Displacement Derivatives in Shape Optimization of Thin Shells

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Abstract: In the present paper the framework for the shape sensitivity analysis of systems of equations defined on a surface in $\mathbb{R}^3$ is established. The model of thin shell presented in (Koiter, 1970) is considered. The formulation of the model in a reference domain has been chosen for our analysis. The shape gradients and shape Hessians of associated shape functionals are defined and evaluated using the so–called displacement derivatives.

Key-words: shape optimization, shape gradient, shape Hessian, thin shell, displacement derivative, surface shape functional

(Résumé : tsvp)

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Dérivées de déplacement en optimisation de forme de coques minces

Résumé : Dans cet article on établit le cadre de l'analyse de sensibilité par rapport à la forme de systèmes d'équations définis sur une surface de $\mathbb{R}^3$. On considère le modèle de coque mince présenté par (Koiter, 1970). La formulation du modèle dans un domaine de référence a été choisie pour notre analyse. Les gradients de forme et les Hessiens de forme associés aux fonctionnelles de forme ont été définis et évalués en utilisant la dérivée de déplacement.

Mots-clé : optimisation de forme, gradient de forme, Hessien de forme, coque mince, dérivée de déplacement, fonctionnelle de forme de surface

1 Introduction

Shape optimization is quite indispensable when constructing industrial structures. For example plane and space shuttles have to satisfy in the same time very hard criteria on good mechanical behaviour and have to weight as less as possible.

For the last thirty years, there were many studies developed in this field of structural optimization. They combine the most recent results on mechanical formulation of the problems, on functional analysis of such problems and on control theory. For such results we can refer to e.g. (Duvaut, Lions, 1972) and to (Lions, 1968).

A structure is generally an assemblage between different parts like beams, plates, shells and three dimensional medium. Here we will restrict our attention to the general continuous formulation of such optimization problems for general thin shallow shells.

The geometry of a general thin shell can be characterized by two different mappings:

1. the mapping φ which defines the middle surface S of the shell as the image of the closure of a bounded domain $\mathcal{O}$ of the plane;

2. the mapping e which defines the thickness of the shell at any point of the middle surface along the normal of this surface.

The shape optimization problem for such a shell consists in finding the geometry of the shell (middle surface and thickness) which minimizes a given functional (for example, the weight of the shell) and satisfies some constraints (for example, bounds on the thickness, on the strain energy, on the displacements).

We consider the following shape functional as an example,

$$J(S) = \int_S \mathfrak{F}(x, \vec{u}(S)(x), \tilde{e}(S)(x))d\Gamma(x) \tag{1.1}$$

where $\vec{u}(S)(x), x \in S$, is the displacement field of the shell.

We denote by $\mathfrak{J}(\varphi, e; \vec{u}) = J(S)$ the integral functional defined on the reference domain $\mathcal{O}$,

$$\mathfrak{J}(\varphi, e; \vec{u}) = \int_{\mathcal{O}} \mathfrak{F}(\varphi(\xi), \mathcal{L} \cdot \vec{u}(\xi), e(\xi))dS(\xi) \tag{1.2}$$

with

$$\mathcal{L} \cdot \vec{u}(\xi) \equiv \vec{u}(S)(\varphi(\xi)),\ x = \varphi(\xi) \in S, \tag{1.3}$$

$$e(\xi) = \tilde{e}(S)(\varphi(\xi)) \quad \xi \in \mathcal{O}, \tag{1.4}$$

where $\vec{u}(\xi) = \mathrm{col}(u_1(\xi), u_2(\xi), u_3(\xi))$ denotes covariant components of the displacement field $\vec{u}(S) \circ \varphi$ of the middle surface $\overline{S}$, and linear mapping $\mathcal{L}$ is defined in terms of contravariant basis on S, i.e. $\mathcal{L} \cdot \vec{u} = u_i \vec{a}^i$.

Computational algorithms usually require that we can compute the derivatives of shape functionals with respect to the geometry, i.e., with respect to $S = \varphi(\mathcal{O})$ and e. This is a difficult problem since in general shape functionals depend on S and e not only explicitely but also implicitly through the dependence of the displacement field $\vec{u}(S)$ on S and e. To circumvent this difficulty we make use of the classical adjoint state method.

We use the shell model presented in (Koiter, 1970). A detailed discussion related to numerical analysis of such problems can be found in (Bernadou, Ciarlet, 1976), (Bernadou, Boisserie, 1982) or (Bernadou, 1995).

2 Shell model

The following hypothesis are assumed to be satisfied.

1. the shell is clamped on its boundary ∂S;
2. the shell is loaded by a distribution of forces whose resultant has density $\vec{p}$ on S,
3. the shell is elastic, homogenous and isotropic.

   Moreover, according to (Koiter, 1970), we assume that
4. the normals to the middle surface remain normals to the deformed middle surface;
5. the stresses are approximatively plane and parallel to the tangent plane to the middle surface.

Under the assumptions (1) to (5) above, the problem takes the following variational form.

$$\text{Find } \vec{u} = (u_1, u_2, u_3) \in \vec{\mathcal{H}} = (H_0^1(\mathcal{O}))^2 \times H_0^2(\mathcal{O}) \text{ such that} \tag{2.1}$$

$$a(\vec{u}, \vec{v}) = f(\vec{v}) \qquad \forall \vec{v} \in \mathcal{H} \ ,$$

where

$$a(\vec{u}, \vec{v}) = \int_{\mathcal{O}} e E^{\alpha\beta\lambda\iota} \{\gamma_{\alpha\beta}(\vec{u})\gamma_{\lambda\iota}(\vec{v}) + \frac{e^2}{12}\rho_{\alpha\beta}(\vec{u})\rho_{\lambda\iota}(\vec{v})\} dS, \tag{2.2}$$

$$f(\vec{v}) = \int_{\mathcal{O}} \vec{p} \cdot \vec{v} dS, \tag{2.3}$$

$$E^{\alpha\beta\lambda\iota} = \frac{E}{2(1+\mu)}[a^{\alpha\lambda}a^{\beta\iota} + a^{\alpha\iota}a^{\beta\lambda} + \frac{2\mu}{1-\mu}a^{\alpha\beta}a^{\lambda\iota}], \tag{2.4}$$

$$dS = \sqrt{a} d\xi^1 d\xi^2 \ ,$$

$$E = \text{ Young modulus ;} \qquad \nu = \text{ Poisson's coefficient.}$$

$$\gamma_{\alpha\beta}(\vec{u}) = \frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta}u_3 \tag{2.5}$$

$$\overline{\rho}_{t\alpha\beta}(\vec{u}_t) = u_{t3|\alpha\beta} - b^{\lambda}_{t\alpha}b_{t\lambda\beta}u_{t3} + b^{\lambda}_{t\alpha|\beta}u_{t\lambda} + b^{\lambda}_{t\alpha}u_{t\lambda|\beta} + b^{\lambda}_{t\beta}u_{t\lambda|\alpha} \tag{2.6}$$

These expressions can be simplified by taking into account deformations of general thin shallow shells. This allows to keep $\gamma_{\alpha\beta}$ unchanged and to replace $\overline{\rho}_{\alpha\beta}(\vec{u})$ by

$$\rho_{\alpha\beta}(\vec{u}) = u_{3|\alpha\beta} \tag{2.7}$$

Theorem 2.1 *Problem (2.1) has a unique solution.*

The proof of Theorem 2.1 is given in (Bernadou, Ciarlet, 1976), we refer the reader also to (Bernadou, Ciarlet and Miara, 1994) for a more simple proof.

Theorem 2.2 *Problem (2.1) with $\overline{\rho}_{\alpha\beta}(\vec{u})$ replaced by $\rho_{\alpha\beta}(\vec{u})$ (see (2.7)) has a unique solution.*

The proof of Theorem 2.2 is given in (Bernadou, Lalane, 1986) or in (Bernadou, 1994 or 1995).

3 Displacement derivatives

Basic assumption we make in this section is that the shape functional $J(S) = \mathfrak{J}(\varphi, e; \vec{u})$ under consideration depends only on $S = \varphi(\mathcal{O})$ and $\tilde{e}$ and is independent of the parametrization φ for a fixed surface S and the given reference domain $\mathcal{O}$.

Under this assumption, by an application of the Hadamard formula (see section 4 for details), we can obtain the form of shape gradient of the shape functional $J(S) = \mathfrak{J}(\varphi, e; \vec{u})$. In particular, we can use displacement derivatives of solutions to shell equations to evaluate the Eulerian derivative $dJ(S;V)$ in the direction V.

Notation

Let $\mathcal{O}$ be a bounded domain of a plane $\mathbb{R}^2$ with boundary $\Gamma = \partial\mathcal{O}$. We assume that the middle surface $\overline{S}$ of the shell is the image of the set $\overline{\mathcal{O}}$ under a regular mapping φ, i.e. $\overline{S} = \varphi(\overline{\mathcal{O}})$,

$$\varphi : \ (\xi^1, \xi^2) \in \overline{\mathcal{O}} \subset \mathbb{R}^2 \mapsto \varphi(\xi^1, \xi^2) \in \overline{S} \subset \mathbb{R}^3 \tag{3.1}$$

We define two local bases in $\overline{S}$:

1. *the covariant basis* $(\vec{a}_i, \ i = 1, 2, 3)$

$$\vec{a}_\alpha = \varphi_{,\alpha} = \frac{\partial\varphi}{\partial\xi^\alpha}, \qquad \vec{a}_3 = \frac{\vec{a}_1 \times \vec{a}_2}{||\vec{a}_1 \times \vec{a}_2||_{\mathbb{R}^3}} \ ; \tag{3.2}$$

2. *the contravariant basis* $(\vec{a}^i, \ i = 1, 2, 3)$

$$\vec{a}^\beta \cdot \vec{a}_\alpha = \delta_{\alpha\beta} \qquad ; \qquad \vec{a}^3 = \vec{a}_3 \ . \tag{3.3}$$

From now on small greek indices take values 1 and 2 while small latin indices take values 1,2 and 3. We use Einstein's summation convention for repeated indices at higher and lower positions.

To the covariant and contravariant bases we assign the first $(a_{\alpha\beta}, a^{\alpha\beta})$ and the second $(b_{\alpha\beta}, b_\alpha^\beta, b^{\alpha\beta})$ fundamental forms of the middle surface (note that $b_\alpha^{\cdot\beta} = b^\beta_{\cdot\alpha} = b_\alpha^\beta$) , respectively,

$$a_{\alpha\beta} = \vec{a}_\alpha \cdot \vec{a}_\beta \ ; \quad a^{\alpha\beta} = \vec{a}^\alpha \cdot \vec{a}^\beta \ ; \quad a = \det(a_{\alpha\beta}) \tag{3.4}$$

$$b_{\alpha\beta} = \vec{a}_3 \cdot \vec{a}_{\alpha,\beta} \ ; \quad b_\alpha^\beta = a^{\beta\lambda} b_{\lambda\alpha} \ ; \qquad b^{\alpha\beta} = a^{\alpha\lambda} b_\lambda^\beta .$$

It is also convenient to introduce the covariant derivatives

$$u_{\alpha|\beta} = u_{\alpha,\beta} - \Gamma^\lambda_{\alpha\beta} u_\lambda \ ; \qquad u_{3|\lambda} = u_{3,\alpha} \ ; \qquad u_{3|\alpha\beta} = u_{3,\alpha\beta} - \Gamma^\lambda_{\alpha\beta} u_{3,\lambda} \ , \tag{3.5}$$

where the Christoffel symbols are given as

$$\Gamma^\lambda_{\alpha\beta} = \vec{a}^\lambda \cdot \vec{a}_{\alpha,\beta} = \Gamma^\lambda_{\beta\alpha} \ . \tag{3.6}$$

Subsequently, the same notation will be used for the surface S_t.

The thickness of the shell can be defined as a regular mapping

$$e : (\xi^1, \xi^2) \in \overline{\mathcal{O}} \subset \mathbb{R}^2 \mapsto e(\xi_1, \xi_2) \in \{ \ x \in \mathbb{R} : \ x > 0 \ \} \ . \tag{3.7}$$

Then the shell $\mathcal{S}$ is the set

$$\begin{aligned} \mathcal{S} \ = \ & \{M \in \mathbb{R}^3 \ : \ \overrightarrow{OM} = \varphi(\xi^1, \xi^2) + \xi^3 \vec{a}_3(\xi^1, \xi^2), \ (\xi^1, \xi^2) \in \overline{\mathcal{O}}, \\ & -\frac{1}{2} e(\xi^1, \xi^2) \leq \xi^3 \leq \frac{1}{2} e(\xi^1, \xi^2)\} \end{aligned}$$

The family of surfaces S_t

By t we denote a real parameter which belongs to $[0, \delta]$, $\delta > 0$. Let S_t be a family of surfaces defined as the images of the reference plane domain $\mathcal{O}$ under regular mappings φ_t :

$$\varphi_t : \ (\xi^1, \xi^2) \in \overline{\mathcal{O}} \subset \mathbb{R}^2 \mapsto \varphi_t(\xi^1, \xi^2) \in \overline{S_t} \subset \mathbb{R}^3 \tag{3.8}$$

For $t = 0$ we recover the original surface S, i.e.,

$$\varphi_0 = \varphi, \qquad \text{and} \qquad S_0 = S.$$

It follows that

$$\frac{\partial \varphi_t}{\partial t} = c_t^\alpha \vec{a}_{t\alpha} + w_t \vec{a}_{t3},$$

and for convected parametrizations, i.e. when $c_t^\alpha = 0$ we have

$$\frac{\partial \varphi_t}{\partial t} = w_t \vec{a}_{t3} \ , \tag{3.9}$$

where w_t is the normal speed of the surface S_t, and $\vec{a}_{t3}$ is the unit normal vector to S_t.

For any parametrization of the surface S_t we have

$$\frac{\partial \varphi_t}{\partial t} = c_t^\alpha \vec{a}_{t\alpha} + \frac{\delta \varphi_t}{\delta t},$$

where $\frac{\delta}{\delta t}$ denotes the displacement derivative defined below (Definition 3.1).

Definition 3.1 *Let $z_t \ : \mathcal{O} \mapsto \mathbb{R}$, $t \in [0, \delta]$ be a family of functions and let φ_t, $t \in [0, \delta)$, be a family of regular mappings, $\overline{S}_t = \varphi_t(\overline{\mathcal{O}})$. The displacement derivative $\frac{\delta z_t}{\delta t}$ for this family of functions is defined by*

$$\frac{\delta z_t}{\delta t} = \frac{\partial z_t}{\partial t} - z_{t,\alpha} g^{\alpha\beta} \varphi_{t,\beta} \cdot \frac{\partial \varphi_t}{\partial t} \ . \tag{3.10}$$

Remark 3.2 *Using the material derivative method a mapping $T_t = T_t(V) : \mathbb{R}^3 \mapsto \mathbb{R}^3$ is constructed where V is a given vector field. The mapping, in the case of thin shell, assigns the surface S_t to the reference surface S, i.e.,*

$$S_t = T_t(S)$$

For

$$\varphi_t = T_t \circ \varphi = T_t(V) \circ \varphi, \quad t \in [0, \delta)$$

we have

$$\frac{\partial \varphi_t}{\partial t} = \left[\frac{\partial T_t}{\partial t} \circ T_t^{-1}\right] \circ \varphi = V(t) \circ \varphi_t = V(t, \varphi_t) \tag{3.11}$$

therefore

$$\frac{\delta z_t}{\delta t} = \frac{\partial z_t}{\partial t} - z_{t,\alpha} g^{\alpha\beta} \varphi_{t,\beta} \cdot V(t, \varphi_t) \ . \tag{3.12}$$

It is easily seen that the displacement derivative of φ_t takes the form

$$\frac{\delta \varphi_t}{\delta t} = \langle V(t, \varphi_t), \vec{a}_{3t} \rangle_{\mathbb{R}^3} \vec{a}_{3t} = w_t \vec{a}_{3t} \ . \tag{3.13}$$

Now we determine the displacement derivative of the restriction of a function to S. Let $\psi : [0, \delta) \times D \mapsto \mathbb{R}$. Denoting $u_t = \psi(t, \varphi_t)$, we get

$$\begin{aligned} \frac{\partial u_t}{\partial t}(\xi) &= \frac{\partial \psi}{\partial t}(t, \varphi_t(\xi)) + \langle \nabla_x \psi(t, \varphi_t(\xi)), V(t, \varphi_t(\xi)) \rangle_{\mathbb{R}^3} \ , \\ \frac{\delta u_t}{\delta t}(\xi) &= \frac{\partial \psi}{\partial t}(t, \xi) + \langle \nabla_x \psi(t, \varphi_t(\xi)), \vec{a}_{t3} \rangle_{\mathbb{R}^3} w_t(\xi) \ . \end{aligned}$$

If $z'(S;V)$ is the boundary shape derivative of $z(S_t) = \psi_{|S_t}$ in the direction of a vector field $V(t) = \frac{\partial T_t}{\partial t} \circ T_t^{-1}$, it follows that the following relation is obtained

$$\frac{\delta u_t}{\delta t}_{|t=0}(\xi) = z'(S;V)(\varphi(\xi)) \ .$$

For $t > 0$ we use the following notation,

$$\frac{\delta u_t}{\delta t}(\xi) = z'(S_t; V(t))(\varphi_t(\xi)) \ .$$

It is clear that if $\frac{\partial \psi}{\partial t} = \frac{\partial \psi}{\partial n} = 0$ on S and $u_t = \psi(t, \varphi_t)$, then $\frac{\delta u_t}{\delta t} = 0$ for $t = 0$.

Remark 3.3 *Given mappings* φ_t , φ*, we can define the mapping*

$$T_t : S \mapsto S_t$$

of the form $T_t = \varphi_t \circ \varphi^{-1}$*, and the vector field*

$$V(t, x) = \left[\frac{\partial T_t}{\partial t} \circ T_t^{-1}\right](x) = \left[\frac{\partial \varphi_t}{\partial t} \circ \varphi_t^{-1}\right](x) \quad x \in S_t \ .$$

The normal and tangent components of the field V on S are given by

$$\begin{aligned} &w(t,x) = \langle V(t,x), n(t,x)\rangle_{\mathbb{R}^3} \quad x \in S_t \ , \\ &V_\tau(t,x) = V(t,x) - w(t,x)n(t,x), \\ &w_t(\xi) = w(t, \varphi_t(\xi)), \\ &n(t,x) = \vec{a}_{t3}(\varphi_t^{-1}(x)) \quad x \in S_t \ . \end{aligned}$$

Therefore, without any loss of generality we assume that there is given a mapping φ and an admissible vector field V such that $\varphi_t = T_t(V) \circ \varphi$.

A tangent vector on $\partial S_t = \{x \in \mathbb{R}^3 | x = \varphi(\xi) \quad \xi \in \partial\mathcal{O}\}$ is denoted by $\vec{b}_t(\xi)$; $\vec{l}_t(\xi) = \Sigma_{i=1}^2 \alpha_t^i(\xi)\vec{a}_{it}(\xi)$ is the unit vector normal to ∂S_t such that $\langle \vec{l}_t(\xi), \vec{b}_t(\xi)\rangle_{\mathbb{R}^3} = 0$. For $t = 0$ we denote $\vec{b}(\xi) = \vec{b}_0(\xi), \vec{l}(\xi) = \vec{l}_0(\xi)$, respectively; $\vec{l}(t,x) = \vec{l}_t(\varphi_t^{-1}(\xi))$ for $x = \varphi_t(\xi) \in \partial S_t$, $\vec{b}(t,x)$ is defined in the same way for $x \in \partial S_t$.

4 Derivatives of shape functionals

We recall here the basic notions of the shape calculus which are used in the paper. For further results on the material derivative method we refer the reader to (Sokolowski and Zolesio, 1992). Shape optimization problems in solid mechanics are considered in (Khludnev and Sokolowski, to appear)

Suppose we are given an open set D in $\mathbb{R}^N$, a measurable subset Ω of D, an admissible vector field $V \in C(0,\varepsilon; C^k(\overline{D}; \mathbb{R}^N))$, $k \geq 1$, and an associated transformation $T_t(V)$ from $\overline{D}$ onto $\overline{D}$.

Let $J(\Omega)$ be a well defined functional for any measurable subset Ω of D. Assume that $\Omega_t = T_t(V)(\Omega)$, $t \in [0,\delta)$, is a family of deformations of Ω. The set Ω_t is a measurable subset of D for any $t \in [0,\delta)$.

Definition 4.1 *For an admissible vector field $V \in C(0,\varepsilon; C^k(\overline{D}; \mathbb{R}^N))$, the Eulerian derivative of the domain functional $J(\Omega)$ at Ω in the direction of V is the limit*

$$dJ(\Omega; V) = \lim_{t\downarrow 0}(J(\Omega_t) - J(\Omega))/t, \tag{4.1}$$

where

$$\Omega_t = T_t(V)(\Omega) \ .$$

Definition 4.2 *A functional $J(\Omega)$ is shape differentiable (or simply differentiable) at Ω if*

1. *there exists the Eulerian derivative $dJ(\Omega; V)$ for all directions V,*

2. *the mapping $V \mapsto dJ(\Omega; V)$ is a linear and continuous mapping from $C(0,\varepsilon; C^k(D; \mathbb{R}^N))$ into $\mathbb{R}$.*

Gradients of shape differentiable functionals can be characterized as follows.

Theorem 4.3 *Let $J(\Omega)$ be a shape differentiable functional at every domain Ω of class C^k, $\Omega \subset D$. Assume that $\Omega \subset D$ is a domain with the boundary of class C^{k-1}. There exists a scalar distribution*

$$g(\Gamma) \in \mathcal{D}^{-k}(\Gamma)$$

such that the gradient of the functional J at Ω, $G(\Omega) \in \mathcal{D}^{-k}(\Omega; \mathbb{R}^N)$, with $\mathrm{spt} G(\Omega) \in \Gamma$*, is given by*

$$G(\Omega) = {}^*\gamma_\Gamma(g \cdot n), \tag{4.2}$$

where $\gamma_\Gamma \in \mathcal{L}(\mathcal{D}(\overline{D}; \mathbb{R}^N), \mathcal{D}(\Gamma; \mathbb{R}^N))$ is the trace operator and ${}^*\gamma_\Gamma$ is the transpose of γ_Γ, n is a unit normal vector on Γ directed into the exterior of Ω.

From (4.2) it follows that

$$dJ(\Omega; V) = \langle g, V \cdot n \rangle_{\mathcal{D}^{-k}(\Gamma) \times \mathcal{D}^k(\Gamma)},$$

where $V \cdot n = \langle V(0,x), n(x) \rangle_{\mathbb{R}^N}$, $x \in \Gamma$.

In general, $g = g(\Omega) \in \mathcal{D}^{-k}(\Gamma)$. However, for some classes of shape functionals it can be assumed that $g(\Omega)$ is an integrable function on Γ and then

$$dJ(\Omega; V) = \int_\Gamma g(x) \langle V(0,x), n(x) \rangle_{\mathbb{R}^N} d\Gamma \tag{4.3}$$

and we denote

$$\mathfrak{D}J(\Omega; V(0)) = \int_\Gamma g(x) \langle V(0,x), n(x) \rangle_{\mathbb{R}^N} d\Gamma \ .$$

Let $\overline{S} = \varphi(\overline{\mathcal{O}})$, and let $J(S)$ be a given differentiable shape functional. For any vector field V such that $\mathrm{spt} V(0) \cap \partial S = \emptyset$, the Eulerian derivative $dJ(S; V) = \lim_{t \downarrow 0} (J(S_t) - J(S)))/t$ of the shape functional takes the following form.

Corollary 4.4 *There exists a distribution $\mathcal{G}_S \in \mathcal{D}^{-k}(S)$ such that*

$$dJ(S; V) = \langle \mathcal{G}_S, w \rangle_{\mathcal{D}^{-k}(S) \times \mathcal{D}^k(S)} \ , \tag{4.4}$$

where, w is the normal speed of the surface S_t at $t = 0$, i.e. $w(x) = w(\varphi(\xi)) = \langle V(0, \varphi(\xi)), \vec{a}_3(\xi) \rangle_{\mathbb{R}^3}$, for $x = \varphi(\xi) \in S$ and $\xi = \varphi^{-1}(x) \in \mathcal{O}$.

Remark 4.5 *If* $sptV(0) \cap \partial S \neq \emptyset$*, then in the above formula for* $dJ(S;V)$ *an additional term related to the boundary* ∂S *may appear. The term takes the form*

$$\langle \vec{\mathcal{G}}_{\partial S}, V_\ell(0) \rangle_{\mathcal{D}^{-k}(\partial S;\mathbb{R}^3)\times\mathcal{D}^k(\partial S;\mathbb{R}^3)} , \tag{4.5}$$

where $V_\ell(0)$ *is a component of tangent vector field* $\gamma_{\partial S} V_\tau(0)$ *orthogonal to the tangent vector* $\vec{b}$ *on* C^1 *curve* ∂S*, i.e.* $V_\ell(0,\varphi(\xi)) = \langle V(0,\varphi(\xi)), \vec{l}(\xi)\rangle_{\mathbb{R}^3} \vec{l}(\xi)$ *for* $\xi \in \partial\mathcal{O}$*.* $\gamma_{\partial S} V_\tau(0)$ *denotes the trace of* $V_\tau(0) = V - \langle V, n\rangle_{\mathbb{R}^3} n$ *on* ∂S*.*

We recall the formulae for the derivatives of integrals. Given a family of shape differentiable functions $y(\Omega_t)$, $y(\Gamma_t)$, $\Omega_t = T_t(\Omega)$, then

$$\frac{d}{dt}\left[\int_{\Omega_t} y(\Omega_t)dx\right]_{|t=0} = \int_\Omega y'(\Omega;V)dx + \int_{\partial\Omega} y(\Omega)\langle V(0), n\rangle_{\mathbb{R}^N} d\Gamma$$
$$\frac{d}{dt}\left[\int_{\Gamma_t} z(\Gamma_t)dx\right]_{|t=0} = \int_\Gamma z'(\Gamma;V) - 2\kappa z(\Gamma)\langle V(0), n\rangle_{\mathbb{R}^N} d\Gamma$$

where $\kappa = -\frac{1}{2}\mathrm{div}_\Gamma n$ is the mean curvature on $\partial\Omega$.

Let us consider a surface integral $J(S) = \int_S G d\Gamma$. For $S = \varphi(\mathcal{O}) \subset \mathbb{R}^3$ it follows that $J(S) = \int_{\mathcal{O}} G \circ \varphi dS$, $dS = a^{\frac{1}{2}} d\xi$, where $a = \det[a_{\alpha,\beta}]$.

Given vector field V and the transformation $T_t = T_t(V) : S \mapsto S_t$, we denote $\varphi_t = T_t \circ \varphi \ : \ \mathcal{O} \mapsto S_t$, $V(t,x) = \left[\frac{\partial T_t}{\partial t} \circ T_t^{-1}\right](x)$ for $x \in S_t$. Consider the shape functional defined on S_t,

$$\begin{aligned} J(S_t) &= \int_{S_t} F(t) d\Gamma = \int_{S_t} z(S_t) d\Gamma \\ &= j(t) = \int_{\mathcal{O}} F_t dS_t \end{aligned}$$

where $F : \mathbb{R} \times \mathbb{R}^3 \mapsto \mathbb{R}$ is a given sufficiently smooth function and the following notation is used,

$$\begin{aligned} & z(S_t) = F(t,\cdot)_{|S_t} , \\ & F_t(\xi) = F(t,\varphi_t(\xi)) \quad \xi \in \mathcal{O}, \\ & dS_t = \det([DT_t]\circ\varphi) \| \left[{}^*DT_t^{-1} \circ \varphi\right] \cdot \vec{a}_{t3} \|_{\mathbb{R}^3} dS \ . \end{aligned}$$

We are going to evaluate the second order shape derivative of the shape functional $J(S_t)$ in the direction of vector fields V, W. For the first order shape derivative we have the following representations

$$\begin{aligned} dJ(S_t;V(t)) &= \int_{S_t} z'(S_t;V(t)) - 2\kappa(t)z(S_t)\langle V(t), n(t)\rangle_{\mathbb{R}^3} d\Gamma_t \\ &= \int_{S_t} \frac{\partial F}{\partial t}(t) + \left(\frac{\partial F}{\partial n}(t) - 2\kappa(t)F(t)\right)\langle V(t), n(t)\rangle_{\mathbb{R}^3} d\Gamma_t \\ &= \frac{dj}{dt}(t) = \int_{\mathcal{O}} \left[\frac{\delta F_t}{\delta t} - 2\kappa_t F_t w_t\right] dS_t \end{aligned}$$

where we denote

$$\begin{aligned} &\kappa(t,x) \text{ is the mean curvature at } x \in S_t \ , \\ &n(t) = \vec{a}_{t3} \circ \varphi_t^{-1} \text{ the normal vector at } x = \varphi_t^{-1}(\xi) \in S_t \ , \xi \in \mathcal{O}, \\ &\frac{\partial F}{\partial n}(t) = \langle \nabla F(t), n(t)\rangle_{\mathbb{R}^3} \ , \\ &\kappa_t(\xi) = \kappa(t, \varphi_t(\xi)), \\ &w_t(\xi) = \left\langle \frac{\delta \varphi_t}{\delta t}(\xi), \vec{a}_{t3}(\xi)\right\rangle_{\mathbb{R}^3}, \ \xi \in \mathcal{O} \ . \end{aligned}$$

Remark 4.6 *In general, we have the following formula for derivative of a surface integral, we refer the reader to (Sokołowski, Zolesio, 1992) for the proof.*

Let Σ be a connected surface contained in Γ with C^2 boundary $\partial\Sigma$, $\vec{l}$ denote the unit normal vector to $\partial\Sigma$ that is perpendicular to the surface normal n to Γ and directed into the exterior of Σ. For $\Sigma_t = T_t(\Sigma) \subset \Gamma_t$ it follows that,

$$\begin{aligned} &\frac{d}{dt}\left[\int_{\Sigma_t} z(\Gamma_t) d\Gamma_t\right]_{|t=0} = \frac{d}{dt}\left[\int_{\Sigma} z(\Gamma_t) \circ T_t(V)\omega(t) d\Gamma\right]_{|t=0} \\ = &\int_\Sigma \dot{z}(\Gamma;V) + \dot{\omega}(\Gamma;V)z(\Gamma)d\Gamma \\ = &\int_\Sigma z'(\Gamma;V) + \nabla_\Gamma z(\Gamma)\cdot V_\tau(0) - 2\kappa z(\Gamma)V(0)\cdot n + z(\Gamma)\mathrm{div}_\Gamma V_\tau(0) d\Gamma \\ = &\int_\Sigma z'(\Gamma;V) - 2\kappa z(\Gamma)V(0)\cdot n d\Gamma + \int_\Sigma \mathrm{div}_\Gamma\left(z(\Gamma)V_\tau(0)\right) d\Gamma \\ = &\int_\Sigma z'(\Gamma;V) - 2\kappa z(\Gamma)V(0)\cdot n d\Gamma + \int_{\partial\Sigma} z(\Gamma)\langle V_\tau(0), \vec{l}\rangle_{\mathbb{R}^3} dl, \end{aligned}$$

where $\omega(t) = \gamma(t)\|{}^*DT_t^{-1}\cdot n(t)\|_{\mathbb{R}^3}$ and $\dot{\omega} = -2\kappa V(0)\cdot n + \mathrm{div}_\Gamma V_\tau(0)$ in the notation of (Sokołowski, Zolesio, 1992).

To differentiate the shape functional

$$I(S) = \mathfrak{D}J(S; V(0))$$

in the direction of a vector field W we need the following notation, $r \in (-\epsilon, \epsilon)$ is a parameter,

$$\begin{aligned}
&S_r = T_r(S), \text{ where } T_r = T_r(W) : S \mapsto S_r \ , \\
&V(r) = V(r, \varphi_r(\xi)), \\
&\varphi_r = T_r(W) \circ \varphi, \\
&w_r(\xi) = \left\langle \frac{\delta \varphi_r}{\delta r}(\xi), \vec{a}_{r3}(\xi) \right\rangle_{\mathbb{R}^3} \ , \\
&F_r(\xi) = F(r, \varphi_r(\xi)), \\
&\kappa_r(\xi) = \kappa(r, \varphi_r(\xi)), \\
&\frac{\delta F_r}{\delta t}(\xi) = \frac{\partial F}{\partial t}(r, \varphi_r(\xi)) + \langle \nabla_x F(r, \varphi_r(\xi)), \vec{a}_{r3}(\xi) \rangle_{\mathbb{R}^3}, \ \xi \in \mathcal{O} \ .
\end{aligned}$$

With the notation we have

$$\begin{aligned}
I(S_r) &= dJ(S_r; V(r)) \\
&= \int_{S_r} z'(S_r; V(r)) - 2\kappa(r) z(S_r) \langle V(r), n(r) \rangle_{\mathbb{R}^3} d\Gamma \\
&= \int_{S_r} \frac{\partial F}{\partial t}(r) + \left(\frac{\partial F}{\partial n}(r) - 2\kappa(r) F(r) \right) \langle V(r), n(r) \rangle_{\mathbb{R}^3} d\Gamma \\
&= \frac{dj}{dt}(r) = \int_{\mathcal{O}} \left[\frac{\delta F_r}{\delta t} - 2\kappa_r F_r w_r \right] dS_r
\end{aligned}$$

For $u(S_r) = z'(S_r; V(r)) - 2\kappa(r) z(S_r) \langle V(r), n(r) \rangle_{\mathbb{R}^3}$ and $u_r = u(S_r) \circ \varphi_r = \frac{\delta F_r}{\delta t} - 2\kappa_r F_r w_r$ it follows that

$$\begin{aligned}
&\mathfrak{D}I(S; W(0)) = \mathfrak{D}^2 J(S; V(0), W(0)) \\
&= \int_S u'(S; W) - 2\kappa u(S) \langle W(0), n \rangle_{\mathbb{R}^3} d\Gamma \\
&= \left[\frac{d}{dr} \frac{dj}{dt}(r) \right]_{|r=0} = \int_{\mathcal{O}} \frac{\delta u_r}{\delta r}_{|r=0} - 2 \left(\kappa_r u_r w_r \right)_{|r=0} dS
\end{aligned}$$

In the present paper the second order derivatives of shape functionals are evaluated for the fields $W = V = \frac{\partial \varphi_t}{\partial t} \circ \varphi_t^{-1}$.

The second order derivative $d^2J(\Omega; V, W)$ of the shape functional $J(\Omega)$ in the direction of vector fields V, W is defined as follows

$$d^2J(\Omega; V, W) = \lim_{s\downarrow 0} \frac{1}{s}[dJ(\Omega_s; V(s)) - dJ(\Omega; V)] \ ,$$

where $\Omega_s = T_s(W)(\Omega)$, $V(s) = V(s, x)$ with $x = T_s(W)(X)$ for $X \in \Omega$.

It can be shown, that we have the following representation of the second order shape derivative, if the shape derivative exists,

$$d^2J(\Omega; V, W) = \mathfrak{d}^2J(\Omega; V(0), W(0)) + \mathfrak{D}J(\Omega; \dot{V}(0)) \ ,$$
$$\mathfrak{D}J(\Omega; \dot{V}(0)) = \mathfrak{D}J(\Omega; \partial_t V(0)) + \mathfrak{D}J(\Omega; [DV \cdot W](0))$$

where $\dot{V}(0)$ denotes the material derivative of the velocity field V in the direction of the field W,

$$\dot{V}(0, \cdot) \quad = \quad \frac{\partial V}{\partial t}(0, \cdot) + [DV \cdot W](0, \cdot) = \partial_t V(0) + [DV \cdot W](0) \ .$$

Definition 4.7 *The linear operator $\mathcal{D}(D; \mathbb{R}^N) \mapsto \mathcal{D}'(D; \mathbb{R}^N)$ associated with the symmetric bilinear form $\mathfrak{d}^2J(\Omega; \cdot, \cdot)$ is called the Shape Hessian.*

Whenever it exists the *Shape Hessian* is a symmetric operator. The form of the *Shape Hessian* can be identified from the second order shape derivative $d^2J(\Omega; V, W)$ by taking the vector fields V such that $\dot{V}(0) = 0$. Therefore, at least at the first stage, the material derivatives can be used in order to evaluate the second order shape derivative of a specific shape functional.
On the other, the second order Eulerian derivative

$$\mathfrak{D}^2J(\Omega; V(0), W(0)) \equiv \mathfrak{d}^2J(\Omega; V(0), W(0)) + \mathfrak{D}J(\Omega; [DV \cdot W](0)) \tag{4.6}$$

can be evaluated by taking the shape derivative of the shape functional $I(\Omega) = \mathfrak{D}J(\Omega; V(0))$ in the direction of a vector field W, we refer the reader to eg. (Delfour, Zolesio, 1991) for related results. In particular, our definition of the *Shape Hessian* is stricly different from the definition given in (Delfour, Zolesio, 1991), where the nonsymmetric shape Hessian is introduced. It is also a difficult task to evaluate the symmetric part of the shape Hessian directly from the Eulerian derivative $\mathfrak{D}^2J(\Omega; V(0), W(0))$ as it was observed in (Novruzi, Roche, 1994) where the relation similar to (4.6) was established for a specific problem of shape optimization. The *Shape Hessian* is required for the applications to the Newton method (Novruzi, Roche, 1994) as well as to the stability analysis of the shape optimization problems (Sokolowski, 1993).

Remark 4.8 *For $T_t = I + t\Theta$, the associated velocity field takes the form $V = \frac{\partial T_t}{\partial t} \circ T_t^{-1} = \Theta \circ (I + t\Theta)^{-1}$, whence the material derivative $\dot{V}(0) = \frac{d\Theta}{dt} = 0$ in any direction W.*

Hence for vector fiels $V(t,x) = [\Theta \circ (I + t\Theta)^{-1}](x)$, $W(t,x) = [\Psi \circ (I + t\Psi)^{-1}](x)$, we have

$$d^2 J(\Omega; V, W) = \mathfrak{d}^2 J(\Omega; V(0), W(0)) = \mathfrak{d}^2 J(\Omega; \Theta, \Psi),$$

and the second order Frechet derivative evaluated by the method of perturbation of identity is symmetric, hence

$$\mathfrak{d}^2 J(\Omega; \Theta, \Psi) = \mathfrak{d}^2 J(\Omega; \Psi, \Theta) \qquad \textit{for all admissible vector fields } \Psi, \Theta$$

Therefore, the second order derivative $\mathfrak{d}^2 J(\Omega; V(0), W(0))$ is symmetric with respect to directions $V(0), W(0)$ whenever the second order shape derivative $d^2 J(\Omega; V, W)$ exists.

Results on the second order differentiability of shape functionals were derived by several authors, eg. by N. Fujii, Z. Mróz and H. Petryk, J. Simon, P. Guillaume and M. Masmoudi.

5 The first order shape sensitivity analysis

The displacement derivative of solutions to (2.1) is given by the solutions to the following system.

Theorem 5.1 *Assume we are given the displacement derivatives $\frac{\delta e_t}{\delta t}$ and $\frac{\delta \varphi_t}{\delta t}$.*

The derivative $\vec{u}' = \frac{\delta \vec{u}_t}{\delta t}_{|t=0}$ of solution to equation (3.3) at $t = 0$ is the unique solution to the equation

$$\textit{find } \vec{u}' \in \mathcal{H} = (H_0^1(\mathcal{O}))^2 \times H_0^2(\mathcal{O}) \textit{ such that}$$

$$a(\vec{u}', \vec{v}) + \frac{\partial a_t}{\partial \varphi_t}(\vec{u}, \vec{v}) \frac{\delta \varphi_t}{\delta t}_{|t=0} + \frac{\partial a_t}{\partial e_t}(\vec{u}, \vec{v}) \frac{\delta e_t}{\delta t}_{|t=0} = \frac{\delta f_t}{\delta t}(\vec{v})_{|t=0}$$

$$\forall \vec{v} \in (H_0^1(\mathcal{O}))^2 \times H_0^2(\mathcal{O})$$

where the subsequent terms are given below and $\vec{u}$ denotes a solution to (2.1).

We have

$$\frac{d}{dt}[f_t(\vec{v})]_{|t=0} = \int_{\mathcal{O}} [\frac{\partial \vec{p}_t}{\partial t}_{|t=0} \cdot \vec{v} + w_t \frac{\partial \vec{p}_t}{\partial n}_{|t=0} \cdot \vec{v} - 2w\kappa \vec{p}_{t|t=0} \cdot \vec{v}] dS \ .$$

Here we use the following notation.

$$\frac{\partial a_t}{\partial u_t}(\frac{\delta \vec{u}_t}{\delta t}, \vec{v}) = \int_{\mathcal{O}} e_t E_t^{\alpha\beta\iota\mu} \left\{ \gamma_{t\alpha\beta}\left(\frac{\delta \vec{u}_t}{\delta t}\right) \gamma_{t\lambda\mu}(\vec{v}) \right.$$
$$\left. + \frac{e_t^2}{12} \overline{\rho}_{t\alpha\beta}\left(\frac{\delta \vec{u}_t}{\delta t}\right) \overline{\rho}_{t\lambda\mu}(\vec{v}) \right\} dS_t$$

$$\frac{\partial a_t}{\partial \varphi_t}(\vec{u}_t, \vec{v}) \frac{\delta \varphi_t}{\delta t}$$
$$= \int_{\mathcal{O}} 2e_t w_t \Big\{ \frac{E}{1+\nu}\left[a_t^{\alpha\lambda} b_t^{\beta\mu} + a_t^{\alpha\mu} b_t^{\beta\lambda} + \frac{\nu}{1+\nu}\left(a_t^{\alpha\beta} b_t^{\lambda\mu} + a_t^{\lambda\mu} b_t^{\alpha\beta}\right)\right]$$
$$-\kappa_t E_t^{\alpha\beta\lambda\mu}\Big\} \Big\{ \gamma_{t\alpha\beta}(\vec{u}_t)\, \gamma_{t\lambda\mu}(\vec{v}) + \frac{e_t^2}{12} \overline{\rho}_{t\alpha\beta}(\vec{u}_t)\, \overline{\rho}_{t\lambda\mu}(\vec{v}) \Big\}$$
$$-e_t E_t^{\alpha\beta\lambda\mu} \left\{ \left[u_{t\iota} \frac{\delta}{\delta t}\left(\gamma^{\iota}_{t\alpha\beta}\right) + u_{t3} \frac{\delta}{\delta t}(b_{t\alpha\beta}) \right] \gamma_{t\lambda\mu}(\vec{v}) \right.$$
$$+ \gamma_{t\alpha\beta}(\vec{u}_t) \left[v_\iota \frac{\delta}{\delta t}\left(\Gamma^{\iota}_{t\lambda\mu}\right) + v_3 \frac{\delta}{\delta t}(b_{t\lambda\mu}) \right]$$
$$+ \frac{e_t^2}{12} \Big[u_{t3,\iota} \frac{\delta}{\delta t}\left(\Gamma^{\iota}_{t\alpha\beta}\right) + u_{t3} \frac{\delta}{\delta t}(b^{\omega}_{t\alpha} b_{t\omega\beta}) - u_{t\omega} \frac{\delta}{\delta t}\left(b^{\omega}_{t\alpha}|_\beta\right) - u_{t\omega,\beta} \frac{\delta}{\delta t}(b^{\omega}_{t\alpha})$$
$$+ u_{t\iota} \frac{\delta}{\delta t}\left(b^{\omega}_{t\alpha}\Gamma^{\iota}_{t\omega\beta}\right) - u_{t\omega,\alpha} \frac{\delta}{\delta t}\left(b^{\omega}_{t\beta}\right) + u_{t\iota} \frac{\delta}{\delta t}\left(b^{\omega}_{t\beta}\Gamma^{\iota}_{t\omega\alpha}\right) \Big] \overline{\rho}_{t\lambda\mu}(\vec{v})$$
$$+ \frac{e_t^2}{12} \overline{\rho}_{t\alpha\beta}(\vec{u}_t) \Big[v_{3,\iota} \frac{\delta}{\delta t}\left(\Gamma^{\iota}_{t\lambda\mu}\right) + v_3 \frac{\delta}{\delta t}(b^{\omega}_{t\lambda} b_{t\omega\mu}) - v_\omega \frac{\delta}{\delta t}\left(b^{\omega}_{t\lambda}|_\mu\right) - v_{\omega,\mu} \frac{\delta}{\delta t}(b^{\omega}_{t\lambda})$$
$$\left. + v_\iota \frac{\delta}{\delta t}\left(b^{\omega}_{t\lambda}\Gamma^{\iota}_{t\omega\mu}\right) - v_{\omega,\lambda} \frac{\delta}{\delta t}\left(b^{\omega}_{t\mu}\right) + v_\iota \frac{\delta}{\delta t}\left(b^{\omega}_{t\mu}\Gamma^{\iota}_{t\omega\lambda}\right) \Big] \right\} dS_t$$

$$\frac{\partial a_t}{\partial e_t}(\vec{u}_t, \vec{v}) \frac{\delta e_t}{\delta t}$$
$$= \int_{\mathcal{O}} E_t^{\alpha\beta\lambda\iota} \frac{\delta e_t}{\delta t} \{ \gamma_{t\alpha\beta}(\vec{u}_t) \gamma_{t\lambda\iota}(\vec{v}) + \frac{e_t^2}{4} \overline{\rho}_{t\alpha\beta}(\vec{u}_t) \overline{\rho}_{t\lambda\iota}(\vec{v}) \} dS_t \ .$$

Remark 5.2 *The form of displacement derivatives used in the above formulae can be obtained by direct computations. We refer the reader for such formulae to (Kosinski, 1986) and (Khludnev, Sokolowski, to appear).*

Let $\mathfrak{I}(\varphi_t, e_t; \vec{u}_t)$ be a functional that we are going to optimize. We assume that

$$\mathfrak{I}(\varphi_t, e_t; \vec{u}_t) = J(S_t), \text{ where } S_t = \varphi_t(\mathcal{O}) \tag{5.1}$$

and $J(S_t)$ is a shape functional.

Put

$$j(t) = \mathfrak{J}(\varphi_t, e_t; \vec{u}_t)$$

Under the assumption that data are sufficiently smooth and $\mathfrak{J}$ is differentiable, we can compute (Chenais, 1987)

$$\frac{dj}{dt}(0) = \lim_{t\to 0} \frac{j(t) - j(0)}{t} \tag{5.2}$$

$$= \Big[\frac{\partial \mathfrak{J}}{\partial \varphi_t}(\varphi, e_t; \vec{u}_t)\frac{\partial \varphi_t}{\partial t} + \frac{\partial \mathfrak{J}}{\partial e_t}(\varphi, e_t; \vec{u}_t)\frac{\partial e_t}{\partial t} + \frac{\partial \mathcal{J}}{\partial u_t}(\varphi, e_t; \vec{u}_t)\frac{\partial \vec{u}_t}{\partial t}\Big]_{t=0} \ .$$

Denote

$$\frac{\delta j}{\delta t}(0) = \Big[\frac{\partial \mathfrak{J}}{\partial \varphi_t}(\varphi, e_t; \vec{u}_t)\frac{\delta \varphi_t}{\delta t} + \frac{\partial \mathfrak{J}}{\partial e_t}(\varphi, e_t; \vec{u}_t)\frac{\delta e_t}{\delta t} + \frac{\partial \mathcal{J}}{\partial u_t}(\varphi, e_t; \vec{u}_t)\frac{\delta \vec{u}_t}{\delta t}\Big]_{t=0} \tag{5.3}$$

By applying the structure theorem 4.1, it follows that,

$$\frac{dj}{dt}(0) = \frac{\delta j}{\delta t}(0) \ .$$

Since, in general, we are not able to compute $\frac{\delta \vec{u}_t}{\delta t}$ for any $\frac{\delta \varphi_t}{\delta t}$, we introduce the adjoint state equation

Find $\vec{q} \in (H_0^1(\mathcal{O}))^2 \times H_0^2(\mathcal{O})$ such that

$$a(\vec{q}, \vec{v}) = \Big[\frac{\partial \mathcal{J}}{\partial u_t}(\varphi_t, e_t; \vec{u}_t)\Big]_{|t=0}\vec{v}, \quad \forall \vec{v} \in (H_0^1(\mathcal{O}))^2 \times H_0^2(\mathcal{O}) \ . \tag{5.4}$$

Clearly, equation (5.4) is uniquely solvable and expression (5.2) takes the form

$$\frac{dj}{dt}(0) = -\Big[\frac{\partial a_t}{\partial u_t}(\vec{u}_t, \vec{q})\frac{\delta \varphi_t}{\delta t}\Big]_{|t=0} - \Big[\frac{\partial a_t}{\partial e_t}(\vec{u}_t, \vec{q})\frac{\delta e_t}{\delta t}\Big]_{|t=0}$$

$$+\Big[\frac{\delta f_t}{\delta t}(\vec{q})\Big]_{|t=0} + \Big[\frac{\partial \mathfrak{J}}{\partial \varphi_t}(\varphi_t, e_t; \vec{u}_t)\frac{\delta \varphi_t}{\delta t}\Big]_{|t=0} + \Big[\frac{\partial \mathfrak{J}}{\partial e_t}(\varphi_t, e_t; \vec{u}_t)\frac{\delta e_t}{\delta t}\Big]_{|t=0} \ ,$$

where the subsequent terms of the right–hand side can be obtained in the explicite form (Khludnev, Sokolowski, to appear). One should note that the right–hand side is actually a linear mapping with respect to $(w = \frac{\delta \varphi_t}{\delta t}, \varepsilon = \frac{\delta e_t}{\delta t}, t = 0)$.

6 The second order shape sensitivity analysis

For any $t \geq 0$, (5.2) can be written as

$$\frac{dj}{dt}(t) = \frac{\partial \mathfrak{J}}{\partial \varphi_t}(\varphi, e_t; \vec{u}_t)\frac{\delta \varphi_t}{\delta t} + \frac{\partial \mathfrak{J}}{\partial e_t}(\varphi, e_t; \vec{u}_t)\frac{\delta e_t}{\delta t} + \frac{\partial \mathcal{J}}{\partial u_t}(\varphi, e_t; \vec{u}_t)\frac{\delta \vec{u}_t}{\delta t} \tag{6.1}$$

and for any $t \geq 0$ we can define the following adjoint state equation.

Find $\vec{q}_t \in (H_0^1(\mathcal{O}))^2 \times H_0^2(\mathcal{O})$ such that

$$a_t(\vec{q}_t, \vec{v}) = \frac{\partial \mathcal{J}}{\partial u_t}(\varphi, e_t; \vec{u}_t)\vec{v}, \quad \forall \vec{v} \in (H_0^1(\mathcal{O}))^2 \times H_0^2(\mathcal{O}) \ . \tag{6.2}$$

As in Theorem 4.1, we get the following result.

Theorem 6.1 *The displacement derivative $\frac{\delta \vec{u}_t}{\delta t}$ of a solution to (3.3) is a unique solution to the following equation given $\left(\frac{\delta e_t}{\delta t}, \frac{\delta \varphi_t}{\delta t}\right)$, find $\frac{\delta \vec{u}_t}{\delta t} \in (H_0^1(\mathcal{O}))^2 \times H_0^2(\mathcal{O})$ such that*

$$a(\frac{\delta \vec{u}_t}{\delta t}, \vec{v}) + \frac{\partial a_t}{\partial \varphi_t}(\vec{u}_t, \vec{v})\frac{\delta \varphi_t}{\delta t} + \frac{\partial a_t}{\partial e_t}(\vec{u}_t, \vec{v})\frac{\delta e_t}{\delta t} = \frac{\delta f_t}{\delta t}(\vec{v}) \tag{6.3}$$

$$\forall \vec{v} \in (H_0^1(\mathcal{O}))^2 \times H_0^2(\mathcal{O}) \ ,$$

where the subsequent terms are given in theorem 5.1

For $t = 0$ we denote

$$a(\vec{u}', \vec{v}) + \frac{\partial a_t}{\partial \varphi_t}(\vec{u}, \vec{v})\frac{\delta \varphi_t}{\delta t}_{|t=0} + \frac{\partial a_t}{\partial e_t}(\vec{u}, \vec{v})\frac{\delta e_t}{\delta t}_{|t=0} = \frac{\delta f_t}{\delta t}(\vec{v})_{|t=0}$$

Furthermore, the displacement derivative $\frac{\delta \vec{q}_t}{\delta t}$ of a solution to (6.2) is given as a unique solution to the equation

find $\frac{\delta \vec{q}_t}{\delta t} \in (H_0^1(S))^2 \times H_0^2(S)$ such that

$$a(\frac{\delta \vec{q}_t}{\delta t}, \vec{v}) + \frac{\partial a_t}{\partial \varphi_t}(\vec{q}_t, \vec{v})\frac{\delta \varphi_t}{\delta t} + \frac{\partial a_t}{\partial e_t}(\vec{q}_t, \vec{v})\frac{\delta e_t}{\delta t} \tag{6.4}$$

$$= \frac{\partial^2 J}{(\partial u_t)^2}(\varphi_t, e_t; \frac{\delta \vec{u}_t}{\delta t})\vec{v} + \frac{\partial^2 J}{\partial u_t \partial \varphi_t}(\varphi_t, e_t; \vec{u}_t)\left(\vec{v}, \frac{\delta \varphi_t}{\delta t}\right)$$

$$+ \frac{\partial^2 J}{\partial u_t \partial e_t}(\varphi_t, e_t; \vec{u}_t)\left(\vec{v}, \frac{\delta e_t}{\delta t}\right)$$

$$\forall \vec{v} \in (H_0^1(\mathcal{O}))^2 \times H_0^2(\mathcal{O}) \ .$$

Therefore, by relations (6.1)–(6.3), we obtain

$$\frac{dj}{dt}(t) = \int_{\mathcal{O}} G_t dS_t = -\frac{\partial a_t}{\partial \varphi_t}(\vec{u}_t, \vec{q}_t)\frac{\delta\varphi_t}{\delta t} - \frac{\partial a_t}{\partial e_t}(\vec{u}_t, \vec{q}_t)\frac{\delta e_t}{\delta t} \tag{6.5}$$

$$+\frac{\delta f_t}{\delta t}(\vec{q}_t) + \frac{\partial \mathfrak{J}}{\partial \varphi_t}(\varphi, e_t; \vec{u}_t)\frac{\delta\varphi_t}{\delta t} + \frac{\partial \mathfrak{J}}{\partial e_t}(\varphi, e_t; \vec{u}_t)\frac{\delta e_t}{\delta t} \ .$$

Using (5.1) it follows that

$$\frac{dj}{dt}(t) = dJ(S_t; V(t)), \quad V(t) = \frac{\partial \varphi_t}{\partial t} \circ \varphi_t^{-1}$$

therefore,

Corollary 6.2 *The second order Eulerian derivative of shape functional (5.1) (see section 4 for the definition) is given by*

$$\mathfrak{D}^2 J(S; V(0), V(0)) = \frac{d^2 j}{dt^2}(0) = \int_{\mathcal{O}} \left[\frac{\delta G_t}{\delta t}_{|t=0} - 2w\kappa G \right] dS$$

We obtain the form of the *Shape Hessian* using the latter formula.

Corollary 6.3 *We have*

$$\begin{aligned}
&\mathfrak{d}^2 J(S; V(0), V(0)) = -2\int_{\mathcal{O}} w\kappa G dS \\
&+\frac{\partial^2 J}{(\partial\varphi_t)^2}(\varphi, e_t; \vec{u}_t)\left(\frac{\delta\varphi_t}{\delta t}\right)^2 + 2\frac{\partial^2 J}{\partial\varphi_t \partial e_t}(\varphi, e_t; \vec{u}_t)\left(\frac{\delta\varphi_t}{\delta t}, \frac{\delta e_t}{\delta t}\right) \\
&+\frac{\partial^2 J}{\partial\varphi_t \partial u_t}(\varphi, e_t; \vec{u}_t)\left(\frac{\delta\varphi_t}{\delta t}, \frac{\delta\vec{u}_t}{\delta t}\right) + \frac{\partial^2 J}{(\partial e_t)^2}(\varphi, e_t; \vec{u}_t)\left(\frac{\delta e_t}{\delta t}\right)^2 \\
&+\frac{\partial^2 J}{\partial e_t \partial u_t}(\varphi, e_t; \vec{u}_t)\left(\frac{\delta e_t}{\delta t}, \frac{\delta\vec{u}_t}{\delta t}\right) + \frac{d}{dt} f_t\left(\frac{\delta\vec{q}_t}{\delta t}\right) \\
&-\frac{\partial^2 a_t}{(}\partial\varphi_t)^2(\vec{u}_t, \vec{q}_t)\left(\frac{\delta\varphi_t}{\delta t}\right) - 2\frac{\partial^2 a_t}{\partial\varphi_t \partial e_t}(\vec{u}_t, \vec{q}_t)\left(\frac{\delta\varphi_t}{\delta t}, \frac{\delta e_t}{\delta t}\right) \\
&-\frac{\partial a_t}{\partial\varphi_t}\left(\frac{\delta\vec{u}_t}{\delta t}, \vec{q}_t\right)\frac{\delta\varphi_t}{\delta t} - \frac{\partial a_t}{\partial\varphi_t}\left(\vec{u}_t, \frac{\delta\vec{q}_t}{\delta t}\right)\frac{\delta\varphi_t}{\delta t} \\
&-\frac{\partial^2 a_t}{(\partial e_t)^2}(\vec{u}_t, \vec{q}_t)\left(\frac{\delta e_t}{\delta t}\right)^2 - \frac{\partial a_t}{\partial e_t}\left(\frac{\delta\vec{u}_t}{\delta t}, \vec{q}_t\right)\frac{\delta e_t}{\delta t} - \frac{\partial a_t}{\partial e_t}\left(\vec{u}_t, \frac{\delta\vec{q}_t}{\delta t}\right)\frac{\delta e_t}{\delta t}
\end{aligned}$$

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