Essential Faces and Stability Conditions of Multiclass Networks with Priorities

Vincent Dumas

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Essential faces and stability conditions of multiclass networks with priorities

Vincent Dumas

N ° 3030

Décembre 1996

THÈME 2

Rapport de recherche

ISSN 0249-6399

Essential faces and stability conditions of multiclass networks with priorities

Vincent Dumas

Thème 2 — Génie logiciel
et calcul symbolique
Projet Algorithmes

Rapport de recherche n ° 3030 — Décembre 1996 — 22 pages

Abstract: It is now well-known that multiclass networks may be unstable even under the "usual conditions" of stability (when the loads are less than one at all queues), but the proofs of transience (in the Markovian case) generally require a complex work based on the dynamics of an associated "fluid model". Here we develop a sample-path argument introduced in a previous paper, which provides new ergodicity conditions for networks ruled by *priorities*; when one of these conditions is violated, the network diverges at linear speed. Our approach is based on the identification of the *essential faces*, which are the sets of classes that can be simultaneously occupied in stationary regime. A graph is associated with the network, the existence of unessential faces being equivalent to the presence of *cycles* in this graph, in which case the usual conditions are not sufficient conditions of stability. As a by-product of our results, we recover the stability conditions and complete the analysis of two exemplary models, the Rybko-Stolyar network and the Lu-Kumar network.

Key-words: multiclass queueing networks, preemptive resume priorities; ergodicity, stability conditions, irreducibility, essential states; space homogeneity, faces.

(Résumé : tsvp)

Unité de recherche INRIA Rocquencourt
Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
Téléphone : (33) 01 39 63 55 11 – Télécopie : (33) 01 39 63 53 30

Faces essentielles et conditions de stabilité des réseaux multiclasses avec priorités

Résumé : On sait maintenant que les réseaux multiclasses peuvent être instables sous les "conditions habituelles" de stabilité (c'est-à-dire quand la charge est inférieure à 1 à chaque file), mais prouver que le modèle est transient (dans le cas markovien) exige généralement un travail fastidieux fondé sur l'étude de la dynamique du "modèle fluide" associé. Nous développons ici un argument purement trajectoriel introduit dans un article précédent, lequel fournit de nouvelles conditions d'ergodicité pour les réseaux régis par des *priorités* ; si l'une de ces conditions est violée, le réseau diverge à vitesse linéaire. Notre approche est fondée sur l'identification des *faces essentielles*, c'est-à-dire des ensembles de classes pouvant être occupées simultanément en régime stationnaire. L'existence de faces non essentielles équivaut à la présence de *cycles* dans un graphe associé au réseau, auquel cas les conditions habituelles ne sont pas suffisantes pour rendre le réseau stable. Comme corollaire de nos résultats, nous retrouvons les conditions de stabilité de deux modèles de référence, le réseau de Rybko-Stolyar et celui de Lu-Kumar.

Mots-clés : Réseaux de files d'attente multiclasses, priorités préemptives ; ergodicité, conditions de stabilité ; irréductibilité, états essentiels ; homogénéité spatiale, faces.

1 Introduction.

One of the most remarkable properties of multiclass queueing networks is that transience (in a Markovian setting) may occur even when traffic intensities are smaller than one at all queues, that is under what we shall call the *usual conditions* of stability. This phenomenon is however hard to detect, the only general approach consisting in finding diverging paths in a deterministic "fluid model" of the network. Such paths have been described for networks working under priority policies (see Dai and Weiss [6], Dai and Meyn [5]) or FIFO discipline (see Dumas [8]), but to prove that the original, stochastic model is then unstable still requires a complex work based on large deviation estimates (see Rybko and Stolyar [15] for priorities, Bramson [2, 3] for FIFO), or a global analysis of the fluid model exploited via an associated transience criterion (see Botvitch and Zamyatin [1] and Dumas [9] for networks with priorities).

However, in [9], we introduced a simple, sample-path argument that provided an unexpected stability condition of the special model under consideration, a model ruled by *priorities*. Multiclass networks with priorities have provided most of the examples of unstable models analyzed so far. Their success is primarily due to the simplicity of the state variable, which is finite-dimensional. Moreover, regardless of the models that have been proved stable under the usual conditions, the few multiclass, stochastic networks whose stability conditions have been explicitly determined were with priorities: the first one was analyzed in [15, 1], the second one in [13, 6] (instability here was only demonstrated at the fluid level), both of them deriving from deterministic models studied by Kumar and Seidman under "clear-a-fraction" policies [12]; the last one was analyzed in [9].

Here we will extend the argument of [9] to general, multiclass networks with fixed routes and class priorities, thus obtaining new conditions of stability for this family of models. The approach is based on the characterization of the "unessential states" of the system, which may be characterized as the states that cannot be reached from the empty network. The existence of unessential states was previously noticed on special models by Rybko and Stolyar [15], Botvitch and Zamyatin [1], and more recently by Harrison and Nguyen [10]. But surprisingly enough, no connection had yet been established between this phenomenon and the stability conditions (extensions of our results for "global stability" have however been obtained by Dai and Vande Vate [7] while this paper was in preparation, see the conclusion).

In section 2, we first present the general model of multiclass networks with fixed customer routes and class priorities, and the associated Markov state variable, which is finite-dimensional and discrete. We define the notion of *essential states* of the model (Definition 2.2), the state process being irreducible iff all the states are essential, and we show that the asymptotic behavior of the system may be analyzed in terms of the *irreducible* Markov process obtained by restricting the state space to essential states. In section 3, we adopt the hypothesis that there is "no feed-back with priority" (Assumption 3.3), and we prove that the distinction between essential and unessential states only relies on which classes they occupy. We explain how to test whether a given set of occupied classes (that is a given *face*) corresponds to essential states or not (Proposition 3.6); a graph $\mathcal{G}$ on the set of classes is associated with the network, the state process being irreducible iff $\mathcal{G}$ acyclic; conversely, any cycle of $\mathcal{G}$ corresponds to an unessential face (Proposition 3.9).

In section 4, as a direct consequence of our preliminary work, we show our main result (Proposition 4.2): any face Λ is associated with a stability condition of the form: $\rho_\Lambda < n(\Lambda)$, where ρ_Λ is the total traffic intensity in Λ, and $n(\Lambda)$ is an integer depending on Λ; in particular, if there exist unessential faces and Λ is a minimal cycle of graph $\mathcal{G}$, then condition $\rho_\Lambda > n(\Lambda)$, which implies *linear divergence* of the system, is compatible with the usual conditions (Corollary 4.4). Finally, we treat several examples, including the two models analyzed respectively in [15, 1] and [13, 6], whose instability phenomenon may be completely explained in terms of unessential faces (Example-4.5 and section 5). In the appendix, we describe a natural classification of unessential faces (Definition-Lemma A.1), which is then used to prove that the set of essential states is hit in finite time under the usual conditions (Proposition 2.5).

2 Markovian model. Essential states.

We consider networks with K single-server queues and I types of customers. Type i customers enter the network according to a Poisson process at rate ν_i; they arrive into queue $k_{i1} \in \{1, \ldots, K\}$, first stage of their route, which comprises l_i stages; at the s^{th} stage of their route ($1 \leq s \leq l_i$), they are in queue k_{is} ($1 \leq k_{is} \leq K$), where they require independent, exponential service times of mean $1/\mu_{is}$; after service, they immediately reach the next stage (they leave the network if $s = l_i$). All the parameters ν_i and μ_{is} are assumed finite and positive. The various service sequences and arrival processes are independent. This general model derives from Kelly networks [11]. In the conclusion, we will explain how the results of this paper can be generalized to the case of non-exponential variables.

Type i customers at the s^{th} stage of their route form *class* (i, s). Denote by $\mathcal{C}$ the set of all the classes. Inside each class, customers are served according to the order of arrivals. Now we say that class (i, s) has *preemptive resume priority* over class (i', s'), and we note: $(i, s) > (i', s')$, if $k_{is} = k_{i's'}$ and (i', s') customers are served only when there are no (i, s) customers in the queue; if an (i', s') customer is being served and an (i, s) customer arrives, the service is interrupted and the residual service time will be executed after all the (i, s) customers will have departed the queue.

Denote by:

$$\mathcal{C}_k = \{(i, s) \in \mathcal{C} / k_{is} = k\}, \quad 1 \leq k \leq K,$$

the set of the classes that belong to queue k. The service discipline in any queue k is assumed to be based on priorities between classes, which means that $\mathcal{C}_k$ may be ordered according to a sequence: $\mathcal{C}_k = \{(i_1, s_1), (i_2, s_2), \ldots, (i_n, s_n)\}$, with $(i_1, s_1) > (i_2, s_2) > \ldots > (i_n, s_n)$.

Example 2.1 *Figure 1 pictures a special model of multiclass network with priorities. It is characterized by:* $K = I = 2$, $l_1 = l_2 = 2$, $k_{11} = k_{22} = 1$, $k_{12} = k_{21} = 2$, $(2, 2) > (1, 1)$ *and* $(1, 2) > (2, 1)$. *This model was analyzed by Rybko/Stolyar [15] and Botvitch/Zamyatin [1].*

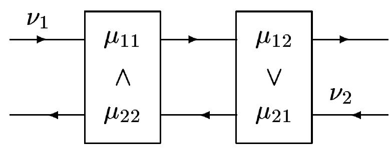

Figure 1: Multiclass network with priorities.

Denote by $Q_{is}(t)$ the number of customers in class (i, s) at time $t \geq 0$, and: $Q(t) = (Q_{is}(t))_{(i,s)\in\mathcal{C}}$. Obviously $(Q(t))_{t\geq 0}$ is a Markov jump process in state space $\mathbb{Z}_+^{\mathcal{C}}$. We will denote by: $\rho_{is} \triangleq \nu_i/\mu_{is}$, the *traffic intensity* of class (i, s) ($(i, s) \in \mathcal{C}$). Traffic intensities are the parameters that actually determine the behavior of the model.

Some auxiliary processes will be repeatedly used in the analysis of $Q(t)$. With each class (i, s) we associate the cumulative service time that will be required at stage s by *all* type i customers present at time t in the network; we call it the *potential load of class* (i, s), and denote it by $\boldsymbol{W}_{is}(t)$. This is a random function of $Q(t)$, but by the law of large numbers:

$$\frac{1}{t}\left(\boldsymbol{W}_{is}(t) - \frac{\sum_{r=1}^{s} Q_{ir}(t)}{\mu_{is}}\right) \longrightarrow 0 \quad \text{a.s. when } t \to \infty. \tag{1}$$

There is a natural decomposition of $\boldsymbol{W}_{is}(t)$ as:

$$\boldsymbol{W}_{is}(t) = \boldsymbol{\Omega}_{is}(t) - B_{is}(t), \tag{2}$$

where $\boldsymbol{\Omega}_{is}(t)$ is the cumulative load brought to class (i, s) up to time t (each type i customer which enters the network brings his future service time at stage s), and $B_{is}(t)$ is the time devoted by server k_{is} to class (i, s) up to time t. By laws of large numbers again:

$$\frac{1}{t}\left(\boldsymbol{\Omega}_{is}(t) - \frac{N_i(t)}{\mu_{is}}\right) \longrightarrow 0 \quad \text{and} \quad \frac{N_i(t)}{t} \longrightarrow \nu_i \quad \text{a.s. when } t \to \infty,$$

where $N_i(t)$ denotes the number of type i external arrivals into the network up to time t. In consequence:

$$\frac{\boldsymbol{\Omega}_{is}(t)}{t} \longrightarrow \rho_{is} \quad \text{a.s. when } t \to \infty. \tag{3}$$

On the other hand, in view of the service discipline:

$$B_{is}(t) = \int_0^t \mathbb{1}_{\{Q_{is}(u)>0,\ Q_{jr}(u)=0 \text{ for all } (j,r)>(i,s)\}} du. \tag{4}$$

We shall repeatedly use the notation: $f_\Lambda \triangleq \sum_{(i,s)\in\Lambda} f_{is}$ for various functions f_{is} of (i, s) and subsets Λ of $\mathcal{C}$.

Now the main point is that the Markov process $Q(t)$ is *not* irreducible in general. However it is always possible to obtain an irreducible process by restricting the state space to the set of "essential states". This notion is introduced below.

Definition 2.2 *A state $q \in \mathbb{Z}_+^{\mathcal{C}}$ is called* unessential *if there exists a state q' that can be reached (with positive probability) from q, and q cannot be reached from q'. A state that is not unessential is called* essential.

The partition of $\mathbb{Z}_+^{\mathcal{C}}$ between essential states and unessential states is independent of the values of parameters ν_i, μ_{is}, $(i,s) \in \mathcal{C}$ (they are positive anyway). More generally, it is obvious that unessential states (if there exist some) are always transient, and that the set E of the essential states is a *closed* set (no unessential state can be reached from any essential state). Moreover, state 0 (which corresponds to an empty network) can obviously be reached from any other state; hence the essential states are those that can be reached from state 0, and they form an *irreducible* set E.

If the initial state of the network belongs to E, $Q(t)$ thus behaves like an irreducible Markov process, which we shall denote by $Q^*(t)$, and is obtained by restricting the state space to E. It is hence natural to define stability as follows.

Definition 2.3 *We say that the queueing network is* stable *is $Q^*(t)$ is ergodic.*

It is however necessary to address the question of the behavior of $Q(t)$ when the initial state is unessential. If $\rho_{\mathcal{C}_k} > 1$ for some queue k ($\rho_{\mathcal{C}_k} = \sum_{k_{is}=k} \rho_{is}$ according to our notation), it is not necessarily true that set E is reached in finite time, but in any case the evolution of the network can be easily described.

Proposition 2.4 *If $\rho_{\mathcal{C}_k} > 1$ for some k, then:*

$$\liminf_{t\to\infty} \frac{\boldsymbol{W}_{\mathcal{C}_k}(t)}{t} \geq \rho_{\mathcal{C}_k} - 1 > 0 \quad a.s.,$$

hence $Q(t)$ diverges at linear rate in view of (1).

Proof :
For $1 \leq k \leq K$, we have:

$$\forall t \geq 0: \quad B_{\mathcal{C}_k}(t) = \int_0^t \mathbb{1}_{\{Q_{\mathcal{C}_k}(u)>0\}} du, \tag{5}$$

because the classes in $\mathcal{C}_k$ share the same server. If $\rho_{\mathcal{C}_k} > 1$ for some k, the result of divergence is then an immediate consequence of (2) and (3). ♠

Under a mild condition on the topology (no "feedback with priority", see Assumption 3.3 and the comment below), we will be able to prove the following result in the Appendix.

Proposition 2.5 *Under the conditions: $\rho_{\mathcal{C}_k} \leq 1$, $1 \leq k \leq K$, the set E of the essential states is reached in finite time.*

Hence the asymptotic behavior of $Q(t)$ coincides with that of $Q^*(t)$ in this case.

In consequence, from now on, unless otherwise stated, *we assume that the state space is restricted to* E, that is we consider $Q^*(t)$ instead of $Q(t)$ (but we forget the mark *).

In case of stability, we can derive some elementary results that will later prove useful to get necessary stability conditions.

Lemma 2.6 *If the network is stable, then for any subset* Λ *of* $\mathcal{C}$*:*

$$\frac{B_\Lambda(t)}{t} \longrightarrow \rho_\Lambda \quad \textit{in probability,} \tag{6}$$

and if π *denotes the limit distribution:*

$$\frac{1}{t}\int_0^t \mathbb{I}_{\{Q_\Lambda(u)>0\}} du \longrightarrow \pi[q_\Lambda > 0] < 1 \quad \textit{a.s.} \tag{7}$$

Proof :

If $Q(t)$ is ergodic, then for any class (i,s): $Q_{is}(t)/t \to 0$ in probability; hence for any subset Λ of $\mathcal{C}$: $\boldsymbol{W}_\Lambda(t)/t \to 0$ in probability (by (1)), which implies (6) in view of (2) and (3). In (7), we just need to show that $\pi[q_\Lambda > 0] < 1$; but since 0 is an essential state: $\pi[q = 0] > 0$, which completes the proof. ♠

As an immediate consequence, we mention the following, well-known stability conditions.

Corollary 2.7 *If the network is stable, then* $\rho_{\mathcal{C}_k} < 1$ *for all* k*: these are the* usual conditions *of stability.*

The proof immediately derives from Lemma 2.6 and formula (5).

Example 2.8 *For the network of figure 1, the usual conditions are:*

$$\rho_{\mathcal{C}_1} = \rho_{11} + \rho_{22} < 1, \quad \rho_{\mathcal{C}_2} = \rho_{12} + \rho_{21} < 1.$$

In the next section, we address the question of identifying the essential states.

3 Essential faces.

We are going to show that the distinction between essential and unessential states only relies on the "face" they belong to, and then we will explain how to identify the "essential faces". First we define faces.

Definition 3.1 *With any subset* Λ *of* $\mathcal{C}$ *we associate the subset* F_Λ *of* $\mathbb{Z}_+^{\mathcal{C}}$*, defined by:*

$$(q \in F_\Lambda) \Longleftrightarrow (q_{is} > 0 \textit{ if } (i,s) \in \Lambda, \quad q_{is} = 0 \textit{ if } (i,s) \notin \Lambda).$$

The sets F_Λ*,* $\Lambda \subset \mathcal{C}$*, are called the* faces *of* $\mathbb{Z}_+^{\mathcal{C}}$.

Notice that the faces form a partition of $\mathbb{Z}_+^{\mathcal{C}}$. We will occasionally call a subset $\Lambda \subset \mathcal{C}$ a "face" if there is no risk of confusion.

There is a finite number of transitions that may occur from any state $q = (q_{is})_{(i,s)\in\mathcal{C}}$; they may be described as follows:

transition t_{i0}: external arrival of a type i customer, with intensity ν_i;

transition t_{is} $(1 \leq s \leq l_i)$: end of service of an (i,s) customer (with immediate arrival into class $(i, s+1)$ if $s < l_i$, or immediate departure from the network if $s = l_i$), with intensity:

$$\mu_{is}\mathbb{1}_{\{q_{is} > 0,\ q_{i's'} = 0 \text{ for all } (i',s') > (i,s)\}}.$$

Hence the intensity of transition t_{is} only depends on the face F_Λ that q belongs to (this is "space homogeneity" in the sense of Malyshev and Menshikov [14]); in particular, in face F_Λ, transition t_{is} $(1 \leq s \leq l_i)$ is, we shall say, *allowed* (i.e. it has positive intensity) iff $(i,s) \in \Lambda$ and $\Lambda \cap \{(i',s')/(i',s') > (i,s)\} = \emptyset$; transitions t_{i0} are always allowed.

In the following lemma, e_Λ denotes the state with exactly one customer per class of Λ.

Lemma 3.2 **(i)** *A state $q \in \mathbb{Z}_+^{\mathcal{C}}$ is essential iff there is a sequence of (allowed) transitions t_{is}, $s < l_i$, that leads from 0 to q.*

(ii) *If q is essential, so is any state $q' \leq q$ componentwise.*

(iii) *Assume that there exists no class (i,s) such that $(i,s+1) > (i,s)$. If e_Λ is essential for some $\Lambda \subset \mathcal{C}$, so is any state $q \in F_\Lambda$.*

Proof :

(i) Remember that state q is essential iff it can be reached from 0, hence iff there is a sequence of (allowed) transitions that leads from 0 to q. Assume that this sequence includes a transition t_{il_i}. Think of state q as a set of customers positioned in different classes. Transition t_{il_i} makes a customer leave the network, hence he does not belong to the set of customers corresponding to q; in consequence, all the transitions t_{is} $(0 \leq s \leq l_i)$ that affected this customer may be eliminated, the remaining transitions being still valid (to suppress a customer cannot block other customers), and leading to q as well: it is of no use to introduce additional customers, or equivalently to use transitions t_{il_i}, $1 \leq i \leq I$.

(ii) Assume that state q is essential, and consider a state $q' \leq q$: it can be obtained by suppressing some of the customers associated with q. Take a sequence of (allowed) transitions t_{is}, $s < l_i$, that lead from 0 to q, and get rid of the transitions that move suppressed customers. Then the remaining transitions are still valid (again: to suppress some customers cannot block the remaining ones), and they form a sequence that leads from 0 to q'.

(iii) Assume that e_Λ is essential. Take a sequence of, say, N transitions t_{is}, $s < l_i$, that lead from 0 to e_Λ, and denote by $e(n)$ the state reached after the n^{th} transition (hence $e(0) = 0$, $e(N) = e_\Lambda$). In state $e(n)$, the customer destined to class $(i,s) \in \Lambda$ is currently in class (i,s'), $s' \le s$ ($s' = 0$ if he hasn't been introduced in the network yet). Consider a state $q \in F_\Lambda$, and denote by $q(n)$ the state with q_{is} customers in class (i,s') associated with $(i,s) \in \Lambda$ ($q_{is} \ge 1$ since $q \in F_\Lambda$), for all $(i,s) \in \Lambda$ (hence $q(0) = 0$, $q(N) = q$). It is sufficient to prove that state $q(n+1)$ can be reached from $q(n)$. State $e(n+1)$ must be obtained by moving the customer destined to a class $(i,s) \in \Lambda$ from (i,s') (in $e(n)$) to $(i,s'+1)$. A natural way to reach $q(n+1)$ consists in moving the corresponding q_{is} customers (in $q(n)$) consecutively from (i,s') to $(i,s'+1)$: let us check that this procedure is valid. First notice that $q(n)$ belongs to the same face as $e(n)$, hence transition $t_{is'}$, which moves the first (i,s') customer, must still be valid. Moreover, the only reason why the next (i,s') customers couldn't be moved forward like the first one is that they be blocked be him, that is: $(i,s'+1) > (i,s')$. Since this possibility is excluded, property **(ii)** is proved.

♠

In case there exists a class (i,s) such that $(i,s+1) > (i,s)$, property **(iii)** is not satisfied: it is possible to bring one customer into class $(i,s+1)$, but no more. However, if there exists no class (i',s') such that $(i,s+1) > (i',s') > (i,s)$, classes (i,s) and $(i,s+1)$ may be treated as a single class with i.i.d. services of mean: $1/\mu_{is} + 1/\mu_{is+1}$. Though these services are not any longer exponentially distributed, all the coming results may be generalized to this case (see the conclusion). Otherwise, our arguments must be carefully adapted in view of the model.

In the remainder of this paper:

Assumption 3.3 *We assume that there exists no class (i,s) such that $(i,s+1) > (i,s)$, that is* no feed-back with priority.

In view of the above lemma, for any $\Lambda \subset \mathcal{C}$, either all the states $q \in F_\Lambda$ are essential or all are unessential; moreover, if Λ is unessential, then so is any face Λ' with $\Lambda \subset \Lambda'$. The following definitions are hence natural.

Definition 3.4 *Let $\Lambda \subset \mathcal{C}$.*

- *Face Λ is called* essential *(resp.* unessential*) if all the states in F_Λ are essential (resp. unessential). Besides, it is said* simple *if any queue comprises at most one class of Λ.*
- *If Λ is unessential and all the faces $\Lambda' \subset \Lambda$, $\Lambda' \neq \Lambda$, are essential, we call Λ a* minimal *(unessential) face.*

Minimal faces (if there exist some) allow for a simple description of the (un)essential states: obviously, a state $q \in \mathbb{Z}_+^{\mathcal{C}}$ is essential iff:

$$\forall \Lambda \subset \mathcal{C},\ \Lambda \text{ minimal face, } \exists\, (i,s) \in \Lambda / q_{is} = 0.$$

They are also associated with new conditions of stability, as will be shown in section 4. Before attending to this issue, let us explain how to determine whether a face is essential or not. We need to introduce the notion of "antecedent".

Definition 3.5 • *Let $q \in \mathbb{Z}_+^{\mathcal{C}}$. An* antecedent *of q is a state q' such that there is an (allowed) transition t_{is}, $s < l_i$, that leads from q' to q.*

- *Let $\Lambda \subset \mathcal{C}$. An* antecedent *of Λ is a face Λ' such that e_Λ admits an antecedent in $F_{\Lambda'}$.*

Once antecedents are characterized, it is easy to test whether a given face is essential or not.

Proposition 3.6 *1. If $q \in F_\Lambda$, any antecedent q' is obtained by moving backward a customer from a class $(i,s) \in \Lambda$ to class $(i,s-1)$; the new state is actually an antecedent iff $\Lambda \cap \{(i',s') > (i,s-1)\} = \emptyset$.*

2. **Test of essentiality.** *Face Λ is essential iff there exists a chain of antecedents that leads from Λ to $\emptyset$. The length of any chain of antecedents is upper bounded by:*

$$L(\Lambda) = \sum_{(i,s)\in\Lambda} s.$$

Proof :

1. It is clear that any antecedent of q is obtained by moving backward a customer from a class $(i,s) \in \Lambda$ to class $(i,s-1)$. We thus get a state: $q' = q - e_{is} + e_{is-1}$ ($e_{i0} = 0$), which belongs to some face $F_{\Lambda'}$. This a real antecedent iff transition t_{is-1} is allowed in q', or equivalently iff $\Lambda' \cap \{(i',s') > (i,s-1)\} = \emptyset$. But Λ and Λ' may only differ by classes (i,s) and $(i,s-1)$, and both of them cannot have priority on class $(i,s-1)$ (Assumption 3.3). In consequence:
$$\Lambda' \cap \{(i',s') > (i,s-1)\} = \Lambda \cap \{(i',s') > (i,s-1)\}.$$
2. **Test.** By lemma 3.2, a state q is essential iff there is a chain of antecedents that leads from q to 0. Hence by Definition 3.4, a face Λ is essential iff there exists a chain of antecedents that leads from Λ to $\emptyset$. Moreover, for any antecedent Λ' of Λ, we have: $L(\Lambda') \leq L(\Lambda) - 1$, which completes the proof.

♠

Example 3.7 *Consider face $\Lambda = \{(1,1),(1,2),(2,1)\}$ in the network of figure 1. A chain of antecedents is:*

$$\{(1,1),(2,1)\}, \quad \{(2,1)\}, \quad \emptyset.$$

Hence Λ is essential, and by symmetry of the network, $\Lambda' = \{(2,1),(2,2),(1,1)\}$ is essential too.

The test of essentiality of Proposition 3.6 potentially allows for the detection of all the minimal faces. However, in case there exist unessential faces, we would like to be able to identify the minimal faces without having to test all the faces. This is possible at least for an important category of minimal faces, namely those *without antecedents.* They may be derived through a graph associated with the network. This graph also provides a test of existence of unessential faces.

Definition 3.8 *The* associated graph $\mathcal{G}$ *is a directed graph whose set of vertices is* $\mathcal{C}$*; we put a directed edge* $(i, s) \to (i', s')$ *iff* $(i', s') > (i, s - 1)$.

A cycle *is a face* $\Lambda \subset \mathcal{C}$ *that may be ordered in a sequence* $\Lambda = \{(i_1, s_1), ..., (i_N, s_N)\}$ *such that*

$$(i_n, s_n) \to (i_{n+1}, s_{n+1}),\ 1 \leq n \leq N,\ \textit{with the convention: } (i_{N+1}, s_{N+1}) = (i_1, s_1).$$

A minimal cycle *is a cycle without subcycle.*

Notice that no self-loop $(i, s) \to (i, s)$ is possible by Assumption 3.3.

Proposition 3.9

1. *Any minimal cycle is a* simple *face* without antecedent *(hence unessential).*
2. *If there exist unessential faces, there must exist minimal faces* without antecedents. *Any such face is a* minimal cycle.
3. *In particular:*

$$(\textit{all the faces are essential}) \Longleftrightarrow (\mathcal{G} \textit{ is acyclic}).$$

Proof :

1. Consider a cycle $\Lambda = \{(i_1, s_1), ..., (i_N, s_N)\}$, with:

   $$(i_n, s_n) \to (i_{n+1}, s_{n+1}),\ 1 \leq n \leq N \quad ((i_{N+1}, s_{N+1}) = (i_1, s_1)).$$

   For any class $(i_n, s_n) \in \Lambda$, we have: $(i_n, s_n - 1) < (i_{n+1}, s_{n+1})$. Hence, in view of Proposition 3.6, e_Λ (or equivalently Λ) admits no antecedent.

   Then assume that two distinct classes (i_p, s_p) and (i_q, s_q) of Λ belong to the same queue, with say $(i_p, s_p) > (i_q, s_q)$. Then by definition of $\mathcal{G}$:

   $$(i_{q-1}, s_{q-1} - 1) < (i_q, s_q) < (i_p, s_p),$$

   hence there is a directed edge $(i_{q-1}, s_{q-1}) \to (i_p, s_p)$. In consequence, the cycle is not minimal.

2. Assume that there exist unessential faces, and consider an unessential face Λ that minimizes $L(\Lambda)$ (hence Λ is minimal). If Λ admits an antecedent Λ', then $L(\Lambda') \leq L(\Lambda) - 1$; in consequence, Λ' is essential, and so is Λ since Λ' is an antecedent: this is absurd. Hence Λ is a minimal face without antecedent. To complete the proof, in view of 1, it is sufficient to prove that any face without antecedent contains a cycle. Consider such a face Λ. In view of Proposition 3.6, for any class $(i,s) \in \Lambda$, there exists a class $(i',s') \in \Lambda$ such that $(i,s) \to (i',s')$. The conclusion is easy.

3. This is an easy consequence of the preceding results.

♠

The associated graph is hence a practical tool first to determine whether there exist unessential faces (there must be cycles in $\mathcal{G}$), then to identify the minimal faces without antecedent (they must be minimal cycles of $\mathcal{G}$). Though inversely minimal cycles are unessential faces, we have not been able to prove that they are minimal faces yet; hence it is necessary to check that their subfaces are essential. In the appendix, we will present a classification of minimal faces that potentially allows for a recursive identification of all the minimal faces from those without antecedent (Proposition A.1).

Example 3.10 *An elementary network which exhibits a cycle is that of figure 1. Graph $\mathcal{G}$ is simply :*

$$(1,1) \quad (1,2) \longleftrightarrow (2,2) \quad (2,1)$$

There is a single, minimal cycle $\Lambda = \{(1,2),(2,2)\}$, hence the unique minimal face without antecedent. Since we have already checked that faces $\mathcal{C}\backslash\{(1,2)\}$ and $\mathcal{C}\backslash\{(2,2)\}$ are essential, Λ is in fact the only minimal face. In consequence, the essential states are the states in which classes $(1,2)$ and $(2,2)$ are not both occupied.

With minimal faces are associated new conditions of stability, which are introduced in the following section.

4 Necessary conditions of stability.

As a first stage, we are going to associate a necessary condition of stability with *each* face Λ (Proposition 4.2). It is expressed in terms of a special integer $n(\Lambda)$, which is defined below. After deriving basic properties of function $n(\Lambda)$ (Lemma 4.3), it becomes clear that this set of conditions includes the usual conditions, but that it is stronger if there exist unessential faces, because at least the conditions associated with minimal cycles are not implied by the usual ones; proceeding, we show that Λ need not be taken into account if it is not a *union of simple, minimal faces* (Corollary 4.4).

Definition 4.1 *For any face $\Lambda \subset \mathcal{C}$, $\Lambda \neq \emptyset$, set $n(\Lambda)$ the maximal number of classes of Λ that may be treated simultaneously in the set of essential states. Since two classes inside the* *same queue cannot be treated simultaneously, denoting by $K(\Lambda')$ the number of non-empty queues in face $F_{\Lambda'}$, we have:*

$$n(\Lambda) = \max\{K(\Lambda'),\ \Lambda' \subset \Lambda,\ \Lambda' \textit{ essential}\},$$

or equivalently, if $|\Lambda'|$ denotes the cardinality of Λ':

$$n(\Lambda) = \max\{|\Lambda'|,\ \Lambda' \subset \Lambda,\ \Lambda' \textit{ essential and simple}\}.$$

Now the elementary results on potential loads obtained in section 2 enable us to obtain the following results easily.

Proposition 4.2 *If there exists a face Λ such that $\rho_\Lambda > n(\Lambda)$, then*

$$\liminf_{t\to\infty} \frac{\boldsymbol{W}_\Lambda(t)}{t} \geq \rho_\Lambda - n(\Lambda) > 0 \quad a.s., \tag{8}$$

hence $Q(t)$ diverges at linear rate in view of (1).

If the network is stable, then for any face Λ: $\rho_\Lambda < n(\Lambda)$.

Proof :

For any face Λ we have:

$$\forall t \geq 0 : \quad B_\Lambda(t) \leq n(\Lambda) \int_0^t \mathbb{I}_{\{Q_\Lambda(u)>0\}} du,$$

since at most $n(\Lambda)$ classes of Λ may be treated simultaneously.

If $\rho_\Lambda > n(\Lambda)$, then in view of (2) and (3), we get (8).

On the other hand, if $(Q(t))_{t\geq 0}$ is ergodic, then in view of (6) and (7) we get:

$$\rho_\Lambda \leq n(\Lambda)\pi[q_\Lambda > 0] < n(\Lambda),$$

and the proof is complete. ♠

In order to assess the power of Proposition 4.2, we need to investigate some properties of function $n(\Lambda)$.

Lemma 4.3 *Function $n(\Lambda)$ satisfies the following properties:*

(i) *For any queue $k \in \{1, ..., K\}$, and any face $\Lambda \subset \mathcal{C}_k$, $\Lambda \neq \emptyset$: $n(\Lambda) = 1$.*

(ii) *For any simple, minimal face Λ: $n(\Lambda) = |\Lambda| - 1$.*

(iii) *If $\Lambda \not\subset \mathcal{C}_k$, $1 \leq k \leq K$, and Λ is not a union of simple, minimal faces, then there exists a partition of Λ into non-empty subsets Λ_1 and Λ_2 such that: $n(\Lambda) = n(\Lambda_1) + n(\Lambda_2)$.*

Proof :

(i) This property is obvious (for any class (i, s), face $\{(i, s)\}$ is essential).

(ii) If Λ is an unessential face, then obviously: $n(\Lambda) \leq |\Lambda| - 1$; if Λ is moreover minimal and simple, then any strict subface is essential and simple, hence: $n(\Lambda) \geq |\Lambda| - 1$.

(iii) Assume that Λ is not a union of simple, minimal faces, and $\Lambda \not\subset \mathcal{C}_k$, $1 \leq k \leq K$. Then there exists $(i, s) \in \Lambda$ such that for any simple, minimal face Λ', either $(i, s) \notin \Lambda'$ or $\Lambda' \not\subset \Lambda$. Set:

$$\Lambda_1 = \{(i', s') \in \Lambda / \ k_{i's'} = k_{is}\}, \quad \Lambda_2 = \Lambda \backslash \Lambda_1.$$

Face Λ_1 contains at least (i, s), and Λ_2 is not empty, otherwise we would have $\Lambda \subset \mathcal{C}_{k_{is}}$. Now choose $\Lambda'' \subset \Lambda_2$, Λ'' essential and simple, such that $n(\Lambda_2) = |\Lambda''|$. Set $\Lambda' = \Lambda'' \cup \{(i, s)\}$. This is a simple face included in Λ, and above all it is essential, otherwise either Λ'' would be unessential, or (i, s) would belong to a minimal face included in Λ. In consequence: $n(\Lambda) \geq |\Lambda'| = |\Lambda''| + 1$. On the other hand: $n(\Lambda_2) = |\Lambda''|$, and $n(\Lambda_1) = 1$ by **(i)**. Hence: $n(\Lambda) \geq n(\Lambda_1) + n(\Lambda_2)$. Moreover, from the definition of $n(\Lambda)$, it comes that for any partition of Λ into subsets Λ_1 and Λ_2, we have: $n(\Lambda) \leq n(\Lambda_1) + n(\Lambda_2)$, which completes the proof.

♠

The meaning of Proposition 4.2 may now be enlightened in view of the above results.

Corollary 4.4 *Denote by $\mathcal{U}$ the set of all the faces that are unions of simple, minimal faces. Then conditions:*

$$\textbf{(i)}\ \rho_\Lambda < n(\Lambda),\ \Lambda \subset \mathcal{C}, \quad \textbf{(ii)}\ \exists \Lambda \subset \mathcal{C} / \rho_\Lambda > n(\Lambda),$$

are respectively equivalent to:

$$\textbf{(i)}\ \rho_{\mathcal{C}_k} < 1,\ 1 \leq k \leq K,\ \rho_\Lambda < n(\Lambda),\ \Lambda \in \mathcal{U}, \quad \textbf{(ii)}\ \exists k \in \{1, ..., K\} / \rho_{\mathcal{C}_k} > 1 \ \textit{or}\ \exists \Lambda \in \mathcal{U} / \rho_\Lambda > n(\Lambda).$$

In particular, if all the faces are essential, conditions **(i)** *are just the usual conditions of stability. Otherwise, for any minimal cycle Λ, condition $\rho_\Lambda > n(\Lambda)$ is compatible with the usual conditions: hence* the usual conditions are not sufficient *if there exist unessential faces.*

Proof :
Assume that $\Lambda \neq \mathcal{C}_k$, $1 \leq k \leq K$, and $\Lambda \notin \mathcal{U}$. By lemma 4.3, **(i)** and **(iii)**:

- if $\Lambda \subset \mathcal{C}_k$ for some k, then $\rho_\Lambda > n(\Lambda)$ implies $\rho_{\mathcal{C}_k} > 1$, and $\rho_{\mathcal{C}_k} < 1$ implies $\rho_\Lambda < n(\Lambda)$;
- if $\Lambda \not\subset \mathcal{C}_k$ for any k, then $\rho_\Lambda > n(\Lambda)$ implies that $\rho_{\Lambda'} > n(\Lambda')$ for some $\Lambda' \subset \Lambda$, $\Lambda' \neq \Lambda$, and conditions $\rho_{\Lambda'} < n(\Lambda')$, $\Lambda' \subset \Lambda$, $\Lambda' \neq \Lambda$ imply $\rho_\Lambda < n(\Lambda)$.

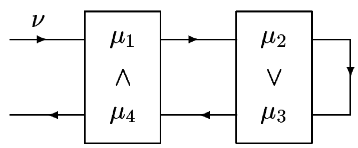

Figure 2: Lu and Kumar's network.

In the latter case, we may repeat the above arguments for all the strict subfaces of Λ. Finally, we see that the only faces to be taken into account are faces $\mathcal{C}_k$, $1 \le k \le K$, and the unions of simple, minimal faces.

If there exist unessential faces, Proposition 3.9 shows that graph $\mathcal{G}$ admits cycles. Any cycle Λ being unessential, we have $n(\Lambda) \le |\Lambda| - 1$. Moreover, any *minimal* cycle is *simple*, hence condition $\rho_\Lambda > |\Lambda| - 1$ is compatible with the usual conditions, which completes the proof. ♠

Example 4.5 *We have previously shown that the network of figure 1 admits a unique minimal face:* $\Lambda = \{(1,2),(2,2)\}$. *By Lemma 4.3, the associated stability condition is:*

$$\rho_\Lambda = \rho_{12} + \rho_{22} < |\Lambda| - 1 = 1.$$

Botvitch and Zamyatin [1] have proved that, under this additional condition, the network is stable under any conservative discipline. They had also shown that the priority network is transient if $\rho_{12} + \rho_{22} > 1$, *by exploiting Malyshev and Menshikov's criteria for random walks with reflection [14]. This result may here be obtained as a direct consequence of Proposition 4.2, which moreover shows that divergence occurs at linear rate.*

Our approach is applied to other typical models in the following section.

5 Examples.

Consider the network introduced by Lu and Kumar [13], which is pictured in figure 2 (since there is a unique route, we omit the index of type).

The associated graph is:

$$(1) \quad (2) \longleftrightarrow (4) \quad (3).$$

There is a unique cycle: $\Lambda = \{(2),(4)\}$, hence the unique minimal face without antecedent. Thanks to the test of Proposition 3.6, one can easily check that faces $\mathcal{C}\backslash\{(2)\}$ and $\mathcal{C}\backslash\{(4)\}$ are essential, hence Λ is the only minimal face. The associated stability condition is:

$$\rho_\Lambda = \rho_2 + \rho_4 < |\Lambda| - 1 = 1.$$

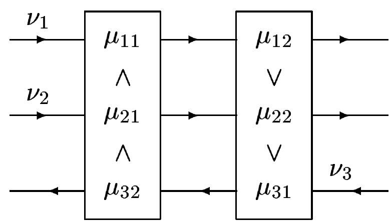

Figure 3: Necessary stability condition: $\rho_{12} + \rho_{22} + \rho_{32} < 1$.

By a method derived from Botvitch and Zamyatin's, Dai and Weiss [6] have shown that, under this additional condition, the network is stable under any conservative discipline. They have also identified a diverging, "fluid" path that appears when: $\rho_2 + \rho_4 > 1$, but without being able to prove that the network is then transient. Here it is a consequence of Proposition 4.2.

Then consider the network of figure 3.

Graph $\mathcal{G}$ is:

$$(1,1) \quad (2,1) \longleftarrow (1,2) \longleftrightarrow (3,2) \longleftrightarrow (2,2) \quad (3,1)$$

There are two minimal cycles: $\Lambda_1 = \{(1,2),(3,2)\}$ and $\Lambda_2 = \{(2,2),(3,2)\}$, which are obviously minimal faces. It can be checked that there are no other minimal faces. The associated stability conditions are:

$$\rho_{12} + \rho_{32} < |\Lambda_1| - 1 = 1 \quad \text{and} \quad \rho_{22} + \rho_{32} < |\Lambda_2| - 1 = 1.$$

Consider the union of simple, minimal faces: $\Lambda = \Lambda_1 \cup \Lambda_2$. It is easy to check that $n(\Lambda) = 1$, hence stability implies that:

$$\rho_{12} + \rho_{22} + \rho_{32} < 1.$$

This condition is stronger than the two previous ones, which shows that not even simple, minimal faces are always worth considering.

The example of figure 4 presents a simple, minimal face Λ with $n(\Lambda) > 1$ (to show that the new conditions are not necessarily of the form: $\rho_\Lambda < 1$), and another face Λ' that is not simple (hence the distinction of simple faces matters). The associated graph $\mathcal{G}$ is pictured in figure 5. There is a single, minimal cycle: $\Lambda = \{(1,2),(3,2),(4,2)\}$, hence the unique minimal face without antecedent. The associated stability condition is:

$$\rho_\Lambda = \rho_{12} + \rho_{32} + \rho_{42} < |\Lambda| - 1 = 2.$$

Now consider face $\Lambda' = \{(1,3),(2,2),(3,2),(4,2)\}$. It can be checked that all the (strict) subfaces are essential. Moreover, its only antecedent is face: $\Lambda'' = \{(1,2),(2,2),(3,2),(4,2)\}$, which contains Λ; in consequence, Λ'' is unessential, hence so is Λ'. We have thus exhibited a minimal face which is not simple, hence which we need not take into account for the stability conditions.

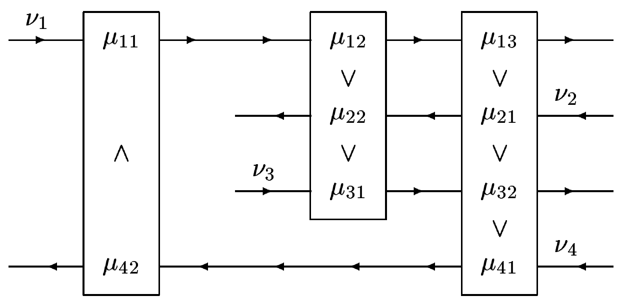

Figure 4: Non-simple, minimal face: $\Lambda = \{(1,3),(2,2),(3,2),(4,2)\}$.

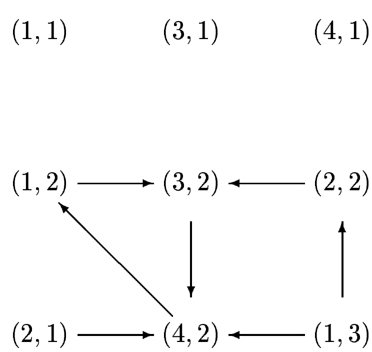

Figure 5: Graph $\mathcal{G}$.

6 Conclusion.

In order to expose our results in the familiar context of discrete state variables, we assumed that the random variables be exponentially distributed. Most of our results may however be easily generalized to the "i.i.d." framework described by Dai in [4] (services are i.i.d. in each class, and external arrivals form renewal processes with "unbounded and spread-out" interarrivals). The states that cannot be reached from zero in the exponential case can *a fortiori* not be reached in a given i.i.d. model; moreover, in this i.i.d. framework, when the model is ergodic (in the sense of Harris), then the stationary probability that the network be empty is positive: all the arguments used for Proposition 4.2 are therefore still valid.

¿From a practical point of view, it would be preferable to identify all the minimal faces in a constructive way (preferably based on graph $\mathcal{G}$, like for minimal faces without antecedent), rather than to have to test all the faces. The classification of minimal faces that is presented in the appendix (Definition-Lemma A.1) suggests that it should be possible to build the minimal faces by increasing "order". Such a procedure is however hard to elaborate, and we are still investigating this issue.

In section 5, we saw that our sample-path argument accounts for the instability phenomena that occur in the exemplary cases of Rybko/Stolyar and Lu/Kumar networks. Other applications may be expected on the question of "global stability", where disciplines based on priorities seem to play the role of "extreme disciplines". Developing our results in the special context of two-station networks, Dai and Vande Vate [7] have obtained the global stability conditions of such models. In particular, they introduce a new argument stating that the service rate of the classes of a given face Λ may be reduced due to the presence of other classes with higher priority in their queues; it then seems possible to replace the linear conditions $\rho_\Lambda < n(\Lambda)$ by stronger, quadratic conditions.

In any case, the connection between stability and essential states should be further investigated, the conditions obtained in Proposition 4.2 being presumably not optimal. Nevertheless, the reader should already be convinced that the identification of the essential states is not a secondary step in the stability analysis of multiclass networks, but that it must be paid great attention since it can lead to unexpected stability conditions. It should however not be expected that the linear conditions of Proposition 4.2, or even better quadratic conditions based on the arguments of [7], can provide the exact stability conditions of general networks with priorities. A counterexample is indeed provided by the model analyzed in [9], for which only a part of the stability conditions may be interpreted in terms of essential states. In such cases, a deeper insight into the dynamics of the system is necessary in order to complete the stability conditions, and the main tool remains the associated "fluid limit model".

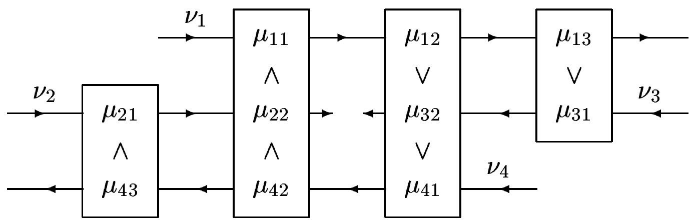

Figure 6: Faces of order 0, 1 and 2.

A Classification of minimal faces.

The following definition allows for a classification of minimal faces according to their "order". This classification will then enable us to show that the set of essential states is reached in finite time, at least when $\rho_{\mathcal{C}_k} \leq 1$ for all k (this is Proposition 2.5).

Definition-Lemma A.1 *Minimal faces without antecedent are said to be* of order 0. *By induction, we define faces* of order $n+1$ *($n \in \mathbb{N}$) as the minimal faces that are not of order $\leq n$, but whose all antecedents contain a subface of order $\leq n$. Denote $\mathcal{O}_n$ the set of all the faces of order n. If $\mathcal{O}_n$ is empty, then so is $\mathcal{O}_{n+1}$. Moreover, the sets $\mathcal{O}_n$ form a partition of the set of all the minimal faces.*

Remember that Proposition 3.9 shows that $\mathcal{O}_0$ is not empty if there exist unessential faces. Moreover, any antecedent of a minimal face must be unessential, hence it must admit a minimal subface. We have already met a minimal face of order 1 in section 5: for the network of figure 4, we exhibited a minimal face Λ' with a unique antecedent Λ'' that contained a minimal cycle Λ (a minimal face of order 0).

Proof :

First notice that according to our definition by induction, the sets $\mathcal{O}_n$ are disjoint.

Denote $\mathcal{O} = \bigcup_{n\in\mathbb{N}} \mathcal{O}_n$, $\mathcal{N} = \{\Lambda \text{ minimal}/\Lambda \notin \mathcal{O}\}$. Assume that $\mathcal{N}$ is non-empty, set: $L = \min_{\Lambda\in\mathcal{N}} L(\Lambda)$, and consider $\Lambda \in \mathcal{N}$ such that $L(\Lambda) = L$. Face Λ must admit an antecedent (otherwise $\Lambda \in \mathcal{O}_0$), and any antecedent Λ' of Λ must be unessential, hence it admits a minimal subface Λ''. Clearly we have: $L(\Lambda'') \leq L(\Lambda') < L(\Lambda) = L$, hence $\Lambda'' \in \mathcal{O}$. Let N be the maximum of the orders of all the minimal faces Λ'' included in the various antecedents of Λ. Then Λ satisfies the definition of faces of order $N+1$, hence a contradiction. In consequence, $\mathcal{N}$ is empty, and the sets $\mathcal{O}_n$ actually form a partition of the set of the minimal, unessential faces.

Finally assume that $\mathcal{O}_n = \emptyset$, but there exists $\Lambda \in \mathcal{O}_{n+1}$. Then Λ is not of order $\leq n-1$, but any antecedent contains a subface of order $\leq n-1$: this is precisely the definition of faces of order n, hence a contradiction. ♠

$$
\begin{tikzcd}
(1,1) & (1,2) \arrow[d] \arrow[ddr, leftrightarrow] & (1,3) & \\
(2,1) & (2,2) \arrow[d] & (3,2) \arrow[u] & (3,1) \\
 & (4,3) & (4,2) \arrow[u] & (4,1)
\end{tikzcd}
$$

Figure 7: Graph $\mathcal{G}$.

In order to better understand this classification, let us consider the network pictured in figure 6. The associated graph $\mathcal{G}$ is represented in figure 7. Let us show that this network admits faces of order 0, 1 and 2.

There a unique, minimal cycle: $\Lambda(0) = \{(1,2),(4,2)\}$, hence the unique face of order 0. Faces $\Lambda_1(1) = \{(1,3),(3,2),(4,2)\}$ et $\Lambda_2(1) = \{(4,3),(2,2),(1,2)\}$ admit unique antecedents, resp. faces $\Lambda_1^-(1) = \{(1,2),(3,2),(4,2)\}$ and $\Lambda_2^-(1) = \{(4,2),(2,2),(1,2)\}$, which contain $\Lambda(0)$. Moreover, it may be checked that all the subfaces of $\Lambda_1(1)$ and $\Lambda_2(1)$ are essential. Faces $\Lambda_1(1)$ and $\Lambda_2(1)$ are hence minimal faces of order 1 (it might be checked that these are the only ones). Now consider face $\Lambda(2) = \{(1,3),\ (2,2),\ (3,2),\ (4,3)\}$. Its only antecedents are faces $\Lambda_1^-(2) = \{(1,3),(2,2),(3,2),(4,2)\}$ and $\Lambda_2^-(2) = \{(1,2),(2,2),(3,2),(4,3)\}$, which resp. contain $\Lambda_1(1)$ and $\Lambda_2(1)$ (and not $\Lambda(0)$). Finally, it may be checked that all its subfaces are essential: hence it is a face of order 2 (the only one in fact). There exists no face Λ of order 3 (and a fortiori no face of greater order), because an antecedent of Λ should contain a face of order 2 exactly, hence $\Lambda(2)$, and all the classes of $\Lambda(2)$ are at the end of their route.

B Hitting time of the set of essential states.

In order to prove Proposition 2.5, we first need to show:

Lemma B.1 *Assume that $\rho_{\mathcal{C}_k} \leq 1$ for some queue k. Then from any (essential or unessential) state, queue k empties infinitely often.*

Proof :
Set: $\tau_0^k = \inf\{t \geq 0 / Q_{\mathcal{C}_k}(t) = 0\}$. The evolution of $\boldsymbol{W}_{\mathcal{C}_k}(t)$ until τ_0^k coincides with the load of an $M/M/1/\infty$ queue with traffic intensity $\rho_{\mathcal{C}_k} \leq 1$, which is recurrent. Hence τ_0^k is almost surely finite. By the strong Markov property, this means that queue k empties infinitely often. ♠

Proof of Proposition 2.5 :

With any *minimal* face Λ, let us associate the set:

$$F_\Lambda^+ = \bigcup_{\Lambda' \supset \Lambda} F_{\Lambda'} = \{q \in \mathbb{Z}_+^{\mathcal{C}} / q_{is} > 0, (i,s) \in \Lambda\}.$$

Any unessential state belongs to (at least) one set F_Λ^+. Assume there exists a minimal face Λ of order $n+1$, and consider a state $q \in F_\Lambda^+$. Assume there exists a state $q' \notin F_\Lambda^+$ and a transition that leads from q' to q. This means that q' can be obtained from q by moving backward a customer $(i,s) \in \Lambda$, with $q_{is} = 1$. In consequence, q' belongs to some face Λ' which contains an antecedent of Λ, hence admits a minimal subface Λ'' of order $\leq n$.

In conclusion, the states in F_Λ^+ (Λ of order $n+1$) can only be reached (in one transition) from states that belong to the union of F_Λ^+ and all the $F_{\Lambda''}^+$, Λ'' of order $\leq n$. If Λ is of order 0, the states in F_Λ^+ can only be reached (in one transition) from states that belong to F_Λ^+.

We may now argue by induction on n. According to the above lemma, each queue empties infinitely often. In particular if Λ is of order 0, $Q(t)$ leaves F_Λ^+ in finite time, and it does not come back to this set since it cannot not be reached from outside. In consequence, after some finite time $\tau(0)$, there is no return into the sets $F_{\Lambda'}^+$, Λ' of order 0. Assume that after some finite time $\tau(n)$, there is no more return into the sets $F_{\Lambda'}^+$, Λ' of order $\leq n$. Whichever face Λ of order $n+1$ we consider, $Q(t)$ will leave F_Λ^+ at some finite time after $\tau(n)$, and it cannot come back to it since F_Λ^+ may not be reached from outside the union of F_Λ^+ and the $F_{\Lambda'}^+$, Λ' of order $\leq n$. In consequence, after some time $\tau(n+1)$, $Q(t)$ will not return into any set $F_{\Lambda'}^+$, Λ' of order $\leq n+1$.

Since there is a finite number of minimal faces, the set of the unessential states will be left in finite time. ♠

References

[1] Botvitch, D., and Zamyatin, A. Ergodicity of conservative communication networks. Rapport de recherche 1772, INRIA, October 1992.

[2] Bramson, M. Instability of FIFO queueing networks. *Annals of Applied Probability 4* (1994), 414–431.

[3] Bramson, M. Instability of FIFO queueing networks with quick service times. *Annals of Applied Probability 4* (1994), 693–718.

[4] Dai, J. On positive Harris recurrence of multiclass queueing networks: a unified approach via fluid limit models. *Annals of Applied Probability 5*, 1 (1993), 49–77.

[5] Dai, J., and Meyn, S. Stability and convergence of moments for multiclass queueing networks via fluid limit models. *IEEE Transactions on Automatic Control 40*, 11 (November 1995).

[6] Dai, J., and Weiss, G. Stability and instability of fluid models for re-entrant lines. *Mathematics of Operations Research 21*, 1 (February 1996), 115–134.

[7] Dai, J. G., and Vate, J. H. V. Virtual stations and the capacity of two-station queueing networks. Preprint, March 1996.

[8] Dumas, V. Instability cycles in fluid Bramson networks. Submitted to *IEEE Transactions on Automatic Control*, May 1995.

[9] Dumas, V. A multiclass network with non-linear, non-convex, non-monotonic stability conditions. To appear in *Queueing Systems, Theory and Applications*, 1996.

[10] Harrison, J., and Nguyen, V. Brownian models of multiclass queueing networks: current status and open problems. *Queueing Systems, Theory and Applications 13* (1993), 5–40.

[11] Kelly, F. Networks of queues. *Advances in Applied Probability 8* (1976), 416–432.

[12] Kumar, P., and Seidman, T. Dynamic instabilities and stabilization methods in distributed real-time scheduling of manufacturing systems. *IEEE Transactions on Automatic Control 35*, 3 (March 1990), 289–298.

[13] Lu, S., and Kumar, P. Distributed scheduling based on due dates and buffer priorities. *IEEE Transactions on Automatic Control 36*, 12 (December 1991), 1406–1416.

[14] Malyshev, V., and Menshikov, M. Ergodicity, continuity, and analyticity of countable Markov chains. *Trans. Moskow Math. Soc. 1* (1981), 1–47.

[15] Rybko, A., and Stolyar, A. Ergodicity of stochastic processes describing the operation of open queueing networks. *Problemy Peredachi Informatsii 28*, 3 (July-September 1992), 3–26.

Unité de recherche INRIA Lorraine, Technopôle de Nancy-Brabois, Campus scientifique,
615 rue du Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY
Unité de recherche INRIA Rennes, Irisa, Campus universitaire de Beaulieu, 35042 RENNES Cedex
Unité de recherche INRIA Rhône-Alpes, 655, avenue de l'Europe, 38330 MONTBONNOT ST MARTIN
Unité de recherche INRIA Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex
Unité de recherche INRIA Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex

Éditeur
INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
http://www.inria.fr
ISSN 0249-6399