Approximation of the Value Function for a Class of Differential Games with Target

Odile Pourtallier, Mabel M. Tidball

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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Approximation of the value function for a class of differential games with target

Odile POURTALLIER [*]
and Mabel TIDBALL [**]

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Abstract: We consider the approximation of a class of differential games with target by stochastic games. We use Kruzkov transformation to obtain discounted costs. The approximation is based on a space discretization of the state space and leads to consider the value function of the differential game as the limit of the value function of a sequence of stochastic games. To prove the convergence, we use the notion of viscosity solution for partial differential equations. This allows us to make assumptions only on the continuity of the value function and not on its differentiability. This technique of proof has been used before by M. Bardi, M. Falcone and P. Soravia for another kind of discretization. Under the additional hypothesis that the value function is Lipschitz continuous, we prove that the rate of convergence of this scheme is of order $\sqrt{h}$ where h is the space parameter of discretization. Some numerical experiments are presented in order to test the algorithm for a problem with discontinuous solution.

Key-words: Hamilton Jacobi Bellman Isaacs equation, dynamics programming, approximation schemes, viscosity solutions.

(Résumé : tsvp)

[*] INRIA, Centre Sophia Antipolis, 2004 route des Lucioles BP 93, 06902 Sophia-Antipolis, France
[**] Department of Mathematics, University of Rosario, Pellegrini 250, 2000 Rosario, Argentina

Unité de recherche INRIA Sophia-Antipolis
2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex (France)
Téléphone : (33) 93 65 77 77 – Télécopie : (33) 93 65 77 65

Approximation de la fonction valeur pour une classe de jeux différentiels avec cible

Résumé : On considère l'approximation d'une classe de jeux différentiels avec cible par des jeux stochastiques. On utilise la transformation de Kruzkov pour obtenir des coûts avec taux d'actualisation. Le schéma d'approximation utilisé est basé sur la discrétisation de l'espace d'état et revient à considérer la fonction Valeur du jeu différentiel comme la limite des fonctions Valeur d'une suite de jeux stochastiques. Pour prouver la convergence du schéma, on utilise la notion de solution de viscosité pour les équations aux derivées partielles du premier ordre. Cela nous permet de restreindre les hypothèses sur la fonction Valeur à la continuité, sans se préoccuper de sa différentialité. Les techniques de preuve ont été déjà utilisées par M. Bardi, M. Falcone et P. Soravia pour un autre schéma de discrétisation. Sous l'hypothèse additionnelle de Lipschitzianité de la fonction Valeur, on prouve que la vitesse de convergence du shéma étudié est de l'ordre de $\sqrt{h}$ où h est le paramètre de discrétisation de l'espace. On présente quelques expériences numériques pour tester l'algorithme dans un cas où la fonction valeur est non continue.

Mots-clé : Equation de Hamilton Jacobi Bellman Issacs, programmation dynamique, schéma d'approximation, solution de viscosité

1 Introduction

We study an approximation scheme of the value function for a differential game with target. The value function is the "minmax" time for the state to reach the target, and it is defined in the sense of Varaiya, Roxin, Elliot and Kalton (see [26], [23], [14] and [15]). Under a controllability assumption (the usable part (BUP) of the target is the boundary of the target) Bardi and Soravia proved [5], that the value function is the unique viscosity solution of the Isaacs' equation associated to the game.

The approximation scheme presented in this paper is an adaptation to games of methods developed by H.Kushner, see [19], for stochastic control. The main point is that the scheme uses a space discretization and an approximation of partial derivatives in such a way that transition probabilities appear naturally. The continuous game is then approximated by stochastic discrete state games. A large literature exists on numerical methods (see [22] for a survey). This scheme was used previously for discounted game in open state space without boundary conditions, see [21].

This scheme is basically the same used previously for control problem, see [16] and for stopping time game, see [25]. In these papers the interpretation in terms of controlled Markov chain or stochastic games is not explicited.

Another scheme of discretization has been deeply studied. It uses first a discretization of time, and then a discretization of the state space. This scheme has been used first for control problems, see [9], [10], [2], [3], [6] for the time discretization, [17] for fully discrete problems, and [6], [4] [1] for games with target.

In this paper we prove the convergence of the value function of the discrete game to the viscosity solution of Isaacs' equation of the continuous game. The proof of the convergence uses similar arguments to the ones in [4].

Under the assumption that the value function is Lipschitz continuous, we prove that the rate of convergence of this scheme is of order $\sqrt{h}$ (h being the space discretization parameter). The technique is similar to the one in [25].

We also make a few remarks concerning the comparison on the two schemes and numerical results.

2 The continuous problem

We consider the class of differential games defined by

- the dynamic

$$\begin{cases} \dot{y}(t) = f(y(t), u(t), v(t)), & t > 0 \\ y(0) = x. \end{cases}$$

  Where

  - $y(t) \in I\!R^M$ is the state of the game,
  - $u(t) \in U$, U compact, is the control of the minimizer at time t,

– $v(t) \in V$, V compact, is the control of the maximizer at time t.

We assume that the dynamic function f is a continuous function from $I\!R^M \times U \times V$ to $I\!R^M$ that satisfies :

$$\| f(x,u,v) - f(y,u,v) \| \leq L \| x - y \|, \quad \forall u, v, x, y \tag{1}$$

$$\| f(y,u,v) \| \leq F, \quad \forall u, v, y. \tag{2}$$

($\| \, . \, \|$ denotes the $L_1(I\!R^M)$ norm and L, F are constants). These assumptions insure that the differential equation has an unique solution.

- a closed target $\mathcal{T} \in I\!R^M$. The game will stop whenever the state reaches the target.
- a cost function associated to each pair of functions $(u(.), v(.))$,

$$J(x, u(), v()) = \inf\{t \quad | \quad y(t) \in \mathcal{T}\}.$$

In other words, $J(x,u(),v())$ is the smallest time such that the state reaches the target, if the initial state is x. We set $J(x,u(),v()) = +\infty$ if the state never reaches the target.

- We define the function $T(x)$ as the lower value function of the game in the sense of Varaiya, Roxin, Elliot and Kalton [26], [23], [14]. Namely

$$T(x) = \inf_{\alpha \in A} \sup_{v() \in \mathcal{V}} J(x, \alpha(v()), v()),$$

where $\mathcal{V}$ is the set of measurable functions from $I\!R^+$ to the control set V, and A is the set of non anticipative strategies of the minimizer.

We could define, in the same way, the upper value function of the game, and all the results of this paper would remain valid.

In order to obtain a discounted differential game (which will be important to prove the existence of the approximated solution), we make use of the following Kruzkov transformation

$$V(x) = \begin{cases} 1 - e^{-T(x)} & \text{if } T(x) < +\infty \\ 1 & \text{if } T(x) = +\infty, \end{cases}$$

and the following proposition can be proved (see [18]):

Proposition 2.1 *Assume that the whole boundary of the target is usable (or equivalently that $T(x)$ is continuous on $\partial\mathcal{T}$), then the function V is the unique viscosity solution of the Hamilton-Jacobi-Isaacs equation*

$$\begin{cases} V(x) + \min_{v \in V} \max_{u \in U} \{- < \nabla V(x), f(x,u,v) > -1\} = 0, & \text{if } x \in I\!R^M / \mathcal{T} = \Omega \\ V(x) = 0, \quad \text{if } x \in \mathcal{T} \end{cases} \tag{3}$$

We note

$$H(x, \nabla V(x)) = \min_v \max_u \{- < \nabla V(x), f(x, u, v) > -1\}$$

the hamiltonian of the boundary value problem and recall the definition of a viscosity solution of (3), see [11], [12], [20], [2] and [6].

Definition 2.1 *w is said to be a viscosity subsolution (supersolution) of the boundary value problem (3) if it is a upper (lower) semicontinuous function and if for all $\varphi \in C^1(\overline{\Omega})$ such that $w - \varphi$ attains a local maximum (minimum) point x we have*

$$\begin{array}{ll} w(x) + H(x, \nabla\varphi(x)) \leq 0, & \textit{if } x \in \Omega \\ w(x) + H(x, \nabla\varphi(x)) \leq 0, \quad \textit{or} \quad w(x) \leq 0, & \textit{if } x \in \partial\Omega \end{array} \tag{4}$$

$$\left(\begin{array}{ll} w(x) + H(x, \nabla\varphi(x)) \geq 0, & \textit{if } x \in \Omega \\ w(x) + H(x, \nabla\varphi(x)) \geq 0, \quad \textit{or} \quad w(x) \geq 0, & \textit{if } x \in \partial\Omega \end{array} \right) \tag{5}$$

We say that w is a viscosity solution of (3) if it is both sub and super viscosity solution.

3 Approximation of the differential game by a stochastic game

In this section we describe the stochastic game that will approximate the differential game. The reasoning that leads to this approximation is described in details in [21]. It is an adaptation to the case of deterministic games of a scheme introduced by H.Kushner for stochastic control. The basic idea is to discretize the space and approximate the partial derivative of the dynamics function in such a way that transition probabilities appear.

Let us consider the stochastic game composed of the following elements

- A state space Ω^h.

  Let $\mathbb{R}_h^M$ be the discrete grid defined by :

  $$\mathbb{R}_h^M = \{x \mid x = \sum_{i=1}^{M} \alpha_i h e_i, \alpha_i \in \mathbb{N}\},$$

  where $(e_1, \ldots, e_M)$ is a normal basis of $\mathbb{R}^M$.

  Define the sets $\partial\Omega^h$ and $\overset{o}{\Omega}{}^h$ in the following way :

  $$\partial\Omega^h = \partial\mathcal{T}^h = \{x \in \mathbb{R}_h^M, \qquad x \in \partial\mathcal{T}, \text{ or } \exists i \mid x + e_i h \in \mathcal{T} \text{ or } x - e_i h \in \mathcal{T}\}$$

  $$\overset{o}{\Omega}{}^h = \overline{\mathcal{T}}^c \bigcap \mathbb{R}_h^M - \partial\Omega^h$$

The state space of the stochastic game will be

$$\Omega^h = \overset{o}{\Omega}{}^h \bigcup \partial\Omega^h.$$

and the discrete target is defined by

$$\mathcal{T}^h = I\!R_h^M - \Omega^h.$$

– A transition probability $p(x,y|u,v)$ defined by :

  If $x \in \overset{o}{\Omega}{}^h$ and $\sum_i \mid f_i(x,u,v) \mid \neq 0$

$$\begin{aligned} p(x, x+e_i h|u,v) &= \frac{f_i^+(x,u,v)}{\sum_i \mid f_i(x,u,v) \mid} \\ p(x, x-e_i h|u,v) &= \frac{f_i^-(x,u,v)}{\sum_i \mid f_i(x,u,v) \mid} \\ p(x,y|u,v) &= 0 \text{ for every other } y, \end{aligned} \tag{6}$$

  where $f_i^+ = \sup(0, f_i)$ and $f_i^- = \sup(0, -f_i)$.
  If $x \in \partial\mathcal{T}^h$ or $\sum_i \mid f_i(x,u,v) \mid = 0$

$$\begin{array}{l} p(x,x|u,v) = 1 \\ p(x,y|u,v) = 0, \forall y \neq x \end{array} \tag{7}$$

  $p(x,y|u,v)$ is the probability that the next state of the game will be y, if the present state is x and if the players use the controls u and v at the present time. Let us note that each x of $\partial\mathcal{T}^h$ is an absorbing state. If the system happens to be in a state $x \in \partial\mathcal{T}^h$, then, it will stay on this state indefinitely since the transition probability to any other state is null.

– An instantaneous reward given by

$$\begin{aligned} k^h(x,u,v) &= \frac{h}{\sum_i \mid f_i(x,u,v) \mid + h}, \text{ for } x \in \overset{o}{\Omega}{}^h \\ k^h(x,u,v) &= 0 \text{ for } x \in \partial\Omega^h \end{aligned} \tag{8}$$

  Note that when the system stops on the boundary the total reward does not increase any more since the instantaneous reward is null.

– A discount factor $\beta^h(x,u,v)$ given for each triple (x,u,v)

$$\beta^h(x,u,v) = \frac{\sum_i \mid f_i(x,u,v) \mid}{\sum_i \mid f_i(x,u,v) \mid + h}. \tag{9}$$

This last discount factor is not classical in the stochastic game or controlled Markov chain literature. Nevertheless the main point is that it does not affect the convergence of the classical algorithms such as Shapleys' algorithm, since $\| f \|$ being bounded, $\beta^h(x,u,v)$ is bounded by a scalar strictly smaller than one.

We will denote $V^h(x)$ the value function of this stochastic game, and it is a classical result that it satisfies following equation :

$$V^h(x) = \max_v \min_u \left\{ k^h(x,u,v) + \beta^h(x,u,v) \sum_y p(x,y|u,v) V^h(y) \right\} \tag{10}$$

Note that if x belongs to $\partial\Omega^h$ (10) gives

$$V^h(x) = 0$$

which is compatible with the continuous equation (3). Since the state never reaches the interior of the target (because of the definition of the probabilities), we can set $V^h(x) = 0$ to be compatible with the continuous game.

Proposition 3.1 *The solution V^h of equation (10) exists and is unique.*

Proof
V^h is showed to be the unique fixed point of a contractive operator T^h defined by

$$\begin{array}{cccc} T^h & : & \mathrm{B}(\mathbb{R}_h^M) & \longrightarrow & \mathrm{B}(\mathbb{R}_h^M) \\ & & U & \longrightarrow & T^h U \end{array}$$

with

$$T^h U(x) = \max_v \min_u \left\{ k^h(x,u,v) + \beta^h(x,u,v) \sum_y p(x,y|u,v) U(y) \right\}$$

and $\mathrm{B}(\mathbb{R}_h^M)$ denoting the set of all bounded real-valued functions defined on $\mathbb{R}_h^M$.

Clearly T^h is well defined since $\beta^h(.,.,.)$ and $k^h(.,.,.)$ are bounded :

$$\beta^h(x,u,v) = \frac{\sum_i \mid f_i(x,u,v) \mid}{\sum_i \mid f_i(x,u,v) \mid + h} \leq \frac{1}{1 + \frac{h}{F}} < 1 \tag{11}$$

and

$$k^h(x,u,v) = \frac{h}{\sum_i \mid f_i(x,u,v) \mid + h} \leq 1 \tag{12}$$

Also T^h is a contractive mapping because for all U and W in $\mathrm{B}(I\!\!R_h^M)$

$$\begin{aligned}(T^hU - T^hW)(x) = \quad & \max_v \min_u \left\{ k^h(x,u,v) + \beta^h(x,u,v) \sum_y p(x,y|u,v) U(y) \right\} \\ & -\max_v \min_u \left\{ k^h(x,u,v) + \beta^h(x,u,v) \sum_y p(x,y|u,v) W(y) \right\} \\ \leq \quad & \beta^h(x,\bar{u},\bar{v}) \sum_y p(x,y|\bar{u},\bar{v})(U(y) - W(y))\end{aligned}$$

where $\bar{v}$ satisfies

$$\max_v \min_u \left\{ k^h(x,u,v) + \beta^h(x,u,v) \textstyle\sum_y p(x,y|u,v) W(y) \right\} = \min_u \left\{ k^h(x,u,\bar{v}) + \beta^h(x,u,\bar{v}) \textstyle\sum_y p(x,y|\bar{u},v) W(y) \right\}$$

and $\bar{u}$ is the argument of the following minimization

$$\min_u \left\{ k^h(x,u,\bar{v}) + \beta^h(x,u,\bar{v}) \sum_y p(x,y|u,\bar{v}) U(y) \right\}$$

That leads to

$$(T^hU - T^hW) \leq \beta^h \parallel U - W \parallel$$

where β^h defined by (11) is strictly smaller that one, and the norm used is the *sup* norm : $\parallel U \parallel = \sup_{x \in I\!\!R_h^M} \mid U(x) \mid$.

Similarly it is easy to show that $(T^hW - T^hU) \leq \beta^h \parallel U - W \parallel$. T^h is then a contracting mapping and therefore has an unique fixed point V^h in $\mathrm{B}(I\!\!R_h^M)$ that satisfies equation (10), and this completes the proof.

$\diamond\diamond$

4 Main theorem

We prove in this part the convergence of the value function of the stochastic game to the value function of the differential game. For that we first introduce $\tilde{V}^h$ an interpolation on the closure of the set Ω in the following way. $\tilde{V}^h$ is the restriction to $\overline{\Omega}$ of the affine interpolation $\tilde{\tilde{V}}^h$ of V^h that is

- $\tilde{\tilde{V}}^h(x) = V^h(x)$ for $x \in \Omega^h$

- $\tilde{\tilde{V}}^h$ is affine on each simplex defined by its vertices $\{x, x + e_i h, x \in \overset{o}{\Omega}{}^h, i = 1, \ldots, M\}$ or
$\{x, x - e_i h, x \in \overset{o}{\Omega}{}^h, i = 1, \ldots, M\}$.

For sake of completeness we first state a comparison theorem that we will use later in the proof of the main result of this paper. The proof of this theorem can be found in [6].

Theorem 4.1 *Assume that $\mathcal{T}$ is the closure of an open set and that $\partial\mathcal{T}$ is a Lipschitz surface. Let u_1 and u_2 be bounded functions from $\overline{\Omega}$ to $I\!R$ such that :*

- *i) u_1 is a viscosity subsolution of $u(x) + H(x, \nabla u(x)) = 0$ in the open set Ω, and is continuous and non positive at each point of $\partial\Omega$,*

- *ii) u_2 is a viscosity supersolution of the equation with boundary condition (3).*

Then

$$u_1(x) \leq u_2(x), \quad x \in \overline{\Omega}.$$

The same conclusion holds if u_1 is a viscosity subsolution of (3), and u_2 is a viscosity supersolution of $u(x) + H(x, \nabla u(x)) = 0$ in Ω and is continuous non negative at each point of $\partial\Omega$.

Let us now state and prove the main theorem of this paper

Theorem 4.2 *Assume that the regularity assumptions (1) and (2) hold and that $\mathcal{T}$ is the closure of an open set and that $\partial\mathcal{T}$ is a Lipschitz surface. We have : $\tilde{V}^h$ converges, uniformly on each compact of Ω, to the value function of the continuous game.*

Proof of theorem 4.2 :

As in [4], let us introduce the functions $\overline{V}$ and $\underline{V}$ defined by

$$\overline{V}(x) = \limsup_{\substack{y \to x \\ h \to 0}} \tilde{V}^h(y) \text{ and } \underline{V}(x) = \liminf_{\substack{y \to x \\ h \to 0}} \tilde{V}^h(y),$$

and let us state, and admit for a while, a lemma. The proof of this lemma is the main part of the proof of the theorem.

Lemma 4.1 *$\overline{V}$ and $\underline{V}$ are respectively sub and super viscosity solutions of equation (3).*

On one hand, by definition of $\overline{V}$ and $\underline{V}$, we know that

$$\overline{V} \geq \underline{V}$$

On the other hand, to prove the reverse inequality, we use the same argument as in [4]. We remind it here to make the proof selfcontaining.

Since V, the solution of the continuous problem is continuous and null on the boundary it satisfies the condition $i)$ of theorem 4.1. Applying this theorem we obtain that $V \leq \underline{V}$. With the same argument and the second part of the theorem, we prove that $\overline{V} \leq V$ and then

$$V = \overline{V} = \underline{V}$$

which proves that $\tilde{V} = \lim_{h\to 0} \tilde{V}^h$ is a viscosity solution of (3) and it is the solution of the continuous problem.

$\diamond\diamond$

Let us state and prove a modified version of lemma 6.1 of [13] that we will use in the proof of lemma 4.1.

Lemma 4.2 *Let φ be a $\mathcal{C}^1(\mathbb{R}^M)$ function. Let u^h be a sequence of functions defined on $\mathbb{R}_h^M$, and $\overline{u}$ be the function defined by*

$$\overline{u}(x) = \limsup_{\substack{y\to x\\ h\to 0}} u^h(y)$$

Define x^+ a strict local maximum of $\overline{u} - \varphi$, (let assume that it is the unique maximum in $B(x^+, r)$ for a fixed r),

Then
there exists sequences h_j, x_j such that

- *x_j is a maximum of $u^{h_j} - \varphi$ in $B^{h_j} = B(x^+, r) \bigcap \mathbb{R}_{h_i}^M$,*
- $\lim_{j\to+\infty} x_j = x^+$
- $\lim_{j\to+\infty} u^{h_j}(x_j) = \overline{u}(x^+)$

Proof of the lemma 4.2.
By definition of $\overline{u}(x^+)$, there exists sequences h_n ($h_n \to 0$ *whenn* $\to \infty$) and x_n such that

$$\lim_{n\to+\infty} x_n = x^+, \text{ and } \lim_{n\to+\infty} u^{h_n}(x_n) = \overline{u}(x^+). \tag{13}$$

Notice that for any sequences h_k, x_k such that $\lim_{n\to+\infty} x_k = x^+$, we have

$$\limsup_{k\to+\infty} u^{h_k}(x_k) \leq \overline{u}(x^+). \tag{14}$$

Now, for any h let us define $x^{+,h}$ as the maximum ot $u^h - \varphi$ in B^h, in other words we have

$$u^h(x) - \varphi(x) \leq u^h(x^{+,h}) - \varphi(x^{+,h}), \forall x \in B^h. \tag{15}$$

In particular this equality is true for $h = h_n$ and $x = x_n$, that is :

$$u^{h_n}(x_n) - \varphi(x_n) \leq u^{h_n}(x^{+,h_n}) - \varphi(x^{+,h_n}). \tag{16}$$

Now the sequence x^{+,h_n} is defined on a the compact set, and then we can extract a convergent sub sequence, that we note again x^{+,h_n}. Let us note y the limit of this subsequence. We take the inferior limit in the inequality (16) and obtain,

$$\overline{u}(x^+) - \varphi(x^+) \leq \lim \inf_{j \to +\infty} u^{h_n}(x^{+,h_n}) - \varphi(y).$$

and now using inequality (14) for the sequence x^{+,h_n} we obtain

$$\overline{u}(x^+) - \varphi(x^+) \leq \lim \inf_{j \to +\infty} u^{h_n}(x^{+,h_n}) - \varphi(y) \leq \lim \sup_{j \to +\infty} u^{h_n}(x^{+,h_n}) - \varphi(y) \leq \overline{u}(y) - \varphi(y)$$

and finaly since x^+ is the maximum of $\overline{u} - \varphi$ we can deduce that $y = x^+$ and then

$$\overline{u}(x^+) = \lim_{j \to +\infty} u^{h_n}(x^{+,h_n})$$

which ends the proof the the lemma.

$\diamond\diamond$

Proof of the lemma 4.1

We will only prove that $\overline{V}$ is a viscosity subsolution, since the proof that $\underline{V}$ is a viscosity supersolution is basically the same.

From its definition, it is easy to see that $\overline{V}$ is uper semi continuous. Now, let φ be a $\mathcal{C}^1(\Omega)$ function, and x^+ a local maximum of $\overline{V} - \varphi$. Let us assume that x^+ is a strict local maximum, so we can find $r > 0$ such that x^+ is the unique maximum of $\overline{V} - \varphi$ in the open ball $B = B(x^+, r)$. (If x^+ is not strict, we can modify slightly the test function φ). Two possibilities occur : either x^+ belongs to $\overset{o}{\Omega}$, or it belongs to the boundary $\partial\Omega$.

- Assume first that x^+ belongs to $\overset{o}{\Omega}$. We want to prove that

$$\overline{V}(x^+) + \min_v \max_u \big(- < \nabla\varphi(x^+), f(x^+, u, v) > -1\big) \leq 0.$$

  Let h_n be a sequence such that the conclusion of lemma 4.2 holds, and let x^n be a sequence of B such that x^n is a maximum of $V^{h_n} - \varphi$ in $B^{h_n} = B \cap I\!R^M_{h_n}$, with h_n a sequence such that h_n tends to zero when n tends to infinity. In order to simplify the notation we will write V^n instead of V^{h_n}, and B^n instead of B^{h_n}.

  We have

$$(V^n - \varphi)(x^n) \geq (V^n - \varphi)(x),$$

for all $x \in B^n$. In particular this equality is true for all $x^n + e_i h_n$ and $x^n - e_i h_n$ (which are elements of B^n for n large enough because of lemma 4.2). We can write

$$\sum_y p(x^n, y|u, v)(V^n - \varphi)(x^n) \geq \sum_y p(x^n, y|u, v)(V^n - \varphi)(y)$$

or equivalently

$$\sum_y p(x^n, y|u, v)\left(\varphi(y) - \varphi(x^n)\right) \geq \sum_y p(x^n, y|u, v)\left(V^n(y) - V^n(x^n)\right). \tag{17}$$

On the other hand, developing slightly equation (10) we obtain that for all $x^n \in \overset{o}{\Omega}{}^h$ we have

$$\max_v \min_u \left\{ \frac{\sum_i |f_i(x^n,u,v)|}{h_n + \sum_i |f_i(x^n,u,v)|} \sum_y p(x^n, y|u, v) V^n(y) + \frac{h_n}{h_n + \sum_i |f_i(x^n,u,v)|} - V(x^n) \right\} = 0$$

that is

$$\max_v \min_u \left\{ \frac{\sum_i |f_i(x^n,u,v)|}{h_n} \sum_y p(x^n, y|u, v) V^n(y) - \frac{h_n + \sum_i |f_i(x^n,u,v)|}{h_n} V(x^n) + 1 \right\} = 0$$

and finally

$$\max_v \min_u \left\{ \frac{\sum_i \mid f_i(x^n, u, v) \mid}{h_n} \sum_y p(x^n, y|u, v)\left(V^n(y) - V^n(x^n)\right) - V^n(x^n) + 1 \right\} = 0,$$

and using the monotonicity of the "minmax" function and the inequality (17) we obtain for all n

$$\max_v \min_u \left\{ \frac{\sum_i \mid f_i(x^n, u, v) \mid}{h_n} \sum_y p(x^n, y|u, v)\left(\varphi(y) - \varphi(x^n)\right) - V^n(x^n) + 1 \right\} \geq 0,$$

and again developing the transitions probabilities,

$$\max_v \min_u \left\{ \sum_i f_i^+(x^n, u, v) \frac{\varphi(x^n + e_i h_n) - \varphi(x^n)}{h_n} - \sum_i f_i^-(x^n, u, v) \frac{\varphi(x^n) - \varphi(x^n - e_i h)}{h_n} - \overline{V}(x^n) + 1 \right\} \geq 0$$

Taking the *limsup* when n tends to infinity (that is h_n tends to zero), and using the lemma 4.2 we obtain that

$$\max_v \min_u \left\{ \sum_i f_i^+(x^+,u,v)\frac{\partial \varphi}{\partial x_i}(x^+) - \sum_i f_i^-(x^+,u,v)\frac{\partial \varphi}{\partial x_i}(x^+) - \overline{V}(x^+) + 1 \right\} \geq 0$$

that is

$$\max_v \min_u < \nabla\varphi(x^+), f(x^+,u,v) > -\overline{V}(x^+) + 1 \geq 0.$$

and finally, multiplying by -1, we obtain the required inequality

$$\overline{V}(x^+) + \min_v \max_u \big(- < \nabla\varphi(x^+), f(x^+,u,v) > -1\big) \leq 0.$$

- If x^+ belongs to $\partial\mathcal{T} = \partial\Omega$ we want to prove that one of the following inequalities holds,

  (a) $\quad \overline{V}(x^+) \leq 0,$
  or
  (b) $\quad \overline{V}(x^+) + \min_v \max_u \big(- < \nabla\varphi(x^+), f(x^+,u,v) > -1\big) \leq 0$

  Again we construct the sequence x^n such that x^n is a maximum of $V^{h_n} - \varphi$ in B^{h_n}. Two different cases occur. Either there exists n_0 such that for all $n > n_0$, x^{h_n} belongs to $\overset{o}{\Omega}{}^{h_n}$, in this case the previous reasoning holds and we obtain (b), either there exits a subsequence $(h_m)_m$ of the sequence $(h_n)_n$, such that h_m tends to 0 if m tends to infinity and such that x^{h_m} does not belong to $\overset{o}{\Omega}{}^{h_m}$. In this case $V^{h_m}(x^{h_m}) = 0$ by definition of V^{h_m} on the boundary. Then, since according to the second conclusion on lemma 4.2, the sequence $V^{h_m}(x^{h_m})$ converges to $V(x^+)$, we obtain that $V(x^+) = 0$, and then satisfies (a).

$\diamond\diamond$

5 Convergence rate of the discrete problem

In this section we want to obtain an estimate on the rate of convergence of the approximation scheme. This rate of convergence is computed with the additional hypothesis that V is Lipschitz continuous (let us note L_V it Lipschitz constant). A set of hypothesis that insures that V is Lipschitz continuous can be found in [7]. Again, these hypothesis concern regulatity of $\partial\mathcal{T}$ and dynamics.

In order to simplify the notations, let us introduce the operators W and W^h defined by :

$$(WF)(x) = F(x) + \min_v \max_u (- < \nabla F(x), f(x,u,v) > -1),$$

and

$$(W^h F)(x) = F(x) + \min_v \max_u \left(-\frac{\partial F}{\partial x_f}(x,u,v) - 1 \right),$$

Here $\frac{\partial F}{\partial x_f}(x,u,v)$ stands for the approximation of $\nabla F(x) f(x,u,v)$, that is,

$$\frac{\partial F}{\partial x_f}(x,u,v) \stackrel{\text{def}}{=} \sum_i \left\{ \frac{F(x+e_i h) - F(x)}{h} f_i^+(x,u,v) - \frac{F(x) - F(x+e_i h)}{h} f_i^-(x,u,v) \right\}.$$

With the notations introduced, $V(.)$ is solution of the boundary value problem

$$\begin{cases} WV(x) = 0 \text{ if } x \in \mathbb{R}^M / \mathcal{T} = \Omega, \\ V(x) = 0 \text{ if } x \in \mathcal{T}, \end{cases} \tag{18}$$

and V^h is the solution of the discrete space boundary value problem

$$\begin{cases} W^h V^h(x) = 0 \text{ if } x \in \overset{o}{\Omega}{}^h, \\ V^h(x) = 0 \text{ if } x \in \mathcal{T}^h \cup \partial \mathcal{T}^h. \end{cases} \tag{19}$$

Note that with this notation V^h verifies (10).

To obtain the rate of convergence of the scheme, we want to obtain an upper bound of $\sup_{x \in \mathbb{R}^M_h} \mid V(x) - V^h(x) \mid$. To this aim we use the following decomposition

$$\mid V - V^h \mid \leq \mid V - V^h_\rho \mid + \mid V^h_\rho - V^h \mid, \tag{20}$$

where V_ρ is the regularization of function V, and V^h_ρ is the affine interpolation of the restriction of V_ρ on the discrete space state. With classical results we obtain a upper bound of the first term of the decomposition. A result from [16] together with the interpretation of the function V^h_ρ as a solution of an auxiliary boundary value problem which is "almost" the problem (19), give an upper bound of the second term of the right hand side of (20). Combining these results we obtain the rate of convergence of the problem.

Let $\chi_1(\cdot)$ be the function such that:

$$\chi_1(\cdot) \in \mathcal{C}^\infty(\mathbb{R}^M),$$

$$\chi_1(x) \geq 0,$$

$$\text{if } \parallel x \parallel > 1 \text{ then } \quad \chi_1(x) = 0,$$

$$\int_{\mathbb{R}^M} \chi_1(s) ds = 1.$$

For a strictly positive scalar ρ, we define:

$$\chi_\rho(x) = \frac{1}{\rho^M}\chi_1(x/\rho)$$

and the regularization $V_\rho(.)$ of $V(.)$,

$$V_\rho(x) = (V * \chi_\rho)(x), \quad \forall x \in \Omega \tag{21}$$

where "$*$" means the convolution product, that is

$$(f*g)(x) = \int_{I\!R^M} f(x-y)g(y)dy.$$

We also define the function V_ρ^h as the continuous piecewise affine function of Ω such that :

$$V_\rho^h(x) = V_\rho(x), \quad \forall x \in I\!R_h^M.$$

It is well known that $V_\rho(.) \in \mathcal{C}^\infty(I\!R^M)$, and furthermore, if we assume that $V(.)$ is Lipschitz continuous, we have the classical properties (see [8] for example) :

$$i) \qquad \frac{\partial}{\partial x}V_\rho(x) = \left(\frac{\partial}{\partial x}V * \chi_\rho\right)(x),$$

$$ii) \quad \|\frac{\partial}{\partial x}V_\rho\| \leq \|\frac{\partial}{\partial x}V\| \leq L_V \quad ,$$

$$iii) \quad \|\frac{\partial^2}{\partial x_i \partial x_j}V_\rho\| \leq C\frac{1}{\rho}\|\frac{\partial}{\partial x}V\| \leq C\frac{1}{\rho}L_V,$$

$$iv) \quad |V_\rho(x) - V(x)| \leq L_V\,\rho$$

In (iii), C is a constant that we do not want to precise. We have furthermore the following property, for each x in $I\!R^M$:

$$v) \quad |WV_\rho(x) - (WV * \chi_\rho)(x)| \leq C\rho$$

and from the definition of V_ρ^h and the property (iv) it follows that for $x \in I\!R_h^M$,

$$|V_\rho^h(x) - V(x)| \leq L_V\rho, \tag{22}$$

which gives an upper bound for the first term of our decomposition. The next two theorems, Theorems 5.1 and 5.2, give an upper bound for the second term of the decomposition (20). Theorem 5.1 applies for x in $\overset{o}{\Omega}{}_\rho^h$, and theorem 5.2 applies for x in $\mathcal{T}_\rho^h$, where, the set $\overset{o}{\Omega}{}_\rho^h$ is the complementary in the discrete space of the enlarged discrete target $\mathcal{T}_\rho^h$, that is

$$\mathcal{T}_\rho^h = \left\{x \;/\; d(x,\mathcal{T}^h) \leq \rho\right\}, \text{ and } \overset{o}{\Omega}{}_\rho^h = I\!R_h^M - \mathcal{T}_\rho^h.$$

Theorem 5.1 *For $V^h(.)$ and $V^h_\rho(.)$ as defined previously, and $x \in \overset{o}{\Omega}{}^h_\rho$, we have :*

$$| V^h(x) - V^h_\rho(x) | \leq C\left(\rho + \frac{h}{\rho}\right) \tag{23}$$

where C is a constant, independent of x, that we will not precise.

Before starting the proof of the theorem we want to state a technical lemma:

Lemma 5.1 *For all $x \in I\!\!R^M_h$ we have the following upper bounds*

$$|W^h V^h_\rho(x) - WV_\rho(x)| \leq C\,\frac{h}{\rho}, \tag{24}$$

$$|W^h V^h_\rho(x) - (WV * \chi_\rho)(x)| \leq C\left(\rho + \frac{h}{\rho}\right). \tag{25}$$

Proof of lemma : The proof of (24) By iii) we have that:

$$|\frac{\partial V^h_\rho(x)}{\partial x_f} - < \frac{\partial V_\rho(x)}{\partial x}, f(x,u,v) > | \leq C \parallel \frac{\partial^2}{\partial x_i x_j} V_\rho \parallel h \leq C\frac{h}{\rho}$$

This ends the proof of equation (24). Notice that this proof uses the fact that V is Lipschitz continuous.

Equation (25) is a direct consequence of (24) and of property (v) of V_ρ.

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Proof of theorem : To prove this theorem we first interpret $V^h_\rho(.)$ as the solution of the following auxiliary boundary value problem :

$$\begin{cases} W^h V^h_\rho(x) - \phi(x,h,\rho) = 0, & \text{if } x \in \overset{o}{\Omega}{}^h_\rho, \\ V^h_\rho(x) = \overline{\phi}(x,h,\rho) & \text{if } x \in \mathcal{T}^h_\rho. \end{cases} \tag{26}$$

Here,

$$\phi(x,h,\rho) = (W^h V^h_\rho)(x) - WV * \chi_\rho(x) + WV * \chi_\rho(x).$$

Using equation (25) of lemma 5.1, the fact that $WV * \chi_\rho(x)$ is null on $\overset{o}{\Omega}{}^h_\rho$ because of (18), it comes

$$|\phi(x,h,\rho)| \leq C\left(\rho + \frac{h}{\rho}\right), \quad \forall x \in \overset{o}{\Omega}{}^h_\rho\ .$$

In the same vein, $\overline{\phi}$ has to be written as

$$\overline{\phi}(x,h,\rho) = V^h_\rho(x) - V(x) + V(x) - V(\xi) + V(\xi),$$

for $x \in \mathcal{T}_\rho^h$, where $\xi \in \mathcal{T}^h$ is such that $\| x - \xi \| \leq \rho$. We have $V(\xi) = 0$ from equation (18) and using the fact that $|V(x) - V(\xi)| \leq L_V\ \rho$, we obtain :

$$|\bar{\phi}(x, h, \rho)| \leq C\rho,$$

where again, C is a constant that we do not want to precise. Developing the operator W^h and using definitions (8) of k^h and (9) of β^h, $V_\rho^h(x)$ can be written for all x in Ω^h

$$V_\rho^h(x) = \max_v \min_u \left\{ \tilde{k}^h(x, u, v) + \beta^h(x, u, v) \sum_y p(x, y \mid u, v) V_\rho^h(y) \right\},$$

where

$$\tilde{k}^h(x, u, v) = k^h(x, u, v)(1 - \phi(x, h, \rho)).$$

Combining this last equation with equations (19) and (10), and reminding that for functions $g_1(.,.)$ and $g_2(.,.)$, there exists $\bar{u}$ and $\bar{v}$ such that

$$\max_v \min_u g_1(u, v) - \max_v \min_u g_2(u, v) \leq g_1(\bar{u}, \bar{v}) - g_2(\bar{u}, \bar{v}),$$

we can write for x in Ω^h :

$$\begin{aligned} V^h(x) - V_\rho^h(x) \leq \quad & k^h(x, \bar{u}, \bar{v}) + \beta^h(x, \bar{u}, \bar{v}) \sum_y p(x, y \mid \bar{u}, \bar{v}) V^h(y) \\ & - k^h(x, \bar{u}, \bar{v}) + \beta^h(x, \bar{u}, \bar{v}) \sum_y p(x, y \mid \bar{u}, \bar{v}) V_\rho^h(y). \end{aligned}$$

It follows that

$$\begin{aligned} & V^h(x) - V_\rho^h(x) \leq \\ & \quad \| k^h(., \bar{u}, \bar{v}) - \tilde{k}^h(., \bar{u}, \bar{v}) \| + \beta^h(x, \bar{u}, \bar{v}) \sum_y p(x, y \mid \bar{u}, \bar{v}) \| V^h(.) - V_\rho^h(.) \| . \end{aligned} \tag{27}$$

In the same way we obtain the reverse inequality

$$\begin{aligned} & V_\rho^h(x) - V^h(x) \leq \\ & \quad \| k^h(., \tilde{u}, \tilde{v}) - \tilde{k}^h(., \tilde{u}, \tilde{v}) \| + \beta^h(x, \tilde{u}, \tilde{v}) \sum_y p(x, y \mid \tilde{u}, \tilde{v}) \| V^h(.) - V_\rho^h(.) \|, \end{aligned} \tag{28}$$

and these two last equations together lead to the inequality

$$\begin{aligned} & \| V^h(.) - V_\rho^h(.) \| \leq \\ & \quad \frac{1}{1 - \| \beta^h \|} \| k^h - \tilde{k}^h \| \leq \frac{F + h}{h} \| k^h - \tilde{k}^h \| . \end{aligned} \tag{29}$$

We finally use the fact that

$$\| k^h - \tilde{k}^h \| \leq \| k^h \| \ \| \phi \| \leq \frac{h}{F+h}(\rho + \frac{h}{\rho}),$$

to obtain that for x in $\overset{o}{\Omega}{}^h_\rho$,

$$\| V^h(.) - V^h_\rho(.) \| \leq C(\rho + \frac{h}{\rho}).$$

This ends the proof of theorem 5.1.

$\diamond\diamond$

Remark 5.1 *For a given space discretization h, the optimal value of ρ in (23) is $\rho = \sqrt{h}$.*

Theorem 5.2 *Let $V^h(.)$ and $V^h_\rho(.)$ be as defined previously, if $x \in \mathcal{T}^h_\rho$, we have :*

$$|V^h(x) - V^h_\rho(x)| \leq C\rho.$$

where C is a constant independent of x.

Proof of theorem : The proof of this theorem differs from the proof of the previous theorem only in the definition of the auxiliary boundary value problem. We want to estimate the expresion:

$$\max_v \min_u \left\{ k^h(x,u,v) + \beta^h(x,u,v) \sum_y p(x,y|u,v) V^h_\rho(y) - V^h_\rho(x) \right\} \tag{30}$$

which can be rewritten

$$\begin{aligned} &\max_v \min_u \Big\{ k^h(x,u,v) + \beta^h(x,u,v) \sum_y p(x,y|u,v) V^h_\rho(y) - \\ &\qquad\qquad V^h_\rho(x) + \beta^h(x,u,v) V^h_\rho(x) - \beta^h(x,u,v) V^h_\rho(x) \} \\ &= \max_v \min_u \Big\{ k^h(x,u,v) + \beta^h(x,u,v) \sum_y p(x,y|u,v)(V^h_\rho(y) - V^h_\rho(x)) + \\ &\qquad\qquad (\beta^h(x,u,v) - 1) V^h_\rho(x) \} \end{aligned} \tag{31}$$

As V^h_ρ is a Lipschitz function and $\| x - y \| \leq h$, (since for $\| x - y \| \geq h$, $p(x,y \mid u,v) = 0$), we have $\mid V^h_\rho(y) - V^h_\rho(x) \mid \leq L_V h$. On an other hand, from (26), $|V^h_\rho(x)| \leq C\rho$ for x in $\mathcal{T}^h_\rho$. From that, it follows that for all $x \in \mathcal{T}^h_\rho$, and for all u and v :

$$\begin{aligned} \max_v \min_u |k^h(x,u,v) + \beta^h(x,u,v) \sum_y p(x,y|u,v) V^h_\rho(y) - V^h_\rho(x)| \\ \leq |k^h(x,u,v)| + |\beta(x,u,v)| L_V h + C\rho. \end{aligned}$$

So, for x in $\mathcal{T}_\rho^h$, we have,

$$W^h V_\rho^h(x) - \tilde{\psi}(x, h, \rho) = 0, \qquad |V_\rho^h(x)| \leq C\rho, \tag{32}$$

with $|\tilde{\psi}(x, h, \rho)| \leq C\rho$, or again,

$$V_\rho^h(x.u.v) = \tilde{k}^h(x, u, v) + \beta^h(x, u, v) \sum_y p(x, y \mid u, v) V_\rho^h(y),$$

where

$$\tilde{k}^h(x, u, v) = k^h(x, u, v)(1 - \tilde{\psi}(x, h, \rho)).$$

Now the proof ends with exactly the same computation that previously, using the fact that $V^h(x)$ satisfies the boundary value problem (19).

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The proof of the next theorem that gives the convergence rate of the scheme is now a direct consequence of theorems 5.1 and 5.2.

Theorem 5.3 *If V is a Lipschitz continuous function then:*

$$\|V^h - V\| \leq C\sqrt{h}$$

6 Comparison with another discrete scheme

In [4] the numerical approximation of the same problem is studied. This discretization is done in two steps : first the continuous equation is approximated by a discrete time equation (see [6]) (using an Euler's approximation of the dynamic), and then this equation is approximated by a discrete space equation to obtain the fully discret problem. Thus two parameters are associated to this method. A time parameter k, and a space parameter h. The important feature is that the time parameter is fixed. We show in this section that the scheme of approximation presented in this paper can be interpreted as an extension of this last scheme, considering that the time parameter can be a function of the state. As a matter of fact when using Euler's approximation, it is possible to consider time as a function of the state point and controls.

Let us first describe briefly the fully discrete dynamic programming equation obtained in [4]. We will consider only the equation in the interior of the state space and neglect the boundary problems for this comparison. For x a node of the discretized space we have

$$V(x) = \max_v \min_u e^{-k} \left\{ \sum_i \lambda_i V(x_i) \right\} + 1 - e^{-k}, \tag{33}$$

where the λ_i's are such that $\sum_{i \in S} \lambda_i = 1$, $\lambda_i \geq 0$. S is the set of indices of vertices x_i of the simplex that contains the point $x' = x + kf(x,u,v)$. As in the method presented in this paper, this time and space discretization scheme can also be interpreted in terms of stochastic game. As a matter of fact, the process which leads to equation (33) is the following (see figure 1) : start at node x and let the dynamic evolves for a duration k. The new state, $x' = x + kf(x,u,v)$, is not necessarily a node of the discret space. Let x_i be the vertices of the simplex which contains the point x'. The convex combination of x' in the system of points x_i is considered. The coefficients λ_i defined previously can be interpreted as transition probabilities to go from the point x to the points x_i of the discret space. Nevertheless this last stochastic game is somewhat different to the one obtained with the time discretization scheme. Indeed, in the stochastic game obtained in this paper, the transition probability to go from a point to a non direct neighbor is null. Even if we consider the time parameter k small enough, so that x and x' always belong to the same simplex, the two schemes are differents. Indeed, in the first scheme (the one studied in this paper), the probability to go from x to x is null (or equal to one is the dynamics is null)(see figure 2), which is in general not the case in the second scheme.
Note furthermore that equation (10) can be rewritten as :

$$V(x) = \max_v \min_u \left(1 + \frac{h}{\sum_i |f_i|}\right)^{-1} \left(\sum_i p(x,y|u,v)V(y)\right) + \frac{\frac{h}{\sum_i |f_i|}}{1 + \frac{h}{\sum_i |f_i|}}.$$

We have exactly the same form of equation as equation (33), if we keep in mind that $\frac{1}{1+k}$ is an approximation of e^{-k} and $\frac{k}{1+k}$ an approximation of $1 - e^k$, and if we replace the time parameter k by the state and controls dependent expression : $h/(\sum_i \mid f_i(x,u,v) \mid)$. Note that $h/(\sum_i \mid f_i(x,u,v) \mid)$ can be interpreted as the time necessary for the system to go from the point x to a point at distance h, according to the controls used by the players.

Another remark concerns the convergence condition. As a matter of fact, in [4] the condition

$$lim_{n \to \infty} \frac{h_n}{k_n} = 0, \tag{34}$$

is necessary to prove the convergence of the scheme. In the method presented here this condition is not satisfied since

$$\frac{h}{\frac{h}{\sum_i |f_i|}} = \sum_i |f_i|.$$

A question arises : Is condition (34) a necessary condition for the scheme to converge or is it only a technical condition to make the proof of the convergence simpler ?

Some other points of comparison such that convergence rate, together with comparison with other methods will be presented in a forthcoming paper.