A Minimax Optimal Control Problem with Infinite Horizon

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Abstract: A minimax optimal control problem with infinite horizon is considered. Some properties of the optimal cost function u are studied. Among them, the issue of regularity and the characterization of u in terms of the associated Hamilton-Jacobi-Bellman (HJB) equation. Relations between subsolutions and supersolutions of the HJB equation are also analyzed.

Key-words: minimax optimal control problem, infinite horizon, worst case, sub-solutions, supersolutions, Hamilton-Jacobi-Bellman equation.

(Résumé : tsvp)

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Un problème de contrôle optimal de type minimax avec horizon infini

Résumé : On considère ici un problème de contrôle optimal de type minimax avec horizon infini. On étudie des propriétés générales de la fonction u, en particulier, la régularité et l'identification de u comme solution de l'équation de HJB associée. On analyse les relations entre les sur-solutions et les sous-solutions de la même équation.

Mots-clé : problèmes de contrôle optimal de type minimax, horizon infini, sous-solutions, sur-solutions, équation de Hamilton-Jacobi-Bellman.

AMS Classification: Primary: 49A40, 49B36 Secondary: 49C05, 49C20, 49C99

Contents

1 Introduction

Minimax optimal control problems have received considerable interest in the last years. Although they describe several problems in a better way than those with cumulative costs, these problems are not so widespread known as cumulative cost problems due to the difficulty of its analysis.

Barron and Ishii studied the minimax optimal control problem with finite horizon from its continuous point of view in [2]. In [7, 8] we have developed a numerical procedure to approximate the solution of that problem.

This paper deals with the infinite horizon case. General properties of the optimal cost function are studied, among them, the issue of regularity and the approximation with finite horizon problems. We analyze the relations between subsolutions and supersolutions of the associated Hamilton-Jacobi-Bellman equation and we prove that the optimal cost function is the minimum supersolution and the maximum element of a special class of subsolutions.

1.1 Description of the problem

We consider a dynamic system which evolves according to the ordinary differential equation

$$\begin{cases} y'(t) = g(y(t), \alpha(t)) & \forall\ 0 \le t < \infty, \\ \\ y(0) = x, & x \in \Omega \subset \mathbb{R}^m,\ \Omega \text{ an open domain.} \end{cases} \tag{1}$$

The problem consists in minimizing the functional J

$$\begin{array}{rcl} J : \Omega \times \mathcal{A} & \mapsto & \mathbb{R} \\ (x, \alpha(\cdot)) & \mapsto & J(x, \alpha(\cdot)) = \operatorname{ess\,sup}_{t \in [0,\infty)} f(y(t), \alpha(t)). \end{array} \tag{2}$$

The set of admissible control policies is denoted by $\mathcal{A}$,

$$\mathcal{A} = L^\infty((0,\infty); A), \qquad A \subset \mathbb{R}^\nu. \tag{3}$$

We study the optimal cost function

$$u(x) = \inf \{ J(x, \alpha(\cdot)) : \alpha(\cdot) \in \mathcal{A} \}, \tag{4}$$

the regularity of u and the associated Hamilton–Jacobi–Bellman equation in its integral form.

1.2 General assumptions

Let $BUC(\Omega \times A)$ be the family of bounded and uniformly continuous on $\Omega \times A$.

We assume the following hypotheses are satisfied

(H_1) $g : \Omega \times A \mapsto \mathbb{R}^r$, $g \in BUC(\Omega \times A)$,

$$\|g(x,a)\| \leq M_g \,, \quad \|g(x,a) - g(\widehat{x},a)\| \leq L_g \, \|x - \widehat{x}\| \,, \quad \forall\, x,\, \widehat{x} \in \Omega \,,\ \forall\, a \in A \tag{5}$$

(H_2) $f : \Omega \times A \mapsto \mathbb{R}$, $f \in BUC(\Omega \times A)$,

$$0 \leq f(x,a) \leq C_f \,, \quad |f(x,a) - f(\widehat{x},a)| \leq L_f \, \|x - \widehat{x}\| \,, \quad \forall\, x,\, \widehat{x} \in \Omega \,,\ \forall\, a \in A \tag{6}$$

(H_3) The control set A is compact in $\mathbb{R}^\nu$

(H_4) The trajectory $y(t)$ remains in Ω, $\forall\, t \in [0,\infty)$, $\forall\, \alpha \in \mathcal{A}$.

2 The optimal cost function u

As it is usual with infinite horizon problems, the optimal cost function u defined in (4) has poor properties of regularity. In fact, under general hypotheses, it is only possible to prove that u is bounded. Moreover, no semicontinuity property holds as it is shown in the examples presented below.

2.1 Boundedness properties

$B(\Omega)$ denote the family of the bounded functions, in symbols,

$$B(\Omega) = \left\{ v : \Omega \to \mathbb{R} \ : \ \sup_{x \in \Omega} |v(x)| < \infty \right\}.$$

Proposition 2.1 *$u \in B(\Omega)$.*

Proof. From (2) and (4), it follows that u is bounded by the bounds of f, then $0 \leq u \leq C_f$.

$\square$

2.2 Regularity properties

We will present here two examples which show that, in general, the cost u is not upper or lower semicontinuous.

Example 2.1 u **is not upper semicontinuous**

We consider a dynamic system which evolves according to the following differential equation

$$\begin{cases} x'(s) = x(s)(1 - x^2(s)) & s \in [0, \infty)\,, \\ x(0) = x_\circ & x_\circ \in \mathbb{R}\,. \end{cases} \tag{7}$$

The solution is

$$x(s) = \left| \begin{array}{ll} \left(1 + (x_\circ^{-2} - 1)e^{-2s}\right)^{-1/2} & \text{if } x_\circ > 0, \\ -\left(1 + (x_\circ^{-2} - 1)e^{-2s}\right)^{-1/2} & \text{if } x_\circ < 0, \\ 0 & \text{if } x_\circ = 0. \end{array} \right. \tag{8}$$

Let $f(x(s)) = |x(s)|$. It is easy to check that f verifies the hypothesis (6). Moreover, if $x_\circ = 0$, $f(x(s)) = 0$ and if $x_\circ \neq 0$, $|x(s)| \to 1$ when $s \to \infty$. Then, we have

$$J(x_\circ) = u(x_\circ) = \left| \begin{array}{ll} 1 & x_\circ \neq 0, \\ 0 & x_\circ = 0. \end{array} \right. \tag{9}$$

Therefore, in this example, the function u is not upper semicontinuous; see Fig. 1.

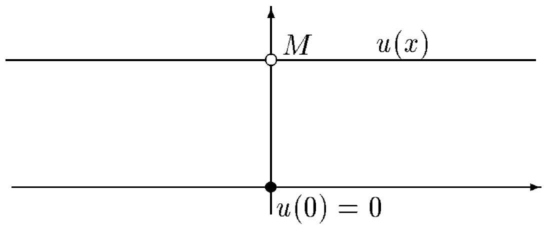

Figure 1: u is not upper semicontinuous

Example 2.2 ***u* is not lower semicontinuous**

Let us consider a dynamic system which evolves according to the differential equation

$$\begin{cases} x_1'(s) = x_1(s)(1 - x_1^2(s)), \\ x_2'(s) = 1/2(x_2(s) + x_3^2(s))(1 - x_2^2(s)), \\ x_3'(s) = \alpha(s), \\ x(0) = (a, 0, 0), \qquad a \in \mathbb{R} \end{cases} \tag{10}$$

where $\alpha(\cdot) \in \mathcal{A}$, $A = \{-1, 1\}$.

Let J be the functional

$$J(x, \alpha(\cdot)) = \sup_{t \in [0,\infty)} \left(-M\,|x_1(t)| + |x_2(t)| + M\right). \tag{11}$$

where $M > 1$.

The component x_1 has the following temporal evolution

$$x_1(s) = \left| \begin{array}{ll} \left(1 + (a^{-2} - 1)e^{-2s}\right)^{-1/2} & \text{if } a > 0, \\ -\left(1 + (a^{-2} - 1)e^{-2s}\right)^{-1/2} & \text{if } a < 0, \\ 0 & \text{if } a = 0. \end{array} \right. \tag{12}$$

It can be easily checked that $x_2(s)$ is an increasing function and $\lim_{s \to \infty} x_2(s) = 1$.

Hence, if $a = 0$,

$$u(0,0,0) = J((0,0,0),\alpha(\cdot)) = \sup_{t\in[0,\infty)} (|x_2(t)| + M) = 1 + M. \tag{13}$$

If $a \neq 0$, $|x_1(\cdot)|$ is an increasing function and $\lim_{s\to\infty} |x_1(s)| = 1$.

Let $\mu \in \mathbb{N}$ and let us define $h = T/\mu$ and the sequence of controls $\alpha_\mu(\cdot)$

$$\left\{\alpha_\mu \in \mathcal{A} : \alpha_\mu(s) = (-1)^i \text{ if } s \in [ih,(i+1)h), i \in \mathbb{N}_\circ\right\},$$

It can be proved that α_μ is a minimizing sequence for the functional J. Then,

$$\lim_{\mu\to\infty} J((a,0,0),\alpha_\mu(\cdot)) = u(a,0,0).$$

Let $0 < |a| < 1 - 1/M$. For μ large enough, $x_3^2(s) < |x_1(s)|$, $\forall\, s \in [0,\infty)$, then, from (10) we have $|x_1(t)| > x_2(t)$, $\forall\, t \in [0,\infty)$, in consequence

$$J((a,0,0),\alpha_\mu(\cdot)) = \sup_{t\in[0,\infty)} (-M\,|x_1(t)| + |x_2(t)| + M) = M(1-|a|). \tag{14}$$

Therefore, the optimal cost u is

$$u(a,0,0) = \left|\begin{array}{ll} 1+M & \text{if } a = 0, \\ & \\ M(1-|a|) & \text{if } a \neq 0, \end{array}\right. \tag{15}$$

function which is not lower semicontinuous, see Fig. 2.

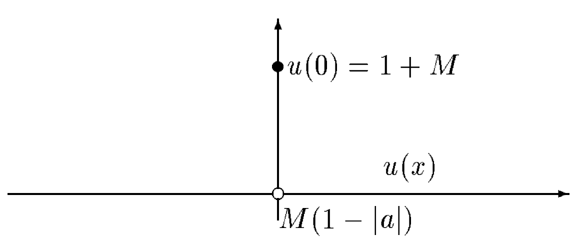

Figure 2: u is not lower semicontinuous

3 Dynamical programming principle

3.1 Auxiliary definitions: Finite horizon problems

We will use some elements and properties of the finite horizon problem studied in [2, 8]. So, we consider the truncated control set $\mathcal{A}_t$, where $t \in [0, \infty)$ and

$$\mathcal{A}_t = L^\infty((0,t);A)\,,$$

the cost functional J_t

$$J_t(x,\alpha(\cdot)) = \operatorname{ess\,sup}_{\tau\in[0,t)} f(y(\tau),\alpha(\tau))\,,$$

where $y(\cdot)$ is the solution of the differential equation (1), and the optimal cost function u_t

$$u_t(x) = \inf\left\{J_t(x,\alpha(\cdot)) : \alpha(\cdot) \in \mathcal{A}_t\right\}. \tag{16}$$

3.2 Dynamical programming equation for u

The optimal cost function satisfies the following dynamical programming principle.

Theorem 3.1 *For all $t \in [0,\infty)$, the function u is a solution of the following dynamical programming equation,*

$$u(x) = \inf_{\alpha(\cdot)\in\mathcal{A}_t} \{\max\{J_t(x,\alpha(\cdot))\,,\ u(y(t))\}\}\,. \tag{17}$$

Proof. For all $x \in \Omega$, $\alpha(\cdot) \in \mathcal{A}$ and $t \in [0,\infty)$, it is valid that

$$J(x,\alpha(\cdot)) = \max\left\{J_t(x,\alpha(\cdot)), \operatorname{ess\,sup}_{s\in[t,\infty)} f(y(s),\alpha(s))\right\}. \tag{18}$$

Since

$$\operatorname{ess\,sup}_{s\in[t,\infty)} f(y(s),\alpha(s)) \geq u(y(t)), \tag{19}$$

we have

$$J(x,\alpha(\cdot)) \geq \max\{J_t(x,\alpha(\cdot)), u(y(t))\}. \tag{20}$$

In the right side of (20), α is involved until time t. Since its restriction to $[0,t)$, which is also denoted by α, belongs to $\mathcal{A}_t$, it is valid that

$$J(x,\alpha(\cdot)) \geq \inf_{\alpha(\cdot)\in\mathcal{A}_t} \{\max\{J_t(x,\alpha(\cdot)), u(y(t))\}\}. \tag{21}$$

As (21) is valid for all α, we have that

$$u(x) \geq \inf_{\alpha(\cdot)\in\mathcal{A}_t} \{\max\{J_t(x,\alpha(\cdot)), u(y(t))\}\}. \tag{22}$$

On the other hand, for all $x \in \Omega$, $\varepsilon > 0$, there exists α_ε such that

$$J(y(t),\alpha_\varepsilon(\cdot)) \leq u(y(t)) + \varepsilon\,. \tag{23}$$

Then,

$$\max\{J_t(x,\alpha(\cdot)), u(y(t))\} \geq \max\{J_t(x,\alpha(\cdot))\,,\; J(y(t),\alpha_\varepsilon(\cdot)) - \varepsilon\}\,. \tag{24}$$

By defining

$$\overline{\alpha}(s) = \begin{cases} \alpha(s) & \text{if } s \in [0,t)\,, \\ \alpha_\varepsilon(s) & \text{if } s \in [t,\infty)\,, \end{cases}$$

we have

$$\max\{J_t(x,\alpha(\cdot))\,, J(y_x(t),\alpha_\varepsilon(\cdot)) - \varepsilon\} \geq J(x,\overline{\alpha}(\cdot)) - \varepsilon \geq u(x) - \varepsilon\,. \tag{25}$$

Since α is arbitrary, it follows that

$$\inf_{\alpha(\cdot)\in\mathcal{A}_t} \{\max\{J_t(x,\alpha(\cdot)), u(y(t))\}\} \geq u(x) - \varepsilon\,. \tag{26}$$

As ε is also arbitrary,

$$\inf_{\alpha(\cdot)\in\mathcal{A}_t} \{\max\{J_t(x,\alpha(\cdot)), u(y(t))\}\} \geq u(x)\,. \tag{27}$$

$\square$

Remark 3.1 It is clear that the dynamical programming equation has not an unique solution. In effect, every constant function greater than C_f is a solution. Then, in this problem, this equation is not enough to characterize the function u.

4 Finite horizon approximations

Proposition 4.1 *The family $\{u_t\}_{t>0}$ defined in (16) is non-decreasing when $t \to \infty$ and it is upper bounded by u.*

Proof. $J_t(\cdot)$ is non-decreasing function of t. Then, if $t < t'$, we have

$$J_t(\cdot) \leq J_{t'}(\cdot) \leq J(\cdot).$$

Since $\mathcal{A}$, $\mathcal{A}_t$ and $\mathcal{A}_{t'}$ have their images on the same set A, it results

$$u_t(\cdot) \leq u_{t'}(\cdot) \leq u(\cdot).$$

Consequently, $\{u_t\}_{t>0}$ is non-decreasing family, with upper bound. Then, it has a limit $\underline{u}$. It is obvious that $\underline{u}(x) \leq u(x)$.

$\square$

Remark 4.1 Generally, it is not true that $\underline{u}(x) = u(x)$ as we show in the following example.

Example 4.1 Let (x_1, x_2) be the dynamic of the system, given by

$$\begin{cases} x_1'(s) = \alpha(s), \\ \\ x_2'(s) = (1 - (x_2(s))^2)(x_2(s) + (x_1(s))^2), \\ \\ (x_1(0), x_2(0)) = x, \quad x \in \mathbb{R} \times \mathbb{R}^+. \end{cases} \tag{28}$$

We define the functional J to be

$$J(x, \alpha(\cdot)) = \sup_{t \in [0,\infty)} x_2(t)\,, \tag{29}$$

where $\alpha(\cdot) \in \mathcal{A}$, and the control set is $A = \{-1, 1\}$.

From (28), for the initial conditions $x = (0, 0)$, it follows that $x_2(\cdot)$ is increasing function which verifies

$$\lim_{t\to\infty} x_2(t) = 1. \tag{30}$$

Therefore, $\forall\, \alpha(\cdot) \in \mathcal{A}$ it results

$$J(0, \alpha(\cdot)) = 1, \tag{31}$$

which implies that

$$u(0) = 1. \tag{32}$$

We consider now the finite horizon problem

$$J_T(x, \alpha(\cdot)) = \max_{t\in[0,T]} x_2(t). \tag{33}$$

Let $\mu \in \mathbb{N}$ and let us define $h = T/\mu$ and the sequence of controls

$$\left\{\alpha_\mu \in \mathcal{A} : \alpha_\mu(s) = (-1)^i \;\; \forall\, s \in [ih, (i+1)h),\; i \in \mathbb{N}_\circ\right\}.$$

Since $x_2(\cdot)$ is increasing function, it follows that

$$J_T(0, \alpha(\cdot)) = x_2(T)\,. \tag{34}$$

Besides, from (28) we have

$$x_2(T) = \int_0^T (1 - x_2^2(t))(x_2(t) + x_1^2(t))\, dt \le \int_0^T (x_2(t) + x_1^2(t))\, dt\,. \tag{35}$$

By using one of the Gronwall inequalities, it results

$$x_2(T) \le e^T \int_0^T x_1^2(t)\, dt \le e^T T \max_{t\in[0,T)} x_1^2(t)\,. \tag{36}$$

By considering only the controls α_μ , we have that

$$\max_{t\in[0,T)} x_1^2(t) \leq h^2,$$

so

$$x_2(T) \leq e^T T h^2. \tag{37}$$

Since $h = T/\mu$, if we take the limit of the right side of (37) when $\mu \to \infty$, it results

$$x_2(T) \leq e^T T h^2 \to 0\,. \tag{38}$$

Then, from (34)

$$u_T(0) = 0 \tag{39}$$

and in consequence,

$$\lim_{T\to\infty} u_T(0) = 0 \neq u(0) = 1\,. \tag{40}$$

Taking in mind the behaviour of u shown in Remark 4.1, we conclude that the infinite horizon problem cannot be approximated by a sequence of finite horizon problems as it is usually done in other control problems. In the following sections, other analytical tools will be employed to study and characterize the optimal cost u.

5 Subsolutions and supersolutions

5.1 The operator M_t and their properties

We define $\forall\ t \in [0,\infty)$, the operator

$$\begin{aligned} M_t : B(\Omega) &\mapsto B(\Omega) \\ w &\mapsto M_t\, w, \end{aligned}$$

such that

$$(M_t\, w)(x) = \inf_{\alpha(\cdot)\in\mathcal{A}_t} \{\max\{J_t(x,\alpha(\cdot))\,,\, w(y(t))\}\}\,. \tag{41}$$

The right side of (41) is the optimal cost corresponding to the finite horizon problem where the functional (42) is minimized

$$\max\{J_t(x,\alpha(\cdot))\,,\, w(y(t))\}\,. \tag{42}$$

Proposition 5.1 *M_t is monotone $\forall\, t\in[0,\infty)$, i.e. $w \geq v$ implies $M_t\, w \geq M_t\, v$.*

Proof. The proof is immediate from the definition of M_t.

□

Now, let us see that $\{M_t\}_{t\geq 0}$ has the following semigroup property.

Proposition 5.2 *If $t<\tau$, it is valid that*

$$M_t\, M_{\tau-t} \;= M_\tau\,. \tag{43}$$

Proof. It is immediate from the definition of $\mathcal{A}_\tau$, that this set can be decomposed in the following way

$$\mathcal{A}_\tau = \mathcal{A}_t \times \mathcal{C}\,, \tag{44}$$

where

$$\mathcal{C} = L^\infty\left((t,\tau);A\right).$$

We consider an operator $F:\mathcal{A}_\tau \to \mathbb{R}$ and we define,

$$\begin{array}{rcl} \overline{F}:\mathcal{A}_t\times\mathcal{C} & \to & \mathbb{R} \\ (\alpha_t,\tilde{\alpha}) & \to & \overline{F}(\alpha_t,\tilde{\alpha}) = F(\alpha_t\cup\tilde{\alpha}), \end{array}$$

where $\cup$ is the concatenation between two functions. With these definitions and taking in mind that any element of $\mathcal{A}_\tau$ can be seen as the concatenation of elements of $\mathcal{A}_t$ and $\mathcal{C}$, it is clear that

$$\inf_{\mathcal{A}_\tau}(F(\alpha)) = \inf_{\mathcal{A}_t}\left(\inf_{\mathcal{C}}\overline{F}(\alpha_t, \tilde{\alpha})\right). \tag{45}$$

Then we have

$$(M_\tau\, w)(x) = \inf_{\mathcal{A}_t}\left\{\inf_{\mathcal{C}}\left\{\max\left\{J_\tau(x,\alpha(\cdot)), w(y(\tau))\right\}\right\}\right\}. \tag{46}$$

Due to

$$J_\tau(x,\alpha(\cdot)) = \max\left\{J_t(x,\alpha(\cdot))\,,\operatorname{ess\,sup}_{\,\theta\in[t,\tau)} f(y(\theta),\alpha(\theta))\right\}, \tag{47}$$

(46) is equivalent to

$$\inf_{\mathcal{A}_t}\left\{\max\left\{J_t(x,\alpha(\cdot)), \inf_{\mathcal{C}}\left\{\max\left\{\operatorname{ess\,sup}_{\,\theta\in[t,\tau)} f(y(\theta),\alpha(\theta)), w(y(\tau))\right\}\right\}\right\}\right\}. \tag{48}$$

Since

$$\inf_{\mathcal{C}}\left\{\max\left\{\operatorname{ess\,sup}_{\,\theta\in[t,\tau)} f(y(\theta),\alpha(\theta))\,,\ w(y(\tau))\right\}\right\} = (M_{\tau-t}\, w)\,(y(t)), \tag{49}$$

from (48) and (49), it results

$$\begin{aligned} M_\tau\, w(x) &= \inf_{\alpha(\cdot)\in\mathcal{A}_t}\left\{\max\left\{J_t(x,\alpha(\cdot))\,,(M_{\tau-t}\, w)(y(t))\right\}\right\} \\ &= M_t\,(M_{\tau-t}\, w)\,(x). \end{aligned} \tag{50}$$

$\square$

5.2 Subsolutions and supersolutions

Definition 5.1 For each $t\in[0,\infty)$, we define the *t–supersolution set* for the equation (41) to be

$$S_t = \{s\in\ B(\Omega) : M_t\, s \le s\}\,,$$

the *t–subsolution set* for the same equation to be

$$W_t = \{w \in B(\Omega) : M_t w \geq w\},$$

and the *t–solution set* to be $S_t \bigcap W_t$.

We also define the *supersolutions* (S) and *subsolutions* (W) sets.

- $S = \bigcap_{t \in [0,\infty)} S_t$

  the function set consisting of t–supersolutions, $\forall\, t \in [0, \infty)$.

- $W = \bigcap_{t \in [0,\infty)} W_t$

  the function set consisting of t–subsolutions, $\forall\, t \in [0, \infty)$.

Remark 5.1 By virtue of Theorem 3.1, $\forall\, t \in [0, \infty)$ the function u is a solution of the fixed point problem

$$(M_t v)(x) = v(x), \tag{51}$$

so, u is always a t-subsolution, a t-supersolution, a subsolution and a supersolution.

6 Characterization of the optimal cost

6.1 In the supersolutions set

The central result of this paragraph is the characterization of the optimal cost as the minimum supersolution.

6.1.1 Preliminary results

To get that property, first we obtain some auxiliary results.

Proposition 6.1 *The following properties are valid*

1. S *is non empty.*
2. $\forall\, s \in S$, *it results* $s \geq 0$.
3. $M_t S \subset S$, $\forall\, t \in [0, \infty)$.

4. *M_t is non-increasing with respect to t when the operator is restricted to S.*

Proof.

1. $S \neq \emptyset$; for C_f (any upper bound of f) it is valid that $M_t C_f = C_f$.

2. Because of the definition of s and $f \geq 0$, we have $\forall\, t \geq 0$,

$$s(x) \geq M_t\, s(x) \geq \inf_{\mathcal{A}_t} \{\max\{0, s(y(t))\}\} \geq 0. \tag{52}$$

3. If $s \in S$, then $s \geq M_t\, s$, $\forall\, t \geq 0$. By the monotony property, we have $\forall\, t \geq 0$, $M_\tau\, s \geq M_\tau\,(M_t\, s) = M_t\,(M_\tau\, s)$, $\forall\, \tau \geq 0$; which means that $M_\tau\, s \in S$, $\forall\, \tau \geq 0$.

4. Let $s \in S$ and t, $\delta \geq 0$. By Proposition 5.2 and by Definition 5.1, it results

$$M_{t+\delta}\, s = M_t\,(M_\delta\, s) \leq M_t\, s\,. \tag{53}$$

□

From 3 and 4 of the previous Proposition, for each $s \in S$, $\{M_t\, s\}_{t \geq 0}$ is a non-increasing set of functions, which is lower bounded. These properties allow us to introduce the following definition.

Definition 6.1 For all $s \in S$, we define the element Ms in the following form

$$Ms = \lim_{t \to \infty} M_t\, s\,. \tag{54}$$

Obviously, it results

$$Ms \leq s, \forall\, s \in S. \tag{55}$$

Proposition 6.2 *The operator M defined in* (54) *holds the following properties*

1. *$MS \subseteq S$.*

2. *M is monotone, i.e. if $s \leq \tilde{s}$, then $Ms \leq M\tilde{s}$.*

3. *Let $\underline{s} = \inf\{s : s \in S\}$. Then, $M\underline{s} = \underline{s}$.*

Proof.

1. Let $s \in S$. By definition of M, we have that $M_t M s \leq M_t M_\tau s$, $\forall\, \tau$. Then, from Proposition 5.2, $M_t M s \leq M_{t+\tau} s$, $\forall\, \tau$. Since $M_{t+\tau} s \leq M_\tau s$, $\forall\, \tau$, we obtain that $M_t M s \leq M_\tau s$, $\forall\, \tau$. By definition of M, it results $M_t M s \leq M s$, $\forall\, t$, that means $M s \in S$.

2. This result follows from (54) and the monotony of M_t.

3. Since $\underline{s} = \inf \{s \,:\, s \in S\}$, it is valid that

$$\underline{s} \leq s,\ \forall\, s \in S.$$

By virtue of monotony property, we have that

$$M_t \underline{s} \leq M_t s \leq s,\ \forall\, s \in S,\ \forall\, t \geq 0.$$

As the previous inequality is valid $\forall\, s \in S$, we get, by definition of $\underline{s}$,

$$M_t \underline{s} \leq \underline{s}, \quad \forall\, t \geq 0.$$

In consequence, $\underline{s} \in S$. Then, from property 1, it results $M\underline{s} \in S$.

As $\underline{s}$ is the infimum of the set S, we obtain

$$\underline{s} \leq M\underline{s}.$$

This property and (55), imply

$$\underline{s} = M\underline{s}. \tag{56}$$

□

6.1.2 The optimal cost as the minimum supersolution

Now, we characterize the optimal cost as the minimum supersolution. Characterizations of this type are frequently used in the field of HJB equations, when there are not available results of uniqueness (see e.g. [4, 5]).

Theorem 6.1 $\underline{s}(x) = u(x)$.

Proof.

- As it was highlighted in Remark 5.1, $\forall\, t \in [0,\infty)$, $M_t\, u = u$, then $u \in S$ and consequently it results $u \geq \underline{s}$.

- To prove $u \leq \underline{s}$, let $s \in S$, $t \in [0,\infty)$ and $t_\nu = \nu t$, $\forall\, \nu \in \mathbb{N}$.

  By definition, we have
$$M_t\, s(x) \leq s(x).$$
  By definition of M_t, given $\varepsilon > 0, \exists\, \alpha_\varepsilon \in \mathcal{A}_t$, such that,
$$\max\left\{s(y(t)), \operatorname{ess\,sup}_{\tau\in[0,t)} f(y(\tau), \alpha_\varepsilon(\tau))\right\} \leq s(x) + \varepsilon\,. \tag{57}$$

  Let $\varepsilon_\nu = \varepsilon/2^\nu$, where $\nu \in \mathbb{N}$ and
$$y(t_\nu) = y(t_{\nu-1}) + \int_{t_{\nu-1}}^{t_\nu} g(y(s), \alpha_{\varepsilon_\nu}(s))\, ds, \tag{58}$$
  we denote by α_{ε_ν} the control such that
$$\max\left\{s(y(t_\nu)), \operatorname{ess\,sup}_{\tau\in[t_{\nu-1},t_\nu)} f(y(\tau), \alpha_{\varepsilon_\nu}(\tau))\right\} \leq s(y(t_{\nu-1})) + \varepsilon_{\nu-1}\,. \tag{59}$$
  We define
$$\overline{\alpha}(\tau) = \alpha_{\varepsilon_\nu}(\tau), \quad \forall\, \tau \in [t_{\nu-1}, t_\nu).$$

  By induction, we can see that
$$\max\{J_{t_\nu}(x, \overline{\alpha}(\cdot)), s(y(t_\nu))\} \leq s(x) + \sum_{i=0}^{\nu-1} \varepsilon_i\,. \tag{60}$$

  For $\nu = 1$, (60) becomes (57). Let us assume that (60) holds for ν and let us prove that it holds for $\nu+1$. By virtue of (59) and by hypothesis of induction, we write,

$$\max\{J_{t_{\nu+1}}(x,\overline{\alpha}(\cdot)), s(y(t_{\nu+1}))\}$$

$$\begin{aligned}
&= \max\left\{J_{t_\nu}(x,\overline{\alpha}(\cdot)), \max\left\{s(y(t_{\nu+1})), \operatorname{ess\,sup}_{[t_\nu,t_{\nu+1})} f(y(\tau),\alpha_{\varepsilon_{\nu+1}}(\tau))\right\}\right\} \\
&\leq \max\{J_{t_\nu}(x,\overline{\alpha}(\cdot)), s(y(t_\nu)) + \varepsilon_\nu\} \\
&\leq \max\{J_{t_\nu}(x,\overline{\alpha}(\cdot)), s(y(t_\nu))\} + \varepsilon_\nu \\
&\leq \max\{J_{t_\nu}(x,\overline{\alpha}(\cdot)), s(y(t_\nu))\} \leq s(x) + \sum_{i=0}^{\nu} \varepsilon_i\,.
\end{aligned}$$

Then, $\forall\ n \in \mathbb{N}$,

$$J_{t_n}(x,\overline{\alpha}(\cdot)) \leq s(x) + \sum_{i=0}^{n-1} \varepsilon_i \leq s(x) + 2\varepsilon. \tag{61}$$

By taking limit in the left side of the previous inequality, it becomes

$$J(x,\overline{\alpha}(\cdot)) \leq s(x) + 2\varepsilon. \tag{62}$$

Therefore,

$$u(x) \leq s(x) + 2\varepsilon\,. \tag{63}$$

Since ε is arbitrary, making it go to zero, it results

$$u(x) \leq s(x)\,. \tag{64}$$

The inequality (64) holds $\forall\ s \in S$, so does it for $\underline{s}$.

$\square$

6.2 In the subsolutions set

In this paragraph we characterize $\underline{s}$ to be the maximum element of a special subset of subsolutions. Similarly to what was done in the previous section, we first obtain some auxiliary results.

Proposition 6.3 *The following properties are verified*

1. $0 \in W$.
2. $M_t W \subseteq W$, $\forall\, t \geq 0$.
3. *Let* $\{w_p\}_{p \in I} \subseteq W$ *and* $\hat{w}(x) = \sup\{w_p(x) : p \in I\}$, *then* $\hat{w} \in W$.
4. M_t *is non-decreasing with respect to* t *when the operator is restricted to* W.

Proof.

1. From the definition of the operator M_t, it is easy to prove that $\forall\, t \geq 0$, $M_t 0 \geq 0$. Then, $0 \in W$.
2. Let $w \in W$ and $t \geq 0$. From the definition of W, it follows that $M_t w \geq w$. From Proposition 5.2 and the monotony of the operator M_τ, we have that $M_\tau\,(M_t w) = M_t\,(M_\tau w) \geq M_t w$, $\forall\, \tau > 0$. Then, $M_t w \in W$ and therefore, $M_t W \subseteq W$.
3. Let $\{w_p\}_{p \in I} \subseteq W$, and $\hat{w}(x) = \sup\{w_p(x) : p \in I\}$, then $w_p \leq \hat{w}$. From the monotony of the operator M_t, $w_p \leq M_t w_p \leq M_t \hat{w}$. Then, $\hat{w} \leq M_t \hat{w}$ and therefore, $\hat{w} \in W$.
4. This property follows from the Proposition 5.2 and the Definition 5.1. Let $w \in W$ and t, $\delta \geq 0$, then
$$M_{t+\delta}\, w = M_t\,(M_\delta\, w) \geq M_t\, w. \tag{65}$$

□

From the properties 2 and 4 of the previous Proposition, the following operator M is well defined on W.

Definition 6.2 On the set W, we define the operator M to be

$$M w = \lim_{t \to \infty} M_t\, w.$$

Proposition 6.4 *M has the following properties on W*

1. $MW \subseteq W$.
2. *M is monotone on W, i.e. $w \geq v$ implies $Mw \geq Mv$.*

The proof follows from the fact that these properties are valid for all M_t.

Remark 6.1 M is well-defined on $W \bigcap S$, $u \in W \bigcap S$ and $Mu = u$.

In the infinite horizon problem analyzed in this paper, it is not possible to consider all the complete subsolution set because it is not upper bounded. Then, we will only deal with a special subsolution class.

6.2.1 A singular class of subsolutions

Let us consider the following family of sets of subsolutions

$$\mathcal{W} = \{W_0 \subseteq W, \text{ such that conditions (C1) – (C3) are satisfied}\}$$

(C1) $0 \in W_0$.

(C2) $MW_0 \subseteq W_0$.

(C3) Let $\{w_p\}_{p \in I} \subseteq W_0$ and $\hat{w} = \sup\{w_p : p \in I\}$, then $\hat{w} \in W_0$.

Proposition 6.5 *The following properties are valid*

1. $\mathcal{W} \neq \emptyset$,
2. $W_{\underline{s}} = \{w \in W : w \leq \underline{s}\} \in \mathcal{W}$
3. $W_m = \left(\bigcap_{W_0 \in \mathcal{W}} W_0 \right)$ *is the minimum class of $\mathcal{W}$.*

Proof. The validity of Property 1 is obvious, because from the Proposition 6.3, $W \in \mathcal{W}$.

Let us prove that $W_{\underline{s}} \in \mathcal{W}$

1. $0 \in W_{\underline{s}}$ because $0 \in W$ and $0 \leq \underline{s}$ as it was shown in Proposition 2.1.

2. Let $w \in W_{\underline{s}}$. By definition, $w \leq \underline{s}$ and, by monotony of the operator M, we have $Mw \leq M\underline{s}$. As $\underline{s}$ is a fixed point of the operator M, we have $M\underline{s} = \underline{s}$. Therefore, $Mw \leq \underline{s}$. Then, $Mw \in W_{\underline{s}}$ and in consequence, $MW_{\underline{s}} \subseteq W_{\underline{s}}$.

3. Let $\{w_p\}_{p \in I} \subseteq W_{\underline{s}}$ and $\hat{w}$ its supreme, so $w_p \leq \underline{s} \; \forall \; p \in I$. Then, $\hat{w} \leq \underline{s}$. Therefore, $\hat{w} \in W_{\underline{s}}$.

To prove 3, it is enough to show that $W_m \in \mathcal{W}$.

1. Since $0 \in W_0$, $\forall \, W_0 \in \mathcal{W}$, we have that $0 \in W_m$.

2. Let $w \in W_m$. From condition (C2), it is valid that $Mw \in W_0$, $\forall \, W_0 \in \mathcal{W}$. Then, $Mw \in W_m$. Therefore, $MW_m \subseteq W_m$.

3. Let $\{w_p\}_{p \in I} \subseteq W_m$ and $\hat{w}$ its supreme. Then, from condition (C3), it follows that $\hat{w} \in W_0$, $\forall \, W_0 \in \mathcal{W}$. Therefore, $\hat{w} \in W_m$.

□

6.2.2 The optimal cost as the maximum element of W_m

Theorem 6.2 $\underline{s} = \sup \{w \, : \, w \in W_m\}$.

Proof. Let $\widetilde{w} = \sup \{w \, : \, w \in W_m\}$.

From condition (C3), we have that $\widetilde{w} \in W_m$. Then, $\widetilde{w} \in W$. Therefore, $\widetilde{w} \leq M\widetilde{w}$. Moreover, by virtue of condition (C2), $M\widetilde{w} \in W_m$ and $\widetilde{w}$ is the supreme of W_m, it results

$$\widetilde{w} \geq M\widetilde{w} \,. \tag{66}$$

Then, $\widetilde{w}$ is a supersolution of the operator M. Since $\underline{s}$ is the minimum supersolution, we have that $\underline{s} \leq \widetilde{w}$.

To prove $\underline{s} \geq \widetilde{w}$, we consider the set

$$W_{\underline{s}} = \{w \in W : w \leq \underline{s}\} \,. \tag{67}$$

It is clear that $\underline{s} = \sup \{w \, : \, w \in W_{\underline{s}}\}$. By Proposition 6.5 $W_{\underline{s}} \in \mathcal{W}$, so it must be $W_m \subseteq W_{\underline{s}}$ because W_m is the minimum class. Hence, it follows that

$$\underline{s} = \sup\{w : w \in W_{\underline{s}}\} \geq \sup\{w : w \in W_m\} = \widetilde{w}\,. \tag{68}$$

□

A direct consequence of Theorem 6.1 and Theorem 6.2 is the following

Corollary 6.1 *The optimal cost is the minimum supersolution and the maximum element in the minimum closed set of subsolutions W_m. In other words,*

$$u(x) = \underline{s} = \sup\{w : w \in W_m\}\,.$$

7 Final comments

In this work, we have considered the minimax optimal control problem with continuous time and infinite horizon. The analysis of the optimal cost function and its approximated computation present considerable difficulties. Particularly, characterizing it through the HJB equation requires a careful treatment because the optimal cost function has poor properties of regularity.

Taking into account these facts, in this first attempt of analysis we have only analyzed the HJB equation in its integral form and not in its differential form.

Even in its integral form, the HJB equation has not unique solution. For that reason, we have identified the optimal cost function as the minimum supersolution and the maximum element of a special subset of subsolutions.

We have seen that the following property holds,

$$\lim_{t\to\infty} u_t = \underline{u} \leq u\,,$$

property upon which is based the study of the approximation procedure via finite horizon problems. This approach brings up several questions, for instance:

1. under which conditions we have that $\underline{u} = u$,
2. how $\underline{u}$ can be interpreted as the optimal cost of an infinite horizon control problem.

In [9] we deal with these subjects. There, we prove that

(a) Under suitable conditions of compactness (weak-$\star$), we have that

   i. there exists an optimal control,

   ii. it is valid that $\underline{u} = u$.

(b) By itself, the existence of an optimal control does not imply the property $\underline{u} = u$.

(c) Compactness condition seems to be essential. See e.g. Example 4.1, for $\underline{u} < u$.

(d) The function $\underline{u}$ can be interpreted as the optimal cost function corresponding to a minimax problem where the original functional is minimized on a relaxed controls set.

Concerning these results, we wish to remark the following facts:

- The analysis of the function $\underline{u}$ as the increasing limit of a subsolutions sequence starting at 0, and its interpretation as the optimal cost function of a special optimal control problem, is a very well known technique in the literature of this area (see e.g. [4]).
- Relaxation methods are commonly used to obtain a problem with better analytical properties in the areas of *Calculus of variations*, *control theory* and *differential games*. For the minimax problem, we have adapted some results contained in [11, 12, 14, 15] and used techniques and concepts which can be seen in [6, 10, 13]. We want to remark that a similar procedure has been employed by Barron and Jensen in [3] for the finite horizon case.

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References

[1] Barron E.N., *Optimal control and calculus of variations in L^∞*, Optimal Control in Differential Equations, N.H. Pavel (Ed.), Marcel Dekker, New York, 1994.

[2] Barron E.N., Ishii H., *The Bellman equation for minimizing the maximum cost,* Nonlinear Analysis, Theory, Methods and Applications, Vol. 13, N°9, pp. 1067-1090, 1989.

[3] Barron E.N., Jensen R., *Relaxed minimax control,* SIAM J. Control and Optimization, Vol. 33, N°4, pp. 1028-1039, 1995.

[4] Bensoussan A., *Inéquations quasivariationneles avec données non bornées et interprétation probabiliste*, Comptes Rendus Acad. Sc. Paris, Série I, Tome 292, pp. 745-751, 1981.

[5] Bensoussan A., Crouhy M., Proth J.-M., *Mathematical theory of production planning,* North-Holland, New York, 1983.

[6] Bourbaki N., *Éléments de mathématiques. Espaces vectoriels topologiques. Chapitres 1 à 5*, Masson, Paris, 1981.

[7] Di Marco S.C., González R.L.V., *A numerical procedure for minimizing the maximum cost,* Rapport de Recherche N°2454, INRIA, 1995.

[8] Di Marco S.C., González R.L.V., *Une procédure numérique pour la minimisation du coût maximum,* Comptes Rendus Acad. Sc. Paris, Série I, Tome 321, pp. 869-874, 1995.

[9] Di Marco S.C., González R.L.V., *Relaxed minimax optimal control problem – Infinite horizon case,* work in progress.

[10] Ekeland I., Temam R. *Analyse convexe et problèmes variationnels,* Dunod, Paris, 1974.

[11] Friedman A., *Differential games,* Wiley-Interscience, New York, 1971.

[12] Lee E.B., Marcus L., *Foundations of optimal control theory,* John Wiley and Sons, New York, 1967.

[13] Schaefer H.H., *Topological vector spaces*, Springer–Verlag, New York, 1980.

[14] Warga J., *Optimal control of differential and functional equations,* Academic Press, New York, 1972.

[15] Young L.C., *Lectures on the calculus of variations and optimal control theory,* W. B. Saunders Company, Philadelphia, 1969.

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