Stability of Discrete-time Systems : New Criteria and Applications to Control Problems

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Abstract: The aim of this article is to present some new stability sufficient conditions for discrete-time nonlinear systems. It shows how to use nonnegative semi-definite functions as Lyapunov functions instead of positive definite ones for studying the stability of a given system. Several examples and some applications to control theory are presented to illustrate the various theorems.

Key-words: difference equations, semi-definite function, stability, feedback, nonlinear control systems.

(Résumé : tsvp)

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Stabilité des Systèmes Discrets : Nouveaux Critères et Applications aux Problèmes de l'Automatique non Linéaire

Résumé : Le but de cet article est de présenter quelques nouveaux resultats de stabilité. Il s'agit de conditions suffisantes de stabilité pour les systèmes non linéaires en temps discret. On montre comment on peut utiliser des fonctions semi-définies comme fonctions de Lyapounov à la place de fonctions definies positives, pour l'etude qualitative d'un système donné. Pour illustrer les differents théorèmes, plusieurs exemples seront donnés, ainsi que des applications à la théorie du contrôle.

Mots-clé : Equations aux différences, fonction semi-définie, stabilité, feedback, systèmes non linéaires contrôlés.

1 Introduction

In this paper, we will extend the basic results in the classical Lyapunov theory for discrete-time nonlinear system. A good introduction to classical Lyapunov theory for difference equation is [9],[6]. LaSalle in [10] goes beyond the developments to be found in these references, and gives a number of results that are very interesting for difference equations. But this classical approach is essentially based on the construction of a suitable positive definite Lyapunov function (ie a positive definite function which is decreasing along the trajectories of the considered system).

It is always not easy to find such function. For differential equations there is an extensive literature which gives alternative methods to explore the stability of contiuous systems. One can mention the methode of semi-definite Lyapunov functions [7], decomposition technique for large-scale systems [14, 12, 13, 2], center manifold machinery [3] etc. However there is a few literature for difference equations.

The purpose of this paper is to propose a new approach to explore the stability of nonlinear discrete time systems. We show, under some conditions, that the results of Lyapunov are still valid with semi definite nonnegative function V as Lyapunov function . The natural interest is that it is usually easier to find a nonnegative function than a positive one.

Our main result is a generalisation of Lyapunov's theorems in the sense that when the function V of our theorems is definite the results of this paper lead to the Lyapunov's theorems .

The paper is organised as follows : In section 2 we recall some notations and definitions, In section 3 we state and prove the main theorems. Section 4 gives some illustrating examples and remarks. In particular, we show how our results can be helpfull to achieve the local stability for cascade systems, systems with constant of motion and especially when the linearisation technique failed. In section 5 we give an application of the results of section 3 to the feedback stabilization and we derive a link between the zero state detectability and the attractivity of the zero solution.

2 Notations and Preliminaries

Let us consider a system of difference equations

$$\begin{cases} x(k+1) = f(x(k)) & x \in \mathcal{U} \\ f(0) = 0 \end{cases} \tag{1}$$

where $\mathcal{U}$ is a neighbourhood of the origin in $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function.

For each $p \in \mathcal{U}$, let us denote by $f^k(p)$ the value at time k of the solution of (1) starting at p. We recall that $f^k(p) = f(f^{k-1}(p))$, $f^0(p) = p$. We recall that continuity of f is sufficient to guarantee existence and uniqueness of solutions for each initial state $p \in \mathcal{U}$.

In general we are interested in points $x_0 \in \mathcal{U}$ such that $f(x_0) = x_0$. They are usually called *equilibrium point* of the system. Corresponding to each equilibrium point x_0, we have a constant solution $f^k(x_0) \equiv x_0$ of (1).

First we recall some usual notations and standard definitions :

$f^+(p) = \{f^k(p), k \in \mathbb{N}\}$.
$L^+(p)$ is the ω-limit set of p .
$B_\epsilon = \{x \in \mathbb{R}^n :\| x \|< \epsilon\}$, $\overline{B_\epsilon} = \{x \in \mathbb{R}^n :\| x \|\leq \epsilon\}$, $S_\epsilon = \{x \in \mathbb{R}^n :\| x \|= \epsilon\}$.

Definition 1 *Let $x_0 \in \mathcal{U}$ be an equilibrium point. We say that* (1) *is Lyapunov stable at x_0 or that x_0 is a Lyapunov stable equilibrium point for* (1), *if for each $\epsilon > 0$ there is a positive δ such that for each x with $\|x - x_0\| < \delta$ and each solution $f^k(x)$, then $\|f^k(x) - x_0\| < \epsilon$ for all $k > 0$. When* (1) *is not Lyapunov stable at x_0, we say that it is unstable at x_0, or that x_0 is unstable for* (1).

Lyapunov stability has the following heuristic meaning. Think of x_0 as a desired steady-state of our system. Because of unpredictable perturbations (which are supposed to be small), the current state of the system is actually $x \neq x_0$. Lyapunov stability guarantees that every state value taken by the system in its future evolution is not too far from the desired one. Physical systems usually have this property. In fact, due to damping forces, what actually happens is that the amplitude of initial perturbations decreases and eventually vanishes. This physically important aspect is not reflected by Definition 1. Thus, we are led to introduce a stronger notion.

Let $\mathcal{A}$ be the set of points $p \in \mathcal{U}$ for which

$$\lim_{k \to +\infty} f^k(p) = x_0$$

holds, for all solutions $f^k(p)$ issuing from p. $\mathcal{A}$ is called the *region of attraction*, or *domain of attraction* of x_0. An equilibrium point is said to be *attractive* if it is an interior point of its region of attraction. We say also that (1) is attractive at x_0. In general, an attractive equilibrium point is not necessarily Lyapunov stable (an example can be found in [4], p. 170).

Definition 2 *We say that* (1) *is locally asymptotically stable at the equilibrium points x_0 or that x_0 is a locally asymptotically stable equilibrium position for* (1) *if it is Lyapunov stable and attractive at x_0.*

In the sequel we will take $x_0 = 0$.

Definition 3 *A set Y is invariant if $f(Y) = Y$, positively invariant if $f(Y) \subset Y$ and negatively invariant if $Y \subset f(Y)$.*

For any positively invariant set Y, $\mathcal{A}_Y$ will denote the relative domain of attractivity i.e, $\mathcal{A}_Y = \mathcal{A} \cap Y$.

Definition 4 *Let Y be a closed positively invariant set such that $0 \in Y$. The origin is said to be :*
(a) *Y–stable if* $\forall \epsilon > 0 \ \exists \delta > 0 : f^+(B_\delta \cap Y) \subset B_\epsilon$.
(b) *Y–asymptotically stable if it is Y–stable and there exists $\delta > 0$ such that*

$$\lim_{n \to +\infty} f^n(x) = 0 \ \ \forall \ x \in Y \cap B_\delta.$$

In the sequel, if the system (1) has a nonnegative Lyapunov function defined in a neighbourhood of the origin $\mathcal{V} \subset \mathcal{U}$, we will denote by G_0 the set where V vanishes, G the set where the difference of V along the trajectories of the system vanishes, and G^* the largest invariant set contained in G.
One can easily show that G_0, G^* and G are closed sets, G_0 is positively invariant and $G_0 \subseteq G^* \subseteq G$.

3 Main results

The first theorem concerns the Lyapunov stability.

Theorem 1 *If there exist a neighbourhood $\mathcal{V} \subset \mathcal{U}$ of the origin and a function $V \in C^0(\mathcal{V}, \mathbb{R})$ such that*

(1) *$V(x) \geq 0$ for all $x \in \mathcal{V}$ and $V(0) = 0$.*

(2) *$\Delta V(x) = V(f(x)) - V(x) \leq 0$ for all $x \in \mathcal{V}$.*

(3) *the origin is G_0–asymptotically stable , where :*
$G_0 = \{x \in \mathcal{V} : V(x) = 0\}$.

then the origin is Lyapunov stable.

Proof Suppose that the origin is not stable. Then there exists $\epsilon > 0$ for which it is possible to construct a sequence $(x_n)_{n \in \mathbb{N}} \subset B_\epsilon$, $\lim_{n \to \infty} x_n = 0$ such that for each $n \in \mathbb{N}$, the positive trajectory $f^k(x_n)$ of x_n does not stay within B_ϵ for all positive times $k \in \mathbb{N}$. In other words there exists a sequence $(k_n)_{n \in \mathbb{N}} \in \mathbb{N}$ in such a way that

$$\left\{ \begin{array}{l} \| f^m(x_n) \| < \epsilon \;\; for \;\; 0 \leq m < k_n \\ \| f^{k_n}(x_n) \| \geq \epsilon \;\; \forall n \in N. \end{array} \right. \tag{2}$$

We take ϵ sufficiently small in order to have $\overline{B_\epsilon} \cap G_0 \subset \mathcal{A}_{G_0}$
We will show that for any posive integer T there exists $m_o \in \mathbb{N}$ such that $k_n \leq 0$ ie, $k_n \to +\infty$ as $n \to +\infty$.

Indeed let T any positive integer, on the one hand, the functions $\{f^m \;,\;\; m \leq T\}$ are continuous so there exists $\eta > 0$ in such a way $f^m(B_\eta) \subset B_\epsilon$ for all $m \leq T$. On the other hand the sequence $(x_n)_{n \in \mathbb{N}}$ converges to the origin so one can find $m_0 \in \mathbb{N}$ such that $x_n \in B_\eta$ for all $n \geq m_0$. Thus for all $n \geq m_0$ and all $m \leq T$ one has $f^m(x_n) \in B_\epsilon$. And so according to the definition of k_n one has $k_n \geq T$.
The origin is G_0–asymptotically stable so there exists $N \in \mathbb{N}$ such that the solutions of (1) satisfy

$$\| f^n(z) \| < \frac{\epsilon}{2} \;\; \forall n \geq N \;\; \text{and} \;\; \forall z \in \overline{B_\epsilon} \cap G_0 \tag{3}$$

(Thanks to the compacity of $\overline{B_\epsilon} \cap G_0$, one can easily check that the integer N can be chosen independently of z.)
The continuity of the solutions with respect to initial conditions ensures the existence of $\delta > 0$ such that

$$\forall (x,y) \in \overline{B_\epsilon} \times \overline{B_\epsilon}\ ,\quad \| x-y \| < \delta \Longrightarrow \| f^n(x) - f^n(y) \| < \frac{\epsilon}{2} \quad \forall n \leq N \tag{4}$$

Now there exists $n_0 \in \mathbb{N}$ such that $\| x_n \| < \delta$ for all $n \geq n_0$ and thus by (4) one has

$$\| f^p(x_n) \| < \frac{\epsilon}{2} \quad \forall p \leq N \quad \text{and} \quad \forall n \geq n_0.$$

The latter shows that $0 < k_n - N < k_n$ for all $n \geq n_0$. Thus using (2) we obtain

$$\| f^{k_n - N}(x_n) \| < \epsilon \quad \forall n \geq n_0$$

which implies that the sequence $(u_n)_{n \geq n_0}$ defined by $u_n = f^{k_n-N}(x_n)$ has a convergent subsequence say $\left(u_{\phi(n)}\right)_{n \geq n_0}$. Let $z = \lim\limits_{n \to +\infty} u_{\phi(n)} \in \overline{B_\epsilon}$. Since V is assumed to be continuous we have

$$0 \leq V(z) = \lim_{n \to +\infty} V(u_{\phi(n)}) = \lim_{n \to +\infty} V(f^{k_{\phi(n)}-N}(x_{\phi(n)})) \leq \lim_{n \to +\infty} V(x_{\phi(n)}) = 0$$

so z belongs to $\overline{B_\epsilon} \cap G_0$ and then (3) yields

$$\| f^N(z) \| < \frac{\epsilon}{2} \tag{5}$$

On the other hand there exists $p \geq n_0$ such that $\| z - f^{k_p - N}(x_p) \| < \delta$ so by (4)

$$\| f^N(z) - f^N(f^{k_p-N}(x_p)) \| < \frac{\epsilon}{2} \tag{6}$$

Finally the combination of (5) and (6) leads to

$$\| f^{k_p}(x_p) \| < \epsilon$$

which is a contradiction to (2). ■
Now we state and prove the asymptotic stability theorem.

Theorem 2 *If there exist a neighbourhood $\mathcal{V} \subset \mathcal{U}$ of the origin and a function $V \in C^0(\mathcal{V}, \mathbb{R})$ such that*

(1) *$V(x) \geq 0$ for all $x \in \mathcal{V}$ and $V(0) = 0$.*

(2) *$\Delta V(x) = V(f(x)) - V(x) \leq 0$ for all $x \in \mathcal{V}$.*

(3) *0 is G^*–asymptotically stable where G^* is the largest positively invariant set contained in $G = \{x \in \mathcal{V} : V(f(x)) - V(x) = 0\}$*

then the origin is asymptotically stable.

Proof The set $G_0 = \{x \in \mathcal{V} : V(x) = 0\}$ is positively invariant so it is contained in G^*. All the assumptions of Theorem (1) are satisfied which implies that the origin is stable, that it is for any positive δ there exists a positive number γ such that any solution of (1) which starts in B_γ remains in B_δ for all integer n.
Let $\mathcal{A}_{G^*}$ be the domain of attractivity relative to G^*. We choose $\delta > 0$, such that $\overline{B_\delta} \cap G^* \subset \mathcal{A}_{G^*}$. To show the attractivity of the origin, we shall prove that B_γ is contained in the domain of attractivity, i.e,

$$\forall x \in B_\gamma \ : \ \lim_{k \to +\infty} f^k(x) = 0 \tag{7}$$

Let $x_0 \in B_\gamma$ and let ϵ be any positive real number. Thanks to the stability of the origin there exists $\eta > 0$ such that :

$$f^n(B_\eta) \subset B_\epsilon \ \ \forall n \in \mathbb{N} \tag{8}$$

Since

$$\overline{B_\delta} \cap G^* \subset \mathcal{A}_{G^*}$$

there exists $N \in \mathbb{N}$ such that

$$\| f^n(y) \| < \frac{\eta}{2} \ \ \forall n \geq N \ , \ \ \forall y \in \overline{B_\delta} \cap G^*. \tag{9}$$

The continuity of the solutions ensures the existence of $\alpha > 0$ such that

$$\forall (x,y) \in \overline{B_\delta} \times \overline{B_\delta} \ , \ \ \| x - y \| < \alpha \Longrightarrow \| f^n(x) - f^n(y) \| < \frac{\eta}{2} \ \ \forall n \leq N \tag{10}$$

Now, let y be an element of $L^+(x_0)$. According to LaSalle Invariance Principle, y belongs to $\overline{B_\delta} \cap G^*$ so by (9)

$$\| f^n(y) \| < \frac{\eta}{2} \ \ \forall n \geq N. \tag{11}$$

On the other hand $y \in L^+(x_0)$ hence

$$\exists p \in \mathbb{N} \ : \ \| f^p(x_0) - y \| < \alpha \tag{12}$$

Using (12), (10) and (11) we get

$$\| f^{N+p}(x_0) \| < \frac{\eta}{2} + \frac{\eta}{2} = \eta \tag{13}$$

and from (8), it follows

$$\| f^n(f^{N+p}(x_0)) \| < \epsilon \ \ \forall n \in \mathbb{N}$$

which proves that $\lim_{k \to +\infty} f^k(x_0) = 0$. So we have shown that there exists a positive real number γ such that B_γ is contained in the domain of attractivity, and thus Theorem (2) is established. ∎

Now suppose that the system (1) is defined on $\mathbb{R}^n$ and there exists a nonnegative function $V \in C^0(\mathbb{R}^n, \mathbb{R}^+)$ which is a Lyapunov function for (1) that is $\Delta V(x) = V(f(x)) - V(x) \leq 0$ for all $x \in \mathbb{R}^n$. As above G^* denotes the largest invariant set contained in $G = \{x \in \mathbb{R}^n \ :$

$\Delta V(x) = V(f(x)) - V(x) = 0\}$. One can ask the following : Does the global asymptotic stability of the system reduced to the invariant set G^* imply the global asymptotic stability of system (1) ? The answer is unfortunatly no as it can be shown in example (3) below. Nevertheless, we have the following global result which is a consequence of LaSalle Invariance Principle and Theorem 2 .

Theorem 3 *If there exists a function $V \in C^0(\mathbb{R}^n, \mathbb{R}^+)$ satisfying*

(1) *$V(x) \geq 0$ for all $x \in \mathbb{R}^n$ and $V(0) = 0$.*

(2) *$\Delta V(x) = V(f(x)) - V(x) \leq 0$ for all $x \in \mathbb{R}^n$.*

(3) *0 is G^*–globally asymptotically stable where G^* is the largest positively invariant set contained in $G = \{x \in \mathbb{R}^n : V(f(x)) - V(x) = 0\}$.*

(4) *All the solutions of system (1) are bounded*

then the origin is globally asymptotically stable.

4 Remarks and Examples

Remark 1 : In the following we recall the result of Aeyels and Sepulchre in [1], and show that it can be deduced as direct consequence of theorem 1. They have studied the stability of continuous time system which has first integrals and have proved ([1], Theorem 1) that the asymptotic stability of the equilibrium with respect to the perturbations belonging to the level sets of the first integrals containing the equilibrium point ensure the stability of the equilibrium with respect to arbitrary perturbations. This result is still valid for discrete time system. the proof of ([1]) is based on qualitative study of the system, here by using the previous theorem we give a simple proof for discret-time systems indeed :
(H) Assume that system (1) possesses k continuous first integrals $h_i, i = 1 \dots k$ that are defined on a neighbourhood $\mathcal{U}$ of the origin and denoted by h the continuous function :

$$h : \mathcal{U} \subset R^n \to R^k$$

given by $h(x) = (h_1(x), \dots, h_k(x))$ and satisfying $h(0) = 0$.
Define a semi definite Lyapunov function V by :

$$V(x) = \sum_{i=1}^{i=k} h_i^2(x)$$

One has

$$\Delta V(x) = \sum_{i=1}^{i=k} h_i^2(f(x)) - h_i^2(x) = 0$$

so V is constant along the motion. Moreover V is nonnegative continuous function, and here we have

$$G_0 = \{x \in \mathcal{U} \quad : \quad V(x) = 0\} = \{x \in \mathcal{U} \quad : \quad h(x) = 0\}$$

As a consequence of Theorem 1, we get the following :

Proposition 1 *Under the assumption (H). If the origin of the system (1) is G_0 asymptotically stable ,then it is a stable equilibrium of the system (1).*

Remark 2 : Stability of cascade systems
Consider nonlinear systems of the following form

$$\left\{ \begin{array}{l} x(k+1) = f(x(k), y(k)) \\ y(k+1) = g(y(k)) \\ (x(k), y(k)) \in \mathbb{R}^n \times \mathbb{R}^m \end{array} \right. \tag{14}$$

Where f, g are continuous and such that $f(0,0) = 0$, $g(0) = 0$.
Consider then the systems :

$$\left\{ \begin{array}{l} x(k+1) = f(x(k), 0) \\ x(k) \in R^n \end{array} \right. \tag{15}$$

$$\left\{ \begin{array}{l} y(k+1) = g(y(k)) \\ y(k) \in \mathbb{R}^m \end{array} \right. \tag{16}$$

Using Theorem (2) we can state the following

Proposition 2 *The system (14) is asymptotically stable if and only if the subsystems (15) and (16) are asymptotically stable .*

Indeed, there exists a Lyapunov function $V(y)$ satisfying

$$V(y) > 0 \quad \forall y \neq 0 \ , \quad V(0) = 0 \quad \text{and} \quad V(g(y)) - V(y) < 0 \quad \forall y \neq 0$$

For the whole system (14), this function is a semi-definite nonnegative Lyapunov function. Moreover, here $G_0 = \{(x, y) \ : V(x, y) = 0\} = \{(x, 0), x \in \mathbb{R}^n\} = G = G^*$ and the origin is assumed to be G^*-asymptotically stable. So all the hypothesis of Theorem (2) are satisfied which implies that the origin is an asymptotically stable equilibrium point of the system (14).

Example 1: (LaSalle) Consider the following system

$$\left\{ \begin{array}{l} x(k+1) = \dfrac{ay(k)}{1 + x(k)^2} \\[2ex] y(k+1) = \dfrac{bx(k)}{1 + y(k)^2} \\[2ex] (x(k), y(k)) \in \mathbb{R}^2 \end{array} \right. \tag{17}$$

LaSalle in [10] has studied this example by using the Lyapunov function below :

$$V(x,y) = \frac{1}{2}(x^2+y^2)$$

He got the following :
The origin is asymptotically stable if

$$\begin{cases} b^2 < 1 \ \text{ and } \ a^2 < 1 \\ or \\ (b^2 \le 1 \ \text{ and } \ a^2 \le 1) \ \ and \ \ a^2+b^2 < 2 \end{cases}$$

Now by theorem 2, with $V(x,y) = (xy)^2$ we have the following, which include the discussion above :
The origin is asymptotically stable if

$$\begin{cases} |ab| < 1 \\ or \\ |ab| = 1 \ \ and \ \ (\ a < 1 \quad or \quad b < 1 \ ) \end{cases}$$

Remark 3 : Theorem (2) states that the asymptotic stability of the origin is equivalent to its G^*-asymptotic stability. This equivalence is no more true for the Lyapunov stability as it can be shown thanks to the following example :

Example 2 :

$$\begin{cases} x(k+1) = x(k) + y(k) \\ y(k+1) = y(k) \\ (x,y) \in \mathbb{R}^2 \end{cases} \tag{18}$$

Let $V(x,y) = y^2$ we have
$G_0 = \{(x,y) \in \mathbb{R}^2 \ : \ y = 0\}$.
The origin is G_0-stable but the solution of the system is : $f^n(x,y) = (x+ny, y)$, which tends to infinity for all initial data (x,y) satisfying $y > 0$. Thus the system is unstable. This shows that G_0-stability of the origin is not sufficient to get the stability of the origin with respect to arbitrary perturbations.

Example 3 :

$$\begin{cases} x(k+1) = \dfrac{x(k)}{2} + \dfrac{3}{2}(y(k)x(k)^2) \\ y(k+1) = \dfrac{y(k)}{2} \\ (x(k), y(k)) \in \mathbb{R}^2 \end{cases} \tag{19}$$

If we take $V(x,y) = y^2$ then $\Delta V(x,y) = -3/4y^2 \le 0$

and $G_0 = \{(x,y) \in \mathbb{R}^2 \ : \ V(x,y) = 0\} = G$ and so $G_0 = G^*$ The origin is G_0 globally asymptotically stable but the system is not globally asymptotically stable. Indeed one can see that the set :

$$\{(x,y) \in R^2 : xy = 1\}$$

is invariant, and so global asymptotic stability can not be expected.

Example 4: It is known that if f is a $\mathcal{C}^1$ function and the linearization of system (1) at zero, namely

$$A = \frac{\partial f}{\partial x}(0) \tag{20}$$

is stable (that is, the eigenvalues λ_i of the matrix A lie in the open unit disk: $|\lambda_i| < 1$), then system (1) is locally asymtoticaly stable and when the linearization (20) has at least one eigenvalue λ outside the closed unit disk (i.e, $|\lambda| > 1$) then system (1) is unstable. But when the linearised system is critical that is the matrix A has all its eigenvalues inside the closed unit disk with at least one eigenvalue λ which satisfies $|\lambda| = 1$ then one can not conclude about the stability of system (1) that is the zero solution of (1) may be stable or unstable. So the results of this article can be helpful to study the stability properties of systems whose linearisation is critical. For instance consider the following example

$$\left\{ \begin{array}{l} x(k+1) = y(k) \\ y(k+1) = \dfrac{y(k)}{1+\beta\, x(k)^2} \\ (x(k), y(k)) \in \mathbb{R}^2 \quad \beta > 0 \end{array} \right. \tag{21}$$

The linearised system around the equilibrium $(0,0)$ is

$$\left\{ \begin{array}{l} x(k+1) = y(k) \\ y(k+1) = y(k). \end{array} \right. \tag{22}$$

Here $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. The linearised system is critical (here it is stable but not asymptotically stable) so the linearisation techniques do not allow to conclude about the stability of system (21).

Let $V(x,y) = y^2$. We have $\Delta V = y^2 \left[\left(\frac{1}{1+\beta\, x(k)^2} \right) - 1 \right] \leq 0$ so V is a nonnegative Lyapunov function for system (21). Moreover we have $G^* = G_0$ and the origin is G^*–asymptotically stable so by Theorem 2 the zero solution of (21) is asymptotically stable.

5 Applications to control problems

5.1 Stabilization

Consider a discrete-time control nonlinear system of the form

$$x(k+1) = f(x(k), u(k)) \tag{23}$$

where $x(k) \in \mathbb{R}^n$ is the state of the system at time k, $u \in \mathbb{R}^m$ is the control, and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a $\mathcal{C}^2$ function satisfying $f(0,0) = 0$. The problem addressed here is how to find a feedback control which stabilizes the system at its equilibrium point ? To be more precise we recall the following definition :

Definition 5 *System (23) is said to be stabilizable if there exists a continuous mapping $u : \mathbb{R}^n \to \mathbb{R}^m$ satisfying $u(0) = 0$ and such that the closed loop system $x(k+1) = f(x(k), u(x(k)))$ is asymptotically stable at the origin.*

Thanks to Theorem 1 and Theorem 2 we shall develop a machinery to constructe a stabilizing feedback for systems of the form (23) and that are Lyapunov stable (but not asymptotically stable) when the control is identically nul. Before stating our stabilization result we need to introduce the following notations :
Let $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^n$ be the $\mathcal{C}^2$ function defined on $\mathbb{R}^n$ by

$$\tilde{f}(x) = f(x, 0) \tag{24}$$

and assume that:

(d1) The unforced dynamic system

$$x(k+1) = \tilde{f}\big(x(k)\big)$$

is Lyapunov-stable and that a $\mathcal{C}^2$ semi definite positive function

$$V(x) \geq 0, \quad x \neq 0, \qquad V(0) = 0$$

is known such that

$$V\big(\tilde{f}(x)\big) \leq V(x), \quad \forall\, x \neq 0$$

(d2) The sets

$$\begin{aligned} W_1 &= \left\{ x \in \mathbb{R}^n \;\middle|\; V\big(\tilde{f}^{k+1}(x)\big) - V\big(\tilde{f}^{k}(x)\big) = 0, \forall k \in \mathbb{N} \right\} \\ W_2 &= \left\{ x \in \mathbb{R}^n \;\middle|\; \frac{\partial V}{\partial x}\big(\tilde{f}^{k+1}(x)\big)\frac{\partial f}{\partial u}\big(\tilde{f}^{k}(x), 0\big) = 0, \forall k \in \mathbb{N} \right\} \end{aligned}$$

where $\tilde{f}^k : \mathbb{R}^n \to \mathbb{R}^n$, $k \geq 0$, are given recursively by

$$\begin{aligned} \tilde{f}^0(x) &= x \\ \tilde{f}^k(x) &= \tilde{f}\big(\tilde{f}^{k-1}(x)\big), \quad \text{for } k \geq 1 \end{aligned}$$

satisfy $W_1 \cap W_2 = G_0$.
(d3) the origin is G_0–asymptotically stable.

Let $\tilde{V} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $\varphi : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ be defined respectively by

$$\tilde{V}(x,u) = V(f(x,u)) \tag{25}$$

$$\varphi(x,u,v) = \int_0^1 (1-t) v^T(x) \frac{\partial^2 \tilde{V}}{\partial u^2}(x, tu(x))\, v(x)\, dt \tag{26}$$

For a fixed $\eta > 0$, let $K_1(x)$ and $K_2(x)$ be any nonnegative continuous real valued functions satisfying $K_1(x) + K_2(x) \neq 0, \quad \forall x \in \mathbb{R}^n$ and

$$K_1(x) \geq \sup_{\|u\| \leq \eta, \|v\|=1} |\varphi(x,u,v)|, \quad \forall x \in \mathbb{R}^n \tag{27}$$

$$K_2(x) \geq \left\| \frac{\partial V}{\partial x}(\tilde{f}(x)) \frac{\partial f}{\partial u}(x,0) \right\|, \quad \forall x \in \mathbb{R}^n \tag{28}$$

and set

$$K(x) = \frac{\eta}{\eta K_1(x) + K_2(x)} \tag{29}$$

Now we can state our stabilization result :

Theorem 4 *If the assumptions* **(d1)** , **(d2)** *and* **(d3)** *hold, then, for any positive constant η, system (23) is asymptotically stabilizable by means of the continuous feedback law*

$$u(x) = -K(x) \left(\frac{\partial V}{\partial x}(\tilde{f}(x)) \frac{\partial f}{\partial u}(x,0) \right)^T \tag{30}$$

which satisfies

$$\|u(x)\| \leq \eta, \quad \forall\, x \in \mathbb{R}^n$$

Proof If one computes the difference of the Lyapunov function V along the trajectories of the closed-loop system (23-30), one gets from (25) and the Taylor expansion formula:

$$\begin{aligned}
\Delta V(x) &= V\big(f(x,u(x))\big) - V(x) \\
&= \tilde{V}\big(x,u(x)\big) - V(x) \\
&= \tilde{V}(x,0) - V(x) + \frac{\partial \tilde{V}}{\partial u}(x,0)\, u(x) + \int_0^1 (1-t) u^T(x) \frac{\partial^2 \tilde{V}}{\partial u^2}\big(x, tu(x)\big)\, u(x)\, dt
\end{aligned}$$

Notice that

$$\tilde{V}(x,0) = V\big(\tilde{f}(x)\big) \tag{31}$$

and

$$\frac{\partial \tilde{V}}{\partial u}(x,0) = \frac{\partial V}{\partial x}\big(\tilde{f}(x)\big)\frac{\partial f}{\partial u}(x,0)$$

so that, from (26) and (30),

$$\Delta V(x) = V\big(\tilde{f}(x)\big) - V(x) - \frac{1}{K(x)}u^T(x)u(x) + \varphi\big(x, u(x), u(x)\big)$$

It follows that, for $x \in \mathbb{R}^n$ such that $u(x) = 0$ one has

$$\Delta V(x) = V\big(\tilde{f}(x)\big) - V(x)$$

and otherwise, $\varphi(x,u,v)$ being homogeneous of degree 2 with respect to v, one gets:

$$\Delta V(x) = V\big(\tilde{f}(x)\big) - V(x) - \frac{1}{K(x)}\|u(x)\|^2 + \|u(x)\|^2\,\varphi\left(x, u(x), \frac{u(x)}{\|u(x)\|}\right)$$

From (27-28) and (29), one has for any $x \in \mathbb{R}^n$, $\|u(x)\| \le \eta$, and so one can deduce that $\Delta V(x) \le 0$. On the other hand, by assumption **(d3)** and according to Theorem 1, the origin is Lyapunov stable.

It remains to prove the asymptotic stability of the origin. To this end, by Theorem 2 it is sufficient to show that the origin is G^*–asymptotically stable. We recall that G^* is the largest invariant set contained in the locus

$$G = \{x \in \mathbb{R}^n \mid \Delta V(x) = V\big(f(x,u(x))\big) - V(x) = 0\}.$$

We have

$$\begin{aligned}\Delta V(x) = 0 \quad &\Leftrightarrow \quad V\big(\tilde{f}(x)\big) - V(x) = 0 \text{ and } u(x) = 0 \\ &\Leftrightarrow \quad V\big(\tilde{f}(x)\big) - V(x) = 0 \text{ and } \frac{\partial V}{\partial x}\big(\tilde{f}(x)\big)\frac{\partial f}{\partial u}(x,0) = 0 \qquad (32)\end{aligned}$$

Let $x(k)$ be a solution of the closed-loop system with $x(0) = x \in G^*$. Since G^* is invariant for the closed-loop system one has $x(k) \in G^*$ for all $k \ge 0$. But, $u(x)$ vanishing on G^*, one has, from (24), $x(k) = \tilde{f}^k(x)$ and so, by (32),

$$V(\tilde{f}^{k+1}(x)) - V(\tilde{f}^k(x)) = 0 \quad \forall k \in \mathbb{N}$$

and

$$\frac{\partial V}{\partial x}(\tilde{f}^{k+1}(x))\frac{\partial f}{\partial u}(\tilde{f}^k(x),0) = 0 \quad \forall k \in \mathbb{N}$$

It follows that $x \in W_1 \cap W_2$ which implies, by assumption **(d2)**, that $G^* \subset G_0$. So by **(d3)** the origin is G^*–asymptotically stable , which ends the proof by using Theorem 2. ■

Example 5. Consider the nonlinear control system evolving on $\mathbb{R}^3$

$$\begin{cases} x_1(k+1) = \dfrac{x_1(k)}{2} + x_2(k) + x_2^2(k) + x_3^2(k) \\ x_2(k+1) = (x_1(k) + 2x_3^2(k))u_1(k) \\ x_3(k+1) = x_3(k) + \dfrac{x_1^4(k)}{1+x_2^2(k)}u_2(k) \end{cases} \tag{33}$$

The unforced dynamic system

$$\begin{cases} x_1(k+1) = \dfrac{x_1(k)}{2} + x_2(k) + x_2^2(k) + x_3^2(k) \\ x_2(k+1) = 0 \\ x_3(k+1) = x_3(k) \end{cases}$$

is Lyapunov-stable but not asymptotically stable so by using Theorem 4 we shall compute a stabilizing feedback law $u(x)$ (here $x = (x_1, x_2, x_3)$). Let V be the nonnegative function $V(x) = x_2^2(k) + x_3^2(k)$. Here $G_0 = \{x_2 = x_3 = 0\}$.
We have $\Delta V(x) = -x_2^2 \leq 0$ so V is a Lyapunov function for the unforced system. On the one hand it is obvious that the origin is G_0–asymptotically stable. On the other hand one may easily check that $W_1 \cap W_2 = \{x_2 = x_3 = 0\} = G_0$. So, one can apply Theorem 4 to system (33) and the procedure for the computation of a stabilizer feedback is the following : With the same notations as above simples computations give :

$$\varphi(x,u,v) = \big((x_1 + 2x_3^2)v_1\big)^2 + \big((\frac{x_1^4}{1+x_2^2})v_2\big)^2$$

so one can take $K_1(x) = (x_1 + 2x_3^2)^2 + (\frac{x_1^4}{1+x_2^2})^2$.

$$\frac{\partial V}{\partial x}\big(\tilde{f}(x)\big)\frac{\partial f}{\partial u}(x,0) = \big(0\ ,\frac{2\,x_1^4\,x_3}{1+x_2^2}\big)$$

so one can take $K_2(x) = 1 + x_1^8\,x_3^2$. Thus we get the following bounded stabilizer :

$$u(x) = \big(0\ , -\frac{2\,\eta\,x_1^4\,x_3\,(1+x_2^2)}{\eta\,\big(x_1^8 + (1+x_2^2)^2(x_1+2x_3^2)^2\big) + (1+x_2^2)^2(1+x_1^8\,x_3^2)}\big)^T$$

where η is an arbitrary positive real constant.

5.2 Detectability and stability

Consider a discrete-time nonlinear system

$$\begin{cases} x(k+1) = f(x(k)) \\ y_k = h(x_k). \end{cases} \tag{34}$$

where $x(k) \in \mathbb{R}^n$ is the state of the system at time k and $y(k)$ is the measurable output of the system.
$f : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^m$ are $\mathcal{C}^0$ functions satisfying $f(0) = 0, h(0) = 0$. For the sequel we need to introduce some definitions :

Definition 6 *System (34)is said to be Zero-state observable if*

$$h(f^n(p)) \equiv 0$$

for all $n \in \mathbb{N}$ then

$$f^n(p) \equiv 0.$$

Definition 7 *System(34) is said to be Zero-state detectable if*

$$h(f^n(p)) = 0$$

for all $n \in \mathbb{N}$ then

$$\lim_{n \to +\infty} f^n(p) = 0$$

Proposition 3 *Assume That the system (34) is Zero-state detectable and there existe a positive real number q such that the series $\sum_{k=0}^{\infty} |h(f^k(x))|^q$ is uniformly convergent for any $x \in \mathcal{U}$ then the zero solution of (34)is asymptotically stable if and only it is Lyapunov stable .*

Proof Under the hypothesis that system (34) is Zero-state detectable and the $\sum_{k=0}^{\infty} |h(x_k)|^q$ is uniformly convergente, we shall prove that the Lyapunov stability of the origin implies its asymptotic stability. Indeed consider the following candidate Lyapunov function $V(x) = \sum_{k=0}^{\infty} |h(f^k(x))|^q$. V is a continuous nonnegative function (but not definite).
Its variation along the solutions of the system (34) is :

$$\Delta V(x) = V(f(x)) - V(x) = -|h(x)|^q \leq 0$$

Here we have :

$$G = \{ \ x \in \mathcal{U} \ \mid \ h(x) = 0 \ \}.$$

Thanks to theorem (2) to proof the asymptotic stability of the origin, it is sufficient to show that it is G^*-asymptotically stable. Now, if x belong to G^* then $h(x) = 0$ and since G^* is invariant and contained in G one has $h(f^k(x)) = 0$ for all $k \in N$ so :

$$G^* = \{ \ x \in \mathcal{U} \ \mid \ h(f^k(x)) = 0, \ \ \forall k \in N \ \}.$$

Now by the Zero-state detectability assumption, we have $\lim_{n \to +\infty} f^n(x) = 0$, so we have shown the G^*-attractivity of the origin and thus it is G^*-asymptotically stable, since it is assumed to be Lyapunov stable.

Remark One of the interesting work for studying the relationship between stability and observability for the linear continuous time system is the result of Morse [11]. He has proved that for an observable continuous linear system the origin is asymptotically stable if and only if the output function is square integrable for all initial condition. Following the straight line of this work, recently in [2] the authors extend and generalise the studies to nonlinear continous time system. An analogue result of the proposition (3) for the continuous time system was given under a stronger assumption than ours : Instead of the Zero-state detectability they suppose that the system is Zero-state observable.

6 Conclusion

The qualitative theory centers around the basic concept of Lyapunov theory and LaSalle invariance principle. In this work the obtained result extend the interest of classical theory, in the sens that we don't require the definiteness of the Lyapunov functions used. For instance the existence of an energy function V can be used in the absence of definiteness on such function.

This is a new tool for the analysis of stability of discrete-time nonlinear systems that has not yet been exploited. An extention to non autonomous difference equations will be considered in a forthcoming paper.

We have applied this tool to the problem of nonlinear stabilization by improving the classical Jurdjevic-Quin theorem [8], and remarks 1,2 illustrate how this new approach may have significant applications.

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