Liveness in Weighted Routed Nets

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Abstract: In this paper, we present an algorithm to check liveness of weighted routed nets. This novel method is based on the total number of event equations for each single input subnet and on a continuous approximation of a Petri net that provides linear equation descriptions.

Key-words: Petri nets, (min,+) algebra, linear algebra

(Résumé : tsvp)

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Vivacité des réseaux pondérés à routage

Résumé : Dans cet article, nous présentons un algorithme pour vérifier la vivacité des réseaux de Petri pondérés avec routage. Cette nouvelle approche s'appuie sur l'équation du nombre total de tirs dans chaque sous-réseau à entrée unique et sur une approximation continue du réseau de Petri qui permet de manipuler des équations linéaires.

Mots-clé : Réseaux de Petri, algèbre (min,+), algèbre linéaire

1 Introduction

The problem of liveness in a Petri Net has been studied as a basic behavioral property since the early work compiled in [Peterson, 1981, Murata, 1989]. Several key results have been obtained over the years. On the negative side, liveness has been proved co-NP-complete for general Petri nets as well as for free choice nets [Jones *et al.*, 1977]. However, liveness is easy to check in marked graphs and in state machines. The Commoner's theorem ([Commoner, 1972]) also provides a characterization for liveness in free choice nets (exp-time in the worse case). More recently, the rank theorem ([Desel and Esparza, 1995]) has provided a polynomial algorithm to check well-formedness (liveness and boundedness) of free choice nets. Some results have been obtained for weighted systems. The case of weighted marked graphs was studied in [Teruel *et al.*, 1992]. The case of weighted equal conflict nets is studied in [Teruel and Silva, 1994] where the rank theorem is extended to weighted systems. In this paper, we present an algorithm to check liveness or structural liveness of weighted nets (possibly unbounded) which is based on the technique developed in [Baccelli *et al.*, 1996, Baccelli and Gaujal, 1996] and a continuous approximation of the net presented in [Cohen *et al.*, 1996, Gaujal, 1996a]. This algorithm also provides a liveness criterion for continuous Petri nets and other properties.

The following is organized as follows. Section 2 presents the definition of weighted timed Petri nets, and the transformations operated on the net to obtain its canonical form the notion of continuous timed Petri nets. Section 3 shows the corresponding counter equations. Section 4 presents the algorithm to test liveness of the continuous model. Section 5 shows how the algorithm is adapted to check structural liveness of the underlying discrete net. Finally, Section 6 shows a condition to check liveness of a discrete net.

2 Weighted Timed Petri Nets

Definition 1 *A weighted Timed Petri net is a 4-tuple $(\mathcal{P}, \mathcal{Q}, C, \mathcal{M}_0)$ where,*

$\mathcal{P}$ is the set of places of size P,

$\mathcal{Q}$ is the set of transitions of size Q,

C is a mapping between places and transitions or between transitions and places into $\mathbb{N}$ the set of integer numbers. $C(p,q)$ (resp. $C(q,p)$) gives the weight on the arc (p,q) (resp. (q,p)). In particular, a zero value corresponds to the absence of arc.

$\mathcal{M}_0$ is the initial marking in the places.

We denote by ${}^\bullet q$ the set $\{p \in \mathcal{P} : C(p,q) \neq 0\}$ (i.e. the set of all input places of q). We define similarly the sets $q^\bullet$, ${}^\bullet p$, $p^\bullet$ as the set of output places of q, the set of input transitions of p and the set of output transitions of p, respectively.

A TPN is *ordinary* if all weights are equal to one.

A *timed* Petri net is a Petri net with temporal data attached to transitions[1] : $\Sigma = \{\sigma_q(n), q \in \mathcal{Q}, n \in \mathbb{N}\}$ where $\sigma_q(n)$ gives the duration of the n-th firing of transition q. If transition q begins to fire for the n-th time at epoch e, this firing will end at epoch $e + \sigma_q(n)$; tokens are then taken out of input places (if they are still available) and put in output places of q according to the firing rule of the un-timed Petri net.

2.1 Routing

When a place p has several output transitions, the presence of tokens in p may create a conflict between its output transitions and one has to choose a policy to solve this conflict. In the following, we will choose the routing policy that solves the conflict by imposing a route to each token entering place p. Place p has a routing sequence $\nu^p : \mathbb{N} \to p^\bullet$, where $\nu^p(n)$ gives the transition $t \in p^\bullet$ to which the n-th token to enter place p is routed.

We say that ν^p is *fair* if p sends tokens to all output transitions infinitely often, *i.e.* $\lim_{k\to\infty} \#\{k, \nu^p(k) = q\} = \infty$, for all $q \in p^\bullet$.

ν^p is compatible with the weights if whenever one token is routed to transition q, $C(p,q)$ consecutive tokens are routed from p to q. Formerly, this means that $\nu^p(k) \neq q, \nu^p(k+1) = q \implies \nu^p(k+i) = q, i = 1, \cdots, C(p,q)$. Note that in this case, we can considered that tokens are routed in bunches of the size of the weights on arcs between places and transitions.

[1] Without loss of modeling power, we assume that the places have no minimal holding times

2.2 Continuous Timed Petri Nets

We now introduce continuous Timed Petri nets (CTPN), in which the marking in places are numbers in $\mathbb{R}^+$ instead of integers. These nets model systems where fluid rather than discrete quantities flow in it. Let $\mathcal{F}$ be a discrete net and F a continuous net with the same topology (calligraphic letters will always refer to discrete nets and capital letters to continuous nets). The difference between the behavior of the two systems is illustrated in the following example (Figure 1).

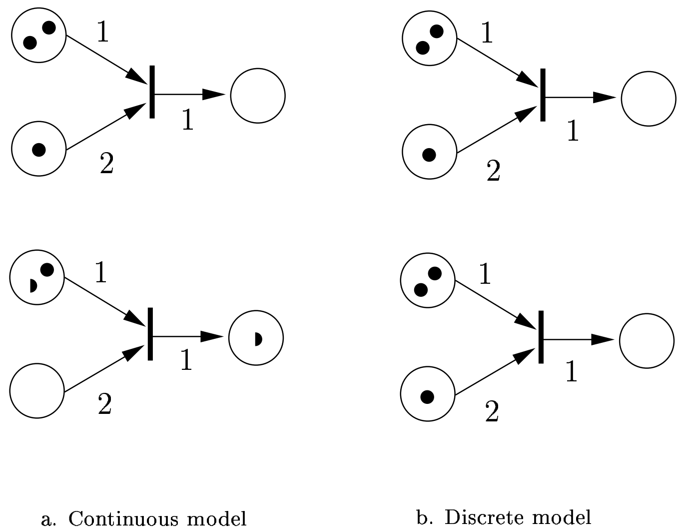

Figure 1: Continuous and discrete nets

In the continuous model, (Figure 1.a), the transition t lets "half" a token go through from the bottom place, while in the discrete model, (Figure 1.b), no token can go through the transition. This is a general property that the continuous model gives an "upper bound" of the discrete one (see [Gaujal, 1996b] for further details on this fact).

2.3 Transformations of the Model

In this section, we present several transformations of the net which will preserve the behavior properties of the net in order to put the net into a canonical form suitable for equational description. In the following figures, ordinary (or simple) arcs have no explicit weights.

- The first transformation $\mathcal{T}_0$ adds a transition and a place for each routing place, to separate the superposition and the choice. This transformation is depicted in Figure 2.
- The next transformation $\mathcal{T}_1$ consists in separating the synchronizations with the weights in incoming arc. Once again, this is done at the net level locally. The transformation is displayed in Figure 3.

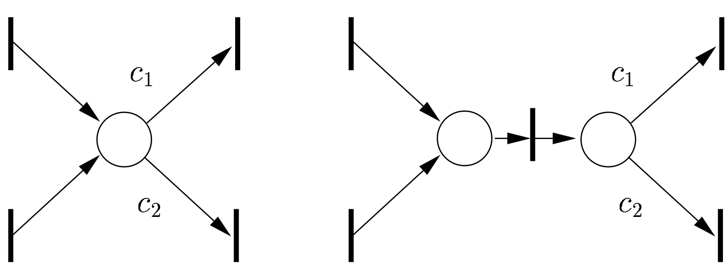

Figure 2: Separation of superposition and choices

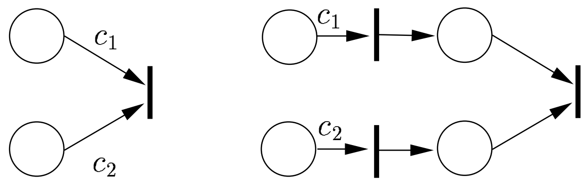

Figure 3: Separation of synchronizations and weights

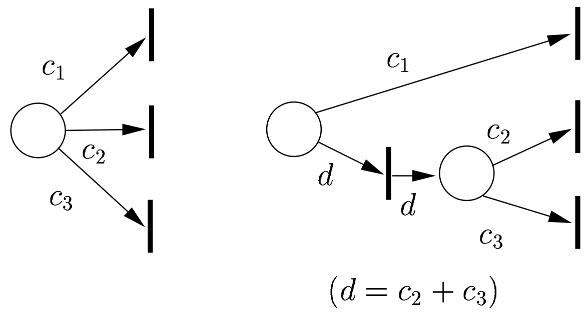

Figure 4: Decomposition into binary choices

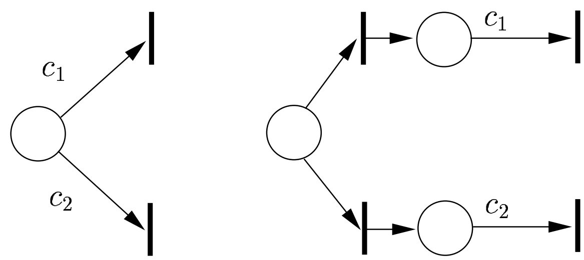

Figure 5: Separation of weights and choices

- A third transformation $\mathcal{T}_2$ of the net decomposes the choices into binary choices by replacing its choice place by a binary tree. The transformation is displayed in Figure 4.
- The last transformation $\mathcal{T}_3$ that we perform on the Petri net is to separate weighted arcs and choices. This can be done by local transformations at the net level, illustrated by Figure 5.

After applying these transformations whenever possible, we obtain a FCNet under *canonical form*. This net has the same behavior as the original net. In the following, we will always consider nets under their canonical form without explicit mention of it.

Note that in canonical nets, non unit weights are only situated on “simple arcs”, that is on arcs from a single output place to a single input transition.

3 Counter Equations

Let S be a timed Petri net. We will assume that S is under canonical form.

In S we partition the set of transitions into two sets:

- $\mathcal{A}$ is the set of all transitions with several input places
- $\mathcal{B}$ is the set $\mathcal{Q} - \mathcal{A}$, the set of all transitions with only one input place.

3.1 The Routing Functions

Definition 2 (routing function) *From the routing sequence ν^p defined for each routing place p, we define for every transition $q \in p^\bullet$, the function $\mathcal{H}_q$ by :*

$$\mathbb{N} \rightarrow \mathbb{N} \tag{1}$$

$$n \rightarrow \left\lfloor \frac{\sum_{i=1}^{n} 1_{\{\nu^{\bullet q}(i)=q\}}}{C(^\bullet q, q)} \right\rfloor. \tag{2}$$

In words, $\mathcal{H}_q(n)$ gives the number of tokens sent to transition q after n tokens have entered place $^\bullet q$.

$\mathcal{H}_q$ is *fair* writes

$$\lim_{n\rightarrow\infty} \mathcal{H}_q(n) = \infty.$$

In CTPN, the routing function $\mathcal{H}$ is replaced by the *filters* h_q.

We define for every transition q, the coefficient h_q that gives the proportion of fluid directed to transition q divided by the weight on arc $^\bullet q, q$.

We say that h is fair if

$$\forall q \in \mathcal{Q},\ h_q \not\equiv 0.$$

CTPN can also be viewed as approximations of discrete systems. If, in the discrete case, the routing function is ergodic, (*i.e.* for all $q \in \mathcal{Q}, \lim_{n\rightarrow\infty} \frac{\mathcal{H}_q(x)}{x}$ exists (noted l_q)), then the *fluid approximation* of the TPN is the CTPN with the same topology and with a filter $h_q = l_q$ for all transition q. In that case, the fluid approximation provides an upper bound for the discrete behavior [Gaujal, 1996a]. The relations between the fluid approximation and its discrete counterpart will also be further discussed in §5 and §6.

3.2 Counters

For a transition q in $\mathcal{A}$, let $Y_q(t)$ denote the *counter* associated with q at time t. $Y_q(t)$ is the number of times transition q has started firing up to time t. Similarly, for a transition $q \in \mathcal{B}$, the counter at time t will be denoted $Z_q(t)$.

We shall consider the case when firing times are all constant, positive, and integer multiples of a common number, which will be taken equal to 1 without loss of generality.

We can now give the dynamic equations satisfied by a TPN. These equations have been established in [Baccelli *et al.*, 1996] for ordinary nets and can be extended to the weighted case.

Theorem 3
Under the above assumptions, for all $k \in \mathbb{Z}$, the counting vectors $\{Y(k), Z(k)\}$ satisfy the following evolution equation:

$$Y(k) = 0, \quad Z(k) = 0, \quad \forall k < 0, \tag{3}$$

and, for $k \geqslant 0$,

$$Y(k) = \bigoplus_{l=1}^{K} (A_l \otimes Y(k-l) \oplus B_l \otimes Z(k-l)) \tag{4}$$

$$Z(k) = \mathcal{H}\left(\sum_{l=1}^{K} (P_l \times Z(k-l) + Q_l \times Y(k-l)) + R\right). \tag{5}$$

In this evolution equation, $(\oplus, \otimes)$ respectively denote matrix products and additions in the $(\min, +)$ semi-ring (see [Baccelli et al.*, 1992]), whereas $(+, \times)$ denote the same operations but in the conventional algebra. The matrices used in the recurrence equations are defined from the net structure as follows:*

- *The $|\mathcal{A}| \times |\mathcal{A}|$ matrix A_l is defined by $A_l(q, q') = m$, if the firing time of $q' \in \mathcal{A}$ is l, and there is a serial place between $q' \in \mathcal{A}$ and q, with $\mathcal{M}_0$-marking equal to m; $\varepsilon(= \infty)$ otherwise. If there are more than one serial places between q' and q, we take c equal to the minimum of their $\mathcal{M}_0$-markings.*

- *The* $|\mathcal{A}| \times |\mathcal{B}|$ *matrix* B_l *is defined by* $B_l(q, q') = m$*, if the firing time of* $q' \in \mathcal{B}$ *is* l*, and if there is a serial place between* $q' \in \mathcal{B}$ *and* $q \in \mathcal{A}$*, with* $\mathcal{M}_0$*-marking equal to* m*;* ε *otherwise.*
- *The* $|\mathcal{B}| \times |\mathcal{B}|$ *matrix* P_l *is defined by* $P_l(q, q') = C(q', (q')^\bullet)$*, if the firing time of* $q' \in \mathcal{B}$ *is* l *and* ${}^\bullet q \in (q')^\bullet$*; 0 otherwise.*
- *The* $|\mathcal{B}| \times |\mathcal{A}|$ *matrix* Q_l *is defined by* $Q_l(q, q') = C(q', (q')^\bullet)$*, if the firing time of* $q' \in \mathcal{A}$ *is* l *and* ${}^\bullet q \in (q')^\bullet$*; 0 otherwise.*
- R *is a* $|\mathcal{B}|$ *vector and* $R_q = \mathcal{M}_0(q)$*.*
- *For all vectors of integers* Z*,* $\mathcal{H}(Z)$ *is the vector of integers:*

$$\mathcal{H}(Z) = (\mathcal{H}_1(Z_1), \dots, \mathcal{H}_q(Z_q)).$$

Proof. These equations are obtained in a way which is similar to that used for establishing the evolution equation for ordinary nets in [Baccelli *et al.*, 1996]. For instance, the number of firings initiated by transition $q \in \mathcal{A}$ at time k cannot exceed the minimum of the number of tokens arrived in the places of ${}^\bullet q$ by time k, which is exactly

$$\bigoplus_{l=1}^{K} \left(A_l \otimes Y(k-l) \oplus B_l \otimes Z(k-l)\right)_q .$$

Furthermore, $Y_q(k)$ is equal to this quantity because transitions are assumed to fire as soon as they are enabled. To obtain Equation (5), the key observation is that, due to our preliminary assumption, a transition q which belongs to $\mathcal{B}$ has at most one input arc, which allows us to write (5), and so the number of firings it initiates by time k is simply the '$\mathcal{H}_q$-filtering' of the total number of arrivals $N_p(k)$ into place $p =^\bullet q$, up to time k, that is

$$N_p(k) = \left(\sum_{l=1}^{K} \left(P_l \times Z(k-l) + Q_l \times Y(k-l)\right) + R(k) \right)_q .$$

□

3.3 Counters For Continuous Timed Petri Nets

As for a CTPN, the counters are defined similarly. If $q \in \mathcal{A}$ (resp. $q \in \mathcal{B}$), we denote by $U_q(k)$ (resp. $V_q(k)$) the quantity of fluid through transition q by time k.

We define the *filter* matrix H which is the fluid counterpart of the vector routing function $\mathcal{H}$,

$$\begin{aligned} H(i,i) &= h_i, \\ H(i,j) &= 0 \ (j \neq i). \end{aligned}$$

Now, the counter evolution equation of a CTPN are:

$$U(k) = \bigoplus_{l=1}^{K} \left(A_l \otimes U(k-l) \oplus B_l \otimes U(k-l)\right) \tag{6}$$

$$V(k) = H \times \left(\sum_{l=1}^{K} \left(P_l \times V(k-l) + Q_l \times U(k-l)\right) + R \right). \tag{7}$$

Note that unlike in the discrete case, Equation (7) is linear in the classical sense.

3.4 Total Firing Equations

Let $Y \stackrel{\text{def}}{=} Y(\infty)$ and $Z \stackrel{\text{def}}{=} Z(\infty)$ denote the vectors counting the total number of firings of the transitions.

Lemma 4 ([Baccelli *et al.*, 1996]) *The integer-valued vectors* Z *and* Y *satisfy the system of equations*

$$Y = A \otimes Y \oplus B \otimes Z \tag{8}$$

$$Z = \mathcal{H}(P \times Z + Q \times Y + R), \tag{9}$$

where $A = \bigoplus_{l=1}^{K} A_l$*,* $B = \bigoplus_{l=1}^{K} B_l$*,* $P = \sum_{l=1}^{K} P_l$ *and* $Q = \sum_{l=1}^{K} Q_l$*.*

As for CTPN, the total firing equation is similar. Let $U \stackrel{\text{def}}{=} U(\infty)$ and $V \stackrel{\text{def}}{=} V(\infty)$ denote the vectors counting the total number of firings of the transitions.

Lemma 5 *The real-valued vectors U and V satisfy the system of equations*

$$U = A \otimes U \oplus B \otimes V \tag{10}$$
$$V = H(P \times V + Q \times U + R), \tag{11}$$

Proof. In [Baccelli *et al.*, 1996] the proof uses the fact that a sequence of increasing integers diverges or remains constant. Here, the proof simply uses the fact that the operators involved are all continuous when we take the limit in k on each side of the equations. □

3.5 Logical Properties

The properties of the underlying un-timed Petri net are called *logical* properties of the TPN. These properties include boundedness, starvation, deadlock freeness, liveness, consistency and many others (see for example [Murata, 1989]) for a survey on un-timed Petri nets. In the following, we will focus on the *liveness* of the net.

The main difference between liveness in a routed Petri net and in a classical Petri net, is the availability of tokens in places with several output transitions. In a routed Petri net, a transition q is enabled if for all place $p \in {}^\bullet q$, the number of tokens present in p and *routed towards* q is larger than $C(p, q)$. Since enabling in routed Petri net is more constrained than in classical nets, it is possible to find Petri nets which are live in the classical sense but not live when routed. An example of such a net is given in Figure 6.

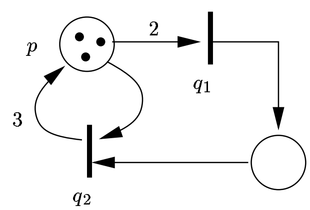

Figure 6: A live Petri net which routed version is not live

The Petri net displayed in Figure 6 is live while its routed version is not, since when all tokens entering place p are routed towards q_1, then transition q_2 will never fire.

However, for *Equal Conflict* nets (defined in [Teruel and Silva, 1994]) both liveness notions coincide. This can be shown using a transformation of any equal conflict net into a free choice net where choices are ordinary, illustrated in Figure 7.

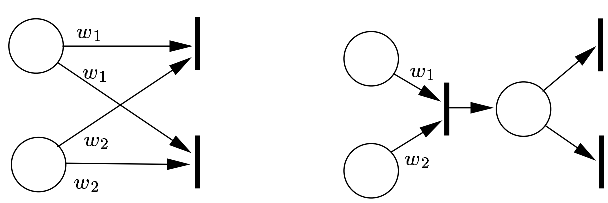

Figure 7: Transformation of an equal conflict into an ordinary free choice

In [Baccelli *et al.*, 1996] the following result is proved.

Lemma 6 *A ordinary Petri net with routing is live if and only if for all fair routing function $\mathcal{H}$, the minimal solution of equations (8)-(9) is (∞, ∞).*

This means that liveness of a routed net can be checked on the total number of firing equations. An algorithm to check liveness based on this property is developed in [Baccelli and Gaujal, 1996].

For weighted Petri nets, we have a similar result:

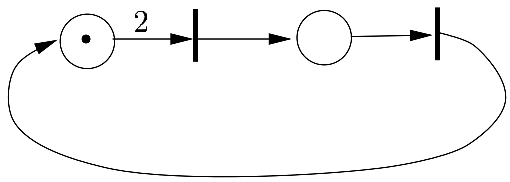

Figure 8: A non-live continuous net with no deadlock

Lemma 7 *A weighted Petri net with routing is live if and only if for all fair routing function $\mathcal{H}$, the minimal solution of equations (8)-(9) is* (∞, ∞).

Proof. The proof is similar to the proof of Lemma 6. Let S be a routed Petri net under canonical form and suppose that some fair $\mathcal{H}$, the minimum solution $X = (Y, Z)$ is not infinite. Let us assume that $X_1, X_2, \cdots, X_h$ are the finite elements of X. So, for this evolution of the net, the transitions of τ corresponding to $\{X_1, \cdots, X_h\}$ fire a finite number of times, for any timed evolution of the net. Let us chose some positive values for the firing times of the transitions, and let d_0 be the epoch when all the transitions of τ stop firing in the corresponding timed evolution of the net. So we have $X_i(u) = X_i$, for all $0 \leqslant i \leqslant h$ and $u \geqslant d_0$. Let $q_i \in \tau$.

- If $q_i \in \mathcal{B}$, then by fairness of $\mathcal{H}$, the place $p_i = {}^\bullet q_i$ receives a finite number of tokens during the evolution of the net. Let $d_1(i)$ be the epoch when the last token enters place p_i. From time $d_1(i) + 1$ on, place p_i does not receive any tokens: if one denotes $\mathcal{M}(u)$ the marking at time u, then $\mathcal{M}(u)(p_i) < C(p_i, q)$ for all $u \geqslant d_1(i) + 1$ and $q_i \in p^\bullet$.
- If $q_i \in \mathcal{A}$, then all its input places are serial. There exists a place $p_i \in {}^\bullet t_i$, such that $\mathcal{M}(u)(p_i) = 0$, for all $u \geqslant d_0$ (otherwise we would have $X_i(u + 1) > X_i(d_0)$).

Choose $d_2 > d_0 \vee (\max_{i=1,\cdots,h}(d_1(i) + 1)$; from what precedes, for all $q_i \in \tau$, there exists at least one place $p_i \in {}^\bullet t_i$ which is empty under the marking $\mathcal{M}(u)$, for all $u \geqslant d_2$ or have a marking smaller than $C(p_i, q_i)$, and this marking does not change over time.

Now, we show that $s = \{p_1, \cdots, p_h\}$ forms a *siphon* (a siphon is a set of places σ verifying the set inclusion ${}^\bullet\sigma \subset \sigma^\bullet$). Let $p_i \in s$. For all $u \geqslant d_2$, then p_i receives no token during the evolution of the net. The total number of tokens received is $\sum_{j \in {}^\bullet p_i} X_j.C(j, p_i)$. Therefore, all those X_j are finite which means that all input transitions of p_i are in τ and have an input place in s. So s is a siphon indeed.

Therefore, there exists a reachable marking $(\mathcal{M}(d_2))$ of the untimed net under which siphon S is deadlocked. But a deadlocked siphon remains deadlocked for all further evolutions of the net as from this marking on, there exists no sequence of firings of the untimed net capable of enabling a transition of τ. Therefore, the net is not live. This concludes the proof of the fact that the liveness property implies that $(Y, Z) = (\infty, \infty)$ for all fair $\mathcal{H}$.

For the converse, we will use the fact that the invariance of total firing holds true for non-constant firing times as well (see [Baccelli *et al.*, 1996]). For each reachable marking $\mathcal{M}$ (in the untimed sense), there exists a choice of firing durations and routing decisions which leads to $\mathcal{M}$ in the timed version of the net as well. Actually, the choice concerns a finite initial subsequence of the routing and the firing sequences. If the minimum solution is $(Y, Z) = (\infty, \infty)$, for all fair $\mathcal{H}$, then it is in particular infinite for any fair continuation of this specific initial subsequence of routing decisions. This implies that for all fair continuations of the $\mathcal{H}$ sequence and for all transition q, q becomes eventually enabled from $\mathcal{M}$. This in turn implies that a marking which enables q is reachable from $\mathcal{M}$, which proves liveness. $\square$

In a CTPN, liveness has to be defined. We define liveness similarly to the characterization given in the previous lemma.

Definition 8 *A CTPN is live if for all fair filter matrix H, the minimal solution of equations (10)-(11) is* (∞, ∞).

This definition implies that a CTPN can have an "infinite" activity on all transitions and still be considered dead. In the CTPN displayed in Figure 8, the quantity of fluid through the transitions is increasing with time, however the net is considered dead because this quantity converges to a finite value ($\lim_{k \to \infty} U_q(k)$ (resp $V_q(k)$) is finite).

3.6 Transformations

The logical behavior of a Petri net (or a CTPN) is preserved by transformations $\mathcal{T}_0$, $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$.

Lemma 9 *A Petri net (resp. continuous Petri net) S is live if and only if the transformed net $\mathcal{T}_i(S)$ is live, $i \in \{0,1,2,3\}$.*

Proof. The proof is similar for all the transformations $\mathcal{T}_i$ and uses the characterization of liveness given in Lemma 7 for the discrete Petri net and Definition 8 for the continuous version. We give the proof for transformation $\mathcal{T}_1$ in the discrete case only. S is live if and only if the minimal solution to the associated equations

$$\begin{aligned} Y &= A \otimes Y \oplus B \otimes Z \\ Z &= \mathcal{H}\left(P \times Z + Q \times Y + R\right), \end{aligned}$$

is infinite.

If the transformation takes place on transition q, let us denote by $\{p_1, \cdots, p_n\}$ the set ${}^\bullet q$. The total firing equation for transition q is:

$$X_q = \min\{N_{p_i}/C(p_i, q)\}.$$

X_q is infinite if and only if all N_{p_i} are infinite.

Now, $\mathcal{T}_1(S)$ contains extra transitions $\{q_1, \cdots q_n\}$ with total firing time equations:

$$\begin{aligned} X_{q_i} &= N_{p_i}/C(p_i, q) \\ X_q &= \min X_{q_i}, \end{aligned}$$

which are all infinite. □

3.7 SI-subnets criterion

We will assume in the following that matrix $A^* = \bigoplus_{i=0}^{\infty} A^i$ exists. This can be done with no loss of generality. In that case, we set $C = A^*B$ and the total counter equations become:

$$\begin{aligned} Y &= C \otimes Z \\ Z &= \mathcal{H}(PZ + QY + R), \end{aligned}$$

in the discrete case, and

$$\begin{aligned} U &= C \otimes V \\ V &= H(PV + QU + R), \end{aligned}$$

in the continuous case.

A SI-subnet of the original net S is any net $S_{<s>}$ with total counter equation

$$Y = C_{<s>} \otimes Z \tag{12}$$
$$Z = \mathcal{H}(PZ + QY + R), \tag{13}$$

in the discrete case, and

$$U = C_{<s>} \otimes V \tag{14}$$
$$V = H(PV + QU + R), \tag{15}$$

in the continuous case, where matrix $C_{<s>}$ is obtained by picking only one non-ε element in each line of C: if for some i and j, $C(i,j) = c < \infty$ and s is such that $C_{<s>}(i,j) = c$, then $C_{<s>}(i,k) = \varepsilon$, for all $k \neq j$. The total number of possible SI-subnets is a combination of all possible choices of picking one finite element per line in C. This number is denoted K.

Lemma 10 *If (Y, Z) is the minimum solution of (8)-(9) (resp. (10)-(11)) then, we can find $s \in \{1, \cdots, K\}$, such that (Y, Z) is the minimum solution of (12)-(13), (resp. (14)-(15)).*

Proof. We choose s_0, or equivalently the non-ε element of line i of C, for all i, as follows:

$$C_{<s_0>}(i,j) = \begin{cases} C(i,j) \geqslant 0 & \text{if } C(i,j) + Z^j \leqslant C(i,k) + Z^k \quad \forall k \\ \varepsilon & \text{otherwise,} \end{cases}$$

(if there are more than one j achieving the minimum, one of them is chosen in an arbitrary way). So $C_{<s_0>} \otimes Z = C \otimes Z$. This and the equations

$$\begin{aligned} Y &= C \otimes Z \\ Z &= \mathcal{H}(P \times Z + Q \times (C \otimes Z) + R), \end{aligned}$$

show that

$$\begin{aligned} Y &= C_{<s_0>} \otimes Z \\ Z &= \mathcal{H}(P \times Z + Q \times (C_{<s_0>} \otimes Z) + R), \end{aligned}$$

and so, (Y, Z) is a solution of $(E_{<s_0>})$ which implies that $(Y, Z) \geqslant (Y_{<s_0>}, Z_{<s_0>})$. Since $(Y, Z) \leqslant (Y_{<s>}, Z_{<s>})$ for all s, then necessarily $(Y, Z) = (Y_{<s_0>}, Z_{<s_0>})$. The proof for the continuous case is similar and is omitted. □

4 Liveness of Continuous Petri Nets

In this section, we will exhibit an algorithm to check liveness of a continuous Petri net

Since liveness is preserved by transformations $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$, we will assume that the CTPN S under study is under canonical form. This implies that matrices P and Q have a special form.

Lemma 11 *The minimum solution of Equations (10),(11) is $(U, V) = (\infty, \infty)$ if and only if for all SI-subnets $F_{<s>}$ of F, $(U_{<s>}, V_{<s>}) = (\infty, \infty)$.*

Proof. The condition is clearly necessary as $(U, V) \leqslant (U_{<s>}, V_{<s>})$, for all $s = 1, \ldots, K$. It is also sufficient as there exists a SI-subnet $S_{<s_0>}$ such that $(U, V) = (U_{<s_0>}, V_{<s_0>})$ (see Lemma 10). □

This lemma allows us to focus on SI-nets. We will assume from now on that the net is a SI-FCNet. When a net is SI, then the matrix C has a single finite element on each line and therefore, Equations (14)-(15) can be simplified into

$$U = C \otimes V \tag{16}$$
$$V = HP'V + HR' \tag{17}$$

(see [Baccelli *et al.*, 1996] for details)

In the following, we will focus on variable V which characterizes completely the infinite behavior of the net and forget about $U = C \otimes V$. Indeed $(U, V) = (\infty, \infty) \Leftrightarrow V = \infty$.

We decompose P' into its strongly connected components, which admit a partial order, referred to as the *reduced order*.

An *initial block* of P' is now an initial strongly connected component. Thus matrix P' has the following block form:

$$P' = \begin{pmatrix} P_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & P_2 & \cdots & 0 & \vdots & & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & P_h & 0 & \cdots & 0 \\ L_{h+1,1} & \cdots & \cdots & L_{h+1,h} & P_{h+1} & \cdots & 0 \\ \vdots & & & & \ddots & \ddots & \vdots \\ L_{c,1} & \cdots & & & & L_{c,c-1} & P_c \end{pmatrix},$$

where $P_1, \cdots, P_h$ are the sub-matrices associated with the initial blocks (these matrices are irreducible) and $P_{h+1}, \cdots, P_c$ are also strongly connected components.

The liveness of a SI-FCNet can be checked on its initial components as shown by the following lemma:

Lemma 12 *The minimum solution $V = (V_1, \cdots V_c)$ of Equation (17) is infinite if and only if it is infinite on all initial blocks, i.e. $V_1 = \infty, \cdots, V_h = \infty$.*

Proof. The "only if" direction is immediate. For the converse suppose that $V_1 = \infty, \cdots V_h = \infty$ Then for all $k > h$, $V_k = H_k(L_{k,1}V_1 + \cdots + L_{k,k-1}V_{k-1} + P_kV_k + R_k)$, where at least one of the matrices $L_{k,i}$ is non-null by fairness of H, so that at least one of the elements in V_k is infinite. By strong connectedness of P_k and fairness of H_k, all the elements of V_k are infinite. □

In the following, we will focus on one initial block, say P_1.

Lemma 13 *The minimal solution of equation $V = H(P_1V + R_1)$ is infinite if and only if $\rho(HP_1) \geqslant 1$ and R_1 is not null, where $\rho(HP_1)$ is the spectral radius of matrix HP_1.*

Proof. If R_1 is null, $v = 0$ is a finite solution of (17). If $\rho(HP_1) < 1$ then the equation also admits a finite non-negative solution: $v = (I - HP_1)^{-1}HR_1$. Conversely, if $\rho(HP_1) \geqslant 1$ then assume $v \geqslant 0$ verifies $v = HP_1v + HR_1$. Since H is non-negative and R_1 is non-negative and not null, then $v \gneqq HP_1v$ coordinate-wise. We divide each side by ρ, and we get

$$\frac{v}{\rho} \gneqq Mv, \tag{18}$$

where $M \stackrel{\text{def}}{=} \frac{HP_1}{\rho}$ has a spectral radius equal to 1 and is strongly connected. Let u be a positive left eigenvector of M, by multiplying each side of (18) by u, we get $\frac{uv}{\rho} > uv$. Since $u > 0$ and $v \geqslant 0$, then $uv > 0$. This says $\rho < 1$. This is a contradiction. Finally, Equation (17) does not have a finite solution when $\rho > 1$. □

At this point, we need to introduce some notations. In the following, we set $M \stackrel{\text{def}}{=} HP_1$. After transformations $\mathcal{T}_i, i = 0..3$, we denote by W the set routing places, of size w. Matrix M has a special form. We focus on the elements of M depending on $\{h_i\}_{i \in W^\bullet}$, and M writes

$$
M=\begin{array}{cc}
 & \begin{array}{ccc} \quad & j & \quad \end{array}\\
\begin{array}{c} \\ \\ i \\ \\ k \\ \\ \\ \end{array} & \begin{pmatrix} & 0 & \\ & \vdots & \\ \cdots & h_i & \cdots \\ & \vdots & \\ \cdots & 1-h_i & \cdots \\ & \vdots & \\ & 0 & \end{pmatrix}
\end{array}
\quad \text{or} \quad
\begin{array}{cc}
 & \begin{array}{ccc} \quad & j & \quad \end{array}\\
\begin{array}{c} \\ \\ i \\ \\ \\ \\ \\ \end{array} & \begin{pmatrix} & 0 & \\ & \vdots & \\ \cdots & h_i & \cdots \\ & \vdots & \\ & \vdots & \\ & \vdots & \\ & 0 & \end{pmatrix}
\end{array}
$$

where $p = {}^\bullet i$ is a place in W, j is the unique transition in ${}^\bullet p$ (transformation $\mathcal{T}_0$) and $p^\bullet = \{i, j\}$ (transformation $\mathcal{T}_2$). In the first case, k belongs to the same strongly connected component as i and in the second case, j does not belong to the initial block.

Therefore, liveness of all CTPN boils down to checking if

$$L \stackrel{\text{def}}{=} \min_{M \in D} \rho(M) \geqslant 1.$$

The domain D is made of all matrices $M(h_1, \cdots, h_w)$ where all the routing binary choices $(h_i, 1 - h_i)$ can vary from $(0, 1)$ to $(1, 0)$ for all routing place ${}^\bullet i$. In other words, D is isomorphic to the hypercube $[0, 1]^w$.

Theorem 14
L is reached in the extremity simplexes of D, namely on $\{0, 1\}^w$.

Proof. To prove the result, we prove that the derivative of $\rho(M)$ with respect to one choice, say h_i is never null in $]0,1[$, or always null. This will prove that the minimum value L can only be reached in 0 or 1 with respect to the variable h_i, by continuity. Repeating the argument for all binary choices will establish the result.

We consider M as a function of h_i.

M is a non-negative matrix. By applying the Perron-Frobenius theorem, $\rho(M)$ is a simple eigenvalue of M. If l and r are left and right (positive) eigenvectors of M respectively with $l_i = 1$ and $r_i = 1$, then, we can apply the derivation formula of M with respect to one coefficient. In the second case ($1 - h_i$ does not appear in M), the derivative writes

$$\frac{\partial \rho(M)}{\partial h_i} = \beta l_i r_j,$$

where β is positive. This is always positive and this ends this case since the minimum for $\rho(M)$ with respect to h_i is reached when $h_i = 0$.

In the first form of matrix M, the derivative is

$$\frac{\partial \rho(M)}{\partial h_i} = \alpha(l_i - l_k) r_j,$$

where α is positive.

Let us suppose that

$$\frac{\partial \rho(M)}{\partial h_i} = 0.$$

This is equivalent to $l_i = l_k$.

In that case, we use the linear system $(S(h_i))$: $\{vM(h_i) = xv, v_i = v_k = 1, \max(x)\}$ on the variables v (vector) and x (real). This system has a unique solution: $x = \rho(M(h_i))$ and v is the associated left eigenvector of $M(h_i)$ with its i-th coordinate set to 1, that is l. If we look at the equality $vM(h_i) = xv$ more closely, it writes:

$$\begin{cases} \sum_m H(m,n) v_m = x v_n & \text{if} \quad n \neq j \\ \sum_{m \neq i,k} H(m,j) v_m + v_i = x v_j & \text{for} \quad j. \end{cases}$$

Note that h_i never appears in any equations.

Now we introduce the linear system $S(1/2)$: $\{uM(1/2) = xu, u_i = u_k = 1, \max(x)\}$ on the variables u (vector) and x (real). $S(1/2)$ admits the same solution as $S(h_i)$. Note however that the solution of $S(1/2)$ does not depend on h_i. This means that the solution x (equal to $\rho(M(h_i))$ does not depend on h_i.

We have shown that $\frac{\partial \rho(M)}{\partial h_i} = 0$ implies that $\rho(M)$ does not depend on h_i.

Finally, either the derivative of $\rho(M)$ is never null or remains null on the interval $]0,1[$. This ends the case. Now, by continuity of $\rho(M)$ with respect to h_i, $\rho(M)$ is minimal for $h_i = 0$ or $h_i = 1$. $\square$

4.1 Computational Issues

The liveness of a continuous SI-FCNet boils down to checking if a rational matrix M has a spectral radius larger than one.

Although existing polynomial tests can be called to check this property that are available in the literature, it is worth mentioning the following criterion.

Lemma 15 *Let M be a non-negative matrix, $\rho(M) < 1$ if and only if $(I - M)^{-1}$ exists and is non-negative.*

Proof. If $\rho(M) < 1$ then, $(I-M)^{-1}$ exists and $(I-M)^{-1} = \sum_{i=0}^{\infty} M^i \geqslant 0$. Conversely, if $(I-M)^{-1}$ does not exist, then $\rho(M) \geqslant 1$. If $(I-M)^{-1}$ exists, then for all n, $\sum_{i=0}^{n-1} M^i = (I-M)^{-1}(I-M^n)$. If $\rho(M) > 1$ then M^n is a matrix where all elements go to infinity when n goes to infinity. The fact that $\sum_{i=0}^{n-1} M^i \geqslant 0$ and that if n is large enough, then $(I - M^n)$ has arbitrarily large negative elements shows that $(I-M)^{-1}$ cannot be non-negative. $\square$

This lemma provides a test in $O(T^3)$ for any matrix M.

However, to find the minimum spectral radius over all possible matrices, this test could be exponential since we have to check over all possible choices for matrix M (2^w different matrices have to be tested).

If we assume that we can compute, for a given rational matrix M, its spectral radius as well as the corresponding right and left eigenvectors in polynomial time (this is a commonly made assumption), then, finding the minimum spectral radius can be done using usual gradient ascents methods, see for example [Ravindran *et al.*, 1987].

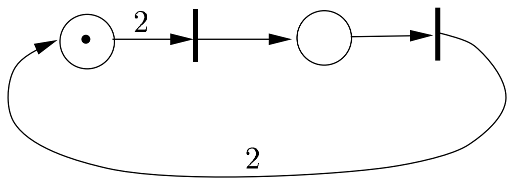

Figure 9: A structurally live but not live net.

5 Structural Liveness of Discrete Nets

The previous section gives a sufficient and necessary condition of liveness for all CTPN associated with a routed net. In this section, we will show that this condition also gives a sufficient and necessary condition of structural liveness of the discrete net.

Lemma 16 *If a canonical CTPN F is live for all routing filter, then its discrete counterpart $\mathcal{F}$ is structurally live.*

Proof. Let $\mathcal{M}_0$ be the initial marking. We denote by Z_0 the total number of firing in $\mathcal{F}$ under $\mathcal{M}_0$. Let us assume that Z_0 is finite. We have $Z_0 = \mathcal{H}(P'Z_0 + R'_0)$. By definition of $\mathcal{H}$ and because $\mathcal{F}$ is under canonical form, we have:

$$\mathcal{H}(P'Z_0 + R'_0)^i = \begin{cases} H_0(i,i)(P'Z_0 + R'_0)^i & \text{if} \quad \text{i is a simple transition,} \\ \left\lfloor \frac{(P'Z_0+R'_0)^i}{C(\bullet i,i)} \right\rfloor & \text{if} \quad \text{i is a weighted transition,} \end{cases}$$

where H_0 is a filter matrix defined by:

$$H_0(i,i) = \frac{\mathcal{H}(P'Z_0 + R'_0)^i}{P'Z_0 + R'_0}.$$

Now, we can choose another marking $\mathcal{M}_1$ such that $R'_1 \geqslant R'_0$. Under $\mathcal{M}_1$, we have

$$Z_1^i = \begin{cases} H_1(i,i)(P'Z_1 + R'_1)^i & \text{if} \quad \text{i is a simple transition,} \\ \left\lfloor \frac{(P'Z_1+R'_1)^i}{C(\bullet i,i)} \right\rfloor & \text{if} \quad \text{i is a weighted transition.} \end{cases}$$

If $\mathcal{M}_1$ is large enough, then we have

$$\begin{array}{ll} H_1(i,i)(P'Z_1 + R'_1)^i \geqslant H_1(i,i)(P'Z_1 + R'_0)^i & \text{if} \quad \text{i is a simple transition,} \\ \left\lfloor \frac{(P'Z_1+R'_1)^i}{C(\bullet i,i)} \right\rfloor \geqslant \frac{(P'Z_1+R'_0)^i}{C(\bullet i,i)} & \text{if} \quad \text{i is a weighted transition.} \end{array}$$

Therefore, we have $Z_1 \geqslant H_1(P'Z_1 + R'_0)$. This implies that Z_1 is larger than the fixed point of the continuous equation $V = H_1(P'V + R'_0)$. By assumption, this fixed point V is infinite. Therefore Z_1 is infinite. □

Theorem 17
A routed net $\mathcal{F}$ is structurally live if and only if its continuous counterpart F is structurally live.

Proof. By lemma 12 we can restrict ourselves to an initial strongly connected component. Then, Let us suppose that the continuous net F is not live. This means that for some filter matrix H, the equation in Z, $Z = H(PZ + R)$ has a finite solution Z_0. Now, we can construct a routing fonction $\mathcal{H}$ such that $\mathcal{H}(PZ_0 + R) \leqslant H(PZ_0 + R)$. This implies that $Z_0 \geqslant \mathcal{H}(PZ_0 + R)$ and therefore the equation in Z, $Z = \mathcal{H}(PZ + R)$ has a finite solution smaller than Z_0. This means that the discrete net is not live. Conversely, we apply Lemma 16 to prove that $\mathcal{F}$ is structurally live if F is structurally live. □

6 Liveness Issues

In the previous sections, we have given a test for structural liveness only. The problem with the initial marking is that it could not allow the discrete net to live only because of its transient initial behavior. In the example of Figure 9, the net is not live because of the initial step only. If the first token could go through the first firing, then the net would remain live.

In the following we will prove that this situation is general, that is, if $\mathcal{F}$ is a SI-FCNet which is structurally live (with $L > 1$), then, $\mathcal{F}$ will be live if the transitions in $\mathcal{F}$ can fire "sufficiently often".

Let us assume that $\mathcal{H}$ is a fair routing function under which the net dies. Then, there exists a finite Z such that $Z = \mathcal{H}(PZ + R)$. This implies that $\mathcal{H}$ does not need to be defined for vectors bigger than $PZ + R$.

We show next that $PZ + R$ can be uniformly bounded by above. This will result from the combination of several lemmas. First we have to introduce some notations. If $\mathcal{H}$ is a routing function, then the routing $\mathcal{H}'$ is a finite *prefix* of $\mathcal{H}$ if both functions coincide up to one point. In this case, we write $\mathcal{H} = \mathcal{H}'|\mathcal{H}''$ for a convenient routing function $\mathcal{H}''$.

Finally, a routing $\mathcal{U}$ is called *unilateral* if for all routing places, all tokens are routed to a single output transition. Note that $\mathcal{U}$ has an associated continuous matrix U such that, for all transition i, $U(i, i) \in \{0, 1\}$.

Lemma 18 *Let $\mathcal{H}$ be a routing function and Z a finite non-negative vector such that $Z = \mathcal{H}(PZ + R)$. Let $\mathcal{H}'$ be an arbitrary routing function that coincide with $\mathcal{H}$ in the point $PZ + R$. Then there exists a vector $Z' \leqslant Z$ such that $Z' = \mathcal{H}'(PZ' + R)$.*

Proof. Since $\mathcal{H}'$ coincide with $\mathcal{H}$ in the point $PZ + R$ we have $Z = \mathcal{H}'(PZ + R)$. This means there exists a minimal finite and positive solution to the equation $X = \mathcal{H}'(PX + R)$. Such a solution Z' verifies $Z' \leqslant Z$ and $Z' = \mathcal{H}'(PZ' + R)$. □

The previous lemma means that is a certain routing kills the net, then all permutations of that routing up to the point $PZ + R$ also kill the net. Therefore, we can write a routing that has a finite fixed point under the form

$$\mathcal{H} = \mathcal{U}_1|\mathcal{U}_2| \cdots |\mathcal{U}_n$$

were the routings $\mathcal{U}_1 \cdots \mathcal{U}_n$ are all unilateral routings. Note that in any case,

$$n \leqslant 2^w. \tag{19}$$

Lemma 19 *Assume that $L > 1$. Let $\beta \stackrel{\text{def}}{=} QM(m^Q(L-1))^{-1}$, where M and m are the biggest output weight and the smallest input weight in the net respectively, then the minimal solution $Z = \mathcal{U}(PZ + R)$, where $\mathcal{U}$ is a unilateral routing verifies $Z \leqslant \beta\mathbf{1}$.*

Proof. As previously, we can define a matrix U such that

$$Z_i = \mathcal{U}(PZ + R)_i = \begin{cases} U(PZ + R)_i & \text{if} \quad i \text{ is simple,} \\ \left\lfloor \frac{(PZ+R)_i}{C(\bullet i, i)} \right\rfloor & \text{if} \quad i \text{ is weighted.} \end{cases} \tag{20}$$

If we define the vector e by

$$e_i = \begin{cases} 0 & \text{if} \quad i \text{ is simple,} \\ \frac{(PZ+R)_i}{C(\bullet i, i)} - \left\lfloor \frac{(PZ+R)_i}{C(\bullet i, i)} \right\rfloor & \text{if} \quad i \text{ is weighted,} \end{cases} \tag{21}$$

we get $Z = UPZ + UR - e$.

Now, we introduce a positive left eigenvector u of matrix UP for eigenvalue $\rho \stackrel{\text{def}}{=} \rho(UP)$ with $\min_i u_i = 1$. We multiply each side by u and we get $u\big((1-\rho)Z + UR - e\big) = 0$. u is a non-negative vector which support forms a strongly connected component of UP.

On that component we have, $u\big((1-\rho)Z + UR - 1\big) \leqslant u\big((1-\rho)Z + UR - e\big) = 0$. This implies that at least one component of Z, say Z_q verifies $Z_q < (\rho - 1)^{-1}$. By strong connectedness, all elements of Z on that component verify $Z_i < (m^Q(\rho - 1))^{-1}$. This implies in turn that the routing has been applied less than $QM(m^Q(\rho - 1))^{-1}$ times on vector Z.

Finally, we can bound all coordinates of Z by

$$\frac{QM}{m^Q(\rho-1)} \leqslant \beta = \frac{QM}{m^Q(L-1)}. \tag{22}$$

□

Theorem 20
Let Z be the minimal finite non-negative solution of equation $Z = \mathcal{H}(PZ+R)$ for some $\mathcal{H}$, then,

$$Z \leqslant \gamma \stackrel{\text{def}}{=} 2^w \beta \mathbf{1}.$$

Proof. As mentioned before, we can write $\mathcal{H} = \mathcal{U}_1|\cdots|\mathcal{U}_n$ with $n \leqslant 2^w$. By applying Lemma 19 to the net reached after the prefix $\mathcal{U}_1\cdots\mathcal{U}_{n-1}$ has been applied, with the routing $\mathcal{U}_n$, then this part of $\mathcal{H}$ necessarily contributes to Z for less than β. Now by permutation of all prefixes in $\mathcal{H}$, which is allowed by Lemma 18, we can apply the same result to all $\mathcal{U}_i$, for all i. □

The last theorem means that for a given structural live net, we only need to check all possible routings up to the point γ. If none of those prefixes have killed the net, then, the net is live.

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