# On the Synthesis of General Petri Nets

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# On the synthesis of general Petri nets

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**Abstract:** A polynomial algorithm was given by the authors and Bernardinello for synthesizing pure weighted Petri nets from finite labeled transition systems. The limitation to pure nets, serious in practice e.g. for modelling waiting loops in communication protocols, may be removed by a minor adaptation of the algorithm, working for general Petri nets fired sequentially. The rule of sequential firing reduces also the expressivity of Petri nets, since it forces a concurrent interpretation on every diamond. This limitation may also be removed by leaving sequential transition systems and lifting the algorithm to step transition systems, which amounts to extract the effective contents of the coreflection between Petri step transition sytems and general Petri nets established by Mukund. By the way, the categorical correspondences between transition systems or step transition systems and nets are re-examined and simplified to Galois connections in the usual setting of ordered sets.

**Key-words:** Synthesis Problem for Nets, Higher-Dimensional Automata, Polynomial Algorithm, Galois Connection

*(Résumé : tsvp)*

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# Sur la synthèse des réseaux de Petri généraux

**Résumé :** Les auteurs ont donné avec Luca Bernardinello un algorithme polynomial pour la synthèse de réseaux de Petri purs à partir de systèmes de transitions. La restriction aux réseaux purs s'avère tres contraignante par exemple lorsque l'on veut modéliser des boucles d'attente dans des protocoles de communication. Cette limitation peut être levée par une adaptation mineure de l'algorithme, utilisable pour des réseaux de Petri généraux exécutés de manière séquentielle. La règle de tirage séquentielle est elle-même limitative puisqu'elle interpréte comme indépendentes en un état toutes les actions qui y sont exécutables dans un ordre arbitraire. Cette seconde limitation peut aussi être levée en adaptant l'algorithme aux systèmes d'hypertransitions (*step transition systems*), ce qui revient à extraire le contenu algorithmique de la coreflection établie par Mukund entre les réseaux de Petri et les systèmes d'hypertransitions de Petri. Nous indiquons également comment les correspondances catégoriques entre réseaux et automates peuvent se présenter plus simplement en terme de connexions galoisiennes.

**Mots-clé :** Synthèse de réseaux, automate de dimension supérieure, algoritmes polynomiaux, connexion galoisienne

# 1 Introduction

Following Ehrenfeucht and Rozenberg's seminal work on *regions* in Partial 2-Structures [ER90a], various correspondences between automata and net systems, based on regions in automata, have been established and analysed. The reference papers for the basic correspondences are [ER90b] or [NRT92] in the frame of C/E-nets or elementary nets, and [Mu92] in the frame of P/T-nets. Variant correspondences have been studied respectively in [NW94] [BD95a] and in [DS93] [DS96]. According to the unified presentation of regions given in [BD95b], a region in a fixed automaton w.r.t. a fixed type of nets is a morphism from the underlying transition system to the marking graph of a one-place net of that type, with one event representative of each typical relationship between places and events. Regions may be seen as places with attached flow arcs, thus nets may be assembled from subsets of regions. An automaton gives rise in this way to an ordered family of subnets of a maximal net whose set of places is just the set of regions of the automaton. The point is to decide whether this maximal net or some of its subnets reproduces in turn the original automaton (up to isomorphism) as the associated graph of reachable markings. The essential contribution of [ER90b] and [NRT92] is the isolation of two characteristic axioms for automata isomorphic to graphs of reachable markings. These axioms require from their models enough regions to separate all different states in an automaton and to rule out all missing events at each state. Based on the separation axioms, various coreflections between categories of automata and nets have been established in the literature. These coreflections may also be obtained in a systematic way as induced restrictions of a general adjunction between automata and net systems, parametric on the type of nets, restricted on models of the separation axioms [BD95b].

The effective or algorithmic contents of the above correspondences have been analysed in a handful of papers. A crucial fact was observed in [DR96]: any set of regions large enough to witness for satisfaction of the separation axioms, and therefore said admissible, is the set of places of a net system which reproduces the given automaton as a graph of reachable markings. Exhibiting the net system when it exists is known as the *synthesis problem*. The synthesis problem for elementary nets, examined further in the case of finite automata in [Ber93] and [Hi94], and proved NP-complete in [BBD95a], was given an approximate solution in algorithmic form in [CKLY95]. The precise problem decided upon by the algorithm is whether the given automaton has some *quotient* isomorphic to the reachable marking graph of some net system. The synthesis problem for *pure* Petri nets, examined further in [BDPV96] in the frame of finite automata, was solved in [BBD95b] in time polynomial in the size of automata.

The resulting algorithm has been used in a few experiments, carried out in view of the automated distribution of protocols. In this context, the requirement to preprocess the genuine automaton which represents a given protocol, so as to split its loops into cycles, was found inconvenient. One goal of the present paper is to lift this impractical constraint: we provide an adaptation of the synthesis algorithm working in polynomial time for *general* Petri nets with the sequential firing rule. The other goal of the paper is to show that the algorithm is robust, by jumping from sequential transition systems to *step transition systems*, as defined by Mukund [Mu92]: we provide a second adaptation of the synthesis algorithm, working in polynomial time for Petri nets with the *step firing rule*.

The remaining sections of the paper are organized as follows. Section 2 recalls from [BD95b] the general definition of regions, parametric on the type of nets, and recasts in this mould the abstract correspondence between sequential transition systems and nets with the sequential firing rule. The categorical correspondence established in [BD95b] is presented here in the simpler form of a Galois connection. Section 3 focuses on Petri nets. A polynomial synthesis algorithm is produced, based on the adaptation of principles inherited from the weaker algorithm for *pure* Petri nets defined in [BBD95b]. Section 4 carries the abstract correspondence between transition systems and nets to step transition systems and nets with the step firing rule. Section 5 presents an upgraded version of the algorithm which solves the synthesis problem in this richer context.

# 2 Galois connection between automata and net systems

A dual adjunction between automata and net systems, parametric on the type of nets, has been constructed in [BD95b]. Our purpose in this section is to account for this correspondence without the help of categories. We shall therefore construct a Galois connection between automata and net systems. We shall identify the kernel of the connection, yielding a relation of duality between separated automata and saturated net systems. Given a separated automaton, the dual net system is assembled from regions in the automaton, seen as places of the net. We shall also isolate admissible subsets of regions characteristic of separated automata, and thereby lay the grounds for the investigation of algorithmic solutions to the synthesis problem, pursued in section 3.

Before stating general definitions for automata, nets and regions parametric on the type of nets, let us indicate a slight difference between the correspondences established below and in [BD95b]. Given a net system with set of events $E$, the set of events of the dual automaton is $E$ in the present paper, while it was a quotient of $E$ in the former work. The import of this distinction is not yet completely clear.

## 2.1 Regions in transition systems

We reserve the appellation of *automata* to initialized transition systems $A = (S, E, T, s_0)$, with set of *transitions* $T \subseteq S \times E \times S$ and *initial state* $s_0 \in S$. The *underlying* transition system $UA = (S, E, T)$ is always assumed to be *deterministic*, meaning that for any *state* $s \in S$ and *event* $e \in E$, $(s \xrightarrow{e} s') \in T \wedge (s \xrightarrow{e} s'') \in T \Rightarrow s' = s''$. Recall that a morphism of transition systems $(\sigma, \eta) : (S, E, T) \rightarrow (S', E', T')$ is a pair of maps $\sigma : S \rightarrow S'$ and $\eta : E \rightarrow E'$ such that $(s \xrightarrow{e} s') \in T \Rightarrow (\sigma(s) \xrightarrow{\eta(e)} \sigma(s')) \in T'$. Automata are always assumed to be *accessible*, meaning that $S$ is the inductive closure of the singleton set $\{s_0\}$ w.r.t. forward transitions in $T$. Morphisms of automata are just the morphisms of the underlying transition systems that preserve initial states. In the sequel, notations $s \xrightarrow{e}$ and $s \not\xrightarrow{e}$ appear as shorthands for $\exists s' \in S\ (s \xrightarrow{e} s') \in T$ and $\forall s' \in S\ (s \xrightarrow{e} s') \notin T$.

We reserve the appellation of *net systems* to initialized nets $N = (P, E, W, M_0)$, with *underlying* net $UN = (P, E, W)$ and *initial marking* $M_0$. In order to give a uniform presentation for all types of nets, we depart from the traditional definition of nets and propose now a general definition encompassing Petri nets as a specific type of nets (fully defined afterwards). A *marking* is a map $M : P \to LS$ from the set $P$ of *places* to a set $LS$ of *local states* (depending on the type of nets). The places in $P$ and the *events* in $E$ are the nodes of a complete bi-partite graph, whose edges are mapped to a set $LE$ of *local events* (depending on the type of nets) by the *weight matrix* $W : P \times E \to LE$. The set of local states, the set of local events, and the partial action of the latter on the former, make up a type of nets.

**Definition 2.1 (Type of nets)** *A type of nets is a (deterministic) transition system $\tau = (LS, LE, \tau)$, where $LS$ and $LE$ are respective sets of local states and local events, and $\tau \subseteq LS \times LE \times LS$ defines the partial action of local events on local states.*

As graphs, nets are of a static nature, but types (of nets) define their dynamics: the partial actions of events on markings may be inferred from the partial actions of local events on local states, using the weight matrix to control synchronized products of local events. The following definition extends in this way the usual *sequential firing rule.*

**Definition 2.2 (Sequential marking graph)** *Given a net $N = (P, E, W)$, of type $\tau = (LS, LE, \tau)$, the (sequential) marking graph of $N$ is the transition system $(LS^P, E, T)$ with set of transitions $T$ defined by :*

$$(M \xrightarrow{e} M') \in T \quad \textbf{iff} \quad \forall x \in P \ \ (M(x) \xrightarrow{W(x,e)} M'(x)) \in \tau \tag{1}$$

*Given a net system $N = (P, E, W, M_0)$, the (sequential) marking graph of $N$ is the automaton $N^* = (S, E, T_S, M_0)$ where $S$ is the inductive closure of $\{M_0\}$ w.r.t. forward transitions in $T$, and $T_S = T \cap (S \times E \times S)$.*

A key example is the type of Petri nets, $\tau_{Petri} = (\mathbb{N}, \mathbb{N} \times \mathbb{N}, \tau_{Petri})$. Here, the local states are natural numbers while the local events are pairs of natural numbers, acting partially on the former according to the transition rule:

$$(n \xrightarrow{(p,q)} n') \in \tau_{Petri} \quad \Leftrightarrow \quad n \geq p \ \wedge \ n' = (n - p) + q$$

If we set now $W(x,e) = (F(x,e), F(e,x))$, the reader recognizes in (1) the classical rule for the sequential firing of Petri nets, namely: $M[e{>}M'$ **iff**

$$\forall x \in P \;\; M(x) \geq F(x,e) \;\; \wedge \;\; M'(x) = M(x) - F(x,e) + F(e,x)$$

A variant example is the type of *pure* Petri nets, $\tau^{pure}_{Petri} = (\mathbb{N}, \mathbb{Z}, \tau^{pure}_{Petri})$. In this weaker type of nets, the local events are relative numbers, acting partially on local states according to the transition rule:

$$n \xrightarrow{z} n' \in \tau^{pure}_{Petri} \quad \Leftrightarrow \quad n' = n + z$$

So, the local action of $z$ is defined at $n$ if and only if $n + z \geq 0$.

The types of pure Petri nets and general Petri nets are in fact connected by morphisms $I : \tau^{pure}_{Petri} \rightarrow \tau_{Petri}$ and $J : \tau_{Petri} \rightarrow \tau^{pure}_{Petri}$ such that $J \circ I$ is the identity on $\tau^{pure}_{Petri}$ : $J = (1_{\mathbb{N}}, j)$ where $j : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{Z}$ : $j(p,q) = q - p$ and $I = (1_{\mathbb{N}}, i)$ where $i : \mathbb{Z} \rightarrow \mathbb{N} \times \mathbb{N}$: $i(z) = (-z, 0)$ $if$ $z < 0$ $else$ $i(z) = (0, z)$.

A synthesis algorithm for *pure* Petri nets was proposed in [BBD95b], based on the principle of *regions as places*. The primary goal of the paper is to extend this algorithm to (general) Petri nets. In order to get an intuitive understanding of regions for Petri nets, let us look back to their sequential firing rule (1). This rule means that for every place $x$, the pair of maps $(\sigma_x, \eta_x)$ defined by $\sigma_x(M) = M(x)$ and $\eta_x(e) = W(x,e)$ is a morphism of transition systems from the marking graph of the net to $\tau_{Petri}$. Let us now forget the internal structure of states in the marking graph, thus identified with any other isomorphic transition system $(S, E, T)$. If in addition we identify places $x$ with their *extensions* $(\sigma_x, \eta_x)$, we can rediscover the places of the original net (and also discover new places) as morphisms $(\sigma, \eta) : (S, E, T) \rightarrow \tau_{Petri}$. This motivates the following definition of regions for arbitrary types of nets.

**Definition 2.3 (Regions)** *Given a transition system $T = (S, E, T)$ and a type of nets $\tau = (LS, LE, \tau)$, the set $\mathcal{R}_\tau(T)$ of $\tau$-type regions in $T$ is the set of morphisms from $T$ to $\tau$.*

By abuse of notations, we extend the above definition to automata by setting $\mathcal{R}_\tau(A) = \mathcal{R}_\tau(UA)$. In the sequel, the subscript $\tau$ is left implicit when it is equal to $\tau_{Petri}$, or abbreviated to *pure* when it is equal to $\tau^{pure}_{Petri}$. So, pure regions and Petri regions are connected by maps $I_A : \mathcal{R}_{pure}(A) \rightarrow \mathcal{R}(A) : I_A(\sigma, \eta) = I \circ (\sigma, \eta)$ and $J_A : \mathcal{R}(A) \rightarrow \mathcal{R}_{pure}(A) : J_A(\sigma, \eta) = J \circ (\sigma, \eta)$ such that $J_A \circ I_A$ is the identity.

## 2.2 Galois connection

We saw above that regions may serve to reverse the production of marking graphs. The reversing process may also be applied to arbitrary transition systems, leading to the following definitions.

**Definition 2.4 (Dual of a transition system)** *Given a transition system $T = (S, E, T)$ and a type of nets $\tau$, the dual of $T$ is the net $T^* = (\mathcal{R}_\tau(T), E, W)$ with weights defined by $W((\sigma, \eta), e) = \eta(e)$. For any subset $R$ of $\mathcal{R}_\tau(T)$, let $T^*_R$ denote the subnet of $T^*$ with restricted set of places $R$.*

**Definition 2.5 (Dual of an automaton)** *Given an automaton $A$ with initial state $s_0$ and a type of nets $\tau$, the dual of $A$ is the net system $A^*$ composed of the underlying net $(UA)^*$ and of the initial marking $M_0$ defined by $M_0(\sigma, \eta) = \sigma(s_0)$ for every $(\sigma, \eta) \in \mathcal{R}_\tau(A)$. For any subset $R$ of $\mathcal{R}_\tau(A)$, let $A^*_R$ denote the subnet system of $A^*$ with restricted set of places $R$.*

We will show that the two $()^*$ operators mapping the automaton $A$ to the net system $A^*$ and the net system $N$ to its marking graph $N^*$ form a Galois connection $A \leq N^* \Leftrightarrow N \leq A^*$. One expects in particular $A \leq N_x^* \Leftrightarrow N_x \leq A^*$ for every region $x = (\sigma_x, \eta_x) \in \mathcal{R}_\tau(A)$ where $N_x$ is the *atomic* subnet system of $A^*$ with the sole place $x$ (i.e. $N_x = A^*_{\{x\}}$) and $N_x^*$ is its marking graph. This particular case will help us to find out the appropriate order relation on automata. Since $N_x$ is a subnet sytem of $A^*$, both $N_x \leq A^*$ and $A \leq N_x^*$ are expected; by definition of regions, if $E$ is the set of events of $A$ then $(\sigma_x, 1_E)$ is an *event preserving* morphism from $A$ to $N_x^*$. Moreover, if there exists an event preserving morphism $(\sigma, 1_E) : A_1 \to A_2$ between two automata with set of events $E$, this morphism is necessarily unique, owing to the strong properties of determinism and accessibility we have assumed from all automata. Therefore, if there exists morphisms $(\sigma_1, 1_E) : A_1 \to A_2$ and $(\sigma_2, 1_E) : A_2 \to A_1$, then $A_1$ and $A_2$ are identical up to the identity of states ($A_1 =_E A_2$), and if we let $\mathbf{Aut}(E)$ be the set of (deterministic and accessible) automata with fixed set of events $E$, quotiented by $=_E$, then

$$A_1 \leq A_2 \quad \textbf{if} \quad \exists \sigma : \ (\sigma, 1_E) : \ A_1 \to A_2$$

defines a partial order on $\mathbf{Aut}(E)$. This partial order is a complete lattice, with greatest lower bounds computed as synchronized products. We remind the reader that the synchronized product $\bigwedge_{i \in I} A_i$ of a family of automata $A_i = (S_i, E, T_i, s_{0,i})$ indexed by $i \in I$ is the automaton $(S, E, T, s_0)$ with components as follows: $s_0 = (s_{0,i})_{i \in I}$, $S$ is the inductive closure of the set $\{s_0\}$ w.r.t. the *synchronized transition rule*

$$(s_i)_{i \in I} \xrightarrow{e} (s'_i)_{i \in I} \quad \textbf{iff} \quad \forall i \in I \quad (s_i \xrightarrow{e} s'_i) \in T_i$$

and $T$ is the set of occurrences of this rule at states $(s_i)_{i \in I} \in S$. So every automaton in $\mathbf{Aut}(E)$ is an idempotent of the binary synchronized product $\wedge$, and the order $\leq$ could as well have been defined as

$$A_1 \leq A_2 \quad \textbf{if} \quad A_1 =_E A_1 \wedge A_2$$

By definition of marking graphs, the automaton $N^*$ dual to a net system $N = (P, E, W, M_0)$ is actually the synchronized product $\bigwedge_{x \in P} N_x^*$ of the marking graphs of its atomic subnet systems.

Our starting point for choosing the order relation on $\mathbf{Aut}(E)$ was to set $A \leq N_x^*$ for every region $x \in \mathcal{R}_\tau(A)$. In fact, any atomic net system $N = (\{x\}, E, W, M_0)$ such that $A \leq N^*$ is isomorphic to $N_x$ for some corresponding region $(\sigma_x, \eta_x) \in \mathcal{R}_\tau(A)$ (i.e. $N$ and $N_x$ are identical up to exchanging $x$ and $(\sigma_x, \eta_x)$).

**Lemma 2.6** *The regions in $\mathcal{R}_\tau(A)$ are in bijective correspondence with the (isomorphism classes of) atomic net systems $N$ such that $A \leq N^*$.*

*Proof:* Let $A \leq N^*$, where $N = (\{x\}, E, W, M_0)$ is an atomic net system of type $\tau = (LS, LE, \tau)$. Let $ls_0 = M_0(x)$, hence $ls_0 \in LS$. By definition of marking graphs, $N^* = (S, E, T, ls_0)$ is the automaton with the least sets of states and transitions, $S$ $(\subseteq LS)$ and $T$ $(\subseteq S \times E \times S)$, such that $ls_0 \in S$ and $(ls \xrightarrow{e} ls') \in T$ whenever $ls \in S \wedge (ls \xrightarrow{W(x,e)} ls') \in \tau$. Let $(\sigma_x, 1_E) : A \to N^*$ be the unique morphism (of transition systems) from $UA$ to $U(N^*)$ acting as the identity on events and mapping the initial state $s_0$ of $A$ to $ls_0$. Now the maps $\iota : S \to LS : \iota(ls) = ls$ and $\eta_x : E \to LE : \eta_x(e) = W(x, e)$ define a morphism $(\iota, \eta_x) : U(N^*) \to \tau$. By composing morphisms one obtains a region in $\mathcal{R}_\tau(A) :$ $(\iota, \eta_x) \circ (\sigma_x, 1_E) = (\sigma_x, \eta_x) : UA \to \tau$. Let $N_x^*$ be the marking graph of the atomic net system $N_x$ derived from the region $(\sigma_x, \eta_x)$. Since $\sigma_x(s_0) =$

$ls_0$ and seeing that $ls \stackrel{W(x,e)}{\longrightarrow} ls'$ in $\tau$ entails $ls \stackrel{e}{\to} ls'$ in $N_x^*$ for $ls \in S$ and $e \in E$, $N_x^*$ is isomorphic to $N^*$. ∎

Concerning the order relation on net systems, the central assumption that $N_x \leq A^*$ for every region $x$ of $A$ leads to choose something akin to the substructure ordering: $N_1 \leq_{sub} N_2$ if $N_1$ is $N_2$ restricted on a subset of places. However *replicated* places may occur in a net system $N = (P, E, W, M_0)$, i.e. places which the initial marking $M_0$ and the weight function $W$ do not happen to distinguish from one another, and we don't care about their degree of multiplicity nor about their identities. Let morphisms of net systems with fixed set of events be defined as follows: a morphism from $N_1 = (P_1, E, W_1, M_{0,1})$ to $N_2 = (P_2, E, W_2, M_{0,2})$ is a map $\beta : P_1 \to P_2$ such that $M_{0,1}(x) = M_{0,2}(x)$ and $W_1(x,e) = W_2(x,e)$ for all $x \in P_1$ and $e \in E$. Two net systems connected by morphisms in both directions are henceforth declared equivalent. Let $\mathbf{Nets}(E)$ denote the set of equivalence classes of net systems with fixed set of events $E$ (replication free nets are the canonical representatives). One can equip $\mathbf{Nets}(E)$ with a partial order relation by setting:

$$N_1 \leq N_2 \quad \textbf{if} \quad \exists \beta : N_1 \to N_2$$

This partial order is a complete lattice, with least upper bounds $\bigvee_{i \in I} N_i$ of families of net systems computed by amalgamation of sets of places. Told in another way, if we identify a place $x$ in a net system $N = (P, E, W, M_0)$ with the pair $(M_0(x), \eta_x)$ such that $\eta_x(e) = W(x,e)$ for $e \in E$ then $\bigvee_{i \in I}(P_i, E, W_i, M_{0,i})$ $= (\bigcup_{i \in I} P_i, E, W, M_0)$ where $W(x,e) = W_i(x,e)$ and $M_0(x) = M_{0,i}(x)$ for $x \in P_i$. A net system $N$ with set of places $P$ is now the least upper bound $\bigvee_{x \in P} N_x$ of its atomic subnet systems $N_x$. In the particular case where $N = A^*$ is dual to the automaton $A$, its set of places $P$ is the set of regions $\mathcal{R}_\tau(A)$, where $\tau$ is the type of $N$, and we get the following.

**Observation 2.7** *Let $P \subseteq R \subseteq \mathcal{R}_\tau(A)$ then $A_P^* = \bigvee_{x \in P} A_{\{x\}}^*$ and $A_P^* \leq A_R^*$.*

We are ready to establish the expected Galois connection between automata and net systems.

**Proposition 2.8** *The two $()^*$ operators, mapping respectively the automaton $A$ to the dual net system $A^*$ and the net system $N$ to its marking graph $N^*$,* *constitute a Galois connection between the ordered sets* $\mathbf{Nets}(E)$ *and* $\mathbf{Aut}(E)$*:* $A \leq N^* \Leftrightarrow N \leq A^*$ *for* $A \in \mathbf{Aut}(E)$ *and* $N \in \mathbf{Nets}(E)$.

*Proof:* By Lem. 2.6, $A \leq N^* \Leftrightarrow N \leq A^*$ if $N$ is an *atomic* net system. Now for a net system $N = \bigvee_{x \in P} N_x$, where $N_x$ is the atomic subnet system of $N$ with the unique place $x$, $N^* = \bigwedge_{x \in P} N_x^*$ by definition of marking graphs. Thus $A \leq N^*$ *if and only if* $A \leq N_x^*$ for all $x \in P$ *if and only if* $N_x \leq A^*$ for all $x \in P$ (because $N_x$ is atomic) *if and only if* $N \leq A^*$. ∎

The relations $A_1 \leq A_2 \Rightarrow A_2^* \leq A_1^*$ (for $A_1, A_2 \in \mathbf{Aut}(E)$) and $N_1 \leq N_2 \Rightarrow N_2^* \leq N_1^*$ (for $N_1, N_2 \in \mathbf{Nets}(E)$) follow immediately from the Galois connection. It is nevertheless instructive to show directly that the $()^*$ operators are decreasing. Let $A_1 \leq A_2$ then every region $(\sigma_2, \eta_2)$ of $A_2$, composed on the left with $(\sigma, 1_E) : A_1 \rightarrow A_2$, yields a corresponding region $(\sigma_2 \circ \sigma, \eta_2)$ of $A_1$; the regions $(\sigma_2, \eta_2)$ and $(\sigma_2 \circ \sigma, \eta_2)$, taken as places of the respective net systems $A_2^*$ and $A_1^*$, are identical since their values are identical in the initial markings $M_{0,1}$ and $M_{0,2}$ ($\sigma$ maps the initial state of $A_1$ to the initial state of $A_2$); hence $A_2^* \leq A_1^*$. Let $N_1 \leq N_2$ then we can assume $P_1 \subseteq P_2$ without loss of generality, where $P_1$ and $P_2$ are the respective sets of places of $N_1$ and $N_2$; every atomic subnet system $N_x$ of $N_1$ is therefore a subnet system of $N_2$; letting $A_x = N_x^*$ (the marking graph of $N_x$), one obtains $N_1^* = \bigwedge_{x \in P_1} A_x \geq \bigwedge_{x \in P_2} A_x = N_2^*$.

Another general property of Galois connections is to produce *closure operators* by conjugated composition of the dual operators. Recall that an operator on $(X, \leq)$, mapping $x$ to $\overline{x}$, is a closure operator if it is increasing ($x_1 \leq x_2 \Rightarrow \overline{x_1} \leq \overline{x_2}$), extensive ($x \leq \overline{x}$), and idempotent ($\overline{\overline{x}} = x$). The double dual operators $()^{**}$ acting respectively on the ordered sets $(\mathbf{Aut}(E), \leq)$ and $(\mathbf{Nets}(E), \leq)$ are therefore closure operators. An automaton $A$ equal to its closure $A^{**}$ is said to be *separated* with respect to the fixed type of nets $\tau$, while a net system $N$ equal to its closure $N^{**}$ is said to be *saturated*. Owing to the Galois connection, the complete lattices of separated automata and saturated net systems are isomorphic.

## 2.3 Representation result

In the end of the section, we state a criterion for the recognition of separated automata, at the basis of the synthesis algorithm proposed afterwards.

By definition, an automaton separated with respect to type $\tau$ is isomorphic to the synchronized product of marking graphs $N_x^*$ of atomic net systems $N_x = A_{\{x\}}^*$ derived from $\tau$-regions $x$ of $A$ (in formulas: $A \cong \bigwedge_{x\in\mathcal{R}_\tau(A)} N_x^*$). Following [DR96], we say that a subset of regions $R \subseteq \mathcal{R}_\tau(A)$ is *admissible* if $A \cong \bigwedge_{x\in R} N_x^*$. So, $A$ is separated if and only if $\mathcal{R}_\tau(A)$ is admissible, and of course every superset of an admissible set of regions is admissible. The marking graph $N^*$ of a net system $N$ is separated because $N^* \cong N^{***}$ follows from the Galois connection. In fact, the extensions $(\sigma_x, \eta_x)$ of places $x$ of $N$ form an admissible set of regions of $N^*$. The following criterion may be used to recognize admissible sets of regions, and consequently separated automata.

**Theorem 2.9** *Given an automaton $A = (S, E, T, s_O)$ and a type of nets $\tau$, a set of regions $R \subseteq \mathcal{R}_\tau(A)$ is admissible if and only if the following* separation properties *are satisfied for all states $s, s' \in S$ and for every event $e \in E$:*

**(SSP)** $s \neq s' \;\Rightarrow\; \exists(\sigma,\eta) \in R:\; \sigma(s) \neq \sigma(s')$

*(read: $(\sigma,\eta)$ solves the* states separation *problem at $(s,s')$)*

**(ESSP)** $s \overset{e}{\not\to} \;\Rightarrow\; \exists(\sigma,\eta) \in R:\; \sigma(s) \overset{\eta(e)}{\not\longrightarrow}$ *w.r.t. $\tau$*

*(read: $(\sigma,\eta)$ solves the* event/state separation *problem at $(s,e)$)*

*When both properties are satisfied, $A \cong (A_R^*)^*$, where $A_R^*$ is the subnet system of $A^*$ with restricted set of places $R$ (also called the net* synthesized *from $R$).*

*Proof:* Let $N_x = A_{\{x\}}^*$ for $x \in R$, and let $N_R = A_R^*$. Seeing that $A \leq N_x^*$ for every region $x$, $A \leq \bigwedge_{x\in R} N_x^* = N_R^*$. Accordingly, there exists a morphism of automata $(\sigma, 1) : A \to N_R^*$. Moreover this morphism is unique. On the other hand, every region $x = (\sigma_x, \eta_x)$ factors into $(\iota, \eta_x) \circ (\sigma_x, 1)$ where $\iota$ acts as the identity on the local states in its domain, and $(\sigma_x, 1)$ lifts to the unique event preserving morphism from $A$ to $N_x^*$. As $N_R^*$ is the synchronized product of $(N_x^*)_{x\in R}$, $\sigma$ must be the map that sends each state $s$ of $A$ to the associated vector $\sigma(s) = (\sigma_x(s))_{x=(\sigma_x,\eta_x)\in R}$ (the $x$-component is computed by evaluating region $x$ at state $s$). Since $(\sigma, 1)$ is the unique morphism of this form from $A$ to $N_R^*$, and seeing that all automata are accessible and deterministic, the assertion $A \cong N_R^*$ is now equivalent to *(i)* $\sigma$ is an injective map, and *(ii)* $s \overset{e}{\to}$ in $A$ whenever $\sigma(s) \overset{e}{\to}$ in $N_R^*$. Now $SSP$ is just another form of assertion *(i)*. By definition of the synchronized product, $\sigma(s) \overset{e}{\to}$ in $N_R^*$ entails $\sigma_x(s) \overset{e}{\to}$ in

$N_x^*$ for all $x \in R$, hence $ESSP$ is just another form of assertion *(ii)*. ∎

# 3 Synthesis of general Petri nets (sequential firing)

## 3.1 The synthesis problem

Assuming the sequential firing rule, the *synthesis problem* for Petri nets consists in *(i)* deciding whether a *finite* automaton $A$ given as input is isomorphic to the marking graph $N^*$ of some net system $N = (UN, M_0)$, where $UN$ is a Petri net fired sequentially from the initial marking $M_0$, and if so, *(ii)* producing as output a net system $N$ such that $A \cong N^*$ and noproper subnet system of $N$ satisfies this property. On the grounds of Theo. 2.9, this amounts to *(i)* deciding whether all instances of the separation problems in $A$ are solved by corresponding regions in $\mathcal{R}(A)$, and if so, *(ii)* synthesizing the desired net system $N = A_R^*$ from a minimal admissible subset of regions $R$, extracted from the set of solutions.

In an automaton $A = (S, E, T, s_0)$, there are at most $|S|^2 - |S|$ possible inputs for the states separation problem:
$SSP_A(s, s')$ : "*construct from $A$ and $s \neq s'$ a region $(\sigma, \eta)$ s.t. $\sigma(s) \neq \sigma(s')$*"
and at most $|S| \times |E|$ instances of the event/state separation problem:
$ESSP_A(s, e)$ : "*construct from $A$ and $(s \overset{e}{\not\to})$ a region $(\sigma, \eta)$ s.t. $(\sigma(s) \overset{\eta(e)}{\not\longrightarrow})$*".
Part $(i)$ of the problem will therefore be solved in time polynomial (in $|S|$ and $|E|$ ) as soon as $SSP_A(s, s')$ and $ESSP_A(s, e)$ are solved in polynomial time. Part $(ii)$ consists in extracting from a set of regions with size polynomial in $|S|$ and $|E|$ a minimal admissible subset and this certainly can be done in polynomial time. So, a polynomial algorithm for the synthesis of Petri nets will follow if we succeed to construct procedures that solve in polynomial time $SSP_A(s, s')$ and $ESSP_A(s, e)$ with respect to the type $\tau_{Petri}$. This is the program of the section. The first stage of the program is to study the algebraic properties of $\mathcal{R}(A)$, the set of Petri regions of $A$. The second stage of the program is to elaborate decision procedures based on these properties.

## 3.2 The structure of Petri regions

From now on, $A = (S, E, T, s_0)$ is a fixed automaton, finite and *reduced* i.e. such that $\forall e \in E \;\; \exists s \in S \;\; s \xrightarrow{e}$. Of major importance for algorithms are the algebraic properties of $\mathcal{R}(A)$, inherited from $\mathcal{R}_{pure}(A)$ through the embedding / projection pair $(I_A, J_A)$ defined in section 2. Recall that $I_A : \mathcal{R}_{pure}(A) \to \mathcal{R}(A)$ maps a pure region $(\sigma, \eta) \in \mathcal{R}_{pure}(A)$ to the region $(\sigma, (\eta_-, \eta_+)) \in \mathcal{R}(A)$ such that for all $e \in E$: $\eta(e) = \eta_+(e) - \eta_-(e)$ and $\eta(e) \leq 0 \Rightarrow \eta_+(e) = 0$ and $\eta(e) \geq 0 \Rightarrow \eta_-(e) = 0$, while $J_A : \mathcal{R}(A) \to \mathcal{R}_{pure}(A)$ maps a region $(\sigma, (\eta_-, \eta_+)) \in \mathcal{R}(A)$ to the pure region $(\sigma, \eta) \in \mathcal{R}_{pure}(A)$ such that for all $e \in E : \;\; \eta(e) = \eta_+(e) - \eta_-(e)$
(where $\sigma : S \to \mathbb{N}$, $\eta : E \to \mathbb{Z}$, $\eta_- : E \to \mathbb{N}$, and $\eta_+ : E \to \mathbb{N}$). Let us focus on the second projections of pure regions, henceforth called *abstract regions*. Let $\mathcal{R}_{abs}(A)$ denote the set of abstract regions of $A$, i.e. $\mathcal{R}_{abs}(A) = \{\eta : E \to \mathbb{Z} \mid \exists \sigma : S \to \mathbb{N} \cdot (\sigma, \eta) \in \mathcal{R}_{pure}(A)\}$. We are primarily interested in the algebraic properties of $\mathcal{R}_{abs}(A)$, with elements $\eta \in \mathcal{R}_{abs}(A)$ seen as vectors in the $\mathbb{Z}$-module $(E \to \mathbb{Z})$ and represented in the sequel as formal sums $\eta = \Sigma \eta_i \cdot e_i$ where $\eta_i = \eta(e_i)$.

Before investigating these properties, let us establish a few notations. In the fixed transition system $UA = (S, E, T)$, let $\partial^0, \partial^1 : T \to S$ and $\ell : T \to E$ denote the respective *source*, *target*, and *labelling* functions given by $\partial^0(t) = s$, $\partial^1(t) = s'$, and $\ell(t) = e$ for $t = s \xrightarrow{e} s' \in T$. A *0-chain* of $UA$ is a vector in the $\mathbb{Z}$-module $C_0(UA) = (S \to \mathbb{Z})$. A *1-chain* of $UA$ is a vector in the $\mathbb{Z}$-module $C_1(UA) = (T \to \mathbb{Z})$. The *boundaries* of the 1-chains are the 0-chains computed by the operator $\partial : C_1(UA) \to C_0(UA) : \partial(\Sigma c_j \cdot t_j) = \Sigma c_j \cdot (\partial^1(t_j) - \partial^0(t_j))$. A *cycle* is a 1-chain with the null boundary. The cycles of $UA$ form a submodule of the $\mathbb{Z}$-module $(T \to \mathbb{Z})$, let $Z(UA)= \{c \in C_1(UA) | \; \partial(c) = 0\}$. The *Parikh images* of cycles form in turn a submodule of the $\mathbb{Z}$-module $(E \to \mathbb{Z})$, where the Parikh image of a 1-chain $c = \Sigma c_j \cdot t_j$ is the vector $\pi(c) = \Sigma c_j \cdot \ell(t_j)$. Given vectors $\alpha = \Sigma \alpha_i \cdot x_i$ and $\beta = \Sigma \beta_i \cdot x_i$ in a finite dimensional $\mathbb{Z}$-module $(X \to \mathbb{Z})$, we let $\alpha \cdot \beta$ denote their *scalar product* $\Sigma \alpha_i \cdot \beta_i$.

**Lemma 3.1** $(\sigma, \eta) \in \mathcal{R}_{pure}(A)$ **iff** $\sigma \cdot \partial(c) = \eta \cdot \pi(c)$ *for all* $c \in C_1(UA)$.

*Proof:* By linearity, the condition $\forall c \in C_1(UA) \;\; \sigma \cdot \partial(c) = \eta \cdot \pi(c)$ is equivalent to the condition $\forall t \in T \;\; \sigma \cdot \partial(t) = \eta \cdot \pi(t)$ where $t$ is identified with the chain

(1.$t$). Now the equation $\sigma \cdot \partial(t) = \eta \cdot \pi(t)$ is valid if and only if $\sigma(\partial^1(t))$ - $\sigma(\partial^0(t))$ $= \eta(\ell(t))$, if and only if $\sigma(\partial^0(t)) \xrightarrow{\eta(\ell(t))} \sigma(\partial^1(t))$ w.r.t. the type $\tau_{Petri}^{pure}$, if and only if $(\sigma, \eta) \in \mathcal{R}_{pure}(A)$ by definition of regions. ∎

**Proposition 3.2** $\eta : E \to \mathbb{Z} \in \mathcal{R}_{abs}(A)$ **iff** $\eta \cdot \pi(c) = 0$ *for every cycle* $c \in Z(UA)$*; the regions* $(\sigma, \eta) \in \mathcal{R}_{pure}(A)$ *which project on* $\eta$ *are then characterized by the condition:* $\sigma(s_0) + (\eta \cdot \pi(c)) \geq 0$ *for every 1-chain* $c \in C_1(UA)$ *such that* $\partial(c) = (s - s_0)$ *for some* $s \in S$.

*Proof:* From Lem. 3.1, the condition on $\eta$ must hold and whenever it does, the scalar product $(\eta \cdot \pi(c))$ takes an identical value for all 1-chains $c$ with an identical boundary. From the definition of regions, the condition on $\sigma(s_0)$ must hold because the local states specified for the type of nets $\tau_{Petri}^{pure}$ are the non negative integers. Now the two conditions taken together guarantee that one does always complete the data $(\sigma(s_0), \eta)$ to a pure region by selecting for each state $s \in S$ a corresponding 1-chain $c_s$ such that $\partial(c_s) = (s - s_0)$ and then setting $\sigma(s) = \sigma(s_0) + \eta \cdot \pi(c_s)$. ∎

Since $A$ is a reduced automaton, $\sigma$ determines $\eta$ for the pure regions $(\sigma, \eta) \in \mathcal{R}_{pure}(A)$, thus $\sigma$ determines $(\eta_+ - \eta_-)$ for the Petri regions $(\sigma, (\eta_-, \eta_+)) \in \mathcal{R}(A)$. Therefore, as observed in [DS96], the latter are fully determined from maps $\sigma$ and $\eta_-$. Now $\sigma$ in turn may be computed from $\sigma(s_0)$ and $\eta = (\eta_+ - \eta_-)$, hence a Petri region is still fully determined from $\sigma(s_0)$ and the maps $\eta_-$ and $(\eta_+ - \eta_-)$. The following corollary to Prop. 3.2 states the recipe for manufacturing regions presented in this form from abstract regions $\eta \in \mathcal{R}_{abs}(A)$.

**Corollary 3.3** $(\sigma, (\eta_-, \eta_+)) \in \mathcal{R}(A)$ *if and only if* $(i)$ $\eta = (\eta_+ - \eta_-) \in \mathcal{R}_{abs}(A)$ *, and* $(ii)$ $\sigma(s_0) + \eta \cdot \pi(c) \geq \eta_-(e)$ *for every chain* $c \in C_1(UA)$ *and* $e \in E$ *such that* $\exists s \in S \;\; \partial(c) = (s - s_0) \;\wedge\; s \xrightarrow{e}$.

In order to complete the machinery, it remains to produce a linear basis for the $\mathbb{Z}$-module of abstract regions $\mathcal{R}_{abs}(A)$. We recall hereafter classical results from graph theory and linear algebra showing that a basis may be computed in time polynomial in $|S|$ and $|E|$ (since $A$ is deterministic, $|S| \times |E|$ is an upper bound for $|T|$). First, one produces a linear basis for the $\mathbb{Z}$-module of cycles $Z(UA)$. According to Prop. 3.4, which specializes results found in e.g. [Lef75] or [GM85], this amounts to construct a *spanning tree* for $A$, i.e. a sub-automaton $\Lambda = (S, E, U, s_0)$ such that: *(i)* $U \subseteq T$ (by definition of a sub-automaton), *(ii)* $u_1 + \cdots + u_n \neq 0$ for $u_i \in U$ and $n \neq 0$ ($n = 0$ indicates the empty chain), and *(iii)* there exists for every state $s \in S$ a unique chain $c_s = u_1 + \cdots + u_n$ such that $u_i \in U$ and $\partial(c_s) = s - s_0$. A spanning tree $\Lambda$ can be computed in polynomial time.

**Proposition 3.4** *Let $\Lambda = (S, E, U, s_0)$ be a spanning tree for $A$, with set of arcs $U = \{u_i \mid i \in I\}$, then each transition $t \in T \setminus U$ determines a unique cycle $c^t = t + \Sigma\, z_i \times u_i$ such that $z_i \in \{-1, 0, 1\}$ for $i \in I$. The set $\{c^t \,|\, t \in T \setminus U\}$ is a basis of the $\mathbb{Z}$-module $Z(UA)$.*

The dimension $\nu\,(A)$ of the module $Z(UA)$ is therefore equal to $|T| - |U|$. Now let $E = \{e_1, \cdots, e_n\}$, $T = \{t_1, \cdots, t_m\}$, and let $\{c_1, \cdots, c_{\nu(A)}\}$ be an arbitrary basis for $Z(UA)$. From Prop. 3.2, $\mathcal{R}_{abs}(A)$ is the kernel of the linear transformation $M_A : \mathbb{Z}^n \to \mathbb{Z}^{\nu(A)}$ defined by the $\nu(A) \times n$ matrix $M_A$ with integral coefficients

$$M_A(i,j) = \Sigma\{c_i(t_k)| \;\; 1 \leq k \leq m \quad \wedge \quad \ell(t_k) = e_j\}$$

Let $k$ be the dimension of $Ker(M_A)$. The algorithm of von zur Gathen and Sieveking (see [Sch86]), given $M_A$ as input, produces in time polynomial in $\nu(A)$ and $n$ (or $|S|$ and $|E|$) a basis $\{\eta_1, \ldots, \eta_k\}$ for $Ker(M_A) = \mathcal{R}_{abs}(A)$.

We have in hand all the elements needed for solving problems $SSP_A(s, s')$ and $ESSP_A(s, e)$ relatively to the type of Petri nets. The data needed are the spanning tree $\Lambda$, or more exactly the application $c_{(.)}$ that maps each state $s \in S$ to the unique path $c_s$ from $s_0$ to $s$ in $\Lambda$, and the basis of abstract regions $\{\eta_1, \ldots, \eta_k\}$.

## 3.3 Solving the states separation problem

Given $s, s' \in S$ such that $s \neq s'$, let us consider the separation problem $SSP_A(s, s')$. Seeing that $J_A(\sigma, (\eta_-, \eta_+)) = (\sigma, \eta_+ - \eta_-)$ according to the correspondence $(I_A, J_A)$ between $\mathcal{R}_{pure}(A)$ and $\mathcal{R}(A)$, $SSP_A(s, s')$ has a solution in $\mathcal{R}(A)$ *iff* it has a solution in $\mathcal{R}_{pure}(A)$. Now from Lem. 3.1 and Prop. 3.2, $SSP_A(s, s')$ has a solution in $\mathcal{R}_{pure}(A)$ *iff* $\eta \cdot \pi(c_s - c_{s'}) \neq 0$ for some abstract region $\eta \in \mathcal{R}_{abs}(A)$ *iff* $\eta_i \cdot \pi(c_s - c_{s'}) \neq 0$ for some $i \in \{1, \cdots, k\}$, and the pure region $(\sigma_i\,,\,\eta_i)$ determined from $\eta_i$ by setting $\sigma_i(s_0) = -min\{\eta_i \cdot \pi(c_s) \mid s \in S\}$ is then a solution. The problem $SSP_A(s, s')$ is thus solved in $\mathcal{R}(A)$ by the Petri region $I_A\,(\sigma_i\,,\,\eta_i)$. Therefore, deciding whether $SSP_A(s, s')$ has a solution and producing it takes time polynomial in $|S|$ and $|E|$.

## 3.4 Solving the event / state separation problem

Given $s' \in S$ and $e' \in E$ such that $(s' \stackrel{e'}{\not\rightarrow})$, let us consider now the separation problem $ESSP_A(s', e')$. From Cor. 3.3, $ESSP_A(s', e')$ has a solution in $\mathcal{R}(A)$ *iff* there exists $\sigma(s_0) \in I\!N$, $\eta \in \mathcal{R}_{abs}(A)$, and $\eta_- : E \rightarrow I\!N$ such that:

1. $\forall e \quad \eta(e) + \eta_-(e) \geq 0$
2. $\forall s \quad \sigma(s_0) + \eta \cdot \pi(c_s) \geq 0$
3. $\forall s \forall e \quad s \stackrel{e}{\rightarrow} \;\Rightarrow\; \sigma(s_0) + \eta \cdot \pi(c_s) \geq \eta_-(e)$
4. $\sigma(s_0) + \eta \cdot \pi(c_{s'}) < \eta_-(e')$

It is worth noting that (2) entails both $s \stackrel{e}{\rightarrow} \;\Rightarrow\; \sigma(s_0) + \eta \cdot \pi(c_s) + \eta(e) \geq 0$ and $s \stackrel{e}{\rightarrow} \;\Rightarrow\; \sigma(s_0) + \eta \cdot \pi(c_s) \geq max\{0, -\eta(e)\}$, hence (1) and (3) hold automatically for $e \neq e'$ if one sets $\eta_-(e) = max\{0, -\eta(e)\}$. What remains after simplification is a system of linear inequations in $k + 2$ variables $x, y, z_i$ $(i = 1 \cdots k)$ where $x = \sigma(s_0) \in I\!N$, $y = \eta_-(e') \in I\!N$, and $\eta = \Sigma\, z_i \eta_i$ (recall that $\{\eta_1, \ldots, \eta_k\}$ is a basis for $\mathcal{R}_{abs}(A)$). In order to obtain a more compact presentation, let $w_{is} = \eta_i \cdot \pi(c_s)$ for $i \in \{1, \cdots, k\}$ and $s \in S$. The system $\Gamma$ to be solved is assembled from the following inequations:

1. $y + \Sigma\, z_i\, \eta_i\,(e') \geq 0$
2. $x + \Sigma\, z_i\, w_{is} \geq 0$ (one inequation for each $s \in S$)
3. $x - y + \Sigma\, z_i\, w_{is} \geq 0$ (one inequation for each $s \in S$ such that $s \stackrel{e'}{\rightarrow}$)
4. $x - y + \Sigma\, z_i\, w_{is'} < 0$

This system augmented by the constraints $x \geq 0$ and $y \geq 0$ is homogeneous, hence it has an integral solution *iff* it has a rational solution. Deciding upon the feasability of $\Gamma$ in $\mathbb{Q}^{k+2}$ and computing a rational solution when it exists may be achieved in time polynomial in $|S|$ and $k$ (and thus in $|S|$ and $|E|$) following the method of Khachiyan (see for instance [Sch86] p.170). Thus $\Gamma$ is solved up to a multiplicative factor or shown unfeasible in polynomial time.

The above reasoning applies to any homogeneous system of linear inequations and it does not take advantage of the particular form of $\Gamma$. The specific analysis presented hereafter allows to optimize the solution. Since $A$ is a reduced automaton, the set $\{s_1, \cdots, s_h\} = \{s \in S \mid s \stackrel{e'}{\to}\}$ is not empty. Therefore, the system $\Delta$ assembled from the linear inequations $\Sigma\, z_i\,(w_{is'} - w_{is}) < 0$ derived from (3) and (4) is not trivial. Let $w_{ij} = w_{is'} - w_{is_j}$ for $i \in \{1, \cdots, k\}$ and $j \in \{1, \cdots, h\}$. Now $\Delta$ may be rewritten as $W^T Z \leq (-1)^k$, where $Z \in \mathbb{Z}^k$ is the vector of the unknown $z_i = Z(i)$ and $(-1)^k = (-1, \cdots, -1)$. A system of this form has an integral solution *iff* it has a rational solution, hence it may be solved or shown unfeasible in time polynomial in $h$ and $k$ following the method of Khachiyan.

We claim that any integral solution of $\Delta$, let $Z$, extends to a solution of $\Gamma$, let $(x, y, Z)$ where $x, y \in \mathbb{N}$. The adequate values for $x$ and $y$ may be chosen as follows. Let $\alpha = min\{\Sigma\, z_i\, w_{is} \mid s \in S\}$, $\beta = min\{\Sigma\, z_i\, w_{is} \mid s \stackrel{e'}{\to}\}$, and $\gamma = \Sigma\, z_i\, \eta_i(e')$. Observe the relations $\alpha \leq 0$ (as $s = s_0 \Rightarrow w_{is} = 0$), $\alpha \leq \beta$, and $\alpha \leq \beta + \gamma$ (as $s \stackrel{e'}{\to} s'' \Rightarrow \Sigma\, z_i\,(w_{is} + \eta_i(e')) = \Sigma\, z_i\, w_{is''}$). Now let $x = -\alpha$ and $y = \beta - \alpha$. The satisfaction of the inequations (1), (2), and (4) follows immediately. Regarding (3), observe that $s \stackrel{e'}{\to} \Rightarrow \Sigma\, z_i\,(w_{is'} - w_{is}) < 0$ since $Z$ is a solution of $\Delta$, hence $\Sigma\, z_i\, w_{is'} < \beta$ as was to show. The problem $ESSP_A(s', e')$ is now solved in $\mathcal{R}(A)$ by the Petri region $(\sigma, (\eta_1, \eta_2))$ defined by $\sigma(s_0) = x$, $(\eta_1(e), \eta_2(e)) = i\,(\Sigma\, z_i\, \eta_i(e))$ for $e \neq e'$, and $(\eta_1(e'), \eta_2(e')) = (y, y + \Sigma\, z_i\, \eta_i(e'))$. Observe finally that $(\sigma, (\eta_1, \eta_2)) = I_A \circ J_A\ (\sigma, (\eta_1, \eta_2))$ *iff* $\alpha \leq \beta + \gamma$, in which case $ESSP_A(s', e')$ is also solved by a pure region. The following theorem summarizes the section.

**Theorem 3.5** *The synthesis problem for Petri net systems with the sequential firing rule is polynomial in the numbers of states and events of the transition systems taken as inputs.*

# 4 Galois connection between step automata and net systems

## 4.1 Step transition systems

We leave now the classical frame of (sequential) transition systems for the more expressive frame of *step transition systems*, defined by Mukund so as to account fully for the independence of events in general Petri nets. A co-reflection was established in [Mu92] between a full subcategory of step transition systems and the category of Petri nets, according to which the co-reflective image of a net is the marking graph of that net equipped with the *step firing rule*. Recalling this rule is a prerequisite for motivating the definition of step transition systems.

**Definition 4.1** *Given a Petri net $N = (P, E, F)$, where $F$ is a partial map from $(P \times E) \cup (E \times P)$ to the positive integers, the* concurrent marking graph *of $N$ is the transition system $(I\!N^P, I\!N^E, T)$ with set of transitions $T \subseteq (I\!N^P \times I\!N^E \times I\!N^P)$ defined as follows: $(M \xrightarrow{\alpha} M') \in T$* **iff** $\forall x \in P \quad M(x) \geq \Sigma_{e \in E}\ \alpha(e) \times F(x, e) \ \wedge \ M'(x) = M(x) + \Sigma_{e \in E}\ \alpha(e) \times (F(e, x) - F(x, e))$. *When these conditions are satisfied we say that the* step $\alpha$ has concession *in state $M$ (written also $M[\alpha>$).*

It follows from Def. 4.1 that whenever $\alpha = \beta + \gamma$ in $E \to I\!N$ and $M \xrightarrow{\alpha} M'$ in $T$, there exists an *intermediate marking* $M''$ such that $M \xrightarrow{\beta} M''$ and $M'' \xrightarrow{\gamma} M'$. Let us inspect the relationship between the sequential and concurrent marking graphs of a net. On the one hand, the sequential marking graph is the induced restriction of the concurrent marking graph on the subset of *atomic* steps, i.e. steps $\alpha$ such that $\Sigma_{e \in E}\, \alpha(e) = 1$. On the other hand, the concurrent marking graph cannot in general be reconstructed up to isomorphism from an arbitrary copy of the sequential marking graph, even though additional data are provided by a binary relation of independence on events, depending on markings, such that $e \|_M e'$ **iff** $M[\{e, e'\}>$. An example borrowed from [HKT96] is shown in Fig. 4.1. This example motivates the following definition.

**Definition 4.2 (Step transition system over an abelian monoid)** *A* step transition system $(S, M, T)$ *over an abelian monoid $M$ consists of a set of states $S$ and a deterministic transition relation $T \subseteq S \times M \times S$, with*

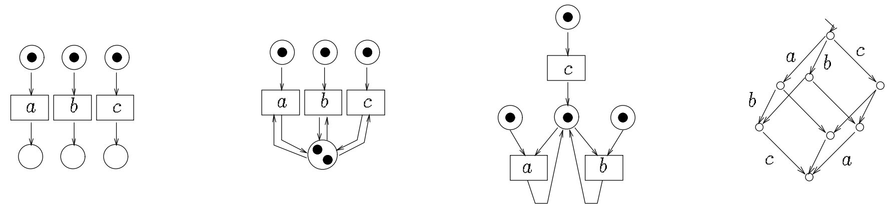


Figure 1: Three nets with an identical sequential marking graph but with different concurrent marking graphs

*distinguished* empty steps*:* $s \stackrel{0}{\rightarrow} s'$ *iff* $s = s'$. *A* step automaton $\mathcal{A}$ *is an initialized step transition system* $(S, M, T, s_0)$ *with initial state* $s_0 \in S$*, such that every state* $s \in S$ *is reachable from* $s_0$ *in* $U\mathcal{A} = (S, M, T)$*. The step automaton* $\mathcal{A}$ *is* finite *if the set of transitions* $T$ *is finite. When* $M = <E>$*, the free abelian monoid freely generated from set* $E$*, the step automaton* $\mathcal{A}$ *is said to be* reduced *if its* skeleton $(S, E, T \cap (S \times E \times S), s_0)$ *is a reduced automaton.*

The definition of step transition systems extends Mukund's original definition, which was restricted to free abelian monoids $M = <E> \cong (E \rightarrow \mathbb{N})$. The extension was not indispensable for the progress of this work, but it allows to accomodate the idea of regions as morphisms to step transition systems which do not present the *intermediate state* property: $s \stackrel{\alpha+\beta}{\rightarrow} s' \not\Rightarrow \exists s'' \in S \; s \stackrel{\alpha}{\rightarrow} s'' \wedge s'' \stackrel{\beta}{\rightarrow} s'$. This aspect will be investigated in another study. The final goal of the present paper is to adapt the synthesis algorithm for Petri nets defined in section 3 so as to fit now step transition systems taken as inputs. This goal will be achieved in section 5. In the meantime, we proceed to adapt regions to step transition systems and to establish a Galois connection between step automata and Petri net systems.

## 4.2 Regions in step transition systems

The definition of regions in step transition systems is parametric on enriched types of nets defined as follows.

**Definition 4.3 (Enriched type of nets)** *An enriched type of nets is a (deterministic) step transition system* $\tau = (LS, LE, \tau)$*, where* $LE$ *is a commutative monoid* $(LE, +, (0, 0))$*.*

Each type of nets determines a specific concurrent firing rule and hence a specific construction of concurrent marking graphs for nets.

**Definition 4.4 (Concurrent marking graph)** *Given a net $N = (P, E, W)$ with (enriched) type $\tau = (LS, LE, \tau)$, the concurrent marking graph of $N$ is the step transition system $(LS^P, <E>, T)$ with set of transitions $T$ defined by :*

$$(M \stackrel{\alpha}{\rightarrow} M') \in T \quad \textbf{iff} \quad \forall x \in P \ \ (M(x) \stackrel{W(x,\alpha)}{\longrightarrow} M'(x)) \in \tau \qquad (2)$$

*where $W(x, e_1 + \cdots + e_n) = W(x, e_1) + \cdots + W(x, e_n)$. Given a net system $N = (P, E, W, M_0)$, the concurrent marking graph of $N$ is the step automaton $N^* = (S, <E>, T_S, M_0)$ where $S$ is the inductive closure of the singleton set $\{M_0\}$ w.r.t. forward transitions in $T$, and $T_S = T \cap (S \times <E> \times S)$.*

Regions may now be introduced, based on the following definition of morphisms of step transition systems.

**Definition 4.5** *A* morphism of step transition systems *from $T$=$(S, M, T)$ to $T$'=$(S', M', T')$ is a pair $(\sigma, \eta)$, made of a map $\sigma : S \rightarrow S'$ and a monoid morphism $\eta : M \rightarrow M'$, such that $s \stackrel{\alpha}{\rightarrow} s' \Rightarrow \sigma(s) \stackrel{\eta(\alpha)}{\rightarrow} \sigma(s')$. The* morphisms of step automata *from $\mathcal{A}$ to $\mathcal{A}'$ are the morphisms from $U\mathcal{A}$ to $U\mathcal{A}'$ that preserve the initial state.*

**Definition 4.6 (Regions in step transition systems)** *Given a step transition system $T = (S, M, T)$ and an enriched type of nets $\tau = (LS, LE, \tau)$, the set $\mathcal{R}_\tau(T)$ of $\tau$-type (extended) regions in $T$ is the set of morphisms of step transition systems from $T$ to $\tau$. The set of $\tau$-type (extended) regions of a step automaton $\mathcal{A}$ is the set $\mathcal{R}_\tau(\mathcal{A}) = \mathcal{R}_\tau(U\mathcal{A})$.*

For the sake of illustration, let us specialize the above definitions to Petri nets, and thereby pave the way for solving their extended synthesis problem (in section 5). The enriched type of Petri nets is just the type $\tau_{Petri}$ = $(\mathbb{N}, \mathbb{N} \times \mathbb{N}, \tau_{Petri})$ enriched with the operation of componentwise addition in $\mathbb{N} \times \mathbb{N}$: $(i, j) + (k, l) = (i + k, j + l)$. This is coherent with Def. 4.6 since $(\mathbb{N} \times \mathbb{N}, +, (0, 0))$ is the free abelian monoid generated from $(0, 1)$ and $(1, 0)$. Now seeing that $n \stackrel{(i,j)}{\longrightarrow}$ in $\tau_{Petri}$ *iff* $n \geq i$, Def. 4.1 may be restored from

Def. 4.4 by specializing $\tau$ to $\tau_{Petri}$. Mukund's definition of regions may be restored similarly from Def. 4.6.

Returning to arbitrary types of nets, observe from equation 2 that the extension $(\sigma_x, \eta_x)$ of a place $x$ in the concurrent marking graph of a net $N$ is a region, defined by $\sigma_x(M) = M(x)$ and $\eta_x(\alpha) = W(x,\alpha) = \Sigma_{e \in E}\ \alpha(e) \times W(x,e)$.

## 4.3 Galois connection and representation result

The technical development about basic transition systems conducted through sections 2.2 and 2.3 may be reproduced nearly intact in the enriched setting of step transition systems over a free abelian monoid. We state here the definitions and results needed for next section, and avoid a remake of the proofs by just indicating the adaptation.

**Definition 4.7 (Dual of a step transition system)** *Given a step transition system over a free abelian monoid $T = (S, <E>, T)$, and an enriched type of nets $\tau$, the dual of $T$ is the net $T^* = (\mathcal{R}_\tau(T), E, W)$ with weights defined by $W((\sigma,\eta),e) = \eta(e)$. For any subset $R$ of $\mathcal{R}_\tau(T)$, $T^*_R$ is the subnet of $T^*$ with restricted set of places $R$.*

**Definition 4.8 (Dual of a step automaton)** *Given a step automaton $\mathcal{A}$ over a free abelian monoid, with initial state $s_0$, and a type of nets $\tau$, the dual of $\mathcal{A}$ is the net system $\mathcal{A}^*$ composed of the underlying net $(U\mathcal{A})^*$ and of the initial marking $M_0$ defined by $M_0(\sigma,\eta) = \sigma(s_0)$ for every $(\sigma,\eta) \in \mathcal{R}_\tau(\mathcal{A})$. For any subset $R$ of $\mathcal{R}_\tau(\mathcal{A})$, $\mathcal{A}^*_R$ is the subnet system of $\mathcal{A}^*$ with restricted set of places $R$.*

Let $\mathbf{SAut}(E)$ denote the set of isomorphism classes of reduced step automata over $<E>$. Seeing that $\mathbf{SAut}(E)$ is a subset of $\mathbf{Aut}(<E>)$, it inherits the induced restriction of the order relation $\mathcal{A}_1 \leq \mathcal{A}_2$ if $\exists \sigma :\ (\sigma, 1_{<E>}) : \mathcal{A}_1 \to \mathcal{A}_2$. This order relation on $\mathbf{SAut}(E)$ may be equivalently defined as: $\mathcal{A}_1 \leq \mathcal{A}_2$ if there exists an event preserving morphism of step transition systems $(\sigma, 1_{<E>}) :\ \mathcal{A}_1 \to \mathcal{A}_2$. Moreover the synchronized product of step automata computed in $\mathbf{Aut}(<E>)$ is a step automaton, and it is in fact their greatest lower bound in $\mathbf{SAut}(E)$. This justifies the following transposition of Prop. 2.8.

**Proposition 4.9** *The two $()^*$ operators, mapping respectively a step automaton $\mathcal{A}$ over the free abelian monoid $<E>$ to the dual net system $\mathcal{A}^*$ and a net system $N$ with set of events $E$ to its concurrent marking graph $N^*$, constitute a Galois connection between the ordered sets* $\mathbf{Nets}(E)$ *and* $\mathbf{SAut}(E)$*:* $\mathcal{A} \leq N^* \Leftrightarrow N \leq \mathcal{A}^*$ *for* $\mathcal{A} \in \mathbf{SAut}(E)$ *and* $N \in \mathbf{Nets}(E)$.

A step automaton $\mathcal{A}$ over $<E>$ is said to be *separated* w.r.t. an enriched type of nets $\tau$ if it is isomorphic to its closure $\mathcal{A}^{**} = \bigwedge_{x \in \mathcal{R}_\tau(\mathcal{A})} N_x^*$ where $N_x = \mathcal{A}^*_{\{x\}}$ is the atomic net system derived from the extended region $x$. A set of extended regions $R \subseteq \mathcal{R}_\tau(\mathcal{A})$ is said to be *admissible* if $\mathcal{A} \cong \bigwedge_{x \in R} N_x^*$. The concurrent marking graph of a net system is always separated, and the extensions of the places of $N$ in $N^*$ form always an admissible set of extended regions (in view of equation 2). Next section is based upon the following counterpart to Theo. 2.9.

**Theorem 4.10** *Given a step automaton over a free abelian monoid, let* $\mathcal{A} = (S, <E>, T, s_O)$*, and an enriched type of nets* $\tau$*, a subset of extended regions* $R \subseteq \mathcal{R}_\tau(\mathcal{A})$ *is admissible if and only if the following* separation properties *are satisfied for all states* $s, s' \in S$ *and for every multiset* $\alpha \in <E>$*:*

($\mathbf{SSP}$) $\quad s \neq s' \;\Rightarrow\; \exists (\sigma, \eta) \in R : \; \sigma(s) \neq \sigma(s')$

*(read:* $(\sigma, \eta)$ *solves the* states separation *problem at* $(s, s')$*)*

($\mathbf{ESSP}$) $\quad s \overset{\alpha}{\not\rightarrow} \;\Rightarrow\; \exists (\sigma, \eta) \in R : \;\; \sigma(s) \overset{\eta(\alpha)}{\not\longrightarrow} \quad$ *w.r.t.* $\tau$

*(read:* $(\sigma, \eta)$ *solves the* event/state separation *problem at* $(s, \alpha)$*)*

*When both properties are satisfied,* $\mathcal{A} \cong (\mathcal{A}^*_R)^*$*, where* $\mathcal{A}^*_R$ *is the subnet system of* $\mathcal{A}^*$ *with restricted set of places* $R$ *(also called the net* synthesized *from* $R$*).*

# 5 Synthesis of general Petri nets (concurrent firing)

## 5.1 The extended synthesis problem

Assuming the concurrent firing rule, the extended *synthesis problem* for Petri nets consists in: $(i)$ deciding whether a *finite* step automaton $\mathcal{A}$ over $<E>$, given as input, is isomorphic to the concurrent marking graph $N^*$ of some net system $N = (UN, M_0)$, and if so: $(ii)$ producing as output a net system $N$

such that $\mathcal{A} \cong N^*$ and no proper subnet system of $N$ satisfies this property.

Henceforth, $\mathcal{A} = (S, <E>, T, s_0)$ is a finite and reduced step automaton. Moreover, seeing that the intermediate state property is always satisfied in the concurrent marking graph of a Petri net, we assume this property from $\mathcal{A}$. Thus, $s \overset{\alpha+\beta}{\to} s' \in T \;\Rightarrow\; \exists s'' \in S \;\; s \overset{\alpha}{\to} s'' \in T \;\wedge\; s'' \overset{\beta}{\to} s' \in T$. The import is that we may further assume a compact representation for $\mathcal{A}$, given by its skeleton and the set of the maximal steps at each state $s \in S$. This makes sense since the set of steps of $\mathcal{A}$ is bounded (from the assumption that $\mathcal{A}$ is finite). In the sequel, $A$ denotes the skeleton of $\mathcal{A}$, and for each state $s \in S$, $Max(s)$ denotes the set of the maximal steps with concession at $s$.

## 5.2 The structure of extended Petri regions

Let us inspect the relationship between $\mathcal{R}(A)$ and $\mathcal{R}(\mathcal{A})$, the respective sets of Petri regions of $A$ and extended Petri regions of $\mathcal{A}$. A monoid morphism $\eta : <E> \to I\!N \times I\!N$ is totally defined from its restriction on generators, let $\eta_E = (\eta_{E-}, \eta_{E+})$. The map which sends $(\sigma, \eta)$ to $(\sigma, \eta_E)$ is thus an injection from $\mathcal{R}(\mathcal{A})$ into $\mathcal{R}(A)$. A Petri region $(\sigma, \eta_E)$ is in the image of this map *iff* $\sigma(s) \geq \Sigma_{e \in E} \; \alpha(e) \times \eta_{E-}(e)$ for every state $s \in S$ and for every multiset $\alpha \in Max(s)$. When these conditions are met, the region $(\sigma, \eta_E)$ may be identified with the extended region $(\sigma, \eta)$ where $\eta$ is the extension of $\eta_E$ to multisets in $<E>$. One may therefore state without further proof the following lemma, extending Cor. 3.3.

**Lemma 5.1** *Let $\sigma$ be a map from $S$ to $I\!N$ and $\eta : <E> \to I\!N \times I\!N$ be a morphism of monoids, restricting on generators to $\eta_E = (\eta_{E-}, \eta_{E+})$ and let $\eta' = \eta_{E+} - \eta_{E-}$, then $(\sigma, \eta) \in \mathcal{R}(\mathcal{A})$ if and only if (i) $\eta' \in \mathcal{R}_{abs}(A)$, and (ii) $\sigma(s_0) + \eta' \cdot \pi(c) \geq \Sigma_{e \in E} \; \alpha(e) \times \eta_{E-}(e)$ for every chain $c \in C_1(UA)$ and for every step $\alpha \in <E>$ such that $\exists s \in S \;\; \partial(c) = (s - s_0) \;\wedge\; s \overset{\alpha}{\to}$.*

## 5.3 Solving the states separation problem

Let us recall from section 3 that computing a basis $\{\eta'_1, \ldots, \eta'_k\}$ for the $Z\!\!\!Z$-module of abstract regions $\mathcal{R}_{abs}(A)$ takes time polynomial in $|S|$ and $|E|$. Now the states separation problem $SSP_{\mathcal{A}}(s,s')$ may be solved in $\mathcal{R}(\mathcal{A})$ *iff* the states separation problem $SSP_A(s,s')$ is solved in $\mathcal{R}(A)$ by a *pure* region of the form $(\sigma_j, i \circ \eta'_j)$, where $1 \leq j \leq k$. Seeing that a pure region of $A$ extends always to a region of $\mathcal{A}$, the separation problem $SSP_{\mathcal{A}}(s,s')$ is then also solved by $(\sigma_j, i \circ \eta'_j)$. Therefore, solving all the instances of the states separation problem in $\mathcal{A}$ takes time polynomial in $|S|$ and $|E|$.

## 5.4 Solving the event / state separation problem

According to Lem. 5.1, an instance $ESSP_{\mathcal{A}}(s',\alpha')$ of the event / state separation problem in $\mathcal{A}$ has a solution in $\mathcal{R}(\mathcal{A})$ if and only if there exist $\sigma(s_0) \in I\!N$, $\eta' \in \mathcal{R}_{abs}(A)$, and $\eta_{E-} : E \to I\!N$ such that the following hold, letting $c_s$ be the unique path from $s_0$ to $s$ in the spanning tree for $A$ defined in section 3 :

1. $\forall e \quad \eta'(e) + \eta_{E-}(e) \geq 0$
2. $\forall s \ \forall \alpha \in Max(s) \quad \sigma(s_0) + \eta' \cdot \pi(c_s) \geq \Sigma_{e \in E} \ \alpha(e) \times \eta_{E-}(e)$
3. $\sigma(s_0) + \eta' \cdot \pi(c_{s'}) < (\Sigma_{e \in E} \ \alpha'(e) \times \eta_{E-}(e))$

The expected solution is then the extended region $(\sigma, \eta)$ such that $\sigma(s) = \sigma(s_0) + \eta' \cdot \pi(c_s)$ and $\eta : <E> \to I\!N \times I\!N$ restricts on $E$ to $(\eta_{E-}, \eta' + \eta_{E-})$. In order to study the existence of solutions to the above constraints, let us introduce variables $x = \sigma(s_0)$, $y_i = \eta_{E-}(e_i)$ with $E = \{e_1, \cdots, e_n\}$, and $z_j$ with $\eta' = z_1 \eta'_1 + \cdots + z_k \eta'_k$. Thus $x \in I\!N$, $y_i \in I\!N$ for $1 \leq i \leq n$, and $z_j \in Z\!\!\!Z$ for $1 \leq j \leq k$. Let us define also integer constants $w_{js} = \eta'_j \cdot \pi(c_s)$ for $1 \leq j \leq k$ and $s \in S$. Assuming w.l.o.g. that $e' = e_1$ and letting $Max$ denote the size of the largest set of maximal steps $Max(s)$ for $s \in S$, then the above constraints may be rewritten into a system of at most $(|E| + Max \times |S| + 1)$ linear inequations in the unknowns $x$, $y_i$ and $z_j$, as follows :

1. $y_i + \Sigma_j z_j \eta'_j(e_i) \geq 0$ (one inequation for each $1 \leq i \leq n$)
2. $x - \Sigma_i \alpha(e_i) \times y_i + \Sigma_j z_j w_{js} \geq 0$ (for each $s \in S$ and $\alpha \in Max(s)$)
3. $x - \Sigma_i \alpha'(e_i) \times y_i + \Sigma_j z_j w_{js'} < 0$
4. $y_i \geq 0$ (for each $1 \leq i \leq n$)

5. $x \geq 0$

The system is homogeneous, hence it has an integral solution *iff* it has a rational solution. The method of Khachiyan may again be used to decide whether the system is feasible and to compute a solution in polynomial time. A region that solves the problem $ESSP_{\mathcal{A}}(s, \alpha)$ solves also all instances of the form $ESSP_{\mathcal{A}}(s, \beta)$ for $\alpha < \beta$ therefore it is sufficient to solve the instances corresponding to *minimal failures*. Thus if we let $Min(s)$ stand for the set of minimal steps having no concession at $s$ and $Min$ the largest size of such set, the following theorem summarizes the section.

**Theorem 5.2** *The synthesis problem for Petri net systems with the step firing rule, taking as inputs finite step transition systems is polynomial in their numbers of states and events, in the size of the largest set of minimal failures in one state, and in the size of the largest set of maximal steps enabled in one state.*

Notice that minimal failures cannot be determined from the maximal enabled steps as illustrated by the third Petri net of Fig. (1) for which $Max(s_0) = \{a + c, b + c\}$ and $Min(s_0) = \{2a, a + b, 2b\}$ and therefore $Min(s) \not\subseteq \{\alpha + e| \alpha \in Max(s)\}$. Let us conclude by a comparison with [Bad96]. Every step automaton may be transformed into an ordinary automaton by splitting its alphabet of events. More precisely the states of the split automaton are the pairs $<s, \alpha>$ where $\alpha$ is a step with concession at $s$ and each transition $s \xrightarrow{e} s'$ gives rise to the pair of transitions $<s, \alpha> \xrightarrow{e^+} <s, \alpha + e>$ and $<s, \alpha + e> \xrightarrow{e^-} <s', \alpha>$ for every step $\beta = \alpha + e$ with concession at $s$. In [Bad96] it is shown that the synthesis problem for Petri nets may be reduced to the synthesis of pure Petri nets by splitting events, and that one can derive from the algorithm of [BBD95a] an algorithm for the synthesis of Petri nets from step automata taking time polynomial in the size of sets of higher-dimensional states. Now if $\beta \in Min(s)$ is a minimal failure, then $<s, \alpha> \stackrel{e^+}{\not\rightarrow}$ for any step $\alpha$ and event $e$ such that $\beta = \alpha + e$, and the problem $ESSP_{\mathcal{A}}(s, \beta)$ is equivalent to the event/state separation problem $ESSP_{\mathrm{Split}(\mathcal{A}}(<s, \alpha>, e^+)$ induced by the event $e^+$ and the higher-dimensional state $<s, \alpha>$. Of course there is more instances of $ESSP$ to be solved in the split automaton since $\beta$ can be decomposed as $\beta = \alpha + e$ in several ways. Nevertheless $Min$ and the number of instances of the event/state separation problem in the split automaton are both exponential in the number of events. Actually if $\mathcal{A}$ is the step automaton whose skeleton $A = (S, E, T, s_0)$ is the $2n$-dimensional hypercube, i.e. $E = \{a_1, \ldots, a_{2n}\}$, $S = 2^E$ with initial state $s_0 = \emptyset$ and $\alpha \xrightarrow{e} \beta$ iff $\beta = \alpha \cup \{e\}$, and such that the concession of steps at states is given by $s \xrightarrow{\alpha}$ iff $s_0 \xrightarrow{s+\alpha}$ and

$$s_0 \overset{\alpha}{\not\to} \quad \Leftrightarrow \quad \bigvee_{k=1}^{n} [j < k \Rightarrow \alpha(a_j) = 0 \quad \wedge \quad \alpha(a_k) = 1 \quad \wedge |\alpha| \geq k + 1]$$

then $|Min(s_0)| = \sum_{k=1}^{n} \; C_{2n-k}^{k} \; \geq \; \sum_{k=1}^{n} \; C_n^k = 2^n - 1$.

# References

[Bad96] Badouel, E., *Splitting of Actions, Higher-Dimensional Automata, and Net Synthesis.* Inria Research Report 3013 (1996).

[BBD95a] Badouel, E., Bernardinello, L., and Darondeau, Ph., *The Synthesis Problem for Elementary Net Systems is NP-complete.* Inria Research Report 2558 (1995) *-to appear in* Theoretical Computer Science.

[BBD95b] Badouel, E., Bernardinello, L., and Darondeau, Ph., *Polynomial Algorithms for the Synthesis of Bounded Nets.* Proceedings of CAAP'95, Lecture Notes in Computer Science vol. 915 (1995) 647-679.

[BD95a] Badouel, E., and Darondeau, Ph., *Trace Nets and Process Automata.* Acta Informatica vol. 32 (1995) 647-679.

[BD95b] Badouel, E., and Darondeau, Ph., *Dualities between Nets and Automata Induced by Schizophrenic Objects.* Proceedings of 6th International Conference on Category Theory and Computer Science, Lecture Notes in Computer Science vol. 953 (1995) 24-43.

[BDPV96] Bernardinello, L., De Michelis, G., Petruni, K., and Vigna, S., *On the Synchronic Structure of Transition Systems.* In “Structures in Concurrency Theory”, J. Desel ed., Springer-Verlag (1996) 11-31.

[Ber93] Bernardinello, L., *Synthesis of Net Systems.* Proceedings of Application and Theory of Petri Nets, Lecture Notes in Computer Science vol. 691 (1993) 89-105.

[CKLY95] Cortadella, J., Kishinevsky, M., Lavagno, L., and Yakovlev, A., *Synthesizing Petri Nets from State-Based Models.* Proceedings of ICCAD'95 (1995) 164-171.

[DR96] Desel, J., and Reisig, W., *The Synthesis Problem of Petri Nets.* Acta Informatica vol. 33 (1996) 297-315.

[DS93] Droste, M., and Shortt, R.M., *Petri Nets and Automata with Concurrency Relations - an Adjunction.* In "Semantics of Programming Languages and Model Theory", M. Droste and Y. Gurevich eds (1993) 69-87.

[DS96] Droste, M., and Shortt, R.M., *From Petri Nets to Automata with Concurrency.* Draft communicated to the authors.

[ER90a] Ehrenfeucht, A., and Rozenberg, G., *Partial (Set) 2-Structures;* Part I: *Basic Notions and the Representation Problem.* Acta Informatica, vol. 27 (1990) 315-342.

[ER90b] Ehrenfeucht, A., and Rozenberg, G., *Partial (Set) 2-Structures;* Part II: *State Spaces of Concurrent Systems.* Acta Informatica, vol. 27 (1990) 343-368.

[GM85] Gondran, M., and Minoux, M., *Graphes et algorithmes.* Eyrolles, Paris (1985).

[Hi94] Hiraishi, K., *Some Complexity Results on Transition Systems and Elementary Net Systems.* Theoretical Computer Science, vol. 135 (1994) 361-376.

[HKT92] Hoogers, P.W., Kleijn, H.C.M., and Thiagarajan, P.S., *A Trace Semantics for Petri Nets.* Proceedings of ICALP'92, Lecture Notes in Computer Science vol. 623 (1992) 595-604.

[HKT96] Hoogers, P.W., Kleijn, H.C.M., and Thiagarajan, P.S., *An Event Structure Semantics for General Petri Nets.* Theoretical Computer Science, vol. 153 (1996) 129-170.

[Lef75] Lefschetz, S., *Applications of Algebraic Topology.* Applied Mathematical Science 16, Spinger-Verlag (1975).

[Mu92] Mukund, M., *Petri Nets and Step Transition Systems.* International Journal of Foundations of Computer Science vol. 3 no. 4 (1992) 443-478.

[NRT92] Nielsen, M., Rozenberg, G., and Thiagarajan, P.S., *Elementary Transition Systems.* Theoretical Computer Science, vol. 96 (1992) 3-33.

[NW94] Nielsen, M., and Winskel, G., *Models for Concurrency.* Handbook of Logic for Computer Science, vol. 3, Oxford University Press (1994) 100-200.

[Sch86] Schrijver, A., *Theory of Linear and Integer Programming.* John Wiley (1986).

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