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***Boundary Control Problems for Shape Memory Alloys  
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## Boundary Control Problems for Shape Memory Alloys under State Constraints

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**Abstract:** We consider two optimal control problems for first order martensitic phase transitions in a deformation-driven experiment on shape memory alloys including state constraints for the total stress and the temperature. We control by the elongation of a thin rod and by the outside temperature. The control problems are stated, and the necessary conditions of optimality are derived.

**Key-words:** shape memory alloys, phase transition, state constraints, optimality conditions

*(Résumé : tsup)*

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## **Problèmes de contrôle de matériaux à mémoire de forme**

**Résumé :** On étudie des problèmes de contrôle frontière avec des contraintes sur l'état pour des modèles non linéaires de transition de phase de matériaux à mémoire de forme. On obtient des conditions d'optimalité pour les problèmes de contrôle.

**Mots-clé :** contraintes sur l'état, condition d'optimalité, transition de phase

## 1 Introduction

In this paper, we consider optimal control problems for a deformation–driven experiment on shape memory alloys (SMA) with state constraints for the total stress and the temperature. SMA exhibit a non–monotone temperature–dependent hysteretic behaviour in their load–deformation cycles leading to interesting industrial applications. In a series of papers (cf. [6],[7],[8], for example), Falk introduced a one–dimensional model that is based on the Landau–Ginzburg theory of phase transitions and uses the linearized shear strain  $\varepsilon = u_x$ , where  $u$  denotes the displacement, as order parameter. The corresponding (Helmholtz–) free energy  $F = F(u, \theta)$ , where  $\theta$  denotes the absolute temperature, is given by

$$F(\varepsilon, \theta) = F_0(\theta) + \theta F_1(\varepsilon) + F_2(\varepsilon), \quad (1.1)$$

where

$$F_0(\theta) = -c_e \theta \log\left(\frac{\theta}{\bar{\theta}}\right) + c_e \theta + C, \quad (1.2)$$

and

$$F_1(\varepsilon) = \frac{1}{2}\gamma\varepsilon^2, \quad F_2(\varepsilon) = -\frac{1}{2}\gamma\theta_1\varepsilon^2 - \frac{1}{4}\beta\varepsilon^4 + \frac{1}{6}\alpha\varepsilon^6, \quad (1.3)$$

with positive constant heat capacity  $c_e$ , a critical temperature  $\theta_1$ , and positive material constants  $\bar{\theta}$ ,  $C$ ,  $\alpha$ ,  $\beta$ , and  $\gamma$ , which have to be determined for each specimen. For thermodynamical reasons, i.e. in order to comply with the second principle, the constitutive equations yield for the total stress:

$$\sigma = \frac{\partial F}{\partial \varepsilon}(\varepsilon, \theta) = -\gamma(\theta - \theta_1)\varepsilon - \beta\varepsilon^3 + \alpha\varepsilon^5. \quad (1.4)$$

In a deformation–driven experiment, a thin rod of a SMA is fixed on one side and pushed and pulled on the other side in the course of time by an elongation  $m$ . In such experiments, the order parameter is taken to be  $\varepsilon = u_x$ ,  $u$  denoting the displacement *in* the direction of the rod. For a detailed description of the physical background, we refer the reader to [2],[3]. Summarizing, we have the following system ( $\Omega := (0, l)$ ,  $Q := \Omega \times (0, T)$ ):

$$\rho u_{tt} - (\gamma(\theta - \theta_1)u_x - \beta u_x^3 + \alpha u_x^5)_x + \delta u_{xxxx} = 0, \quad \text{in } Q, \quad (1.5a)$$

$$c_e \theta_t - \kappa \theta_{xx} - \gamma \theta u_x u_{xt} = g(x, t), \quad \text{in } Q, \quad (1.5b)$$

$$u(0, t) = u_{xx}(0, t) = u_{xx}(l, t) = 0, \quad u(l, t) = m(t), \quad \forall t \in [0, T],$$

$$\theta_x(0, t) = 0, \quad -\kappa \theta_x(l, t) = \bar{\kappa}(\theta(l, t) - \theta_\Gamma(t)), \quad \forall t \in [0, T],$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \theta(x, 0) = \theta_0(x), \quad \forall x \in \bar{\Omega}, \quad (1.5c)$$

The equations (1.5a) and (1.5b) represent the balance laws of momentum and energy, respectively. The physical meanings of the involved quantities are:  $\rho$  – constant mass density,  $\kappa$  – positive constant heat conductivity,  $g$  – density of heat sources or sinks,  $l$  – length of the rod (which is normalized to unity:  $l := 1$ ),  $\bar{\kappa}$  – positive constant heat exchange coefficient,  $\theta_\Gamma$  – temperature of

the surrounding medium. The couple stress leads to the Ginzburg-term  $\delta \cdot u_{xxxx}$ ,  $\delta$  being another positive material constant. The boundary condition for  $u$  at  $x = 1$  reflects the pulling and pushing of the rod in the course of time by a prescribed elongation  $m$ . The other boundary condition for the momentum balance has been taken in analogy to [11]. The boundary condition for the energy balance models a heat exchange with the surrounding temperature at  $x = 1$  using Newton's law. We normalize all physical constants to 1, except for  $\theta_1$  which is set to 0. In order to deal with homogeneous boundary conditions, we transform the system (1.5) by  $\tilde{u}(x, t) := u(x, t) - x \cdot m(t)$ . An additional term  $\rho \cdot x \cdot \ddot{m}(t)$  appears only on the left hand side of the momentum balance. We now have  $\varepsilon = u_x + m(t)$  instead of  $\varepsilon = u_x$ . For simplicity, the tilde for  $u$  and  $u_x$ , respectively, is omitted. We denote by  $\tilde{\sigma}$  the polynom (1.4) where  $\varepsilon = u_x$  is replaced by  $\varepsilon = u_x + m(t)$ .

In this paper, we consider the optimal control of the phase transitions governed by the following weak formulation of (1.5):

$$\int_0^T \langle u_{tt}(s), \phi(s) \rangle_{H^{-1} \times H_0^1} ds + \int_0^T \int_{\Omega} x \ddot{m}(s) \phi dx ds + \int_0^T \int_{\Omega} \tilde{\sigma} \phi_x dx ds - \int_0^T \int_{\Omega} u_{xxx} \phi_x dx ds = 0, \quad \forall \phi \in L^2(0, T; H_0^1(\Omega)), \quad (1.6a)$$

$$\theta_t - \theta(u_x + m(t))(u_{xt} + \dot{m}(t)) - \theta_{xx} = g, \quad \text{a.e. in } Q, \quad (1.6b)$$

$$u(0, t) = u(1, t) = 0, \quad \forall t \in [0, T], \quad u_{xx}(0, t) = u_{xx}(1, t) = 0, \quad \text{a.e. in } (0, T),$$

$$\theta_x(0, t) = 0, \quad -\theta_x(1, t) = \theta(1, t) - \theta_{\Gamma}(t), \quad \text{a.e. in } (0, T),$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \theta(x, 0) = \theta_0(x), \quad \forall x \in \overline{\Omega}, \quad (1.6c)$$

where we want to admit state constraints for the total stress  $\sigma$  defined by (1.4) and the temperature  $\theta$ . Under the following assumptions

$$(H1) \quad m \in H^3(0, T); \quad g \in L^2(0, T; L^2(\Omega)); \quad g(x, t) \geq 0 \quad \text{on } \overline{Q}; \\ \theta_{\Gamma} \in H^1(0, T); \quad \theta_{\Gamma}(t) > 0 \quad \text{on } [0, T]; \quad (1.7)$$

$$(H2) \quad u_0 \in H_E^3(\Omega) := \{u \in H^3(\Omega) \mid u(0) = u''(0) = u(1) = u''(1) = 0\}; \\ u_1 \in H_0^1(\Omega); \quad \theta_0 \in H^1(\Omega); \quad \theta_0(x) > 0 \quad \text{on } \overline{\Omega}, \quad (1.8)$$

the existence and uniqueness of a weak solution has been proved in [4].

**Theorem 1.1** *Suppose that (H1) and (H2) are satisfied. Then the system (1.6) has a solution  $(u, \theta)$  on  $\overline{Q}$  satisfying*

$$u \in X_{1,T} := W^{2,\infty}(0, T; H^{-1}(\Omega)) \cap W^{1,\infty}(0, T; H^1(\Omega)) \cap L^{\infty}(0, T; H_E^3(\Omega)) \quad \text{and} \\ \theta \in X_{2,T} := H^{2,1}(Q) \cap L^{\infty}(0, T; H^1(\Omega)), \quad (1.9)$$

for any  $T > 0$ .

Lemma 3.5 in [4] states uniqueness. We recall that, with stronger assumptions for the data, the existence and uniqueness of a classical solution can be proved (see [11],[2],[4]).

Related optimal control problems have been studied so far in [1] concerning load-driven experiments, state constraints for those problems have been imposed in [9] and [10]. Therein, boundary control problems with state constraints for the transversal displacement and on the shear strain, respectively, were introduced. It has been left out as an open problem whether state constraints for total stress and for the temperature are possible.

Now, in [4] we have shown the differentiability of the observation operator as mapping into the solution space  $X_{1,T} \times X_{2,T}$ , while in [1] only the differentiability into the Banach space

$$\begin{aligned} B &= W^{1,\infty}(0, T; L^2(\Omega)) \cap L^\infty(0, T; H_0^1(\Omega) \cap H^2(\Omega)) \\ &\times L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)) \end{aligned} \quad (1.10)$$

has been proved. Since  $X_{2,T}$  is continuously imbedded in  $C(\overline{\Omega_T})$ , this means that also pointwise constraints on the temperature  $\theta$  and therefore on the stress  $\sigma$ , too, can now be included in the control problem. This was not possible in [9] and [10] where only pointwise constraints on the displacement  $u$  and the strain  $\varepsilon$ , respectively, could be admitted. Note that pointwise constraints for  $\theta$  are very realistic for the particular experimental setup discussed here, where  $\theta$  is kept close to a prescribed (constant) temperature  $\bar{\theta}$  (see also remark 4.1 in [4]). Since we do not want to differentiate with respect to the distributed heat sources and sinks,  $g$ , we even have Fréchet differentiability with the result given in [4].

We define

$$\mathcal{M} := M_m \times M_{\theta_\Gamma}, \quad (1.11)$$

where

$$\begin{aligned} M_m &:= \left\{ m \in H^3(0, T) \mid m(0) = 0, \dot{m}(0) = 0, \ddot{m}(0) = 0 \right\}, \\ M_{\theta_\Gamma} &:= \left\{ \theta_\Gamma \in H^1(0, T) \mid \theta_\Gamma(t) > 0 \text{ on } [0, T] \right\}, \end{aligned} \quad (1.12)$$

and the control space

$$\mathcal{Z} := H^3(0, T) \times H^1(0, T), \quad (1.13)$$

therefore  $\mathcal{M} \subset \mathcal{Z}$ . The solution operator is denoted by

$$\mathcal{G}(\cdot, \cdot) : \mathcal{M} \ni (m, \theta_\Gamma) \mapsto (u, \theta) \in X_{1,T} \times X_{2,T} \subset C(\overline{Q}) \times C(\overline{Q}). \quad (1.14)$$

Note that  $u \in X_{1,T}$  implies  $u_x \in C(\overline{Q})$  and therefore,  $\sigma \in C(\overline{Q})$ , too. From [4] we have the following properties of the solution operator.

**Theorem 1.2**  $\mathcal{G}(\cdot, \cdot)$  is Fréchet differentiable as mapping between the open set  $\mathcal{M}$  and  $X_{1,T} \times X_{2,T}$ , and the Fréchet derivative  $\mathcal{G}'(m, \theta_\Gamma) \cdot (h, l) =: (\phi, \psi)$  of  $\mathcal{G}$  at  $(m, \theta_\Gamma)$  applied to  $(h, l) \in \mathcal{Z}$  is given as the unique solution to the system

$$\begin{aligned} &\int_0^T \langle \phi_{tt}(s), \xi(s) \rangle_{H^{-1} \times H_0^1} ds - \int_0^T \int_\Omega \phi_{xxx} \xi_x dx ds = - \int_0^T \int_\Omega x \ddot{h}(s) \xi dx ds \\ &- \int_0^T \int_\Omega \left( \varepsilon \psi + (\theta + F_2''(\varepsilon)) (\phi_x + h(s)) \right) \xi_x dx ds, \quad \forall \xi \in L^2(0, T; H_0^1(\Omega)), \end{aligned} \quad (1.15a)$$



$$\psi_t - \psi_{xx} = \theta \varepsilon_t (\phi_x + h(t)) + \varepsilon \varepsilon_t \psi + \theta \varepsilon (\phi_{xt} + \dot{h}(t)), \quad a.e. \text{ in } Q, \quad (1.15b)$$

$$\phi(x, 0) = \phi_t(x, 0) = 0 = \psi(x, 0), \quad \forall x \in \overline{\Omega},$$

$$\begin{aligned} \phi(0, t) = \phi(1, t) = 0, \quad \forall t \in [0, T], \quad \phi_{xx}(0, t) = \phi_{xx}(1, t) = 0, \quad a.e. \text{ in } (0, T), \\ \psi_x(0, t) = 0, \quad -\psi_x(1, t) = \psi(1, t) - l(t), \quad a.e. \text{ in } (0, T), \end{aligned} \quad (1.15c)$$

where  $\mathcal{G}(m, \theta_\Gamma) = (u, \theta)$  and  $\varepsilon = u_x + m$ .

Clearly, we have  $(\phi, \psi) \in X_{1,T} \times X_{2,T} \subset C(\overline{Q}) \times C(\overline{Q})$ , and again,  $\phi_x \in C(\overline{Q})$ , too.

Since the strain  $\varepsilon$  plays the role of the order parameter, it is quite natural to consider cost functionals involving  $\varepsilon$ . On the other hand, the natural control variables are  $m$  and  $\theta_\Gamma$ ; in fact, these variables are used to control the processes in actual industrial applications of SMA.

We are going to consider two problems. First, we take the elongation  $m$  as the control variable, and, to simplify, we consider  $\theta_\Gamma$  as given data. We impose state constraints for both the stress and the temperature. Second, we take  $\theta_\Gamma$  as control variable,  $m$  as given data and prescribe constraints for the total stress.

## 2 Control by Elongation

We study the following problem.

**(CP1)** Minimize  $J(m)$ , subject to (1.6),  $(\theta, \sigma) \in \mathcal{C}$  and  $m \in \mathcal{U}_{ad}$ .

Here,  $\mathcal{U}_{ad}$  denotes the set of admissible controls, and is some nonempty, convex, bounded, and closed subset of  $M_m$ .  $\mathcal{C}$  is given by

$$\mathcal{C} := \left\{ (\theta, \sigma) \in C(\overline{Q}) \times C(\overline{Q}) \mid c_1 \leq \theta(x, t) \leq c_2, \quad c_3 \leq \sigma(x, t) \leq c_4, \quad \forall (x, t) \in \overline{Q} \right\}. \quad (2.1)$$

The cost functional is assumed in the form

$$J(m) = \int_0^T \int_\Omega \Phi_1(u_x(x, t), \theta(x, t)) \, dx \, dt + \int_0^T \Phi_2(\ddot{m}(t)) \, dt, \quad (2.2)$$

where  $\Phi_1 \in C^2(\mathbb{R}^2)$ ,  $\Phi_2 \in C^1(\mathbb{R})$ , and where  $\Phi_2$  is convex in its argument. A particular form could be

$$J(m) = \alpha_1 \left( \|\sigma - \bar{\sigma}\|_{L^2(Q)}^2 + \|\theta - \bar{\theta}\|_{L^2(Q)}^2 \right) + \alpha_2 \|\ddot{m}\|_{L^2(0,T)}^2, \quad (2.3)$$

where  $\alpha_1$  and  $\alpha_2$  are non-negative constants, and where  $\bar{\theta}$  and  $\bar{\sigma}$  denote the desired temperature and stress distributions during the evolution of the process, respectively. Of course, also other cost functionals are conceivable in actual applications.

The following existence result can be shown with standard compactness arguments.

**Theorem 2.1** *Assume that there is at least one admissible control  $m$  such that the solution to (1.6) yields  $(\theta, \sigma) \in \mathcal{C}$ . Then there exists an optimal solution to the above control problem.*

The necessary optimality conditions for the control problem are given by the following theorem. Since here  $\theta_\Gamma$  is given, we write  $\mathcal{G}(m)$  instead of  $\mathcal{G}(m, \theta_\Gamma)$ .

**Theorem 2.2** *Let  $m \in \mathcal{U}_{ad}$  denote any solution to the optimal control problem (CP1), and let  $(u, \theta) = \mathcal{G}(m)$ . Then there exist a real number  $\lambda_1 \geq 0$  and Borel measures  $(\mu_1, \mu_2) =: \mu \in (C(\overline{Q}) \times C(\overline{Q}))'$  with  $\lambda_1 + \|\mu\|_{(C(\overline{Q}) \times C(\overline{Q}))'} > 0$  such that  $\int(\hat{\theta} - \theta) d\mu_1 + \int(\hat{\sigma} - \sigma) d\mu_2 \leq 0$ ,  $\forall (\hat{\sigma}, \hat{\theta}) \in \mathcal{C}$ , as well as functions  $(p, q) \in L^2(0, T; H_0^1(\Omega)) \times L^2(0, T; L^2(\Omega))$  satisfying the following optimality conditions.*

*State equations:*

$$\int_0^T \langle u_{tt}(s), \phi(s) \rangle_{H^{-1} \times H_0^1} ds + \int_0^T \int_\Omega x \ddot{m}(s) \phi dx ds + \int_0^T \int_\Omega (\theta(u_x + m(s)) + F_2'(u_x + m(s))) \phi_x dx ds - \int_0^T \int_\Omega u_{xxx} \phi_x dx ds = 0, \quad \forall \phi \in L^2(0, T; H_0^1(\Omega)), \quad (2.4a)$$

$$\theta_t - \theta(u_x + m(t))(u_{xt} + \dot{m}(t)) - \theta_{xx} = g, \quad a.e. \text{ in } Q, \quad (2.4b)$$

$$u(0, t) = u(1, t) = 0, \quad \forall t \in [0, T], \quad u_{xx}(0, t) = u_{xx}(1, t) = 0, \quad a.e. \text{ in } (0, T),$$

$$\theta_x(0, t) = 0, \quad -\theta_x(1, t) = \theta(1, t) - \theta_\Gamma(t), \quad a.e. \text{ in } (0, T), \quad (2.4c)$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \theta(x, 0) = \theta_0(x), \quad \forall x \in \overline{\Omega}. \quad (2.4d)$$

*Adjoint state equations:*

$$\begin{aligned} & \int_0^T \langle \xi_{tt}(s), p(s) \rangle_{H^{-1} \times H_0^1} ds - \int_0^T \int_\Omega \xi_{xxx} p_x dx ds \\ & + \int_0^T \int_\Omega \left( ((\theta + F_2''(\varepsilon)) p_x - \theta \varepsilon_t q) \xi_x - \theta \varepsilon q \xi_{xt} \right) dx ds \\ & = \lambda_1 \int_0^T \int_\Omega D_1 \Phi_1(u_x, \theta) \xi_x dx ds + \int \frac{\partial \tilde{\sigma}}{\partial \varepsilon} \xi_x d\mu_2, \quad \forall \xi \in X_{1,T}, \end{aligned} \quad (2.5a)$$

$$\begin{aligned} & \int_0^T \int_\Omega (q(\varphi_t - \varphi_{xx} - \varepsilon \varepsilon_t \varphi) + \varepsilon p_x \varphi) dx ds \\ & = \lambda_1 \int_0^T \int_\Omega D_2 \Phi_1(u_x, \theta) \varphi dx ds + \int \varepsilon \varphi d\mu_2 + \int \varphi d\mu_1, \quad \forall \varphi \in X_{2,T}. \end{aligned} \quad (2.5b)$$

$$p_t(x, T) = 0, \quad (2.5c)$$

*Optimality conditions:*

$$\begin{aligned} & \int_0^T \int_\Omega \left\{ -\ddot{h}(s) p_x + \dot{h}(s) q \theta \varepsilon - h(s) (p_x (\theta + F_2''(\varepsilon)) + q \theta \varepsilon_t) \right\} dx ds \\ & + \lambda_1 \int_0^T \left\{ \Phi_2'(\ddot{m}(s)) \ddot{h}(s) \right\} ds + \int \frac{\partial \tilde{\sigma}}{\partial m} h d\mu_2 \geq 0, \\ & h = \hat{m} - m, \quad \forall \hat{m} \in \mathcal{U}_{ad}. \end{aligned} \quad (2.6)$$

In addition,  $\lambda_1 = 1$  if the Slater condition is satisfied, i.e. there exists some  $\hat{m} \in \mathcal{U}_{ad}$  such that the unique solution  $(\phi, \psi)$  of the linearized state equations (1.15) corresponding to  $h = \hat{m} - m$  satisfies the condition

$$c_1 < \theta(x, t) + \psi(x, t) < c_2, \quad \text{and} \quad (2.7)$$

$$c_3 < \tilde{\sigma}(x, t) + \psi(x, t) \varepsilon(x, t) + (\phi_x(x, t) + h(t)) \left( \theta(x, t) + F_2''(\varepsilon(x, t)) \right) < c_4, \quad \forall (x, t) \in \overline{Q}.$$

PROOF. Now, a solution to **(CP1)** is denoted by  $m^*$ , and therefore  $h = m - m^*$ . Let us denote by  $J'(m) \in (H^3(0, T))'$  the Fréchet derivative of the cost functional  $J(m)$ , by  $\mathcal{F}(\mathcal{G}(m)) = \tilde{\sigma}$ , and by  $[D_m(\theta, \mathcal{F}(\mathcal{G}(m)))]^*$  the adjoint mapping of the differential. Moreover, let  $\langle \cdot, \cdot \rangle$  denote the dual pairing between the spaces  $(H^3(0, T))'$  and  $H^3(0, T)$ . Applying theorem 5.2 of [5], we conclude that there exist Borel measures  $(\mu_1, \mu_2) = \mu \in (C(\overline{Q}) \times C(\overline{Q}))'$  and some  $\lambda_1 \geq 0$  satisfying

$$\lambda_1 + \|\mu\|_{(C(\overline{Q}) \times C(\overline{Q}))'} > 0, \quad (2.8)$$

$$\langle \mu, z - (\theta^*, \mathcal{F}(\mathcal{G}(m^*))) \rangle \leq 0, \quad \forall z \in \mathcal{C}, \quad (2.9)$$

$$\langle \lambda_1 J'(m^*) + [D_m(\theta^*, \mathcal{F}(\mathcal{G}(m^*)))]^* \mu, m - m^* \rangle \geq 0, \quad \forall m \in \mathcal{K}. \quad (2.10)$$

Furthermore, we have  $\lambda_1 = 1$  if the Slater condition

$$\exists \tilde{m} \in \mathcal{U}_{ad} \quad \text{such that} \quad \mathcal{G}(m) + \mathcal{G}'(m) \cdot (\tilde{m} - m) \in \text{int}(\mathcal{C}) \quad (2.11)$$

is satisfied. Recalling the definition of  $\mathcal{C}$ , we find that this condition is equivalent to (2.7). Now, to continue in a simplified manner, we set  $\lambda_1 = 1$ .

We introduce the linear and bijective operators  $\mathcal{L}_1 : X_{1,T} \times X_{2,T} \rightarrow L^2(0, T; H^{-1}(\Omega))$  and  $\mathcal{L}_2 : X_{1,T} \times X_{2,T} \rightarrow L^2(0, T; L^2(\Omega))$  with

$$\begin{aligned} \int_0^T \langle \mathcal{L}_1(\phi, \psi)(s), \xi(s) \rangle_{H^{-1} \times H_0^1} ds &\equiv \int_0^T \langle \phi_{tt}(s), \xi(s) \rangle_{H^{-1} \times H_0^1} ds \\ &- \int_0^T \int_{\Omega} \phi_{xxx} \xi_x dx ds + \int_0^T \int_{\Omega} (\varepsilon \psi + (\theta + F_2''(\varepsilon)) \phi_x) \xi_x dx ds, \\ &\forall \xi \in L^2(0, T; H_0^1(\Omega)), \quad \text{and} \end{aligned} \quad (2.12)$$

$$\begin{aligned} \int_0^T \int_{\Omega} \mathcal{L}_2(\phi, \psi) \varphi dx ds &\equiv \int_0^T \int_{\Omega} (\psi_t - \psi_{xx} - \theta \varepsilon_t \phi_x - \varepsilon \varepsilon_t \psi - \theta \varepsilon \phi_{xt}) \varphi dx ds, \\ &\forall \varphi \in L^2(0, T; L^2(\Omega)). \end{aligned} \quad (2.13)$$

Furthermore, denoting

$$\mathcal{X} := X_{1,T} \times X_{2,T}, \quad \mathcal{Y} := L^2(0, T; H^{-1}(\Omega)) \times L^2(0, T; L^2(\Omega)), \quad (2.14)$$

$$z := (z_1, z_2) \in \mathcal{Y}, \quad \text{with}$$

$$\begin{aligned} \int_0^T \langle z_1(s), \xi(s) \rangle_{H^{-1} \times H_0^1} ds &:= - \int_0^T \int_{\Omega} x \ddot{h}(s) \xi dx ds \\ &- \int_0^T \int_{\Omega} (\theta + F_2''(\varepsilon)) h(s) \xi_x dx ds, \quad \forall \xi \in L^2(0, T; H_0^1(\Omega)), \quad \text{and} \end{aligned} \quad (2.15)$$

$$\begin{aligned} \int_0^T \int_{\Omega} z_2 \varphi dx ds &:= \int_0^T \int_{\Omega} (\theta \varepsilon_t h(s) + \theta \varepsilon \dot{h}(s)) \varphi dx ds, \quad \forall \varphi \in L^2(0, T; L^2(\Omega)), \\ \mathcal{L} : \mathcal{X} \rightarrow \mathcal{Y}, \quad \text{with} \quad \mathcal{L}[(\phi, \psi)] &:= (\mathcal{L}_1(\phi, \psi), \mathcal{L}_2(\phi, \psi)), \end{aligned} \quad (2.16)$$

the linearized state equations (1.15) take the form

$$\text{Find } (\phi, \psi) \text{ such that } \mathcal{L}[(\phi, \psi)] = z \in \mathcal{Y}, \quad (2.17a)$$

$$\phi(x, 0) = \phi_t(x, 0) = 0 = \psi(x, 0), \quad \forall x \in \overline{\Omega},$$

$$\phi(0, t) = \phi(1, t) = 0, \quad \forall t \in [0, T], \quad \phi_{xx}(0, t) = \phi_{xx}(1, t) = 0, \quad \text{a.e. in } (0, T),$$

$$\psi_x(0, t) = 0, \quad -\psi_x(1, t) = \psi(1, t), \quad \text{a.e. in } (0, T). \quad (2.17b)$$

For any continuous linear form  $\Psi(.,.)$  on  $C(\overline{Q}) \times C(\overline{Q})$  we have an unique element  $\bar{v} \in \mathcal{Y}' = L^2(0, T; H_0^1(\Omega)) \times L^2(0, T; L^2(\Omega))$  such that

$$\left( \mathcal{L}[(r, s)], \bar{v} \right)_y = \Psi(r_x, s), \quad \forall (r, s) \in \mathcal{X}, \quad (2.18)$$

because for an element  $(r, s) \in \mathcal{X}$  we have  $(r, s) = \mathcal{L}^{-1}[(z)]$  for the unique element  $z \in \mathcal{Y}$  and  $\|r_x\|_{C(\overline{Q})} + \|s\|_{C(\overline{Q})} \leq C\|(r, s)\|_{\mathcal{X}}, \forall (r, s) \in \mathcal{X}, C > 0$ . We select the following continuous linear form on  $C(\overline{Q}) \times C(\overline{Q})$

$$\begin{aligned} \Psi(\phi_x, \psi) \equiv & \lambda_1 \left[ \int_0^T \int_{\Omega} \left( D_1 \Phi_1(u_x, \theta) \phi_x + D_2 \Phi_1(u_x, \theta) \psi \right) dx ds + \int \frac{\partial \tilde{\sigma}}{\partial \varepsilon} \phi_x d\mu_2 \right. \\ & \left. + \int \varepsilon \psi d\mu_2 + \int \psi d\mu_1, \quad (\phi, \psi) \in \mathcal{X}. \right. \end{aligned} \quad (2.19)$$

Then there exists a unique adjoint state  $v^* = (p^*, q^*) \in \mathcal{Y}'$  such that the following adjoint state equation is satisfied

$$\left( \mathcal{L}[(r, s)], v^* \right)_y = \Psi(r_x, s), \quad \forall (r, s) \in \mathcal{X}. \quad (2.20)$$

This leads to the adjoint system (2.5), and for any solution  $(\phi, \psi)$  of the linearized state equations we have  $\forall m \in \mathcal{K}, h = m - m^*$ ,

$$\begin{aligned} \langle J'(m^*) + [D_m(\theta^*, \mathcal{F}(\mathcal{G}(m^*)))]^* \mu, h \rangle &= \int_0^T \Phi_2'(\ddot{m}(s)) \ddot{h}(s) ds \\ &+ \int_0^T \int_{\Omega} \left( D_1 \Phi_1(u_x, \theta) \phi_x + D_2 \Phi_1(u_x, \theta) \psi \right) dx ds \\ &+ \int \frac{\partial \tilde{\sigma}}{\partial \varepsilon} \phi_x d\mu_2 + \int \varepsilon \psi d\mu_2 + \int \psi d\mu_1 + \int \frac{\partial \tilde{\sigma}}{\partial m} h d\mu_2 \\ &= \int_0^T \Phi_2'(\ddot{m}(s)) \ddot{h}(s) ds + \Psi(\phi, \psi) + \int \frac{\partial \tilde{\sigma}}{\partial m} h d\mu_2 \\ &= \int_0^T \Phi_2'(\ddot{m}(s)) \ddot{h}(s) ds + \int \frac{\partial \tilde{\sigma}}{\partial m} h d\mu_2 \\ &+ \int_0^T \langle \mathcal{L}_1(\phi, \psi)(s), \xi(s) \rangle_{H^{-1} \times H_0^1} ds + \int_0^T \int_{\Omega} \mathcal{L}_2(\phi, \psi) \varphi dx ds \\ &= \int_0^T \Phi_2'(\ddot{m}(s)) \ddot{h}(s) ds + \int \frac{\partial \tilde{\sigma}}{\partial m} h d\mu_2 \\ &+ \int_0^T \langle z_1(s), \xi(s) \rangle_{H^{-1} \times H_0^1} ds + \int_0^T \int_{\Omega} z_2 \varphi dx ds, \end{aligned} \quad (2.21)$$

whence (2.6) follows from (2.10).  $\square$

### 3 Control by Temperature

Now, we study the following problem.

**(CP2)** Minimize  $J(\theta_\Gamma)$ , subject to (1.6),  $\sigma \in \mathcal{S}$  and  $\theta_\Gamma \in \mathcal{U}_{ad}$ .

Here,  $\mathcal{U}_{ad} \subset M_{\theta_\Gamma}$ .  $\mathcal{S}$  is given by

$$\mathcal{S} := \left\{ \sigma \in C(\overline{Q}) \mid c_5 \leq \sigma(x, t) \leq c_6, \quad \forall (x, t) \in \overline{Q} \right\}. \quad (3.1)$$

The cost functional is assumed in the form

$$J(\theta_\Gamma) = \int_0^T \int_\Omega \Phi_1(u_x(x, t)) \, dx \, dt + \int_0^T \Phi_2(\theta_\Gamma(t)) \, dt, \quad (3.2)$$

where  $\Phi_1 \in C^2(\mathbb{R}^2)$ ,  $\Phi_2 \in C^1(\mathbb{R})$ , and where  $\Phi_2$  is convex in its argument. A particular form could be

$$J(g, \theta_\Gamma) = \alpha_1 \|u_x - \overline{u_x}\|_{L^2(Q)}^2 + \alpha_2 \|\theta_\Gamma\|_{L^2(0, T)}^2, \quad (3.3)$$

where  $\alpha_1$ ,  $\alpha_2$ , and  $\alpha_3$  are non-negative constants, and where  $\overline{u_x}$  denotes the desired strain distribution during the evolution of the process. Again, also other cost functionals are conceivable.

The following existence result can be shown with standard compactness arguments as before.

**Theorem 3.1** *Assume that there is at least one admissible control  $\theta_\Gamma$  such that the solution to (1.6) yields  $\sigma \in \mathcal{S}$ . Then there exists an optimal solution to the above control problem.*

We give the necessary conditions of optimality in the following theorem. Since now  $m$  is given, we write  $\mathcal{G}(\theta_\Gamma)$  instead of  $\mathcal{G}(m, \theta_\Gamma)$ .

**Theorem 3.2** *Let  $\theta_\Gamma \in \mathcal{U}_{ad}$  denote any solution to the optimal control problem **(CP2)**, and let  $(u, \theta) = \mathcal{G}(\theta_\Gamma)$ . Then there exist a real number  $\lambda_2 \geq 0$  and a Borel measure  $\mu_3 \in (C(\overline{Q}))'$  with  $\lambda_2 + \|\mu_3\|_{(C(\overline{Q}))'} > 0$  such that  $\int (\hat{\sigma} - \sigma) \, d\mu_3 \leq 0$ ,  $\forall \hat{\sigma} \in \mathcal{S}$ , as well as functions  $(p, q) \in L^2(0, T; H_0^1(\Omega)) \times L^2(0, T; H^1(\Omega))$  satisfying the following optimality conditions.*

*State equations:*

$$\begin{aligned} & \int_0^T \langle u_{tt}(s), \phi(s) \rangle_{H^{-1} \times H_0^1} \, ds + \int_0^T \int_\Omega x \ddot{m}(s) \phi \, dx \, ds + \int_0^T \int_\Omega \left( \theta(u_x + m(s)) \right. \\ & \left. + F_2'(u_x + m(s)) \right) \phi_x \, dx \, ds - \int_0^T \int_\Omega u_{xxx} \phi_x \, dx \, ds = 0, \quad \forall \phi \in L^2(0, T; H_0^1(\Omega)), \end{aligned} \quad (3.4a)$$

$$\theta_t - \theta(u_x + m(t))(u_{xt} + \dot{m}(t)) - \theta_{xx} = g, \quad a.e. \text{ in } Q, \quad (3.4b)$$

$$u(0, t) = u(1, t) = 0, \quad \forall t \in [0, T], \quad u_{xx}(0, t) = u_{xx}(1, t) = 0, \quad a.e. \text{ in } (0, T),$$

$$\theta_x(0, t) = 0, \quad -\theta_x(1, t) = \theta(1, t) - \theta_\Gamma(t), \quad a.e. \text{ in } (0, T), \quad (3.4c)$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \theta(x, 0) = \theta_0(x), \quad \forall x \in \overline{\Omega}. \quad (3.4d)$$

Adjoint state equations:

$$\begin{aligned} & \int_0^T \langle \xi_{tt}(s), p(s) \rangle_{H^{-1} \times H_0^1} ds - \int_0^T \int_{\Omega} \xi_{xxx} p_x dx ds \\ & + \int_0^T \int_{\Omega} \left( ((\theta + F_2''(\varepsilon)) p_x - \theta \varepsilon_t q) \xi_x - \theta \varepsilon q \xi_{xt} \right) dx ds \\ & = \lambda_1 \int_0^T \int_{\Omega} D_1 \Phi_1(u_x) \xi_x dx ds + \int \frac{\partial \tilde{\sigma}}{\partial \varepsilon} \xi_x d\mu_3, \quad \forall \xi \in X_{1,T} \end{aligned} \quad (3.5a)$$

$$\begin{aligned} & \int_0^T \int_{\Omega} \left( q (\varphi_t - \varepsilon \varepsilon_t \varphi) + q_x \varphi_x + \varepsilon p_x \varphi \right) dx ds \\ & + \int_0^T \varphi(1, s) q(1, s) ds = \int \varepsilon \varphi d\mu_3, \quad \forall \varphi \in X_{2,T}. \end{aligned} \quad (3.5b)$$

$$p_t(x, T) = 0, \quad (3.5c)$$

Optimality conditions:

$$\int_0^T \left\{ \Phi_2'(\theta_{\Gamma}(s)) - q(1, s) \right\} l(s) ds \geq 0, \quad l = \hat{\theta}_{\Gamma} - \theta_{\Gamma}, \quad \forall \hat{\theta}_{\Gamma} \in \mathcal{U}_{ad}. \quad (3.6)$$

Again,  $\lambda_2 = 1$  if the Slater condition is satisfied, i.e. there exists some  $\hat{\theta}_{\Gamma} \in \mathcal{U}_{ad}$  such that the unique solution  $(\phi, \psi)$  of the linearized state equations (1.15) corresponding to  $l = \hat{\theta}_{\Gamma} - \theta_{\Gamma}$  satisfies the condition

$$c_5 < \tilde{\sigma}(x, t) + \psi(x, t) \varepsilon(x, t) + \phi_x(x, t) \left( \theta(x, t) + F_2''(\varepsilon(x, t)) \right) < c_6, \quad \forall (x, t) \in \overline{Q}. \quad (3.7)$$

PROOF. The proof to this theorem is analogue to the last one with the difference that the adjoint variable  $q \in L^2(0, T; H^1(\Omega))$ .  $\square$

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