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Jean-Pierre Hennart*and Edmundo del Valle†

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Abstract: Two new classes of nodal methods, respectively weakly and strongly discontinuous ones, are introduced and applied to the neutron transport equations in X-Y geometry in the discrete ordinates approximation. These methods are then applied to approximate the solution of a well-known benchmark problem of the nuclear engineering literature. The results obtained are finally compared to the ones obtained by recent classical nodal methods.

Key-words: Nodal methods, neutron transport equations, discrete ordinates

(Résumé : tsvp)

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* Jean-Pierre.Hennart@inria.fr
† Instituto Politécnico Nacional, Escuela Superior de Física y Matemáticas, Unidad Profesional "Adolfo López Mateos", 07738 México D.F. (MEXICO), Edmundo@esfm.ipn.mx

Unité de recherche INRIA Rocquencourt
Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
Téléphone : (33 1) 39 63 55 11 – Télécopie : (33 1) 39 63 53 30

Sur les Méthodes Nodales en Transport

Résumé : Deux nouvelles classes de méthodes nodales, faiblement et fortement discontinues, sont introduites et appliquées aux équations du transport de neutrons en géométrie X-Y et en ordonnées discrètes. Ces méthodes sont ensuite utilisées pour l'approximation de la solution d'un problème de référence bien connu de la litérature d'ingénierie nucléaire. Les résultats obtenus sont finalement comparés à ceux obtenus par des méthodes nodales classiques récentes.

Mots-clé : Méthodes nodales, équation du transport neutronique, ordonnées discrètes

1 Introduction

One of the most important partial differential equations in the nuclear engineering field is the neutron transport equation in its discrete-ordinates approximation S_N which in $x-y$ geometry reads:

$$L\psi_k \equiv \mu_k \frac{\partial \psi_k}{\partial x} + \nu_k \frac{\partial \psi_k}{\partial y} + \sigma_t \psi_k = \sigma_s \sum_{l=1}^{M} \omega_l \psi_l + S_k \equiv Q_k, k = 1, \ldots, M, \quad (1)$$

where the unknown is ψ_k, the angular neutron flux corresponding to the k-th ray of the S_N approximation, M being the total number of rays considered which is given in this case by $N(N+2)/2$. The domain to be considered is of the union of rectangles type and boundary conditions must also be imposed.

Classically, with nodal methods, the domain of interest is decomposed in relatively large homogeneous regions or "nodes", over which each angular flux ψ is approximated by a generalized interpolant with interpolation parameters which are cell and/or edge Legendre moments.

This unique interpolant is piecewise continuous using polynomial or exponential shape functions, in the case of the so-called *analytical* nodal methods which depend on transverse integration procedures. See e.g. [2]. For a ray in the first quadrant, the possible left and bottom edge parameters are known from the boundary conditions or from the neighboring left and bottom cells. The unknowns are thus the right and top edge parameters as well as the cell ones.

In this paper, we present two "new" (in a sense to be defined later) classes of polynomial nodal methods. In essence, both classes of methods lead to discontinuous approximations as they at most conserve some edge moments between adjacent nodes, as in the case of the first class of methods which we call *weakly discontinuous*. In this case one or several moments of the angular flux are conserved between a given cell and its upstream neighbors. The second class of methods, called *strongly discontinuous*, is fully discontinuous and only has outgoing (at top and right) edge moments as parameters, in addition to possible cell moments.

Before dealing with these methods in detail, we present in the next section some notation and the basic formalism. The two classes of methods are then

developed in two sections and a third one proposes some numerical results, before presenting some conclusions in a last section.

2 Notation and basic formalism

2.1 Notation

Assuming that the domain Ω of the union of rectangles type has been discretized in N_e nodes or rather *cells* or *elements*, i.e. $\Omega \equiv \Omega_h = \cup_{e=1}^{N_e} \Omega_e$, each cell Ω_e is mapped onto a reference cell $\widehat{\Omega} \equiv [-1, +1 \times [-1, +1]$, as it is traditional with finite element methods. A particular finite element is then defined by a set of degrees of freedom D and a space of functions S with $card(D) = dim(S)$. With degrees of freedom which are cell and/or edge moments as in this paper, we shall speak of *nodal finite elements*. For practical purposes, these moments will be taken as Legendre moments.

To describe D and S in a compact way in the *nodal* case, some notation will be helpful.

Let P_i be the normalized Legendre polynomial of degree i over $[-1, +1]$ which satisfies

$$P_i(+1) = 1, \quad P_i(-1) = (-1)^i, \quad \text{and} \quad \int_{-1}^{+1} P_i(x) P_j(x) dx = N_i \delta_{ij}. \tag{2}$$

with $N_i = 2/(2i + 1)$. Define moreover $P_{ij}(x, y)$ as $P_i(x)P_j(y)$. Assuming that $L\psi = Q$ is the given equation, ψ is approximated by ψ_h and over $\widehat{\Omega}$, cell moments of $\psi_h(x, y) \in S$ are defined as follows

$$\psi_C^{ij} = \int_{-1}^{+1} \int_{-1}^{+1} P_{ij}(x, y) \psi_h(x, y)\, dxdy / N_i \cdot N_j. \tag{3}$$

Edge moments are moreover given by

$$\psi_E^i = \int_{-1}^{+1} P_i(s_E) \psi_h(x_E, y_E) ds_E / N_i, \tag{4}$$

where E is a generic symbol corresponding to L, R, B, and T for the left, right, bottom, and top edges respectively, x_E or y_E is ± 1 depending on the particular edge considered, the other coordinate being s_E, the coordinate along that edge.

S is a space of functions, which are in general *polynomials* at least in this paper. To describe them in a systematic way, let us introduce the spaces of polynomials of degree i in x and j in y, $\mathcal{Q}_{ij}(x,y) \equiv \{x^a y^b \mid 0 \leq a \leq i, 0 \leq b \leq j\}$, with in particular $\mathcal{Q}_i \equiv \mathcal{Q}_{ii}(x,y)$ and also the spaces of polynomials of degree i in x *and* y, $\mathcal{P}_i(x,y) \equiv \{x^a y^b \mid 0 \leq a+b \leq i\}$. For each nodal finite element, we shall call $N_p = dim(D)$ the total number of parameters and N_u the number of unknowns which is less than N_p in the weakly discontinuous case, where the interpolation parameters on the left and bottom edges are taken from the neighboring cells or given by the boundary conditions. In the strongly discontinuous case, there are no left and bottom parameters and we have $N_p = N_u$. In the following, each particular method will be assigned a symbol consisting of two capital letters, WD in the weakly discontinuous case and SD in the strongly discontinuous one, indexed by the two numbers N_p and N_u in the first case, and by N_p or N_u indifferently in the second case. In the WD case, $(N_p - N_u)/2$ is the number of edge moments conserved between adjacent cells. In most practical cases, this number is one or two. In both cases, we have programmed all the methods from two to eight unknowns per cell and applied them to multiplicative and nonmultiplicative problems of the nuclear literature. In the weakly discontinuous case, we have given a constructive algorithm to deduce S if D is known [5]. In that paper, we always assumed that we had the same number of edge moments on each pair of opposite edges. This is clearly not true in the strongly discontinuous case and we had to adapt the earlier algorithm to that situation for applications in neutron transport problems. For second order elliptic equations (the diffusion case), extensions of the basic procedure were also studied to provide transition elements of the p type with progressively more edge moments. These two classes of extensions will be presented elsewhere in a paper in preparation. In two dimensions applications as in this paper, the spaces S can be conveniently described with a Pascal triangle in which the different entries ab correspond to the polynomials $P_{ab}(x,y)$ instead of the more usual monomials $x^a y^b$. The definition of the basic polynomial spaces introduced earlier will be modified accordingly, for instance $\mathcal{P}_i(x,y) \equiv \{span(P_{ab}) | 0 \leq a+b \leq i\}$. With these conventions, it is extremely easy to check the unisolvency with respect to the set D of the space S provided by the algorithm. Due to the normalizations adopted, the basis functions can practically be determined by hand and they

are given in a very compact way in terms of the P_{ab}'s. For the details, at least for the WD case, the readers are referred to [5].

We shall give a concrete example, namely the element WD_{53}. In this case, $D \equiv \{\psi_L^0, \psi_R^0, \psi_B^0, \psi_T^0, \psi_C^{00}\}$ and correspondingly $S \equiv \{P_{00}, P_{01}, P_{10}, P_{02}, P_{20}\}$. On each cell, ψ is approximated by

$$\psi_h = \sum_E \psi_E^0 u_E^0 + \psi_C^{00} u_C^{00} \tag{5}$$

where the basis functions have very compact expressions in terms of the P_{ab}'s. For instance, $u_L^0 = -\frac{1}{2}(P_{10} - P_{20})$ and $u_C^{00} = P_{00} - P_{20} - P_{02}$, the other edge basis functions being obtained by changing the sign of x and/or y or by permuting them.

2.2 Basic formalism

Replacing ψ by ψ_h in each cell, a local residual $L\psi_h - Q_h$ arises where Q_h is evaluated from a previous iterate in the standard source iteration procedure which can be accelerated or not. If the ray (μ_k, ν_k) is in the first quadrant, one proceeds cell by cell by a standard *diagonal sweeping* beginning with the first cell in Ω_h seen by the particular ray considered. Consequently, we know from the boundary conditions at the left or bottom of the domain or from the left or bottom neighbors which have been processed previously the edge moments on the left and bottom edges. If as it is the case with the weakly discontinuous methods, some of these moments are conserved between neighbors, they are directly known in either cell. The moment equations we shall now mention make actually appear edge moments which are not interpolation parameters and it is indispensable to know precisely if we must evaluate them in the current cell or in the previous ones. Locally, Legendre moments of the residual $L\psi_h - Q_h$ are taken to obtain as many equation as unknowns. Since L is a first order partial differential operator, its application to a *discontinuous* approximation ψ_h generates delta distributions and the correct way to take them into account is to derive the moment equations over a cell Ω_e which is shifted upstream by ϵ, in the limit of a vanishing ϵ. For a ray in the first quadrant that means that the cell considered is moved slightly downward and

to the left. As $\epsilon \to 0$, boundary terms arise at the left and the bottom of the cell, connecting it to its upstream neighbors or boundaries.

Going back to the WD_{53} example, there are actually two posibilities: in these two cases the zeroth order moment of the residual should be taken , which ensures balance, i.e. *particles conservation*. Let p_{ij} be P_{ij} scaled to the particular cell considered $[x_L, x_R] \times [y_B, y_T]$. The balance equation corresponds to taking the moment of the residual with respect to p_{00}. The missing equations correspond to taking that moment with respect to p_{10} and p_{01} by symmetry or with respect to p_{20} and p_{02}. Both methods work but the first one is more satisfactory, and in fact better, as it fills in the Pascal triangle from the top by taking moments with respect to $\mathcal{P}_1$.

After taking the limit for ϵ, the general final equations read

$$\begin{aligned} & \int_{y_B}^{y_T} \{\mu p_{ij}(x_L, y)[\psi_h(x_L + 0, y) - \psi_h(x_L - 0, y)]\}\, dy \\ + & \int_{x_L}^{x_R} \{\nu p_{ij}(x, y_B)[\psi_h(x, y_B + 0) - \psi_h(x, y_B - 0)]\}\, dx \\ + & \int_{\Omega_e} p_{ij}(x, y)(L\psi_h - Q_h)\, dxdy \quad = \quad 0. \end{aligned} \tag{6}$$

The first two lines in (6) correspond to the boundary terms at the left and bottom of the cell. For the first of them, since $p_{ij}(x_L, y) = (-1)^i p_j(y)$, we get after some algebra

$$\mu(2i+1)(-1)^i[\psi^j_{L+} - \psi^j_{L-}]/\Delta x, \tag{7}$$

where $\Delta x \equiv x_R - x_L$ while $\psi^j_{L\pm}$ is the jth left edge moment of ψ_h evaluated at $x_L \pm 0$, that is in the cell considered or in the adjacent one on the left. The corresponding expression at the bottom is easily obtained by permuting i and j, x and y, and L and B, to give

$$\nu(2j+1)(-1)^j[\psi^i_{B+} - \psi^i_{B-}]/\Delta y, \tag{8}$$

The third line in (6) corresponds to cell moments of the residual. For $ij = 00$ for instance, we get the cell balance equation

$$\frac{\mu}{\Delta x}[\psi^0_{R-} - \psi^0_{L+}] + \frac{\nu}{\Delta y}[\psi^0_{T-} - \psi^0_{B+}] + \sigma_t \psi^{00}_C = Q^{00}_C. \tag{9}$$

The corresponding $ij = 10$ and $ij = 01$ equations are respectively given by

$$\frac{3\mu}{\Delta x}[\psi^0_{R-} + \psi^0_{L+} - 2\psi^{00}_C] + \frac{\nu}{\Delta y}[\psi^1_{T-} - \psi^1_{B+}] + \sigma_t \psi^{10}_C = Q^{10}_C \tag{10}$$

and

$$\frac{\mu}{\Delta x}[\psi^1_{R-} - \psi^1_{L+}] + \frac{3\nu}{\Delta y}[\psi^0_{T-} + \psi^0_{B+} - 2\psi^{00}_C] + \sigma_t \psi^{01}_C = Q^{01}_C. \tag{11}$$

These last two equations have been obtained using the expression for ψ_h given by (5) with the explicit forms of u^0_E and u^{00}_C, as well as the fact that $P'_3 = 3P_1$. Notice that as soon as i or j are equal to 1, some edge and cell moments of ψ_h appear, which are not among the original interpolation parameters.

With classical nodal methods (see e.g. [2] and the references it contains) polynomial (or exponential) shapes are assumed for the angular flux at the surface and in the interior of each cell *separately*, leading to schemes typically described (in part at least) by two capital letters, like for instance C-Q for Constant-surface-Quadratic-interior, L-L for Linear-surface-Linear-interior, etc.

Here we present "new" nodal methods: with respect to the classical nodal schemes, their main distinctive feature is that all the information contained at the surface *and* in the interior of the cells is integrated into a unique interpolant which is piecewise continuous. One of the advantages of this approach is that *all* the possible edge and cell moments of the resulting representation are *unambiguously* defined in the cell considered (and its upward neighbors if necessary). If we combine (7) and (8) with (9), (10), or (11) following (6), we get a final set of equations which are identical to the former ones except that everywhere ψ^i_{L+} and ψ^i_{B+} with $i = 0$ or 1 are replaced by ψ^i_{L-} and ψ^i_{B-} respectively. It then remains to evaluate in the cell considered or in its upward neighbors the cell and edge moments which are not interpolation parameters, in terms of the interpolation parameters of the common cell. This is a straightforward operation and we have for instance $\psi^{10}_C = (\psi^0_{R-} - \psi^0_{L+})/2$ and $\psi^1_{T-} = (\psi^0_{R-} - \psi^0_{L+})/2$.

Before leaving this section, we want to add a few words about the selection of the N_u moment equations needed. The possible moments are with respect to any P_{ij} included in S. In the strongly discontinuous case, since we have $N_u =$

$N_p = dim(S)$, there is only one possibility but in the weekly discontinuous case several choices of equations are possible as mentioned above for the WD_{53} example. The criteria used are the following ones:

1. The moment (00) must always be included to ensure particle balance.
2. If the moment (ij) is taken, then by symmetry the moment (ji) must also be considered.
3. For some of the spaces S considered where it may happen that the P_{ab}'s do not all appear independently (See [5]), some equations are built by taking the difference between moments (ab) and (ba) of the neutron transport equation (1).
4. The algebraic system obtained for a given set of moments must not be singular in the case of an infinite medium without scattering, i.e. for the equation $\sigma_t \psi_k = Q_k$.
5. The moments of the neutron transport equation considered should ideally fill the Pascal triangle from the top.

If these features are taken into account, the final system generally provides good numerical results as we shall see.

3 Weakly discontinuous methods

In the case of the weakly discontinuous family of schemes, we have programmed all the methods from two to eight unknowns per cell and applied them to different benchmark problems of the nuclear literature. In the following we shall describe them briefly by giving in each case the set D of interpolation parameters and the corresponding space S. In all the following examples, E stands for L, R, B, and T and we have:

$$
\begin{aligned}
WD_{42} &= [D \equiv \{\psi_E^0\}, S \equiv \{\mathcal{P}_1 \oplus (P_{20} - P_{02})\}], \\
WD_{53} &= [D \equiv \{\psi_E^0, \psi_C^{00}\}, S \equiv \{\mathcal{P}_1 \oplus (P_{20}, P_{02})\}],
\end{aligned}
$$

$$
\begin{aligned}
WD_{64} &= [D \equiv \{\psi_E^0, \psi_C^{00}, \psi_C^{11}\}, S \equiv \{\mathcal{P}_2\}],\\
WD_{95} &= [D \equiv \{\psi_E^0, \psi_E^1, \psi_C^{00}\}, S \equiv \{\mathcal{P}_2 \oplus (P_{21}, P_{12}, P_{31} - P_{13})\}],\\
WD_{86} &= [D \equiv \{\psi_E^0, (\psi_C^{ij}, i, j = 0, 1)\}, S \equiv \{\mathcal{P}_2 \oplus (P_{30}, P_{03})\}],\\
WD_{11,7} &= [D \equiv \{\psi_E^0, \psi_E^1, (\psi_C^{ij}, i + j = 0, 1)\}, S \equiv \{\mathcal{P}_3 \oplus (P_{31} - P_{13})\}],\\
WD_{12,8} &= [D \equiv \{\psi_E^0, \psi_E^1, (\psi_C^{ij}, i, j = 0, 1)\}, S \equiv \{\mathcal{P}_3 \oplus (P_{31}, P_{13})\}].
\end{aligned}
$$

With respect to the approximation properties of the schemes described above, all depends on the greatest k such that $\mathcal{P}_k \in S$. With the notation used to describe the different spaces S, it is clear that WD_{42} and WD_{53} contain $\mathcal{P}_1$ but not $\mathcal{P}_2$. Similarly WD_{64}, WD_{95}, and WD_{86} contain $\mathcal{P}_2$ while the last two spaces contain $\mathcal{P}_3$.

A particular scheme is completely defined if we specify the moments of the residual which are taken in (6) to yield a system of linear algebraic equations of order N_u. As mentioned above, these choices are not always unique. In the following, we give the ones chosen to produce the numerical results exhibited later which essentially satisfy the criteria formulated at the end of the previous section. In each case, before the symbol of the method, we give the space M spanned by the Legendre polynomials used to weight the residual in (6), following the conventions defined above. The result is:

$$
\begin{aligned}
M \equiv \{\mathcal{P}_0, \oplus(P_{10} - P_{01})\} &\quad \text{for} \quad WD_{42},\\
M \equiv \{\mathcal{P}_1\} &\quad \text{for} \quad WD_{53},\\
M \equiv \{\mathcal{Q}_1\} &\quad \text{for} \quad WD_{64},\\
M \equiv \{\mathcal{Q}_1 \oplus (P_{31} - P_{13})\} &\quad \text{for} \quad WD_{95},\\
M \equiv \{\mathcal{Q}_1 \oplus (P_{20}, P_{02})\} &\quad \text{for} \quad WD_{86},\\
M \equiv \{\mathcal{P}_2 \oplus (P_{21} - P_{12})\} &\quad \text{for} \quad WD_{11,7},\\
M \equiv \{\mathcal{P}_2 \oplus (P_{21}, P_{12}\} &\quad \text{for} \quad WD_{12,8}.
\end{aligned}
$$

4 Strongly discontinuous methods

In the case of the strongly discontinuous methods, we also programmed all the methods with 2 to 8 (actually 10) unknowns per cell. With the notations of the

previous section, they are very easy to describe. To each weakly discontinuous method $WD_{N_pN_u}$, corresponds a strongly discontinuous method SD_{N_u} with the same number of unknowns. Here $N_u = N_p$ as there are no edge moments on the ingoing edges, that is the left and bottom edges for a ray in the first quadrant. The set D of interpolation parameters for each SD_{N_u} is the same as for $WD_{N_pN_u}$, *provided* E stands for R and T only, corresponding to the outgoing edges. This set will be called $D^*_{N_pN_u}$. For each case, we also need to specify the space S which turns out to be identical to the space M of the weighting Legendre polynomials and we have:

$$
\begin{aligned}
SD_2 &= [D^*_{42}, S \equiv \{\mathcal{P}_0 \oplus (P_{10} - P_{01})\}],\\
SD_3 &= [D^*_{53}, S \equiv \{\mathcal{P}_1\}],\\
SD_4 &= [D^*_{64}, S \equiv \{\mathcal{Q}_1\}],\\
SD_5 &= [D^*_{95}, S \equiv \{\mathcal{Q}_1 \oplus (P_{21} - P_{12})\}],\\
SD_6 &= [D^*_{86}, S \equiv \{\mathcal{P}_2\}],\\
SD_7 &= [D^*_{11,7}, S \equiv \{\mathcal{P}_2 \oplus (P_{21} - P_{12})\}],\\
SD_8 &= [D^*_{12,8}, S \equiv \{\mathcal{P}_2 \oplus (P_{21}, P_{12})\}].
\end{aligned}
$$

With the notation used, it is clear that SD_2 only contains $\mathcal{P}_0$ and is therefore not very interesting. SD_3, SD_4, and SD_5 all contain $\mathcal{P}_1$ as do the last three spaces with $\mathcal{P}_2$.

5 Numerical results

All these methods have been tested on a series of multiplicative and nonmultiplicative benchmark problems of the nuclear literature. A first report on the weakly discontinuous methods appeared in the master thesis of Filio [4]. The strongly discontinuous methods were developed in the master thesis of Delfin [3]. All these results in their final correct form will be a part of the doctoral thesis of one of us (E. del Valle) which is in preparation.

In this paper, some results will be given for a test problem proposed by Azmy [1], in fact problem 2 of his paper. As described by Azmy, it is a modification of Khalil's steel and water problem [6] with a smaller average

scattering ratio to avoid excessively slow convergence of the standard source iteration. The geometric configuration can be described by introducing four square domains $\Omega_i \equiv [0, L_i] \times [0, L_i], i = 0, \ldots, 4$ with $L_1 = 40, L_2 = 50, L_3 = 70$, and $L_4 = 100$. Then four regions I to IV can be defined as follows: $I \equiv \Omega_1, II \equiv \Omega_2 \backslash \Omega_1, III \equiv \Omega_3 \backslash \Omega_2$, and $IV \equiv \Omega_4 \backslash \Omega_3$. Vacuum boundary conditions are specified at the top and right edges, while reflection is imposed at the left and bottom edges. The nuclear data σ_t, σ_s, and the neutron source S are given in Table 1.

Table 1: Nuclear data for Azmy's test problem 2

Region	σ_t	σ_s	S
I	1.0	0.50	1.0
II	0.1	0.01	0.0
III	0.3	0.10	0.0
IV	0.1	0.01	0.0

This problem was solved by Azmy on a sequence of uniform meshes using an S_4 EQN-type angular quadrature, and a pointwise, relative convergence criterion of 10^{-4} on each one of the calculated nodal flux moments. The converged solution was used to calculate the quadrant-averaged scalar fluxes over the four regions. Different methods were used, in particular the Linear Nodal one (LN), which has 7 unknowns ($N_u = 7$) and 11 parameters ($N_p = 11$) per angular direction and node. The method of solution proposed relies on *transverse integration* where zeroth and first order moments of the neutron transport equation (1) are taken on a set of horizontal and vertical slices. The final result is a set of one-dimensional coupled equations, two per horizontal and per vertical slice. These equations are solved *analytically* and exponentials appear in the final expressions which are considerably more complicated than in the purely polynomial case we have considered. Finally no attempt is made to combine the linear edge and cell behaviors, which in the case of the $WD_{11,7}$ method for which results are given below lead to a global interpolant containing $\mathcal{P}_3$ and for which all the extra moments not contained in D are perfectly defined in every cell. The numerical results obtained for different meshes are given in Table 2. The results for the 80×80 mesh can be taken as the reference result.

Table 2: Numerical results obtained for Azmy's test problem 2 using the LN method

Mesh	I	II	III	IV
10×10	1.953	3.650E-1	1.626E-2	2.222E-5
20×20	1.955	3.504E-1	1.556E-2	2.727E-5
40×40	1.957	3 388E-1	1.516E-2	2.696E-5
80×80	1.957	3.339E-1	1.504E-2	2.682E-5

Using different weakly and strongly continuous methods presented in this paper, the Azmy's sample problem 2 was run with the same angular quadrature and stopping criterion. The corresponding numerical results are shown in Table 3. The motivations behind our choice of the results presented is that WD_{64} and SD_4 have the same number of unknowns. On the other hand WD_{64} and SD_6 have the same number of parameters. SD_4 includes $\mathcal{Q}_1$ and thus $\mathcal{P}_1$ but no more. WD_{64}, SD_6, and SD_8 include $\mathcal{P}_2$. $WD_{11,7}$ finally has the same number of parameters and unknowns as LN and includes $\mathcal{P}_3$.

Table 3: Numerical results obtained for Azmy's test problem 2 using the $WD_{64}, WD_{11,7}, SD_4, SD_6$, and SD_8 methods

Mesh	Region	WD_{64}	$WD_{11,7}$	SD_4	SD_6	SD_8
10×10	I	1.957	1.957	1.953	1.955	1.955
	II	3.272E-1	3.329E-1	3.776E-1	3.501E-1	3.493E-1
	III	1.557E-2	1.500E-2	1.463E-2	1.544E-2	1.567E-2
	IV	1.399E-5	2.569E-5	*****	3.636E-5	3.699E-5
20×20	I	1.957	1.957	1.955	1.957	1.956
	II	3.330E-1	3.328E-1	3.546E-1	3.390E-1	3.390E-1
	III	1.508E-2	1.502E-2	1.552E-2	1.511E-2	1.518E-2
	IV	2.873E-5	2.677E-5	2.062E-5	2.722E-5	2.724E-5
40×40	I	1.957	1.957	1.956	1.957	1.957
	II	3.331E-1	3.328E-1	3.418E-1	3.340E-1	3.341E-1
	III	1.504E-2	1.502E-2	1.524E-2	1.503E-2	1.505E-2
	IV	2.695E-5	2.685E-5	2.614E-5	2.684E-5	2.685E-5

6 Conclusions

From Table 3, it is clear that all the methods proposed converge to the reference solution when the mesh is refined. The difference between them is more pronounced in a low flux region like IV when coarse meshes are used. In the case of SD_4, which is clearly the least accurate method, we even get negative (non physical) results in that region. If we compare the results given by WD_{64} and SD_6, with the same number of parameters, 4 unknowns for the first one and 6 for the second one, it turns out that SD_6 is clearly better.

$WD_{11,7}$ on a 40×40 mesh yields much better results than LN on the same mesh, quite comparable with the reference solution on a 80×80 mesh. These results may even be better as LN, at least in regions II and III, does not seem to have converged.

On the same 40×40 mesh, SD_6 and SD_8 which have slightly less or more unknowns are remarkably good, with less parameters and only $\mathcal{P}_2$ included in S: they are actually slightly better than $WD_{11,7}$ with an S containing $\mathcal{P}_3$.

In conclusion, most of the schemes developed are quite valuable. They provide a global representation of the angular fluxes, taking into account all the edge *and* cell information available. Being polynomial, this representation is clearly simpler than the ones based on the exact anlytical solution of the transverse integrated equations. Finally, if moments are needed which do not belong to D, they can always be retrieved in a completely consistent way in terms of the parameters belonging to D.

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Unité de recherche INRIA Lorraine, Technopôle de Nancy-Brabois, Campus scientifique,
615 rue du Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY
Unité de recherche INRIA Rennes, Irisa, Campus universitaire de Beaulieu, 35042 RENNES Cedex
Unité de recherche INRIA Rhône-Alpes, 655, avenue de l'Europe, 38330 MONTBONNOT ST MARTIN
Unité de recherche INRIA Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex
Unité de recherche INRIA Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex

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