kS_k) \tag{8}$$

where (R_g, S_g) is a c.f. of C_g.

The next definition completes Definition 2.1 and Definition 2.2 and ends this section.

Definition 2.3 The set of units obtained by a set of stabilizing compensators Ω.
Let (N_i, D_i) be a c.f. of the system $P_i \in R^p[s]$, and Ω a subset of $S_2(P_i)$. Denote $F(P_i, \Omega)$ the set of units defined as

$$F(P_i, \Omega) := \{\widetilde{U} \in U \textit{ such that } \widetilde{U} = N_iR + D_iS \textit{ with } (R, S) \in \Omega\}.$$

In this definition, $F(P_i, \Omega)$ corresponds to the set of all units obtained with $C = RS^{-1}$ such that $C \in \Omega \subset R[s]$ and C stabilizes $P_i \in R^p[s]$.

3 Simultaneous stabilizability: three approaches.

To clarify the basic ideas of the simultaneous stabilization problem, three approaches for stabilizing simultaneously two plants with the same linear compensator are described in this section. These three approaches are based on Definition 2.1 and Definition 2.2 which are related to two sets of compensators $S_2(P_i)$ and $S_1(P_i, C_i)$.

First Approach Firstly, present the simultaneous stabilization problem as follows: under which condition, for two proper plants P_i and P_j, does a compensator C_s exist such that

$$\Phi(P_i, C_s) \in U \text{ and } \Phi(P_j, C_s) \in U.$$

Or equivalently, under which condition does a compensator C_s exist such that

$$C_s \in S_2(P_j) \text{ and } C_s \in S_2(P_i).$$

Below, this problem is more precisely defined.

Definition 3.1 Simultaneous stabilizability.
Let P_i and P_j be two proper plants. The plants P_i and P_j are simultaneously stabilizable if and only if the set $\Omega_2 := S_2(P_i) \cap S_2(P_j)$ contains at least one compensator C_s.

Then, the following result may be deduced from Definition 3.1.

Theorem 3.1 *Denote P_i and P_j two proper plants. Let $C_i \in S_2(P_i)$ and $C_j \in S_2(P_j)$ be two compensators. Select (N_i, D_i), (N_j, D_j), (R_i, S_i) and (R_j, S_j) the c.f. of P_i, P_j, C_i and C_j respectively. The set Ω_2 is not empty if and only if there are $Q_{si} \in RH_\infty$ and $Q_{sj} \in RH_\infty$ fulfilling*

$$(R_j + Q_{sj}D_j)(S_j - Q_{sj}N_j)^{-1} - (R_i + Q_{si}D_i)(S_i - Q_{si}N_i)^{-1} = 0 \tag{9}$$

with

$$S_j \neq Q_{sj}N_j \text{ and } S_i \neq Q_{si}N_i.$$

Proof. See Appendix A.

Note that (9) uses the set Ω_2 since this expression considers the ratio between the coprime factors and not these factors separately, contrary to the second approach presented now.

Second Approach Let $C_i \in S_2(P_i)$ and $C_j \in S_2(P_j)$ be two compensators. For the two proper plants P_i and P_j does a compensator C_s exist such that

$$\Phi(P_i, C_s) = \Phi(P_i, C_i) \text{ and } \Phi(P_j, C_s) = \Phi(P_j, C_j).$$

Or equivalently, under which condition does a compensator C_s exist such that

$$C_s \in S_1(P_i, C_i) \text{ and } C_s \in S_1(P_j, C_j).$$

From this formulation, the simultaneous stabilization problem of two systems can be defined as follows.

Definition 3.2 Simultaneous stabilizability with the property of preserving the initial units $\Phi(P_i, C_i)$ and $\Phi(P_j, C_j)$.
Denote P_i and P_j two proper plants. Let $C_i \in S_2(P_i)$ and $C_j \in S_2(P_j)$ be two compensators. The plants P_i and P_j are simultaneously stabilizable with the property of preserving the initial units $\Phi(P_i, C_i)$ and $\Phi(P_j, C_j)$ if and only if the set $\Omega_1 := S_1(P_i, C_i) \cap S_1(P_j, C_j)$ contains at least one compensator C_s.

Consequently, the following theorem may be stated.

Theorem 3.2 *Denote P_i and P_j two proper plants. Let $C_i \in S_2(P_i)$ and $C_j \in S_2(P_j)$ be two compensators. Select (N_i, D_i), (N_j, D_j), (R_i, S_i) and (R_j, S_j) the c.f. of P_i, P_j, C_i and C_j respectively. The set Ω_1 is not empty if and only if there are $Q_{si} \in RH_\infty$ and $Q_{sj} \in RH_\infty$ fulfillimg*

$$\left\{ \begin{array}{rcl} R_j + Q_{sj} D_j & = & R_i + Q_{si} D_i \\ S_j - Q_{sj} N_j & = & S_i - Q_{si} N_i \end{array} \right. \tag{10}$$

with

$$S_j \neq Q_{sj} N_j \text{ and } S_i \neq Q_{si} N_i.$$

Proof. See Appendix A with $R'_s = R_s$ and $S'_s = S_s$ then $\overline{U} = 1$ and according to the paragraphs a) and b) of this appendix, we show that the two units $\Phi(P_i, C_i)$ and $\Phi(P_j, C_j)$ are preserved.

Remark 3.1 If the plants P_i and P_j are simultaneously stabilizable with the property of preserving the two initial units $\Phi(P_i, C_i)$ and $\Phi(P_j, C_j)$ then the following relationships may be deduced from (10)

$$\left\{ \begin{array}{rcl} \Phi(P_i, C_i) & = & N_i(R_i + Q_{si} D_i) + D_i(S_i - Q_{si} N_i) \\ \Phi(P_i, C_i) & = & N_i(R_j + Q_{sj} D_j) + D_i(S_j - Q_{sj} N_j) \end{array} \right. \tag{11}$$

$$\left\{ \begin{array}{rcl} \Phi(P_j, C_j) & = & N_j(R_i + Q_{si} D_i) + D_j(S_i - Q_{si} N_i) \\ \Phi(P_j, C_j) & = & N_j(R_j + Q_{sj} D_j) + D_j(S_j - Q_{sj} N_j). \end{array} \right. \tag{12}$$

Remark 3.2 This last approach is more constraining than the first one, since the simultaneous compensator must preserve the two initial units $\Phi(P_i, C_i)$ and $\Phi(P_j, C_j)$. In the state of this research, this problem is set as equivalent to the following one

$$\left\{ \begin{array}{rcl} Q_{si}\Delta & = & -\Phi(P_j, C_i) + \Phi(P_j, C_s) \;\text{ with } S_i \neq Q_{si}N_i \\ Q_{sj}\Delta & = & \Phi(P_i, C_j) - \Phi(P_i, C_s) \;\text{ with } S_j \neq Q_{sj}N_j \end{array} \right. \tag{13}$$

where

$$\Delta = D_i N_j - N_i D_j.$$

Explain the relationships (13). Let (R_s, S_s) be a c.f. of C_s. According to Theorem 3.2, the plants P_i and P_j are simultaneously stabilizable with the property to preserve the initial units $\Phi(P_i, C_i)$ and $\Phi(P_j, C_j)$ if and only if there exists at least one compensator C_s belonging to the set Ω_1 defined by

$$\Omega_1 := S_1(P_i, C_i) \cap S_1(P_j, C_j).$$

Since $C_s \in \Omega_1$, there exist $Q_{si} \in RH_\infty$ and $Q_{sj} \in RH_\infty$ verifying the following relations

$$R_s \;=\; R_j + Q_{sj}D_j \;=\; R_i + Q_{si}D_i \tag{14}$$

and also

$$S_s \;=\; S_j - Q_{sj}N_j \;=\; S_i - Q_{si}N_i. \tag{15}$$

Multiply (14) by N_j and (15) by D_j yields

$$(R_j + Q_{sj}D_j)N_j \;=\; (R_i + Q_{si}D_i)N_j \tag{16}$$
$$(S_j - Q_{sj}N_j)D_j \;=\; (S_i - Q_{si}N_i)D_j. \tag{17}$$

Adding (16) to (17) gives

$$Q_{si}\Delta = -\Phi(P_j, C_i) + \Phi(P_j, C_j) \text{ with } S_i \neq Q_{si}N_i \tag{18}$$

Multiplying (14) by N_i and (15) by D_i, we obtain the pair of equations

$$(R_j + Q_{sj}D_j)N_i \;=\; (R_i + Q_{si}D_i)N_i \tag{19}$$
$$(S_j - Q_{sj}N_j)D_i \;=\; (S_i - Q_{si}N_i)D_i. \tag{20}$$

Adding (19) to (20) yields

$$Q_{sj}\Delta = \Phi(P_i, C_j) - \Phi(P_i, C_i) \;\text{ with } S_j \neq Q_{sj}N_j \tag{21}$$

According to (14) and (15) we get

$$\Phi(P_j, C_j) = \Phi(P_j, C_s) \text{ and } \Phi(P_i, C_i) = \Phi(P_i, C_s).$$

To conclude this remark, we observe that the relations (18) and (21) are similar to the relationships (13).

Third Approach An intermediate approach between the two previous, one, may be formulated as follows: find a compensator that stabilizes the two plants P_i and P_j such that only one of the two initial units $\Phi(P_i, C_i)$ or $\Phi(P_j, C_j)$ is preserved. This problem corresponds to a new approach described now: look for a simultaneous compensator C_s that stabilizes two plants with the property of preserving only one of the two initial units, i.e. such that

$$C_s = C_{si} \in S_1(P_j, C_j) \cap S_2(P_i) \text{ or } C_s = C_{sj} \in S_1(P_i, C_i) \cap S_2(P_j).$$

Hence, the simultaneous stabilization problem is studied in the following way: consider a nominal plant P_j and a second one P_i, let C_j be a compensator stabilizing P_j, try to stabilize each of these two systems by the same compensator while preserving the unit $\Phi(P_j, C_j)$ given on the nominal plant P_j. In this approach, the set of units, which will be defined in Proposition 3.1, is a common stabilization domain for these two systems. In section 4, this domain will be used to derive existence conditions of a simultaneous compensator C_{si}.

Definition 3.3 Simultaneous stabilizability with the property of preserving the initial unit $\Phi(P_j, C_j)$.
Denote P_i and P_j two proper plants. Let $C_j \in S_2(P_j)$ be a compensator. Two plants P_i and P_j are simultaneously stabilizable with the property of preserving the initial unit $\Phi(P_j, C_j)$ if and only if the set $\Omega_3 := S_1(P_j, C_j) \cap S_2(P_i)$ contains at least one compensator C_{si}.

Proposition 3.1 *Let P_i and P_j be two proper plants, and let $C_j \in S_2(P_j)$ be a compensator. The set Ω_3 is not empty if and only if the set $F(P_i, S_1(P_j, C_j))$ contains at least one unit.*

Proof. See Appendix B.

It is interesting to observe that the existence conditions associated with Proposition 3.1 are exactly those required to stabilize two systems with a same compensator without constraints on the initial unit $\Phi(P_j, C_j)$ namely as defined in Definition 3.1. This result is shown in the following corollary.

Corollary 3.1 *Denote P_i and P_j two plants. Let $C_j \in S_2(P_j)$ be a compensator. The systems P_i and P_j are simultaneously stabilizable if and only if there exists a compensator C_{si} such that*

$$C_{si} \in (S_2(P_i) \cap S_1(P_j, C_j)) \tag{22}$$

Proof. See Appendix C.

Remark 3.3 Corollary 3.1 is an intermediate result more tractable than the general form defined by Theorem 3.1 and less constraining than the form described in Theorem 3.2. In section V of this report, we propose to build a simultaneous compensator such that (22) is true.

The Proposition 3.1 claims that if two systems are simultaneously stabilizable with the property to preserve the initial unit, then there exists a unit $\overline{U}$ belonging to $F(P_i, S_1(P_j, C_j))$. According to Proposition 3.1, the unit $\overline{U}$ belongs to $F(P_i, S_2(P_i))$. This unit is a common unit for the stabilization of the plants P_i and P_j since it may be obtained from $S_1(P_j, C_j)$ and $S_2(P_i)$. From this third Approach, a new formulation of the necessary and sufficient conditions to stabilize simultaneously two plants P_i and P_j will be given in the next section.

4 Simultaneous stabilizability of two plants using the third Approach.

4.1 Preliminary results.

Let $P_i \in R^p[s]$ and $P_j \in R^p[s]$ be two plants. Choose any compensator $C_j \in S_2(P_j)$. We want to find a compensator C_{si} belonging to $S_1(P_j, C_j)$ and $S_2(P_i)$ using the third Approach described in the previous section. In this subsection, a complete solution for this problem is expressed as an euclidean division in RH_∞ with a remainder belonging to U. The simultaneous compensator is an affine function of the quotient of this division.

Theorem 4.1 *Let $P_i \in R^p[s]$ and $P_j \in R^p[s]$ be the plants. Choose any compensator $C_j \in S_2(P_j)$. P_i, P_j, C_j are described by their associated c.f. (N_i, D_i), (N_j, D_j), (R_j, S_j) respectively. The plants P_i and P_j are simultaneously stabilizable by a controller C_{si} such that $\Phi(P_j, C_j) = \Phi(P_j, C_{si})$ if and only if there exists $Q_{sj} \in RH_\infty$ such that*

$$(\Phi(P_i, C_j) - Q_{sj}\Delta) \in U \text{ with } S_j \neq Q_{sj}N_j \tag{23}$$

where

$$\Delta = D_i N_j - N_i D_j. \tag{24}$$

Proof. See Appendix D.

The euclidean division in RH_∞ (23) with remainder in U can be solved as an interpolation problem in RH_∞. Some algorithms are given in [3], [6] and [9]. Remark that in Theorem 4.1, (R_j, S_j) is any c.f. of the initial compensator C_j and note that contrary to (23), there are two terms Q_{si} and Q_{sj} to compute in (9).

Now, we are interested by the connection between Proposition 3.1 and Theorem 4.1. According to Proposition 3.1, if P_i and P_j are two plants simultaneously stabilizable with the property of preserving the initial unit $\Phi(P_j, C_j)$, then there exists $\overline{U} \in F(P_i, S_1(P_j, C_j))$. Thus by definition, $F(P_i, S_1(P_j, C_j))$ can be written as

$$F(P_i, S_1(P_j, C_j)) := \{\overline{U} \in U \textit{ such that } \overline{U} = N_i R_{si} + D_i S_{si} \textit{ with } (R_{si}, S_{si}) \in S_1(P_j, C_j)\}.$$

Consequently, there exist an unit $\overline{U}$ and a compensator C_{si} of c.f. (R_{si}, S_{si}) such that

$$\overline{U} = N_i R_{si} + D_i S_{si} \tag{25}$$

As $(R_{si}, S_{si}) \in S_1(P_j, C_j)$, by Definition 2.1 we have

$$\Phi(P_j, C_j) = N_j R_{si} + D_j S_{si} = N_j R_j + D_j S_j. \tag{26}$$

Then according to (25) and (26) there exists a compensator C_{si} such that

$$C_{si} \in S_2(P_i) \cap S_1(P_j, C_j).$$

From Lemma 2.1 and Definition 2.1, Q_{sj} in RH_∞ can be find such that

$$(R_{si}, S_{si}) = (R_j + Q_{sj} D_j, S_j - Q_{sj} N_j).$$

Inserting the expressions of (R_{si}, S_{si}) in (25), this gives

$$Q_{sj}\Delta = \Phi(P_i, C_j) - \overline{U}$$

with

$$\left\{ \begin{array}{rcl} \Delta & = & N_i D_j - N_j D_i \\ \Phi(P_i, C_j) & = & N_i R_j + D_i S_j. \end{array} \right.$$

Comparing these equations with (23) and (24), we remark that the existence of a unit $\overline{U}$ is related directly to the existence of a simultaneous compensator C_{si} by the relationship

$$\overline{U} = (\Phi(P_i, C_j) - Q_{sj}\Delta) = \Phi(P_i, C_{si}). \tag{27}$$

Hence, Proposition 3.1 and Theorem 4.1 are directly connected by expression (27).

In addition, Theorem 4.1 is independent of the choice of a compensator $C_j \in S_2(P_j)$. Indeed, if there exists $Q_{sj} \in RH_\infty$ computed from $C_j \in S_2(P_j)$ and satisfying Theorem 4.1, then it is always possible to deduce a new Q'_{sj} in RH_∞ verifying this theorem from any other $C'_j \in S_2(P_j)$. This result will be presented in the following corollary.

Corollary 4.1 *Let $C_j \in S_2(P_j)$ be a compensator. Consider a unit $\overline{U} \in F(P_i, S_1(P_j, C_j))$. Then for any $C'_j \in S_2(P_j)$, we can find Q'_{sj} in RH_∞ such that (23) holds, namely*

$$\overline{U} = U_j^{-1}\Phi(P_i, C'_j) - Q'_{sj}\Delta \text{ with } U_j \in U \text{ and } U_j = \Phi^{-1}(P_j, C_j)\Phi(P_j, C'_j). \quad (28)$$

Proof. Let (R_j, S_j) and (R'_j, S'_j) be the associated c.f. of $C_j \in S_2(P_j)$ and of $C'_j \in S_2(P_j)$ respectively such that

$$\begin{cases} N_jR_j + D_jS_j &= \Phi(P_j, C_j) \\ N_jR'_j + D_jS'_j &= \Phi(P_j, C'_j). \end{cases}$$

By Lemma 2.2 one obtains

$$(R'_j, S'_j) = (U_jR_j + \tilde{q}_jD_j, U_jS_j - \tilde{q}_jN_j)$$

with

$$U_j = \Phi^{-1}(P_j, C_j)\Phi(P_j, C'_j)$$

then there exists q_j in RH_∞ such that $\tilde{q}_j = q_j\Phi(P_j, C'_j)$ and we have

$$\Phi(P_i, C'_j) = (U_jR_j + \tilde{q}_jD_j)N_i + (U_jS_j - \tilde{q}_jN_j)D_i.$$

Therefore the previous relationship yields

$$\Phi(P_i, C_j) = U_j^{-1}\Phi(P_i, C'_j) - \Phi(P_j, C_j)q_j\Delta. \quad (29)$$

Suppose that the relation (23) holds with $\overline{U} = \Phi(P_i, C_{si})$ given by (27). Substituting (29) into (23) gives

$$\overline{U} = U_j^{-1}\Phi(P_i, C'_j) - Q'_{sj}\Delta$$

with

$$Q'_{sj} = \Phi(P_j, C_j)q_j + Q_{sj} \text{ and } Q'_{sj} \in RH_\infty. \quad (30)$$

End of the proof.

Any compensator $C_j \in S_2(P_j)$ may be an initial compensator in Theorem (4.1) and consequently, all Q'_{sj} in RH_∞ given by (30) can be solution of equation (27).

Remark 4.1 When there exists $Q_{sj} \in RH_\infty$ given by (23), the simultaneous stabilizing compensator C_{si}, obtained from Theorem 4.1, is described by the following c.f. (R_{si}, S_{si})

$$(R_{si}, S_{si}) = (R_j + Q_{sj}D_j, S_j - Q_{sj}N_j). \quad (31)$$

Note that the equation $S_j = Q_{sj}N_j$ may be obtained only if the plant P_j is bi-proper and minimum phase [6].

Remark 4.2 *Reversibility of the simultaneous stabilization problem.*
i) Unlike of Theorem 4.1, the reversibility of Theorem 3.2 is complete. Indeed, if the systems P_j and P_i are simultaneously stabilizable with the property to preserve both units $\Phi(P_j, C_j)$ and $\Phi(P_i, C_i)$, then there exist $Q_{sj} \in RH_\infty$ and $Q_{si} \in RH_\infty$ such that $\Phi(P_i, C_i) = \Phi(P_i, C_s)$ and $\Phi(P_j, C_j) = \Phi(P_j, C_s)$ with $C_s \in S_1(P_i, C_i) \cap S_1(P_j, C_j)$. This reversibility is illustrated by the following representation

$$\Phi(P_j, C_j) \underset{Q_{si}}{\overset{Q_{sj}}{\rightleftarrows}} \Phi(P_i, C_i).$$

ii) The choice of indices i and j is arbitrary in Theorem 4.1, then the simultaneous stabilization problem in this theorem is reversible except for the units to be preserved. This "partial" reversibility can be formulated as follows. Consider a plant P_j with a compensator $C_j \in S_2(P_j)$ and a plant P_i with a compensator $C_i \in S_2(P_i)$. If the plants P_j and P_i are simultaneously stabilizable with the property to preserve the initial units $\Phi(P_j, C_j)$, then we can find $Q_{sj} \in RH_\infty$ and $\Phi(P_i, C_{si}) \in U$ with $C_{si} \in S_2(P_i) \cap S_1(P_j, C_j)$. If the initial unit $\Phi(P_i, C_i)$ is preserved, then we can find $Q_{si} \in RH_\infty$ and $\Phi(P_j, C_{sj}) \in U$ with $C_{sj} \in S_2(P_j) \cap S_1(P_i, C_i)$. This "partial" reversibility is illustrated by the following representation

$$\Phi(P_j, C_j) \xrightarrow{Q_{sj}} \Phi(P_i, C_{si}) \text{ or } \Phi(P_i, C_i) \xrightarrow{Q_{si}} \Phi(P_j, C_{sj}).$$

iii) To end this remark, it remains to study the link between the first Approach and the third Approach. Observe that, contrary to (9) the product $Q_{si}Q_{sj}$ does not appear in (23). In addition, the determination of Q_{sj} in (23) is a necessary and sufficient condition for the existence of a simultaneous compensator C_{si}. Note also that in (23) Q_{si} can be deduced in a straightforward way from Q_{sj}. To show that, let (R_i, S_i), (R_j, S_j), (R_{si}, S_{si}) be c.f. of $C_i \in S_2(P_i)$, $C_j \in S_2(P_j)$ and $C_{si} \in S_1(P_j, C_j)$ respectively. As $C_i \in S_2(P_i)$ then $\Phi(P_i, C_i) \in U$; therefore if (23) holds with $Q_{sj} \in RH_\infty$, $\Phi(P_i, C_{si}) \in U$ and Δ given by (24), Q_{si} in RH_∞ can be chosen as follows

$$Q_{si} = \frac{R_j S_i - R_i S_j + Q_{sj}(D_j S_i + R_i N_j)}{\Phi(P_i, C_i)} \tag{32}$$

such that (9) holds.

The way to pass from a simultaneous compensator to an other is presented in the following subsection.

4.2 Relations between the different forms of simultaneous compensators.

Let C_g, C_{sj}, C_{si} be the simultaneous compensators such that

a) $C_g \in (S_2(P_i) \cap S_2(P_j))$ with (R_g, S_g) a c.f. of C_g.
b) $C_{sj} \in (S_1(P_i, C_i) \cap S_2(P_j))$ with (R_{sj}, S_{sj}), (R_i, S_i) the c.f. of C_{sj} and C_i respectively.
c) $C_{si} \in (S_2(P_i) \cap S_1(P_j, C_j))$ with (R_{si}, S_{si}), (R_j, S_j) the c.f. of C_{si} and C_j respectively.

The compensators C_{si} and C_{sj} can be expressed from the compensator C_g.

The passage from the controller C_g to a controller C_{sj} is given by

$$\left\{ \begin{array}{lcl} R_{sj} & = & R_g \dfrac{U_i}{U_{ig}} \\ S_{sj} & = & S_g \dfrac{U_i}{U_{ig}} \end{array} \right. \tag{33}$$

with $\dfrac{U_i}{U_{ig}} \in U$ and $U_i = N_i R_i + D_i S_i$, $U_{ig} = N_i R_g + D_i S_g$.

In a similar way, the controller C_{si} can be deduced from the controller C_g as follows

$$\left\{ \begin{array}{lcl} R_{si} & = & R_g \dfrac{U_j}{U_{gj}} \\ S_{si} & = & S_g \dfrac{U_j}{U_{gj}} \end{array} \right. \tag{34}$$

with $\dfrac{U_j}{U_{jg}} \in U$ and $U_j = N_j R_j + D_j S_j$, $U_{jg} = N_j R_g + D_j S_g$.

In the next subsection, the conditions for the existence of a simultaneous compensator C_{si} are studied.

4.3 Conditions for the existence of the simultaneous compensator C_{si}.

4.3.1 Divisibility conditions in RH_∞.

From Theorem 4.1 and relation (27), the existence of a simultaneous compensator C_{si} is equivalent to the existence of the terms $Q_{sj} \in RH_\infty$ and $(\Phi(P_i, C_j) - Q_{sj}\,\Delta) \in U$. Hence the problem to study is: what are the conditions to have $\delta(\Phi(P_i, C_j) - Q_{sj}\Delta) = 0$ as Q_{sj} varies over RH_∞ ? Before solving this question with a lemma from Vidyasagar [6], remark that $\Phi(P_i, C_j) \in RH_\infty$ and $\Delta \in RH_\infty$ are coprime over RH_∞. For this define

$$E := \{\sigma_1, \ldots, \sigma_{nr}\}$$

the set of real positive zeroes of Δ with $\sigma_1 = \infty$ if appropriate, without taking the multiplicity into account. The cardinal of E is nr. In a first time, we must show the following result.

Lemma 4.1 *$\Phi(P_i, C_j) \in RH_\infty$ and $\Delta \in RH_\infty$ defined by (24) are coprime over RH_∞.*

Proof. Let (N_i, D_i), (N_j, D_j) and (R_j, S_j) be c.f. of P_i, P_j and $C_j \in S_2(P_j)$ respectively. Let $\sigma_p \in E$. Assume that $\Phi(P_i, C_j)(\sigma_p) = 0$ with $\Delta(\sigma_p) = 0$ and $D_i(\sigma_p) \neq 0$. This implies that $D_j(\sigma_p) \neq 0$. Then we have

$$N_i(\sigma_p)D_i^{-1}(\sigma_p) = N_j(\sigma_p)D_j^{-1}(\sigma_p) = S_j(\sigma_p)R_j^{-1}(\sigma_p)$$

namely

$$\Phi(P_j, C_j)(\sigma_p) = 0.$$

But by assumption

$$C_j \in S_2(P_j).$$

Thus we have

$$\Phi(P_j, C_j)(\sigma_p) \neq 0.$$

Notice that we cannot obtain the following equations simultaneously

$$\Phi(P_i, C_j)(\sigma_p) = 0, \quad \Delta(\sigma_p) = 0 \text{ and } D_i(\sigma_p) = 0$$

because this implies that $\Phi(P_j, C_j)(\sigma_p) = 0$.
Then $\Phi(P_i, C_j)$ and Δ are coprime over RH_∞.
End of the proof.

As $\Phi(P_i, C_i)$ and Δ are coprime over RH_∞, the following lemma may be used.

Lemma 4.2 *[6] Suppose $\Delta \neq 0$. Let λ be the number of sign changes in the sequence $\{\Phi(P_i, C_j)(\sigma_1), \ldots, \Phi(P_i, C_j)(\sigma_{nr})\}$ where $\sigma_p \in E$. Then we have*

$$\min_{Q_{sj}} \delta(\Phi(P_i, C_j) - Q_{sj}\Delta) = \lambda \text{ with } Q_{sj} \in RH_\infty. \tag{35}$$

Proof. See ([6], pp. 27-28).

Consider Theorem 4.1. If two systems P_i and P_j are simultaneously stabilizable with the property to preserve the initial unit $\Phi(P_j, C_j)$, then, from (27), there exist $\Phi(P_i, C_{si}) \in U$ and $Q_{sj} \in RH_\infty$ such that

$$Q_{sj}\Delta = \Phi(P_i, C_j) - \Phi(P_i, C_{si}) \quad \text{with} \quad S_j \neq Q_{sj}N_j.$$

Since $C_{si} \in S_2(P_i)$, we get $\Phi(P_i, C_{si}) \in U$ namely $\delta(\Phi(P_i, C_{si})) = 0$.
Thus, we must have

$$\delta(\Phi(P_i, C_j) - Q_{sj}\Delta) = 0. \tag{36}$$

From Lemma 4.2, the sequence $\{\Phi(P_i, C_j)(\sigma_1), \ldots, \Phi(P_i, C_j)(\sigma_{nr})\}$ must have no sign change to satisfy (36). Conversely, if the sequence has no sign change, by Lemma 4.2 there exists Q_{sj} in RH_∞ such that

$$\Phi(P_i, C_j) - Q_{sj}\Delta = \overline{U} \ \text{ with } \overline{U} \in U.$$

Thus, it is possible to compute from (31) a simultaneous compensator C_{si} such that

$$\Phi(P_j, C_j) = \Phi(P_j, C_{si}) \ \text{ and } \overline{U} = \Phi(P_i, C_{si}).$$

Then it follows from Theorem 4.1 that the systems P_i and P_j are simultaneously stabilizable with the property to preserve the initial unit $\Phi(P_j, C_j)$ if and only if the sequence $\{\Phi(P_i, C_j)(\sigma_1), \ldots, \Phi(P_i, C_j)(\sigma_{nr})\}$ has no sign change. As by definition of $\Phi(P_i, C_j)$ and Δ, $\Phi(P_i, C_j)$ and Δ are coprime in RH_∞, then the necessary and sufficient condition for the existence of an unit $\Phi(P_i, C_{si})$ satisfying (23) is equivalent to the condition presented in the following corollary.

Corollary 4.2 *Let $C_j \in S_2(P_j)$ be a compensator. The two systems P_i and P_j are simultaneously stabilizable with the property to preserve the initial unit $\Phi(P_j, C_j)$ if and only if*

$$\forall \sigma_{p1} \in E, \forall \sigma_{p2} \in E \qquad \Phi(P_i, C_j)(\sigma_{p1})\ \Phi(P_i, C_j)(\sigma_{p2}) > 0. \tag{37}$$

The property (37) is known as the parity interlacing property, denoted p.i.p.

Proof. See Appendix E.

4.3.2 Necessary and sufficient conditions.

In order to introduce a new formulation of the necessary and sufficient conditions for the simultaneous stabilization problem defined in Approach 3, one remark and one definition are presented. These necessary and sufficient conditions are deduced from Corollary 4.2.

Remark 4.3 Let (N_i, D_i) and (N_j, D_j) be the c.f. of the plants P_i and P_j respectively. According to (24), we may write

$$\forall \sigma_p \in E, \ \text{ if } N_i(\sigma_p) = 0 \ \text{ and } \Delta(\sigma_p) = 0 \ \text{ then } N_j(\sigma_p) = 0.$$

Additionally

$$\forall \sigma_p \in E, \ \text{ if } D_i(\sigma_p) = 0 \ \text{ and } \Delta(\sigma_p) = 0 \ \text{ then } D_j(\sigma_p) = 0.$$

Moreover N_j and D_j are coprime over RH_∞, we have

$$\forall \sigma_p \in E, \ \text{ if } N_j(\sigma_p) = 0, \ \text{ then } D_j(\sigma_p) \neq 0.$$

Definition 4.1 *Denote* (N_i, D_i) *and* (N_j, D_j) *the c.f. of two proper plants* P_i *and* P_j *respectively. Define a function* $c : E := \{\sigma_1, \ldots, \sigma_{nr}\} \rightarrow \{-1, 0, 1\}$ *as follows:*

$$\left\{ \begin{array}{lcll} c(\sigma_p) & = & sgn(\dfrac{N_i(\sigma_p)}{N_j(\sigma_p)}) & \text{if } N_j(\sigma_p) \neq 0, \\ c(\sigma_p) & = & sgn(\dfrac{D_i(\sigma_p)}{D_j(\sigma_p)}) & \text{otherwise.} \end{array} \right. \tag{38}$$

Then the following theorem can be stated.

Theorem 4.2 *Let* (N_i, D_i), (N_j, D_j), (R_j, S_j) *be c.f. of* $P_i \in R^p[s]$, $P_j \in R^p[s]$ *and* $C_j \in S_2(P_j)$ *respectively.* P_i *and* P_j *are simultaneously stabilizable by a compensator* C_{si} *that preserves the initial unit* $\Phi(P_j, C_j)$ *if and only if*

$$\left| \sum_{p=1}^{nr} c(\sigma_p) \right| = nr. \tag{39}$$

Proof. See Appendix F.

It follows from Corollary 4.2 that (39) is equivalent to the p.i.p. on an associated plant with a c.f. $(\Delta, \Phi(P_j, C_j))$; thus the third Approach may be seen as a strong stability problem [8]. Due to Corollary 3.1, condition (39) is also necessary and sufficient for the simultaneous stabilization of two plants in the sense of the first Approach.

Remark 4.4 *About Theorem 4.2.*
i) Theorem 4.2 yields that the existence condition of a simultaneous compensator C_{si} for the plants P_i and P_j is independent of the reference plant that may be either P_i or P_j. Similarly the existence condition (39) is independent of the choice of the initial compensator. For any compensators C_j' and C_j belonging to $S_2(P_j)$, testing the condition (39) of Theorem 4.2 directly from the systems P_i and P_j is equivalent to test the p.i.p. on an associated plant of c.f. $(\Delta, \Phi(P_j, C_j))$ or to test the p.i.p. on another associated plant of c.f. $(\Delta, \Phi(P_j, C_j'))$.

ii) When Δ is an unit, the simultaneous stabilization problem is obvious from Theorem 4.1 and Theorem 4.2. Indeed, if there are no unstable zeroes to interpolate, then the problem is reduced to a simple inversion in RH_∞. In this case the simultaneous compensator C_{si} is given for any U_i such that $U_i = \Phi(P_i, C_{si})$.

5 Synthesis: determination of the simultaneous compensator C_{si}.

With the help of Theorems 4.1 and 4.2, we propose the following procedure to find a simultaneous compensator C_{si} that stabilizes the proper plants P_i and P_j and preserves the initial unit $\Phi(P_j, C_j)$.

i) Choose (R_j, S_j) a c.f. of a given controller $C_j \in S_2(P_j)$.
ii) Compute Δ and check condition (39) of Theorem 4.2.
iii) Determine Q_{sj} satisfying the equation (23) (see interpolation techniques in [3], [6]).
iv) Compute the simultaneous compensator C_{si} given by (31).

5.1 Euclidean division in RH_∞.

Suppose that the steps i) and ii) are made and go to step iii). From [6], a synthesis method is proposed to calculate $Q_{sj} \in RH_\infty$ and the unit $\Phi(P_i, C_{si})$. $\Phi(P_i, C_{si})$ is the remainder of the euclidean division of $\Phi(P_i, C_j)$ by Δ.

Consider the expression (27). Set in this expression

$$\left\{\begin{array}{rcl} \Phi(P_i, C_j) & = & F \\ \Delta & = & G \\ Q_{sj} & = & Q \\ \Phi(P_i, C_{si}) & = & W. \end{array}\right. \tag{40}$$

Therefore write (27) as

$$W = F - GQ \text{ with } W \in U. \tag{41}$$

Note that G can be always factorized as $G = G_- G_+$ where G_- and G_+ are the unit part and the nonminimum phase part of G respectively. The relationship (41) becomes

$$G_-^{-1} W = G_-^{-1} F - G_+ Q \text{ with } G_-^{-1} W \in U. \tag{42}$$

Then we obtain the following expression

$$W' = F' - G_+ Q \;\text{ with } W' \in U, W' = G_-^{-1} W \;\text{ and } F' = G_-^{-1} F. \tag{43}$$

As F' and G_+ are coprime over RH_∞, we may write the following Bezout's identity

$$NF' + KG_+ = 1. \tag{44}$$

where N and K belong to RH_∞. Remark that if F' and G_+ would not be coprime over RH_∞, then we should have (see Lemma 4.1) $\delta(W') > 0$ because $Q \in RH_\infty$ and therefore $W' \notin U$.

From the expression (44) determine $N \in RH_\infty$ and $K \in RH_\infty$ which are coprime over RH_∞.

From the expression (44) we deduce

$$\forall \sigma_i \in E \quad (NF')(\sigma_i) = 1.$$

Then the product (NF') has a constant sign for all σ_i in E. Equivalently, this means that the existence conditions affecting F' (see Lemma 4.2) are equivalent to the existence conditions affecting N. On the other hand to say that there exists an unit W' that interpolates the values of F' and the values of these derivatives at the zeroes of G_+, is equivalent to say that there exists an unit V that interpolates the values of N and the values of these derivatives at the zeroes of G_+. Hence there exists an unit V such that $(V - N)$ is divisible by G_+ in RH_∞. Therefore there exists $Z \in RH_\infty$ such that

$$V - N = ZG_+. \tag{45}$$

The Lemma 4.2 states a similar constraint on the unit V in order to allow the existence of a $Z \in RH_\infty$. Namely the unit V must interpolate N at the zeroes of G_+ (if necessary, including the infinity) with multiplicities in order that there exists $Z \in RH_\infty$. Then the expression (23) is equivalent to show that there exists an unit V that interpolates N at the real positive zeroes of G_+ with these multiplicities.

5.2 Interpolation of unit functions.

Recall brieftly, the interpolation method of unit functions proposed in [6].
Let σ_p, $p = 1, \ldots, m$ denote the distinct zeroes of G_+ with positive real part including infinity. These zeroes are arranged in such a way that the real positive zeroes are the first nr ones (nr is the number of real zeroes). Then the set of real zeroes will be the set $\{\sigma_1, \ldots, \sigma_{nr}\}$ and $\sigma_1 = \infty$ may be one of these zeroes and σ_p, $p = nr + 1, \ldots, m$ are distinct complex zeroes of G_+ with positive real part. Let μ_p, $p = 1, \ldots, m$ denote the multiplicity of these zeroes. Now define $r_{p,k}$ as

$$\frac{d^k}{ds^k} N(\sigma_p) = r_{p,k} \tag{46}$$

with

$$k = 0, \ldots, \mu_{p-1} \text{ and } p = 1, \ldots, m$$

where the derivative of order zero of N is taken as N itself.
The problem amounts to find an unit V such that

$$\frac{d^k}{ds^k} V(\sigma_p) = r_{p,k}.$$

In other words, we want to know whether or not there exists an unit V such that its functional and derivative values at specified points σ_p in C_{+e} are equal to the specified values $r_{p,k}$. The Corollary 4.2 gives a simple necessary and sufficient condition for the existence of an unit V satisfying (23). It is established that the interpolation problem stated above has a solution if and only if the numbers $r_{p,0}$ $(r_{p,0} \neq 0)$ with $p = 1, \ldots, nr$ are all of the same sign [6]. This existence condition of the unit V is therefore similar to the existence condition of a simultaneous compensator C_{si}, namely this existence condition is equivalent to the condition (37) or condition (39).

Now consider an unit V[1] that interpolates the values of N and its derivatives at the C_{+e} zeroes of G_+, or equivalently, such that $(V-N)$ is divisible by G_+ in RH_∞, then there exists $Z \in RH_\infty$ such that

$$V - N = ZG_+. \tag{47}$$

Rewrite the relationship (47) as follows

$$N + ZG_+ = V.$$

Dividing both sides of the previous equation by V yields

$$NW' + PG_+ = 1 \tag{48}$$

where $P = \dfrac{Z}{V}$, $W' = \dfrac{1}{V}$ with $W' \in U$. Substracting (48) from (44) gives

$$N(W' - F') = G_+(K - P). \tag{49}$$

Then G_+ divides the product $N(W'-F')$ over RH_∞. From (44), N and G_+ are coprime over RH_∞ then G_+ divides $(W'-F')$ over RH_∞. This is equivalent to write

$$W' = F' - G_+Q \text{ with } Q \in RH_\infty. \tag{50}$$

The relationship (50) is clearly identical to (43).
Therefore we have

$$Q = \frac{F' - \left(\frac{1}{V}\right)}{G_+}. \tag{51}$$

Remark that this division may be written, indeed Q belongs to RH_∞ since $\left(\frac{1}{V}\right)$ interpolates F' to zeroes of G_+. Substituting in (51) the expressions defined in (40), we get

$$Q_{sj} = \frac{\Phi(P_i, C_j) - \left(\left(\frac{1}{V}\right)\Delta_-\right)}{\Delta}. \tag{52}$$

Thus the simultaneous compensator C_{si} is given by

$$\left\{ \begin{array}{rcl} R_{si} & = & R_j + Q_{sj}D_j \\ S_{si} & = & S_j - Q_{sj}N_j. \end{array} \right. \tag{53}$$

[1]The computation of the unit V is not the purpose of this paper. In the literature, methods have been proposed for their determination [3], [6] [9].

In conclusion of this part, verify that the obtained units are given by

$$\begin{aligned}\Phi(P_i, C_{si}) &= N_i R_{si} + D_i S_{si} \\ \Phi(P_j, C_{si}) &= W = (\Delta_-)(V^{-1}) \\ \Phi(P_j, C_{si}) &= N_j R_{si} + D_j S_{si} = N_j R_j + D_j S_j.\end{aligned}$$

Therefore the obtained unit for the final system corresponds to the inverse of the interpolated unit (i.e. V) multiplied by the unit part of Δ, and the unit for the initial system remains unchanged.

5.3 Example.

Consider the systems $P_j(s)$ and $P_i(s)$ with their respective c.f. given by

$$(N_j(s), D_j(s)) = \Big(\frac{2}{s+2}, 1\Big),\ (N_i(s), D_i(s)) = \Big(\frac{s-2}{(s+1)(s+5)}, \frac{s-5}{s+5}\Big).$$

i) A compensator $C_j(s)$ stabilizing $P_j(s)$ has the following c.f.

$$(R_j(s), S_j(s)) = (1, 1).$$

ii) From (24), we obtain $\Delta(s) = \dfrac{(s+0.6904)(s-8.6904)}{(s+1)(s+2)(s+5)}$ and $E := \{\sigma_1, \sigma_2\}$ with $\sigma_1 = \infty$ and $\sigma_2 = 8.6904$. Condition (39) of Theorem 4.2 holds since $c(\sigma_1) = c(\sigma_2) = 1$. Then $P_i(s)$ and $P_j(s)$ are simultaneously stabilizable in the sense of Definition 3.1 and Definition 3.3.

iii) An interpolation procedure applied to equation (23) gives

$$Q_{sj}(s) = \frac{-(s^3 - 241.17s^2 - 866.05s - 759.48)}{s^3 + 19.8s^2 + 438.8s + 284.4}.$$

iv) From (31), a simultaneous compensator $C_{si}(s)$ is given by the following c.f.

$$(R_{si}(s), S_{si}(s)) = \Big(\frac{260.9699(s+1)(s+4)}{(s^3 + 19.8s^2 + 438.8s + 284.4)}, \frac{(s^2 + 17.7894s - 118.7379)(s+4)}{(s^3 + 19.8s^2 + 438.8s + 284.4)}\Big).$$

6 Conclusions.

In this report, the conditions of stabilizability of two single-input single output systems by the same time-invariant compensator have been presented. Three different notions of simultaneous stabilizability by considering constraints on units have been defined. A general framework based on the Youla-parametrization to design the simultaneous compensators have been provided. Finally, an another formulation of the necessary and sufficient conditions than these given in [6] have been derived. Moreover an explicit method have been given to compute simultaneous compensators.

7 Acknowledgments.

The authors thank F. Delebecque (INRIA Rocquencourt) and V. Blondel (Université de Liège) for their helpful and constructive comments.

8 Appendices.

Appendix A. Proof of Theorem 3.1.

Show that the plants P_i and P_j are simultaneously stabilizable if and only if there exist Q_{si} in RH_∞ and Q_{sj} in RH_∞ with $S_j \neq Q_{si}N_i$ such that

$$(R_j + Q_{sj}D_j)(S_j - Q_{sj}N_j)^{-1} - (R_i + Q_{si}D_i)(S_i - Q_{si}N_i)^{-1} = 0.$$

For beginning, we must prove the following technical lemma:

Lemma 8.1 *Denote P a plant and C a compensator belonging to $S_2(P)$. Consider (R,S) a first c.f. of C and (R',S') a second c.f. of C then there exists $\overline{U} \in U$ such that $R = \overline{U}R'$ and $S = \overline{U}S'$.*

Proof. Let

$$C = \frac{R}{S} \text{ and } C = \frac{R'}{S'},$$

then

$$RS' = R'S. \tag{54}$$

Denote R_+, S_+, R'_+ and S'_+ the factors of R, S, R' and S' respectively that contain the unstable zeroes including infinity, and denote R_-, S_-, R'_- and S'_- the factors of R, S, R' and S' respectively that contain the stable zeroes.

In the relationship (54) the nonminimum phase and unit parts can be separated as follows

$$\begin{aligned} R_-S'_- &= R'_-S_- && (55)\\ R_+S'_+ &= R'_+S_+. && (56)\end{aligned}$$

Since R and S are coprime over RH_∞ so R_+, S_+ are coprime over RH_∞ . Then from (56), R_+ and R'_+ (respectively S_+ and S'_+) are not coprime over RH_∞ . It means that either R_+ divides R'_+ or R'_+ divides R_+ in RH_∞. Suppose that R'_+ divides R_+ in RH_∞, then there exists $V \in RH_\infty$ such that

$$R_+ = VR'_+. \tag{57}$$

Replace (57) in (56) gives

$$\begin{aligned} R'_+ S_+ &= V R'_+ S'_+ && (58) \\ R'_+ (S_+ - V S'_+) &= 0. && (59) \end{aligned}$$

So since

$$R'_+ \neq 0 \qquad (60)$$

then

$$S_+ = V S'_+. \qquad (61)$$

From (57) and (61) and the fact that R and S are coprime over RH_∞, we conclude that V is an unit. Multiply by R_- and S_- respectively (57) and (60) yield

$$\begin{cases} R_- R_+ &= V R'_+ R_- \\ S_- S_+ &= V S'_+ S_- \end{cases}$$

or equivalently

$$\begin{cases} R &= V R' \dfrac{R_-}{R'_-} \\ S &= V S' \dfrac{S_-}{S'_-}. \end{cases}$$

Remember that $\dfrac{R_-}{R'_-} \in U$ and $\dfrac{S_-}{S'_-} \in U$.
Equation (55) gives

$$\frac{S_-}{S'_-} = \frac{R_-}{R'_-}.$$

Define $\overline{V}$ as

$$\frac{S_-}{S'_-} = \frac{R_-}{R'_-} = \overline{V}.$$

We obtain

$$\begin{cases} R &= V \overline{V} R' \\ S &= V \overline{V} S'. \end{cases}$$

Let $\overline{U}$ be given by $\overline{U} = V\overline{V}$, thus $\overline{U} \in U$.
This lemma is verified.

Beginning of the proof of Theorem 3.1.
A) Part "$\Leftarrow$" if.

Hypothesis. There exist Q_{si} in RH_∞ and Q_{sj} in RH_∞ with $S_j \neq Q_{sj} N_j$ and $S_i \neq Q_{si} N_i$ such that

$$(R_j + Q_{sj} D_j)(S_j - Q_{sj} N_j)^{-1} - (R_i + Q_{si} D_i)(S_i - Q_{si} N_i)^{-1} = 0. \qquad (62)$$

Conclusion. The plants P_i and P_j are simultaneously stabilizable.

Beginning of the part A). To prove this part, it is necessary to consider two cases: i) and ii).

i) According to (62), we may set

$$C_s = \frac{(R_i + Q_{si} D_i)}{(S_i - Q_{si} N_i)}$$

with

$$\left\{ \begin{array}{lcl} R_s & = & R_i + Q_{si} D_i \\ S_s & = & S_i - Q_{si} N_i \end{array} \right.$$

then

$$N_i R_s + D_i S_s = N_i R_i + D_i S_i = U_i.$$

We remark that $U_i \in U$ since $C_i \in S_2(P_i)$.
From the Bezout's identity (R_s, S_s) are coprime over RH_∞. As $(R_s, S_s) \in S_1(P_i, C_i)$ then (R_s, S_s) stabilizes P_i.

ii) Similarly we have

$$C_s = \frac{(R_j + Q_{sj} D_j)}{(S_j - Q_{sj} N_j)}$$

with

$$\left\{ \begin{array}{lcl} R'_s & = & R_j + Q_{sj} D_j \\ S'_s & = & S_j - Q_{sj} N_j \end{array} \right.$$

then

$$N_j R'_s + D_j S'_s = N_j R_j + D_j S_j = U_j.$$

$U_j \in U$ since $C_j \in S_2(P_j)$.
From this Bezout's identity, we deduce that (R'_s, S'_s) are coprime over RH_∞.
As $(R'_s, S'_s) \in S_1(P_j, C_j)$ then (R'_s, S'_s) stabilizes P_j.

End of i) and ii).

We have $\dfrac{R'_s}{S'_s} = \dfrac{R_s}{S_s}$ according to (62). Consider the Lemma 8.1. Then there exists $\overline{U} \in U$ such that

$$\left\{ \begin{array}{lcl} R_s & = & \overline{U} R'_s \\ S_s & = & \overline{U} S'_s. \end{array} \right.$$

Then we deduce the following Bezout's identity

$$R_s N_j + S_s D_j = \overline{U}(R'_s N_j + S'_s D_j).$$

We conclude that $\overline{U} U_j \in U$ since $\overline{U} \in U$ and $U_j \in U$.
Then (R_s, S_s) stabilizes P_j.
Finally C_s stabilizes P_i and P_j.

End of A).

B) Part "⇒" only if.

Hypothesis. The plants P_i and P_j are defined by the c.f. (N_i, D_i) and (N_j, D_j). Denote (R_i, S_i) and (R_j, S_j) the c.f. of the compensators $C_i \in S_2(P_i)$ and $C_j \in S_2(P_j)$ respectively. Suppose that P_i and P_j are simultaneously stabilizable.

Conclusion. There exist Q_{si} in RH_∞ and Q_{sj} in RH_∞ with $S_j \neq Q_{sj}N_j$ and $S_i \neq Q_{si}N_i$ such that

$$(R_j + Q_{sj}D_j)(S_j - Q_{sj}N_j)^{-1} - (R_i + Q_{si}D_i)(S_i - Q_{si}N_i)^{-1} = 0. \tag{63}$$

Beginning of the part B). The plants P_i and P_j are simultaneously stabilizable then there exists a least one compensator C_g such that $C_g \in S_2(P_i) \cap S_2(P_j)$. Denote (R_g, S_g) a c.f of C_g.
According to Lemma 2.2 and since $C_g \in S_2(P_j)$, there are $\tilde{Q}_i \in RH_\infty$ and $U_i \in U$ such that

$$\left\{ \begin{array}{lcl} R_g & = & R_iU_i + \tilde{Q}_iD_i \\ S_g & = & S_iU_i - \tilde{Q}_iN_i. \end{array} \right. \tag{64}$$

Similarly since $C_g \in S_2(P_j)$, we have $\tilde{Q}_j \in RH_\infty$ and $U_j \in U$ such that

$$\left\{ \begin{array}{lcl} R_g & = & R_jU_j + \tilde{Q}_jD_j \\ S_g & = & S_jU_j - \tilde{Q}_jN_j. \end{array} \right. \tag{65}$$

Then, from (64) and (65), we obtain

$$\frac{R_g}{S_g} = \frac{U_i(R_i + \tilde{Q}_iU_i^{-1}D_i)}{U_i(S_i - \tilde{Q}_iU_i^{-1}N_i)} = \frac{U_j(R_j + \tilde{Q}_jU_j^{-1}D_j)}{U_j(S_j - \tilde{Q}_jU_j^{-1}N_j)}.$$

Thus the equation (63) is verified with Q_{si} and Q_{si} given by

$$\left\{ \begin{array}{lcl} Q_{si} & = & \tilde{Q}_iU_i^{-1} \\ Q_{sj} & = & \tilde{Q}_jU_j^{-1}. \end{array} \right.$$

End of B).

Appendix B. Proof of Proposition 3.1.

A) Part "⇒" if.

Hypothesis. The plants P_i and P_j are simultaneously stabilizable with the property to preserve the initial unit $\Phi(P_j, C_j)$.

Conclusion. There exists an unit $\overline{U}$ such that $\overline{U} \in F(P_i, S_1(P_j, C_j))$.

Beginning of the part A). The plants P_i and P_j are simultaneously stabilizable with the property to preserve the initial unit $\Phi(P_j, C_j)$. Then there exists a compensator $C_{si} \in (S_2(P_i) \cap S_2(P_j))$ with (R_{si}, S_{si}) a c.f. of C_{si}. From the Definition 2.2, there exist $\overline{U} \in U$ and $U_j \in U$ such that

$$\begin{cases} N_i R_{si} + D_i S_{si} &= \overline{U} \\ N_j R_{si} + D_j S_{si} &= U_j. \end{cases}$$

Now express the property to preserve the initial unit $\Phi(P_j, C_j)$ as

$$\begin{cases} U_j &= \Phi(P_j, C_{si}) \\ U_j &= \Phi(P_j, C_j). \end{cases}$$

By definition we have

$$F := F(P_i, S_1(P_j, C_j)) = \{\Phi(P_i, C) \in U \textit{ with } C \textit{ such that } \Phi(P_j, C) = \Phi(P_j, C_j)\}.$$

In our case

$$F := \{\overline{U} \in U; \exists C_{si} \in S_2(P_j) \textit{such that } \Phi(P_i, C_{si}) = \overline{U} \textit{and } \Phi(P_j, C_{si}) = \Phi(P_j, C_j)\}.$$

Consequently $\overline{U} \in F(P_i, S_1(P_j, C_j))$.

End of A).

B) Part "$\Leftarrow$" if.

Hypothesis. There exists an unit $\overline{U}$ such that $\overline{U} \in F(P_i, S_1(P_j, C_j))$.

Conclusion. The plants P_i and P_j are simultaneously stabilizable with the property to preserve the initial unit $\Phi(P_j, C_j)$.

Beginning of the part B). Assume that $\overline{U} \in F(P_i, S_1(P_j, C_j))$. From the definition of $F(P_i, S_1(P_j, C_j))$ we write

$$F := \{\overline{U} \in U; \exists C_{si} \in S_2(P_j) \textit{such that } \Phi(P_i, C_{si}) = \overline{U} \textit{and } \Phi(P_j, C_{si}) = \Phi(P_j, C_j)\}.$$

So there exists a compensator C_{si} such that

$$\begin{cases} \Phi(P_j, C_{si}) &= \Phi(P_j, C_j) \\ \Phi(P_i, C_{si}) &= \overline{U}. \end{cases}$$

Therefore the previous equations yield

$$C_{si} \in S_2(P_i) \cap S_1(P_j, C_j).$$

Finally the plants P_i and P_j are simultaneously stabilizable with the property to preserve the initial unit $\Phi(P_j, C_j)$.

End of B).

Appendix C. Proof of Corollary 3.1.

A) Part "⇒" only if.

Hypothesis. Assume that

$$(S_1(P_j, C_j) \cap S_2(P_i)) \neq 0.$$

Conclusion. Verify that

$$(S_2(P_i) \cap S_2(P_j)) \neq 0.$$

Beginning of the part A). Suppose that

$$(S_1(P_j, C_j) \cap S_2(P_i)) \neq 0.$$

Since

$$S_1(P_j, C_j) \subset S_2(P_j)$$

then we have

$$(S_1(P_j, C_j) \cap S_2(P_i)) \subset (S_2(P_j) \cap S_2(P_i)).$$

End of A).

B) Part "⇐" if.

Hypothesis. Assume that

$$(S_2(P_i) \cap S_2(P_j)) \neq 0.$$

Conclusion. Verify that

$$(S_1(P_j, C_j) \cap S_2(P_i)) \neq 0.$$

Beginning of the part B). Let C_g be a compensator such that $C_g \in (S_2(P_i) \cap S_2(P_j))$ and denote (R_g, S_g) a c.f. of C_g. One obtains

$$\left\{ \begin{array}{rcl} N_i R_g + D_i S_g & = & \Phi(P_i, C_g) \\ N_j R_g + D_j S_g & = & \Phi(P_j, C_g). \end{array} \right.$$

Let $C_j \in S_2(P_j)$ be a compensator with the c.f. (R_j, S_j).
We recall that $N_j R_j + D_j S_j = \Phi(P_j, C_j)$.
Since $\Phi(P_j, C_j) \in U$ and $\Phi(P_j, C_g) \in U$, then there exists $U_j^g \in U$ such that

$$\Phi(P_j, C_g) = U_j^g \, \Phi(P_j, C_j).$$

Thus we may write

$$\begin{cases} N_j R_g (U_j^g)^{-1} + D_j S_g (U_j^g)^{-1} & = & \Phi(P_j, C_j) \\ N_i R_g (U_j^g)^{-1} + D_i S_g (U_j^g)^{-1} & = & \Phi(P_i, C_g)(U_j^g)^{-1}. \end{cases}$$

Let $(R_g(U_j^g)^{-1}, S_g(U_j^g)^{-1})$ a c.f. of a compensator C_{si} that we shall also denote (R_{si}, S_{si}). Notice that R_{si} and S_{si} are coprime over RH_∞. Therefore the previous equations yield

$$\begin{cases} N_j R_{si} + D_j S_{si} & = & \Phi(P_j, C_j) \\ N_i R_{si} + D_i S_{si} & \in & U. \end{cases}$$

So there exists a least one compensator C_{si} described by the c.f. (R_{si}, S_{si}) such that $C_{si} \in S_1(P_j, C_j)$ and $C_{si} \in S_2(P_i)$.

End of B).

Similarly, we may show that two plants P_i and P_j are simultaneously stabilizable if and only if there exists a simultaneous compensator C_{sj} such that

$$C_{sj} \in (S_2(P_j) \cap S_1(P_i, C_i))$$

where C_i is a given compensator belonging to $S_2(P_i)$.

Appendix D. Proof of Theorem 4.1.

A) Part "$\Rightarrow$" only if.

Hypothesis. Let $C_j \in S_2(P_j)$ be a compensator with a c.f. (R_j, S_j). Assume further that the plants P_i and P_j are simultaneously stabilizable by a compensator C_{si} satisfying the following relationship $\Phi(P_j, C_j) = \Phi(P_j, C_{si})$.

Conclusion. There exists $Q_{sj} \in RH_\infty$ with $Q_{sj} N_j \neq S_j$ such that

$$(\Phi(P_i, C_j) - Q_{sj} \Delta) \in U \tag{66}$$

where

$$\Delta = D_i N_j - N_i D_j. \tag{67}$$

Beginning of the part A). By hypothesis, the compensator C_{si} with the c.f. (R_{si}, S_{si}) verifies $\Phi(P_i, C_{si}) \in U$ and $\Phi(P_j, C_{si}) = \Phi(P_j, C_j)$. Using Lemma 2.1, we know that there exists $Q_{sj} \in RH_\infty$ with $Q_{sj} N_j \neq S_j$ such that

$$(R_{si}, S_{si}) = (R_j + Q_{sj} D_j, S_j - Q_{sj} N_j). \tag{68}$$

This expression yields

$$N_i (R_{si} - R_j) + D_i (S_{si} - S_j) + Q_{sj} (-N_i D_j + N_j D_i) = 0$$

or equivalently

$$\Phi(P_i, C_j) - \Phi(P_i, C_{si}) - Q_{sj}\Delta = 0.$$

Since $\Phi(P_i, C_{si}) \in U$, we find $Q_{sj} \in RH_\infty$ with $Q_{sj}N_j \neq S_j$ such that

$$(\Phi(P_i, C_j) - Q_{sj}\Delta) \in U.$$

End of A).

B) Part "$\Leftarrow$" if.

Hypothesis. There exists $Q_{sj} \in RH_\infty$ with $Q_{sj}N_j \neq S_j$ such that

$$(\Phi(P_i, C_j) - Q_{sj}\Delta) \in U$$

where

$$\Delta = D_iN_j - N_iD_j.$$

Conclusion. The plants P_i and P_j are simultaneously stabilizable by a compensator C_{si} that preserves the initial unit $\Phi(P_j, C_j)$.

Beginning of the part B). Let $C_j \in S_2(P_j)$ and C_{si} be the compensators with the c.f. (R_j, S_j) and (R_{si}, S_{si}) respectively such that

$$(R_{si}, S_{si}) = (R_j + Q_{sj}D_j, S_j - Q_{sj}N_j).$$

Firstly, check that C_{si} stabilizes P_j.
We recall that

$$\Phi(P_j, C_{si}) = N_jR_{si} + D_jS_{si}.$$

From the definitions of R_{si} and S_{si}, we write

$$\Phi(P_j, C_{si}) = N_j(R_j + Q_{sj}D_j) + D_j(S_j - Q_{sj}N_j).$$

Then

$$\Phi(P_j, C_{si}) = \Phi(P_j, C_j) \text{ and } C_{si} \in S_1(P_j, C_j).$$

So C_{si} stabilizes P_j.
Secondly prove that C_{si} stabilizes P_i.
From the definitions of R_{si} and S_{si}, we obtain

$$\Phi(P_i, C_{si}) = (\Phi(P_i, C_j) - Q_{sj}\Delta).$$

According to the hypothesis $\Phi(P_i, C_{si}) \in U$ then $C_{si} \in S_2(P_i)$.
Thus C_{si} stabilizes P_i.

Finally, we have shown that C_{si} stabilizes P_i and P_j and that $C_{si} \in S_1(P_j, C_j)$. Hence, the plants P_i and P_j are simultaneously stabilizable by a compensator C_{si} that preserves the initial unit $\Phi(P_j, C_j)$.

End of B).

Appendix E. Proof of Corollary 4.2.

A) Part "$\Rightarrow$" only if.

Hypothesis. Let $C_j \in S_2(P_j)$ be a compensator. Assume further that the plants P_i and P_j are simultaneously stabilizable by a compensator C_{si} that preserves the initial unit $\Phi(P_j, C_j)$.

Conclusion. Verify

$$\forall \sigma_{p1} \in E, \forall \sigma_{p2} \in E \quad \Phi(P_i, C_j)(\sigma_{p1})\Phi(P_i, C_j)(\sigma_{p2}) > 0.$$

Beginning of the part A). By hypothesis, the plants P_i and P_j are simultaneously stabilizable by a compensator C_{si} such that $\Phi(P_j, C_j) = \Phi(P_j, C_{si})$. Then from Theorem 4.1, there exists $Q_{sj} \in RH_\infty$ such that

$$(\Phi(P_i, C_j) - Q_{sj}\Delta) \in U.$$

So we have

$$\delta(\Phi(P_i, C_j) - Q_{sj}\Delta) = 0.$$

From Lemma 4.2, the number of sign changes in the sequence $\{\Phi(P_i, C_j)(\sigma_i)\}_{i=1}^{i=nr}$ is equal to zero. Then we obtain

$$\forall \sigma_{p1} \in E, \forall \sigma_{p2} \in E \quad \Phi(P_i, C_j)(\sigma_{p1})\Phi(P_i, C_j)(\sigma_{p2}) > 0.$$

End of A).

B) Part "$\Leftarrow$" if.

Hypothesis. Let $C_j \in S_2(P_j)$ be a compensator. Assume that

$$\forall \sigma_{p1} \in E, \forall \sigma_{p2} \in E \quad \Phi(P_i, C_j)(\sigma_{p1})\Phi(P_i, C_j)(\sigma_{p2}) > 0.$$

Conclusion. The plants P_i and P_j are simultaneously stabilizable by a simultaneous compensator C_{si} that preserves the initial unit $\Phi(P_j, C_j)$.

Beginning of the part B). From Lemma 4.2, there exists $Q_{sj} \in RH_\infty$ such that

$$\delta(\Phi(P_i, C_j) - Q_{sj}\Delta) = 0.$$

Then

$$(\Phi(P_i, C_j) - Q_{sj}\Delta) \in U.$$

From Theorem 4.1, the plants P_i and P_j are simultaneously stabilizable by a simultaneous compensator C_{si} that preserves the initial unit $\Phi(P_j, C_j)$.

End of B).

Appendix F. Proof of Theorem 4.2.

A) Part "$\Rightarrow$" only if.

Hypothesis. The plants P_i and P_j are simultaneously stabilizable by a compensator C_{si} that preserves the initial unit $\Phi(P_j, C_j)$.

Conclusion. Verify

$$\left|\sum_{p=1}^{nr} c(\sigma_p)\right| = nr \quad \text{where} \quad \sigma_p \in E.$$

Beginning of the part A). Suppose that P_i and P_j are simultaneously stabilizable by a compensator C_{si} that preserves the initial unit $\Phi(P_j, C_j)$. Then according to Theorem 4.1 and relationship (27), there exists $Q_{sj} \in RH_\infty$ such that

$$Q_{sj}\Delta = \Phi(P_i, C_j) - \Phi(P_i, C_{si}).$$

As $\Phi(P_i, C_{si}) \in U$, we have $\Phi(P_i, C_{si})(s) \neq 0$ for $\mathrm{Re}(s) \geq 0$.
In addition, since $\Phi(P_i, C_{si}) \in U$, $\Phi(P_i, C_j)$ and Δ are coprime over RH_∞.
Then for any $\sigma_p \in E$, $\Delta(\sigma_p) = 0$ implies that $\Phi(P_i, C_j)(\sigma_p) \neq 0$.
Distinguish between two cases: i) and ii).

i) Assume that $N_j(\sigma_p) \neq 0$.
Then

$$\Phi(P_i, C_j)N_j = N_i R_j N_j + D_i S_j N_j.$$

Moreover, for any $\sigma_p \in E$, we have

$$\Delta(\sigma_p) = 0 \Leftrightarrow N_i(\sigma_p)D_j(\sigma_p) = N_j(\sigma_p)D_i(\sigma_p).$$

Then

$$\Phi(P_i, C_j)(\sigma_p)N_j(\sigma_p) = N_i(\sigma_p)(N_j(\sigma_p)R_j(\sigma_p) + D_j(\sigma_p)S_j(\sigma_p))$$

or equivalently

$$\Phi(P_i, C_j)(\sigma_p)N_j(\sigma_p) = \Phi(P_j, C_j)(\sigma_p)N_i(\sigma_p)$$

and therefore

$$\frac{\Phi(P_j, C_j)(\sigma_p)N_i(\sigma_p)}{\Phi(P_i, C_j)(\sigma_p)N_j(\sigma_p)} = 1.$$

Then one obtains

$$sgn\Big(\frac{\Phi(P_i, C_j)(\sigma_p)}{\Phi(P_j, C_j)(\sigma_p)}\Big) = sgn\Big(\frac{N_i(\sigma_p)}{N_j(\sigma_p)}\Big) = c(\sigma_p). \tag{69}$$

ii) Assume that $N_j(\sigma_p) = 0$.
According to Remark 4.3, $N_j(\sigma_p) = 0 \Leftrightarrow N_i(\sigma_p) = 0$.
As the pairs (N_j, D_j) and (N_i, D_i) are coprime over RH_∞, then

$$D_j(\sigma_p) \neq 0 \text{ and } D_i(\sigma_p) \neq 0.$$

Based on the definition of $\Phi(P_i, C_j)$, we may write

$$\Phi(P_i, C_j)D_j = N_i R_j D_j + D_i S_j D_j.$$

In addition, for any $\sigma_p \in E$, we verify

$$\Delta(\sigma_p) = 0 \Leftrightarrow N_i(\sigma_p)D_j(\sigma_p) = N_j(\sigma_p)D_i(\sigma_p).$$

Thus

$$\Phi(P_i, C_j)(\sigma_p)D_j(\sigma_p) = D_i(\sigma_p)(N_j(\sigma_p)R_j(\sigma_p) + D_j(\sigma_p)S_j(\sigma_p))$$

or equivalently

$$\Phi(P_i, C_j)(\sigma_p)D_j(\sigma_p) = \Phi(P_j, C_j)(\sigma_p)D_i(\sigma_p)$$

and we get

$$\frac{\Phi(P_j, C_j)(\sigma_p)D_i(\sigma_p)}{\Phi(P_i, C_j)(\sigma_p)D_j(\sigma_p)} = 1$$

or

$$sgn\Big(\frac{\Phi(P_i, C_j)(\sigma_p)}{\Phi(P_j, C_j)(\sigma_p)}\Big) = sgn\Big(\frac{D_i(\sigma_p)}{D_j(\sigma_p)}\Big) = c(\sigma_p).$$

It follows from cases i) and ii) that for any $\sigma_p \in E$,

$$sgn\Big(\frac{\Phi(P_i, C_j)(\sigma_p)}{\Phi(P_j, C_j)(\sigma_p)}\Big) = c(\sigma_p). \tag{70}$$

As $\Phi(P_j, C_j)$ is an unit and the sequence $\Phi(P_i, C_j)(\sigma_p)$ has no sign change for any $\sigma_p \in E$, (see Corollary 4.2), then for any σ_p, the ratio (70) is constant. Consequently, $c(\sigma_p)$ is constant and the following relation holds

$$\left|\sum_{p=1}^{nr} c(\sigma_p)\right| = nr \quad \text{where} \quad \sigma_p \in E.$$

End of i) and ii).

End of A).

B) Part "$\Leftarrow$" if.

Hypothesis. Assume that

$$\left|\sum_{p=1}^{nr} c(\sigma_p)\right| = nr \quad \text{where} \quad \sigma_p \in E.$$

Conclusion. The plants P_i and P_j are simultaneously stabilizable by a compensator C_{si} that preserves the initial unit $\Phi(P_j, C_j)$.

Beginning of the part B). Suppose that

$$\left|\sum_{p=1}^{nr} c(\sigma_p)\right| = nr \quad \text{where} \quad \sigma_p \in E.$$

Let (R_j, S_j) be a c.f. of C_j such that $\Phi(P_j, C_j) \in U$. Thus for all s in C_{+e}, $\Phi(P_j, C_j)(s)$ has no sign change. From Lemma 4.1, it can be stated that $\Phi(P_i, C_j)$ and Δ are coprime over RH_∞. Moreover for any $\sigma_p \in E$ we have

$$\Delta(\sigma_p) = 0 \Leftrightarrow N_j(\sigma_p)D_i(\sigma_p) = N_i(\sigma_p)D_j(\sigma_p). \tag{71}$$

Now examin two cases i) and ii).

Beginning of i) and ii).
i) Consider the relation (71) and suppose that $N_i(\sigma_p) \neq 0$
Multiply $\Phi(P_j, C_j)(\sigma_p)$ by $N_i(\sigma_p)$ then

$$\Phi(P_j, C_j)(\sigma_p)N_i(\sigma_p) = \Phi(P_i, C_j)(\sigma_p)N_j(\sigma_p).$$

ii) Consider the relation (71) and $D_i(\sigma_p) \neq 0$.

Multiply $\Phi(P_j, C_j)(\sigma_p)$ by $D_i(\sigma_p)$ then

$$\Phi(P_j, C_j)(\sigma_p)D_i(\sigma_p) = \Phi(P_i, C_j)(\sigma_p)D_j(\sigma_p).$$

End of i) and ii).
Note that if $N_j(\sigma_p) = 0$, then $D_j(\sigma_p) \neq 0$, so the cases i) and ii) give

$$c(\sigma_p) = sgn\Big(\frac{\Phi(P_i, C_j)(\sigma_p)}{\Phi(P_j, C_j)(\sigma_p)}\Big).$$

As $\Phi(P_j, C_j)$ is an unit then $\Phi(P_j, C_j)$ has no sign change for any s in C_{+e} and the hypothesis becomes:

$$\left|\sum_{p=1}^{nr} c(\sigma_p)\right| = nr \Rightarrow \quad \forall \sigma_{p1} \in E, \forall \sigma_{p2} \in E, \quad \Phi(P_i, C_j)(\sigma_{p1})\,\Phi(P_i, C_j)(\sigma_{p2}) > 0. \tag{72}$$

From the Corollary 4.2, if relationship (72) holds, then the plants P_i and P_j are simultaneously stabilizable by a compensator C_{si} that preserves the initial unit $\Phi(P_j, C_j)$.

End of B).

References

[1] **V. Blondel,** *Simultaneous stabilization of linear systems.* Springer-Verlag, Berlin (1994).

[2] **V. Blondel, G. Campion, M. Gevers,** *A sufficient condition for simultaneous stabilization.* IEEE Trans. Automat. Contr., vol. 38, n^o 8, pp. 1264-1268 (1993).

[3] **P. Dorato, H. Park, Y. Li,** *An algorithm for interpolation with units in H_∞, with application to feedback stabilization.* Automatica, vol. 25, pp. 427-430 (1989).

[4] **A.N. Gündes, M.G. Kabuli,** *Simultaneous stabilization of linear systems under stable additive or feedback perturbations.* IEEE Trans. Automat. Contr., vol. 40, n^o 11, pp. 1926-1931 (1995).

[5] **M. Vidyasagar, N. Viswanadham,** *Algebraic design techniques for reliable stabilization.* IEEE Trans. Automat. Contr., vol. 27, n^o 5 (1982).

[6] **M. Vidyasagar,** *Control system synthesis, a factorization approach.* MIT Press, Cambridge MA (1985).

[7] **K. Wei,** *The solution of a transcendental problem and its applications in simultaneous stabilization problems.* IEEE Trans. Automat. Contr., vol. 37, n^o 9, pp. 1305-1315 (1992).

[8] **D.C. Youla, J.J. Bongiorno and C.N. Lu,** *Single-loop feedback stabilization of linear multivariable dynamical plants.* Automatica, vol. 10, pp. 159-173 (1974).

[9] **R.V. Patel, B. Datta,** *A dual look at unit interpolation in H_∞.* Int. J. Control, vol. 62, n^o 4, pp. 813-829 (1995).

[10] **S. Wang, F. Fairman,** *Simultaneous robust stabilization of two plants having unstructured uncertainty.* Int. J. Control, vol. 59, n^o 2, pp. 485-498 (1995).

Unité de recherche INRIA Lorraine, Technopôle de Nancy-Brabois, Campus scientifique,
615 rue du Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY
Unité de recherche INRIA Rennes, Irisa, Campus universitaire de Beaulieu, 35042 RENNES Cedex
Unité de recherche INRIA Rhône-Alpes, 655, avenue de l'Europe, 38330 MONTBONNOT ST MARTIN
Unité de recherche INRIA Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex
Unité de recherche INRIA Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex

Éditeur
INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
ISSN 0249-6399