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Abstract: An algorithm for minimizing a nonlinear function subject to nonlinear equality and inequality constraints is described. It can be seen as an extension of primal interior point methods to non-convex optimization. The new algorithm applies sequential quadratic programming techniques to a sequence of barrier problems, and uses trust regions to ensure the robustness of the iteration and to allow the direct use of second order derivatives. An analysis of the convergence properties of the new method is presented.

Key-words: Constrained optimization, interior point method, large-scale optimization, nonlinear programming, primal method, SQP iteration, trust region method.

(Résumé : tsvp)

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Une méthode à région de confiance fondée sur les techniques de points intérieurs pour la programmation non linéaire

Résumé : Ce papier décrit un algorithme pour minimiser une fonction non linéaire sous contraintes d'égalité et d'inégalité. Celui-ci peut être vu comme une extension des méthodes de points intérieurs primales à l'optimisation non convexe. Ce nouvel algorithme utilise la Programmation Quadratique Successive sur une suite de problèmes-barrière. Les techniques des régions de confiance assurent la robustesse des itérations et permet l'utilisation des dérivées secondes. Une analyse de la convergence de cette nouvelle méthode est présentée.

Mots-clé : Méthode à région de confiance, méthode de points intérieurs, méthode primale, optimisation sous contraintes, problèmes de grande taille, programmation non linéaire, programmation quadratique successive.

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1 Introduction

Sequential Quadratic Programming (SQP) methods have proved to be very efficient for solving medium-size nonlinear programming problems. They require few iterations and function evaluations, but since they need to solve a quadratic subproblem at every iteration, it is not yet known whether they can be effective for solving problems with large numbers of variables and constraints. On the other hand, interior-point methods have proved to be very successful in solving large linear programming problems, and it is natural to ask whether they can be extended to nonlinear problems. Preliminary computational experience with simple adaptations of primal-dual method interior point methods have given encouraging results on some classes on nonlinear problems (see for example [25, 13, 27, 1]).

In this paper we propose an algorithm for large-scale nonlinear programming that uses ideas from interior point methods *and* sequential quadratic programming. One of its unique features is the use of a trust region framework that allows for the direct use of second derivatives and the inaccurate solution of subproblems.

The new algorithm is designed to solve the nonlinear programming problem

$$\begin{array}{rl} \min & f(x) \\ \text{subject to} & g_{\mathrm{E}}(x) = 0 \\ & g_{\mathrm{I}}(x) \leq 0, \end{array} \tag{1.1}$$

where $f : \mathbf{R}^n \to \mathbf{R}$, $g_{\mathrm{E}} : \mathbf{R}^n \to \mathbf{R}^{m_{\mathrm{E}}}$ and $g_{\mathrm{I}} : \mathbf{R}^n \to \mathbf{R}^{m_{\mathrm{I}}}$ are smooth functions. By introducing slack variables we obtain the following problem in the variables x and s,

$$\begin{array}{rl} \min & f(x) \\ \text{subject to} & g_{\mathrm{E}}(x) = 0 \\ & g_{\mathrm{I}}(x) + s = 0 \\ & s \geq 0. \end{array} \tag{1.2}$$

Even though the general inequality constraints $g_{\mathrm{I}}(x) \leq 0$ have been replaced by the simple bounds $s \geq 0$, formulation (1.2) is as complex as (1.1) in that the set of active constraints – or zero slacks – at the solution needs to be identified. Rather than using an active set approach [10, 9], we follow the strategy of interior point methods (see for example [12, 26, 18]) and associate with (1.2) the following barrier problem in the variables x and s

$$\begin{array}{rl} \min & f(x) - \mu \sum_{i=1}^{m_{\mathrm{I}}} \ln s^{(i)} \\ \text{subject to} & g_{\mathrm{E}}(x) = 0 \\ & g_{\mathrm{I}}(x) + s = 0, \end{array} \tag{1.3}$$

where $\mu > 0$ is a penalty parameter and where the vector $s = (s^{(1)}, \ldots, s^{(m_{\mathrm{I}})})^\top$ is assumed to be positive.

The main goal of this paper is to propose a trust region algorithm, based on sequential quadratic programming, for finding an approximate solution to (1.3) for fixed μ. The algorithm can be applied repeatedly to problem (1.3), for decreasing values of μ, to approximate the solution of the original problem (1.1).

We begin by introducing some notation and by stating the first-order optimality conditions for the barrier problem. The Lagrangian of (1.3) is

$$L(x, s, \lambda_{\mathrm{E}}, \lambda_{\mathrm{I}}) = f(x) - \mu \sum_{i=1}^{m_{\mathrm{I}}} \ln s^{(i)} + \lambda_{\mathrm{E}}^{\top} g_{\mathrm{E}}(x) + \lambda_{\mathrm{I}}^{\top}(g_{\mathrm{I}}(x) + s), \tag{1.4}$$

where $\lambda_{\mathrm{E}} \in \mathbf{R}^{m_{\mathrm{E}}}$ and $\lambda_{\mathrm{I}} \in \mathbf{R}^{m_{\mathrm{I}}}$ are the Lagrange multipliers corresponding to the equality and inequality constraints. At an optimal solution (x, s) of (1.3) we have

$$\nabla_x L(x, s, \lambda_{\mathrm{E}}, \lambda_{\mathrm{I}}) = \nabla f(x) + A_{\mathrm{E}}(x)\lambda_{\mathrm{E}} + A_{\mathrm{I}}(x)\lambda_{\mathrm{I}} = 0 \tag{1.5}$$

$$\nabla_s L(x, s, \lambda_{\mathrm{E}}, \lambda_{\mathrm{I}}) = -\mu S^{-1} e + \lambda_{\mathrm{I}} = 0, \tag{1.6}$$

where

$$A_{\mathrm{E}}(x) = \left(\nabla g_{\mathrm{E}}^{(1)}(x), \ldots, \nabla g_{\mathrm{E}}^{(m_{\mathrm{E}})}(x)\right), \qquad A_{\mathrm{I}}(x) = \left(\nabla g_{\mathrm{I}}^{(1)}(x), \ldots, \nabla g_{\mathrm{I}}^{(m_{\mathrm{I}})}(x)\right) \tag{1.7}$$

are the matrices of constraint gradients, and where

$$e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \qquad S = \begin{pmatrix} s^{(1)} & & \\ & \ddots & \\ & & s^{(m_{\mathrm{I}})} \end{pmatrix}. \tag{1.8}$$

We can now begin the derivation of the new algorithm. To simplify the presentation we define

$$z = \begin{pmatrix} x \\ s \end{pmatrix}, \qquad \varphi(z) = f(x) - \mu \sum_{i=1}^{m_{\mathrm{I}}} \ln s^{(i)}, \tag{1.9}$$

$$c(z) = \begin{pmatrix} g_{\mathrm{E}}(x) \\ g_{\mathrm{I}}(x) + s \end{pmatrix}, \tag{1.10}$$

and rewrite the barrier problem (1.3) as

$$\begin{array}{rl} \min & \varphi(z) \\ \text{subject to} & c(z) = 0. \end{array} \tag{1.11}$$

We now apply the sequential quadratic programming method (see for example [10, 9]) to this problem. At an iterate z, we generate a displacement

$$d = \begin{pmatrix} d_x \\ d_s \end{pmatrix}$$

by solving the quadratic program

$$\begin{array}{rl} \min & \nabla\varphi(z)^{\top} d + \frac{1}{2} d^{\top} W d \\ \text{subject to} & \hat{A}(z)^{\top} d + c(z) = 0, \end{array} \tag{1.12}$$

where W is the Hessian of the Lagrangian of the barrier problem (1.11) with respect to z, and where $\hat{A}^\top$ is the Jacobian of c and is given by

$$\hat{A}(z)^\top = \begin{pmatrix} A_{\mathrm{E}}(x)^\top & 0 \\ A_{\mathrm{I}}(x)^\top & I \end{pmatrix}. \tag{1.13}$$

Note that (1.11) is just a restatement of (1.3), and thus from (1.5)–(1.6) we have that

$$W \equiv \nabla^2_{zz} L(x, s, \lambda_{\mathrm{E}}, \lambda_{\mathrm{I}}) = \begin{pmatrix} \nabla^2_{xx} L(x, s, \lambda_{\mathrm{E}}, \lambda_{\mathrm{I}}) & 0 \\ 0 & \mu S^{-2} \end{pmatrix}. \tag{1.14}$$

To obtain convergence from remote starting points, and to allow for the case when W is not positive definite in the null space of $\hat{A}^\top$, we introduce a trust region constraint in (1.12) to obtain the subproblem

$$\begin{array}{rl} \min & \nabla\varphi(z)^\top d + \frac{1}{2} d^\top W d \\ \text{subject to} & \hat{A}(z)^\top d + c(z) = 0, \\ & \|d\|_T \leq \Delta. \end{array} \tag{1.15}$$

Here $\Delta > 0$ is a trust region radius that is updated at every iteration and $\|\cdot\|_T$ is a norm that will be defined below.

It is well known [24] that the constraints in (1.15) can be incompatible since the steps d satisfying the linear constraints may not lie within the trust region ball. Several modifications to (1.15) have been proposed to make the constraints consistent [5, 4, 22], and in this paper we follow the approach of Byrd [3] and Omojokun [19], which we have found suitable for solving large problems [17].

The strategy of Byrd and Omojokun consists of first taking a vertical (or transversal) step v that lies well inside the trust region and that attempts to satisfy the linear constraints in (1.15) as well as possible. To compute the vertical step v, we choose a contraction parameter $0 < \xi < 1$ (say $\xi = 0.8$) that determines the smaller trust region radius $\xi\Delta$, and then approximately solve the problem

$$\begin{array}{rl} \min & \|\hat{A}(z)^\top v + c(z)\| \\ \text{subject to} & \|v\|_T \leq \xi\Delta, \end{array} \tag{1.16}$$

where $\|\cdot\|$ denotes the Euclidean (or ℓ_2) norm. The vertical step v determines how well the linear constraints in (1.15) will be satisfied. We now compute the total step d by approximately solving the following modification of (1.15)

$$\begin{array}{rl} \min & \nabla\varphi(z)^\top d + \frac{1}{2} d^\top W d \\ \text{subject to} & \hat{A}(z)^\top d = \hat{A}(z)^\top v \\ & \|d\|_T \leq \Delta. \end{array} \tag{1.17}$$

The constraints for this subproblem are always consistent; for example $d = v$ is always feasible. Lalee, Nocedal and Plantenga [17] describe an iterative procedure for approximately solving (1.17) in the case when the number of variables is large.

We now need to decide if the trial step d obtained from (1.17) should be accepted, and for this purpose we introduce a merit function for the barrier problem (1.11). (Recall that our objective at this stage is to solve the barrier problem for a fixed value of the barrier parameter μ.) We follow Byrd and Omojokun and define the merit function to be

$$\phi(z;\nu) = \varphi(z) + \nu\|c(z)\|, \tag{1.18}$$

where $\nu > 0$ is a *penalty parameter*. Since the Euclidean norm in the second term is not squared, this merit function is non-differentiable. It is also exact in the sense that if ν is greater than a certain threshold value, then a Karush-Kuhn-Tucker point of the barrier problem (1.3) is a stationary point of the merit function ϕ. The step d is accepted if it gives sufficient reduction in the merit function; otherwise it is rejected.

We complete the iteration by updating the trust region radius Δ according to standard trust region techniques that will be discussed later on.

Let us now turn our attention to the form of the trust region used in (1.15), (1.16), and (1.17). One role of the trust region can be to ensure the positivity of the slacks, i.e., that $s + d_s > 0$. As in affine scaling methods [12], we can restrict the displacement d_s to be less than a certain fraction of the distance to the first non-negativity constraint. We do this by choosing a parameter τ (say $\tau = 0.99$), and requiring that each component $d_s^{(i)}$ of d_s satisfy

$$\frac{|d_s^{(i)}|}{s^{(i)}} \le \tau, \quad i = 1, \ldots, m_{\mathrm{I}}.$$

This ensures that $s + d_s \ge (1-\tau)s > 0$. Multiplying both sides by Δ/τ, we obtain

$$\frac{\Delta}{\tau}\frac{|d_s^{(i)}|}{s^{(i)}} \le \Delta, \quad i = 1, \ldots, m_{\mathrm{I}},$$

which can also be expressed as

$$\|Dd_s\|_\infty \le \Delta, \tag{1.19}$$

where the diagonal scaling matrix D is defined by

$$D = \frac{\Delta}{\tau}S^{-1}.$$

Relation (1.19) is not a completely adequate trust region constraint, because it does not ensure that d_s decreases with Δ, a crucial property of trust region methods. We could impose the condition $\|d_s\|_\infty \le \Delta$, but following the ideas from affine scaling methods we scale by S^{-1} and require that

$$\|S^{-1}d_s\|_\infty \le \Delta.$$

We can achieve both goals – maintaining the positivity of the slacks and ensuring that Δ influences the size of the step d_s – by redefining D to be

$$D = \max\left(1, \frac{\Delta}{\tau}\right) S^{-1}. \tag{1.20}$$

Then, the trust region constraint on d would become

$$\left\| \begin{pmatrix} d_x \\ Dd_s \end{pmatrix} \right\|_\infty \leq \Delta, \tag{1.21}$$

with D given by (1.20).

Note that it is not necessary to use the infinity norm in (1.21). Other norms, such as the Euclidean norm or a mixture of the two, may turn out to be more effective in practice. To account for this possibility, our analysis will deal with an arbitrary norm, which we denote by $\|\cdot\|_T$. We will write the trust region constraint as

$$\left\| \begin{pmatrix} d_x \\ Dd_s \end{pmatrix} \right\|_T \leq \Delta, \tag{1.22}$$

where D is given by

$$D = \max(\beta, \Delta)\, S^{-1}, \tag{1.23}$$

and where $\beta > 0$ is a constant that has been introduced for more generality. Since τ is constant, the choice (1.20)-(1.21) can be recovered from this framework by appropriately choosing β and the norm $\|\cdot\|_T$. However, by allowing the norm $\|\cdot\|_T$ to be totally arbitrary we can no longer guarantee the positivity of $s+d_s$ for any value of the trust region radius Δ; in such cases our algorithm will enforce positivity by reducing Δ, as will be discussed later on.

We summarize the discussion given so far by presenting a broad outline of the new algorithm for solving the nonlinear programming problem (1.1). We simplify the notation by writing a vector such as z, which has x and s-components, as $z = (x, s)$ instead of $z = (x^\top, s^\top)^\top$. In this way an expression like that in (1.22) is simply written as

$$\left\| \begin{pmatrix} d_x \\ Dd_s \end{pmatrix} \right\|_T \equiv \|(d_x, Dd_s)\|_T \,. \tag{1.24}$$

Algorithm Outline

Choose an initial barrier parameter $\mu > 0$, an initial iterate $z = (x, s)$, and Lagrange multipliers λ_{E}, λ_{I}.

1. If (1.1) is solved to the required accuracy, stop.

2. Apply the Byrd-Omojokun method to solve the barrier problem (1.11), as follows.

   Choose an initial trust region radius $\Delta > 0$, a contraction parameter $\xi \in (0,1)$, and a penalty parameter $\nu > 0$ for the merit function (1.18).

   (a) If the barrier problem (1.11) is solved to the required accuracy, go to 3.

   (b) Compute a vertical step $v = (v_x, v_s)$ by approximately solving the *vertical subproblem*

   $$\begin{aligned} \min \quad & \|\hat{A}(z)^\top v + c(z)\| \\ \text{subject to} \quad & \left\|(v_x, \tilde{D}v_s)\right\|_T \leq \xi\Delta, \end{aligned} \tag{1.25}$$

   where $\tilde{D} = \max(\beta, \xi\Delta)\, S^{-1}$.

(c) Compute the total step $d = (d_x, d_s)$ by approximately solving the *horizontal subproblem*

$$\begin{aligned} \min \quad & \nabla\varphi(z)^\top d + \tfrac{1}{2} d^\top W d \\ \text{subject to} \quad & \hat{A}(z)^\top d = \hat{A}(z)^\top v \\ & \|(d_x, D d_s)\|_T \le \Delta, \end{aligned} \tag{1.26}$$

where D is given by (1.23).

(d) If the step d does not give a sufficient reduction in the merit function (1.18), decrease Δ and go to (b). Otherwise, set $x \leftarrow x + d_x$, $s \leftarrow s + d_s$, $z = (x, s)$, compute new Lagrange multipliers λ_{E}, λ_{I} and go to (a).

3. Decrease the barrier parameter μ and go to 1.

In §2 we discuss in more detail when to accept or reject a step, and how to update the trust region. This will allow us to give a complete description of the algorithm. We now digress to discuss the relationship between our approach and other interior point methods.

1.1 KKT Systems

It is known that Sequential Quadratic Programming, in at least one formulation, is equivalent to Newton's method applied to the optimality conditions of a nonlinear program [9]. This relationship can be used to establish a connection between our approach and other interior point methods.

The KKT conditions for the equality constrained barrier problem (1.3) give rise to the following system of nonlinear equations in $x, s, \lambda_{\mathrm{E}}, \lambda_{\mathrm{I}}$ (see (1.5), (1.6))

$$\begin{pmatrix} \nabla f(x) + A_{\mathrm{E}}(x)\lambda_{\mathrm{E}} + A_{\mathrm{I}}(x)\lambda_{\mathrm{I}} \\ -\mu S^{-1} e + \lambda_{\mathrm{I}} \\ g_{\mathrm{E}}(x) \\ g_{\mathrm{I}}(x) + s \end{pmatrix} = 0. \tag{1.27}$$

Applying Newton's method to this system we obtain the iteration

$$\begin{pmatrix} \nabla^2_{xx} L & 0 & A_{\mathrm{E}}(x) & A_{\mathrm{I}}(x) \\ 0 & \mu S^{-2} & 0 & I \\ A_{\mathrm{E}}^\top(x) & 0 & 0 & 0 \\ A_{\mathrm{I}}^\top(x) & I & 0 & 0 \end{pmatrix} \begin{pmatrix} d_x \\ d_s \\ d_{\lambda_{\mathrm{E}}} \\ d_{\lambda_{\mathrm{I}}} \end{pmatrix} = \begin{pmatrix} -\nabla f(x) - A_{\mathrm{E}}(x)\lambda_{\mathrm{E}} - A_{\mathrm{I}}(x)\lambda_{\mathrm{I}} \\ \mu S^{-1} e - \lambda_{\mathrm{I}} \\ -g_{\mathrm{E}}(x) \\ -g_{\mathrm{I}}(x) - s \end{pmatrix},$$

where we have omitted the argument of $\nabla^2_{xx} L$ for brevity, i.e.,

$$\nabla^2_{xx} L = \nabla^2_{xx} L(x, s, \lambda_{\mathrm{E}}, \lambda_{\mathrm{I}}).$$

Writing $\lambda_{\mathrm{E}}^+ = \lambda_{\mathrm{E}} + d_{\lambda_{\mathrm{E}}}$ and $\lambda_{\mathrm{I}}^+ = \lambda_{\mathrm{I}} + d_{\lambda_{\mathrm{I}}}$, and cancelling terms we obtain

$$\begin{pmatrix} \nabla^2_{xx} L & 0 & A_{\mathrm{E}}(x) & A_{\mathrm{I}}(x) \\ 0 & \mu S^{-2} & 0 & I \\ A_{\mathrm{E}}^\top(x) & 0 & 0 & 0 \\ A_{\mathrm{I}}^\top(x) & I & 0 & 0 \end{pmatrix} \begin{pmatrix} d_x \\ d_s \\ \lambda_{\mathrm{E}}^+ \\ \lambda_{\mathrm{I}}^+ \end{pmatrix} = \begin{pmatrix} -\nabla f(x) \\ \mu S^{-1} e \\ -g_{\mathrm{E}}(x) \\ -g_{\mathrm{I}}(x) - s \end{pmatrix}. \tag{1.28}$$

Note that the current values of the multipliers $\lambda_{\rm E}, \lambda_{\rm I}$ only enter in (1.28) through $\nabla^2_{xx}L$. When the objective function and constraints are linear, we have that $\nabla^2_{xx}L = 0$, and thus the step does not depend on the current values of these multipliers; for this reason a method based on (1.28) is referred to as a *primal* interior point method.

Let us now suppose that the quadratic subproblem (1.12)-(1.14) is strongly convex, i.e., that W is positive definite on the null space of $\hat{A}(z)^\top$. Then it is easy to see that the solution of (1.12) coincides with the step generated by (1.28). Therefore the SQP approach (1.12) with W given by (1.14) is equivalent to a primal interior point method, under the convexity assumption just stated. Several researchers, including Yamashita [25] have noted this relationship.

It is also possible to establish a correspondence between *primal-dual* interior point methods and the SQP approach. Let us multiply the second row of (1.27) by S to obtain the system

$$\begin{pmatrix} \nabla f(x) + A_{\rm E}(x)\lambda_{\rm E} + A_{\rm I}(x)\lambda_{\rm I} \\ S\lambda_{\rm I} - \mu e \\ g_{\rm E}(x) \\ g_{\rm I}(x) + s \end{pmatrix} = 0. \tag{1.29}$$

This may be viewed as a modified KKT system for the inequality constrained problem (1.2), since the second row is a relaxation of the complementary slackness condition (which is obtained when $\mu = 0$). In the linear programming case, primal-dual methods are based on iteratively solving (1.29) in $x, s, \lambda_{\rm E}, \lambda_{\rm I}$. Applying Newton's method to (1.29) results in the iteration

$$\begin{pmatrix} \nabla^2_{xx}L & 0 & A_{\rm E}(x) & A_{\rm I}(x) \\ 0 & \Lambda & 0 & S \\ A_{\rm E}(x)^\top & 0 & 0 & 0 \\ A_{\rm I}(x)^\top & I & 0 & 0 \end{pmatrix} \begin{pmatrix} d_x \\ d_s \\ d_{\lambda_{\rm E}} \\ d_{\lambda_{\rm I}} \end{pmatrix} = \begin{pmatrix} -\nabla f(x) - A_{\rm E}(x)\lambda_{\rm E} - A_{\rm I}(x)\lambda_{\rm I} \\ \mu e - S\lambda_{\rm I} \\ -g_{\rm E}(x) \\ -g_{\rm I}(x) - s \end{pmatrix},$$

where

$$\Lambda = \mathrm{diag}(\lambda_{\rm I}^{(1)}, \ldots, \lambda_{\rm I}^{(m_I)}). \tag{1.30}$$

The iteration matrix is nonsymmetric, but can be symmetrized by multiplying the second block of equations by S^{-1}. Doing this and cancelling terms as in (1.28) yields the equivalent system

$$\begin{pmatrix} \nabla^2_{xx}L & 0 & A_{\rm E}(x) & A_{\rm I}(x) \\ 0 & S^{-1}\Lambda & 0 & I \\ A_{\rm E}(x)^\top & 0 & 0 & 0 \\ A_{\rm I}(x)^\top & I & 0 & 0 \end{pmatrix} \begin{pmatrix} d_x \\ d_s \\ \lambda^+_{\rm E} \\ \lambda^+_{\rm I} \end{pmatrix} = \begin{pmatrix} -\nabla f(x) \\ \mu S^{-1} e \\ -g_{\rm E}(x) \\ -g_{\rm I}(x) - s \end{pmatrix}. \tag{1.31}$$

Now the current value of $\lambda_{\rm I}$ influences the step through the matrix Λ and through $\nabla^2_{xx}L$. We refer to (1.31) as the *primal-dual iteration.*

Consider now the SQP subproblem (1.12) with the Hessian of the Lagrangian W replaced by

$$\tilde{W} = \begin{pmatrix} \nabla^2_{xx}L(x, s, \lambda_{\rm E}, \lambda_{\rm I}) & 0 \\ 0 & S^{-1}\Lambda \end{pmatrix}. \tag{1.32}$$

It is easy to see that if the quadratic program (1.12) is strongly convex, the step generated by the SQP approach coincides with the solution of (1.31). Comparing (1.14) and (1.32) we see that the only difference between the primal and primal-dual SQP formulations is that the matrix μS^{-2} has been replaced by $S^{-1}\Lambda$.

This degree of generality justifies the investigation of SQP as a framework for designing interior point methods for nonlinear programming. Several choices for the Hessian matrix W could be considered, but in this study we focus on the (primal) exact Hessian version (1.14) because of its simplicity. We note, however, that much of our analysis could be extended to the primal-dual approach based on (1.32) if appropriate safeguards are applied.

Many authors, among them Yamashita [25], Herskovits [14], Anstreicher and Vial [2], Jarre and Saunders [16], El-Bakry, Tapia, Tsuchiya, and Zhang [8], Coleman and Li [6], Dennis, Heinkenschloss and Vicente [7], have proposed interior point methods for nonlinear programming based on iterations of the form (1.28) or (1.31). In some of these studies $\nabla^2_{xx}L$ is either assumed positive definite on the whole space or a subspace, or is modified to be so. In our approach there is no such requirement; we can either use the exact Hessian of the Lagrangian with respect to x in (1.28) and (1.31), or *any* approximation B to it. For example, B could be updated by the BFGS or SR1 quasi-Newton formulae. This generality is possible by the trust region framework described in the previous section.

We emphasize that the equivalence between SQP and Newton's method applied to the KKT system holds only if the quadratic subproblem (1.12) is strongly convex, if this subproblem is solved exactly, and if the trust region constraint is inactive. Since these conditions will not hold in most iterations of our algorithm, the approach presented in this paper is distinct from those based on directly solving the KKT system of the barrier problem. However, as the iterates converge to the solution, our algorithm will be very similar to these other interior point methods. This is because near the solution point, the quadratic subproblem (1.12) will be convex and the tolerances of the procedure for solving (1.12) will be set so that, asymptotically, it is solved exactly. Moreover, as the iterates converge to the solution we expect the trust region constraint to become inactive, provided a second order correction is incorporated in the algorithm.

In summary the local behavior of our method is similar to that of other interior point methods, but its global behavior is likely to be markedly different. For this reason the analysis presented in this paper will focus on the global convergence properties of the new method.

Notation. Throughout the paper $\|\cdot\|$ denotes the Euclidean (or ℓ_2) norm, and $\|\cdot\|_T$ stands for an arbitrary vector norm. The vector of slack variables at the k-th iteration is written as s_k, and its i-th component is $s_k^{(i)}$.

2 Algorithm for the Barrier Problem

We now give a detailed description of the new algorithm. To simplify the discussion, we will assume from now on that the nonlinear programming problem (1.1) contains only inequality constraints. We make this assumption because the main ideas we want to convey relate to the treatment of inequality constraints and because the incorporation of equality constraints is straightforward.

The problem we wish to solve is therefore

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & g(x) \leq 0, \end{array} \tag{2.1}$$

and we let m denote the number of inequality constraints. The corresponding barrier problem is given by

$$\begin{array}{ll} \min & f(x) - \mu \sum_{i=1}^{m} \ln s^{(i)} \\ \text{s.t.} & g(x) + s = 0. \end{array} \tag{2.2}$$

If we consider the quadratic problem (1.26) solved in step 2c of the Algorithm Outline we note that the matrix $\hat{A}$ given by (1.13) can be written as

$$\hat{A}(z)^\top = \begin{pmatrix} A(x)^\top & I \end{pmatrix}, \tag{2.3}$$

where we have omitted the subscript in A_{I} for simplicity. It then follows from (1.9), (1.14) and (2.3) that, at an iterate $z_k = (x_k, s_k)$, the problem (1.26) can be expressed as

$$\begin{array}{ll} \min & \nabla f_k^\top d_x - \mu e^\top S_k^{-1} d_s + \frac{1}{2} d_x^\top B_k d_x + \frac{1}{2} \mu d_s^\top S_k^{-2} d_s \\ \text{s.t.} & A_k^\top d_x + d_s = A_k^\top v_x + v_s \\ & \|(d_x, D_k d_s)\|_T \leq \Delta_k, \end{array} \tag{2.4}$$

where $\nabla f_k = \nabla f(x_k)$, B_k stands for $\nabla^2_{xx} L(x_k, s_k, \lambda_k)$ or for a symmetric matrix approximating this Hessian, and D_k has the form (1.23).

Now we will focus on the merit function and, in particular, on how much it is expected to decrease at each iteration. First we extend the definition of the merit function (1.18) to be infinite when the slack variables are negative, in order to provide a mechanism for keeping them positive at all iterates:

$$\phi(x, s; \nu) = \begin{cases} f(x) + \nu \|g(x) + s\| - \mu \sum_{i=1}^{m} \ln s^{(i)} & \text{if } s > 0 \\ +\infty & \text{otherwise.} \end{cases} \tag{2.5}$$

Let us now construct a model m_k of $\phi(\cdot, \cdot; \nu_k)$ around an iterate (x_k, s_k) using the quadratic objective from (2.4) and a linear approximation of the constraints in (2.2),

$$\begin{aligned} m_k(d) \quad = \quad & f_k + \nabla f_k^\top d_x + \frac{1}{2} d_x^\top B_k d_x + \nu_k \|g_k + s_k + A_k^\top d_x + d_s\| \\ & - \mu \left(\sum_{i=1}^{m} \ln s_k^{(i)} + e^\top S_k^{-1} d_s - \frac{1}{2} d_s^\top S_k^{-2} d_s \right). \end{aligned} \tag{2.6}$$

We will show in Lemma 3.1 below that m_k is an accurate local model of ϕ. We define the *predicted reduction* in the merit function ϕ to be the change in the model m_k produced by a step d,

$$\begin{aligned} \mathrm{pred}_k(d) &= m_k(0) - m_k(d) \\ &= -\nabla f_k^\top d_x - \frac{1}{2} d_x^\top B_k d_x \\ &\quad + \nu_k \left(\|g_k + s_k\| - \|g_k + s_k + A_k^\top d_x + d_s\| \right) \\ &\quad + \mu \left(e^\top S_k^{-1} d_s - \frac{1}{2} d_s^\top S_k^{-2} d_s \right). \end{aligned} \tag{2.7}$$

We will always choose the weight ν_k sufficiently large that $\mathrm{pred}_k(d) > 0$, as will be described in §2.3.

The predicted reduction is used as a standard for accepting the step and for updating the trust region. We choose a parameter $\eta \in (0,1)$, and if

$$\phi(x_k + d_x, s_k + d_s; \nu_k) \leq \phi(x_k, s_k; \nu_k) - \eta\, \mathrm{pred}_k(d), \tag{2.8}$$

we accept the step d and possibly increase the trust region radius Δ_k; otherwise we decrease Δ_k by a constant fraction, e.g. $\Delta_k \leftarrow \Delta_k/2$, and recompute d. Since $\mathrm{pred}_k(d) > 0$ this implies the merit function decreases at each step. More sophisticated strategies for updating Δ_k are useful in practice, but this simple rule will be sufficient for our purposes. Note that the extended definition (2.5) of the merit function implies that the inequality (2.8) will only hold when $s_k + d_s > 0$. Therefore the algorithm will reduce Δ_k, and hence d_s, if necessary so that the slacks remain positive.

Next we consider conditions that determine when approximate solutions to the vertical and horizontal subproblems are acceptable. Since these conditions require detailed justification, we consider the vertical and horizontal problems separately.

2.1 Approximate computation of the vertical step

Since we have assumed that only inequality constraints are present in the problem, (1.25) becomes

$$\begin{aligned} \min \quad & \|g_k + s_k + A_k^\top v_x + v_s\| \\ \text{s.t.} \quad & \|(v_x, \tilde{D}_k v_s)\|_T \leq \tilde{\Delta}_k, \end{aligned} \tag{2.9}$$

where we have defined

$$\tilde{\Delta}_k = \xi \Delta_k, \quad \tilde{D}_k = \tilde{\delta}_k S_k^{-1}, \quad \tilde{\delta}_k = \max(\beta, \tilde{\Delta}_k). \tag{2.10}$$

We now present two conditions that an approximate solution v_k of (2.9) must satisfy. To do this we introduce the change of variables

$$u_x = v_x, \qquad u_s = S_k^{-1} v_s, \tag{2.11}$$

so that problem (2.9) becomes

$$\begin{aligned} \min \quad & \|g_k + s_k + A_k^\top u_x + S_k u_s\| \\ \text{s.t.} \quad & \|(u_x, \tilde{\delta}_k u_s)\|_T \le \tilde{\Delta}_k. \end{aligned} \tag{2.12}$$

In the case where the trust region norm is the Euclidean norm on u and $\tilde{\delta}_k = 1$, it is straightforward to show [17] that (2.12) has a solution in the range of

$$\begin{pmatrix} A_k \\ S_k \end{pmatrix}. \tag{2.13}$$

But for other norms, or when $\tilde{\delta}_k \neq 1$, this can only be guaranteed if $\|g_k + s_k\|$ is small enough. Since we regard the Euclidean norm as the most natural metric here, we will ask that u lie in the range of (2.13) whenever (2.12) has a solution in that space. Transforming u back to the original variables v we arrive at our first condition.

Range Space Condition. *An approximate solution v_k of the vertical problem* (2.9) *must be of the form*

$$v_k = \begin{pmatrix} A_k \\ S_k^2 \end{pmatrix} w_k, \tag{2.14}$$

for some vector $w_k \in \mathbf{R}^m$, whenever (2.9) *has a solution of that form.*

The second condition on the vertical step requires that the reduction in the objective of (2.9) be comparable to that obtained by minimizing along the steepest descent direction in u. This direction is the gradient of the objective in problem (2.12) at $u = 0$, which is a multiple of

$$u_k^c \equiv -\begin{pmatrix} A_k \\ S_k \end{pmatrix} (g_k + s_k). \tag{2.15}$$

Transforming back to the original variables we obtain the vector

$$v_k^c \equiv -\begin{pmatrix} A_k \\ S_k^2 \end{pmatrix} (g_k + s_k), \tag{2.16}$$

which we call the *scaled steepest descent direction.* We refer to the reduction in the objective of (2.9) produced by a step $v = (v_x, v_s)$ as the *vertical predicted reduction*:

$$\text{vpred}_k(v) = \|g_k + s_k\| - \|g_k + s_k + A_k^\top v_x + v_s\|, \tag{2.17}$$

and we require that this reduction satisfy the following condition.

Vertical Cauchy Decrease Condition. *An approximate solution v_k of the vertical problem* (2.9) *must satisfy*

$$\text{vpred}_k(v_k) \ge \gamma_1 \, \text{vpred}_k(\alpha_k^c v_k^c), \tag{2.18}$$

for some constant $\gamma_1 > 0$, where α_k^c solves the problem

$$\begin{array}{ll} \min & \|g_k + s_k + \alpha(A_k^\top v_x^c + v_s^c)\| \\ \text{s.t.} & \left\|\alpha(v_x^c, \tilde{D}_k v_s^c)\right\|_T \leq \tilde{\Delta}_k. \end{array} \tag{2.19}$$

Note that the vertical Cauchy decrease condition and the range space condition (2.14) can be satisfied by an optimal solution of (2.9). Both conditions are also satisfied if the step is chosen by truncated conjugate gradient iterations in the variable u on the objective of (2.12) (see Steihaug [23]), and the results are transformed back into the original variables. Also since $\alpha = 0$ is a feasible solution of (2.19), it is clear from (2.18) that

$$\text{vpred}_k(v_k) \geq 0. \tag{2.20}$$

In Lemma 2.2 we give a sharper lower bound on the vertical predicted reduction $\text{vpred}_k(v_k)$ of an approximate solution that satisfies the vertical Cauchy decrease condition. First we give a straightforward but useful generalization of a result by Powell [21], which is obtained by setting $L = I$ and $d = -q$ in formula (2.21) below.

Lemma 2.1. *Consider the quadratic program*

$$\begin{array}{ll} \min & \left(\psi(z) \equiv q^\top z + \dfrac{1}{2} z^\top Q z\right) \\ \text{s.t.} & \|Lz\|_T \leq \Delta, \end{array}$$

where q is a vector, Q is a symmetric matrix, L is a nonzero matrix, Δ is a positive number, and $\|\cdot\|_T$ is a norm. Then, for any direction d such that $q^\top d \leq 0$ and $Ld \neq 0$, the minimum value of ψ along d within the trust region, $\psi_{\min} = \min\{\psi(\alpha d) : \|\alpha L d\|_T \leq \Delta\}$, satisfies

$$-\psi_{\min} \geq \frac{|q^\top d|}{2} \min\left(\frac{\Delta}{\|Ld\|_T}, \frac{|q^\top d|}{(d^\top Q d)^+}\right), \tag{2.21}$$

where $(\cdot)^+ \equiv \max(0, \cdot)$.

Proof. Since it is clear that $\psi_{\min} \leq 0$, the inequality (2.21) holds if $q^\top d = 0$. Therefore we only consider the case when $q^\top d < 0$.

Suppose first that $d^\top Q d > 0$ and define $\varsigma(\alpha) \equiv \psi(\alpha d) = \alpha q^\top d + \frac{1}{2}\alpha^2 d^\top Q d$. The minimizer of ς is given by $\alpha_* = -q^\top d/(d^\top Q d)$, and if $\|\alpha_* L d\|_T \leq \Delta$ we have

$$\psi_{\min} = \psi(\alpha_* d) = -\frac{1}{2}\frac{(q^\top d)^2}{d^\top Q d}.$$

On the other hand if $\|\alpha_* L d\|_T > \Delta$, then the minimum of ς will occur at the boundary of the trust region for a value $\hat{\alpha} = \Delta/\|Ld\|_T$. In this case using the condition $\alpha_* > \hat{\alpha}$ we obtain

$$\psi_{\min} = \psi(\hat{\alpha} d) \leq -\frac{\Delta}{2}\frac{|q^\top d|}{\|Ld\|_T}.$$

Suppose now that $d^\top Qd \le 0$. In this case the minimum of ς will occur at the boundary for the value $\hat\alpha = \Delta/\|Ld\|_T$. Then

$$\psi_{\min} = \psi(\hat\alpha d) \le \Delta \frac{q^\top d}{\|Ld\|_T} \le -\frac{\Delta}{2}\frac{|q^\top d|}{\|Ld\|_T}.$$

Combining all these cases we obtain (2.21). □

Before presenting the next result we recall that, by the equivalence of norms, there exists a constant $\gamma_T \in (0,1)$ such that

$$\gamma_T\|u\| \le \|u\|_T \le \gamma_T^{-1}\|u\|, \tag{2.22}$$

for any vector u. We also define

$$\hat\beta = \max(1,\beta), \tag{2.23}$$

where β is given in (1.23).

Lemma 2.2. *Suppose that* $s_k > 0$ *and that* $v_k = (v_x, v_s)$ *is an approximate solution of* (2.9) *satisfying the vertical Cauchy decrease condition* (2.18). *Then*

$$\begin{aligned}\|g_k + s_k\|\,\mathrm{vpred}_k(v_k) &= \|g_k+s_k\|\Big(\|g_k+s_k\| - \|g_k+s_k+A_k^\top v_x + v_s\|\Big)\\ &\ge \frac{\gamma_1}{2}\left\|\begin{pmatrix}A_k\\ S_k\end{pmatrix}(g_k+s_k)\right\| \min\left(\gamma_T, \frac{\gamma_T\tilde\Delta_k}{\hat\beta}, \frac{\left\|\begin{pmatrix}A_k\\ S_k\end{pmatrix}(g_k+s_k)\right\|}{\|(A_k^\top\ S_k)\|^2}\right),\end{aligned} \tag{2.24}$$

where γ_1 *is defined in* (2.18).

Proof. Inequality (2.24) clearly holds when $g_k + s_k = 0$. Therefore, we now assume that $g_k + s_k \ne 0$.

Since the vertical Cauchy decrease condition holds, by (2.17) and (2.18),

$$\begin{aligned}\|g_k+s_k\|\,\mathrm{vpred}_k(v_k) &\ge \gamma_1\|g_k+s_k\|\Big(\|g_k+s_k\| - \|g_k+s_k+\alpha_k^c(A_k^\top v_x^c + v_s^c)\|\Big)\\ &\ge \frac{\gamma_1}{2}\Big(\|g_k+s_k\|^2 - \|g_k+s_k+\alpha_k^c(A_k^\top v_x^c+v_s^c)\|^2\Big),\end{aligned} \tag{2.25}$$

because of the inequality $2a(a-b) \ge a^2 - b^2$. By the vertical Cauchy decrease condition, $\alpha_k^c v_k^c$ minimizes the quadratic $v = (v_x, v_s) \mapsto \frac{1}{2}\|g_k + s_k + A_k^\top v_x + v_s\|^2$ along the direction v_k^c within the trust region of (2.19). Let us apply Lemma 2.1 to this problem with

$$q = \begin{pmatrix}A_k\\ I\end{pmatrix}(g_k+s_k),\quad Q = \begin{pmatrix}A_k\\ I\end{pmatrix}\begin{pmatrix}A_k^\top & I\end{pmatrix},\quad L = \begin{pmatrix}I & 0\\ 0 & \tilde D_k\end{pmatrix},\quad \text{and}\quad d = v_k^c.$$

Recalling u_k^c defined by (2.15) and observing that $q^\top d = -\|u_k^c\|^2 \le 0$ and $Ld \ne 0$, it follows from (2.25), (2.21), (2.10), and (2.11)

$$\begin{aligned}
\|g_k + s_k\|\,\mathrm{vpred}_k(v_k) &\ge \frac{\gamma_1}{2}\|u_k^c\|^2 \min\left(\frac{\tilde{\Delta}_k}{\|(u_x^c, \tilde{\delta}_k u_s^c)\|_T}, \frac{\|u_k^c\|^2}{\|(A_k^\top \;\; S_k)u_k^c\|^2}\right) \\
&\ge \frac{\gamma_1}{2}\|u_k^c\| \min\left(\frac{\tilde{\Delta}_k \gamma_T}{\max(\hat{\beta}, \tilde{\Delta}_k)}, \frac{\|u_k^c\|}{\|(A_k^\top \;\; S_k)\|^2}\right) \\
&= \frac{\gamma_1}{2}\|u_k^c\| \min\left(\gamma_T, \frac{\gamma_T \tilde{\Delta}_k}{\hat{\beta}}, \frac{\|u_k^c\|}{\|(A_k^\top \;\; S_k)\|^2}\right),
\end{aligned}$$

where the second inequality follows from

$$\|(u_x^c, \tilde{\delta}_k u_s^c)\|_T \le \gamma_T^{-1}\|(u_x^c, \tilde{\delta}_k u_s^c)\| \le \gamma_T^{-1}\max(1, \tilde{\delta}_k)\|u_k^c\| = \gamma_T^{-1}\max(\hat{\beta}, \tilde{\Delta}_k)\|u_k^c\|.$$

Substituting u_k^c by its value in (2.15), we have (2.24). $\square$

2.2 Approximate solution of the horizontal problem

Consider now the horizontal subproblem (2.4). Substituting $d = v_k + h$, where v_k is the approximate solution of the vertical subproblem, and omitting constant terms involving v_k in the objective function, we obtain the following problem in $h = (h_x, h_s)$,

$$\begin{aligned}
\min \quad & (\nabla f_k + B_k v_x)^\top h_x + \frac{1}{2} h_x^\top B_k h_x \\
& - \mu\left(e^\top S_k^{-1} h_s - v_s^\top S_k^{-2} h_s - \frac{1}{2} h_s^\top S_k^{-2} h_s\right) \\
\text{s.t.} \quad & A_k^\top h_x + h_s = 0 \\
& \|(h_x, D_k h_s)\|_T \le \hat{\Delta}_k.
\end{aligned} \tag{2.26}$$

We have modified the trust region constraint $\|(d_x, D_k d_s)\|_T \le \Delta_k$ so that it controls only the horizontal step h; the new trust region radius $\hat{\Delta}_k$ reflects this. For example, suppose that $\|\cdot\|_T$ stands for the Euclidean norm, and that the vector $(h_x, D_k h_s)$ is orthogonal to $(v_x, D_k v_s)$. Then we could define

$$\hat{\Delta}_k = \left(\Delta_k^2 - \|(v_x, D_k v_s)\|^2\right)^{\frac{1}{2}},$$

so that, with the trust region constraint in (2.26), the full step $(d_x, D_k d_s)$ will be in the ball of radius Δ_k. In general, $\hat{\Delta}_k$ will depend on the norm used, but we will always choose it to satisfy

$$\Delta_k - \|(v_x, \tilde{D}_k v_s)\|_T \le \hat{\Delta}_k \le \Delta_k,$$

so that the modification to the original trust region is not great. Note from (2.10) that this implies that

$$\hat{\Delta}_k \ge (1 - \xi)\Delta_k. \tag{2.27}$$

We now describe a condition that an approximate solution of (2.26) must satisfy. For this purpose we define the *horizontal predicted reduction* produced by a step $h = (h_x, h_s)$ as the change in the objective function of (2.26),

$$\begin{aligned}\mathrm{hpred}_k(h) \quad = \quad & -(\nabla f_k + B_k v_x)^\top h_x - \frac{1}{2} h_x^\top B_k h_x \\ & + \mu \left(e^\top S_k^{-1} h_s - v_s^\top S_k^{-2} h_s - \frac{1}{2} h_s^\top S_k^{-2} h_s \right).\end{aligned} \tag{2.28}$$

Next, we let $Z_k = \begin{pmatrix} Z_x^\top & Z_s^\top \end{pmatrix}^\top$ denote a null space basis matrix for the equality constraints in problem (2.26), i.e., Z_k is an $(n+m) \times n$ full rank matrix satisfying

$$\begin{pmatrix} A_k^\top & I \end{pmatrix} Z_k = A_k^\top Z_x + Z_s = 0. \tag{2.29}$$

A simple choice of Z_k is to define $Z_k = \begin{pmatrix} I & -A_k \end{pmatrix}^\top$, but many other choices are possible, and some may have advantages in different contexts. In this paper we will allow Z_k to be any null space basis matrix satisfying

$$\|Z_k\| \le \gamma_Z \quad \text{and} \quad \sigma_{\min}(Z_k) \ge \gamma_Z^{-1}, \quad \text{for all } k, \tag{2.30}$$

where γ_Z is a positive constant and $\sigma_{\min}(Z_k)$ denotes the smallest singular value of Z_k. If $\|A_k\|$ is bounded this condition is satisfied by $Z_k = \begin{pmatrix} I & -A_k \end{pmatrix}^\top$ and by many other choices of Z_k.

Any feasible vector for (2.26) may be expressed as $h = Z_k p$ for some $p \in \mathbf{R}^n$. Thus, writing $h = (h_x, h_s) = (Z_x p, Z_s p)$, the horizontal subproblem (2.26) becomes

$$\begin{aligned}\min \quad & (\nabla f_k + B_k v_x)^\top Z_x p - \mu (S_k^{-1} e - S_k^{-2} v_s)^\top Z_s p \\ & + \frac{1}{2} p^\top (Z_x^\top B_k Z_x + \mu Z_s^\top S_k^{-2} Z_s) p \\ \text{s.t.} \quad & \|(Z_x p, D_k Z_s p)\|_T \le \hat{\Delta}_k.\end{aligned} \tag{2.31}$$

Again, this has the form of a trust region subproblem for unconstrained optimization, and by analogy with standard practice, we will require that the step $h_k = Z_k p_k$ give as much reduction in the objective of (2.31) as a steepest descent step. The steepest descent direction for the objective function of (2.31) at $p = 0$ is given by

$$p_k^c = -Z_x^\top (\nabla f_k + B_k v_x) + \mu Z_s^\top (S_k^{-1} e - S_k^{-2} v_s). \tag{2.32}$$

We are now ready to state the condition we impose on the horizontal step.

Horizontal Cauchy Decrease Condition. *The approximate solution h_k of the horizontal problem* (2.26) *must satisfy*

$$\mathrm{hpred}_k(h_k) \ge \gamma_2 \, \mathrm{hpred}_k(\theta_k^c Z_k p_k^c), \tag{2.33}$$

for some constant $\gamma_2 > 0$, where θ_k^c solves the problem

$$\begin{array}{ll} \min & -\,\mathrm{hpred}_k(\theta Z_k p_k^c) \\ \text{s.t.} & \|\theta(Z_x p_k^c, D_k Z_s p_k^c)\|_T \le \hat{\Delta}_k, \end{array} \tag{2.34}$$

and where Z_k is a null space basis matrix satisfying (2.30).

The horizontal Cauchy decrease condition is clearly satisfied by the optimal solution of (2.31). It is also satisfied if the step is chosen by truncated conjugate gradient iterations in the variable p on the objective of (2.31) (see Steihaug [23]). Note also that since $\theta = 0$ is a feasible solution to (2.34),

$$\mathrm{hpred}_k(h_k) \ge 0. \tag{2.35}$$

The following result establishes a lower bound on the horizontal predicted reduction $\mathrm{hpred}_k(h_k)$ for a step satisfying the horizontal Cauchy decrease condition.

Lemma 2.3. *Suppose that $s_k > 0$ and that $h_k = (h_x, h_s)$ satisfies the horizontal Cauchy decrease condition* (2.33). *Then*

$$\mathrm{hpred}_k(h_k) \ge \frac{\gamma_2}{2}\|p_k^c\| \min\left(\frac{\gamma_T \hat{\Delta}_k}{\|Z_x^\top Z_x + Z_s^\top D_k^2 Z_s\|^{1/2}}, \frac{\|p_k^c\|}{\|Z_x^\top B_k Z_x + \mu Z_s^\top S_k^{-2} Z_s\|}\right), \tag{2.36}$$

where p_k^c and γ_2 are given by (2.32) *and* (2.33).

Proof. The result is obtained by applying Lemma 2.1 to problem (2.31), with $q = -p_k^c$, $Q = (Z_x^\top B_k Z_x + \mu Z_s^\top S_k^{-2} Z_s)$, $L = (\, Z_x^\top \quad Z_s^\top D_k \,)^\top$, and $d = p_k^c$. We have also used the equivalence of norms (2.22). □

2.3 Detailed description of the algorithm

Now that we have specified how the vertical and horizontal subproblems are to be solved, we can give a precise description of our algorithm for solving the barrier problem (2.2).

Algorithm I. Choose the initial iterate $z_0 = (x_0, s_0, \lambda_0)$ with $s_0 > 0$, the initial trust region radius $\Delta_0 > 0$, four constants ξ, η, ρ, and τ in $(0, 1)$, two positive constants β and ν_{-1}, and set $k = 0$.

1. Compute the vertical step $v_k = (v_x, v_s)$ by solving *approximately*

$$\begin{array}{ll} \min & \|g_k + s_k + A_k^\top v_x + v_s\| \\ \text{s.t.} & \|(v_x, \tilde{D}_k v_s)\|_T \le \tilde{\Delta}_k, \end{array} \tag{2.37}$$

   in such a way that v_k satisfies the range space and vertical Cauchy decrease conditions (2.14), (2.18). As before, $\tilde{\Delta}_k = \xi \Delta_k$ and $\tilde{D}_k = \tilde{\delta}_k S_k^{-1}$ with $\tilde{\delta}_k = \max(\beta, \tilde{\Delta}_k)$.

2. Compute the total step $d_k = (d_x, d_s) = v_k + h_k$ by solving *approximately*

$$\begin{aligned} \min \quad & \nabla f_k^\top d_x - \mu e^\top S_k^{-1} d_s + \frac{1}{2} d_x^\top B_k d_x + \frac{\mu}{2} d_s^\top S_k^{-2} d_s \\ \text{s.t.} \quad & A_k^\top d_x + d_s = A_k^\top v_x + v_s \\ & \|(d_x, D_k d_s)\|_T \le \Delta_k, \end{aligned} \tag{2.38}$$

so that h_k satisfies the horizontal Cauchy decrease condition (2.33). Here $D_k = \delta_k S_k^{-1}$ and $\delta_k = \max(\beta, \Delta_k)$.

3. Update the penalty parameter of the merit function (2.5) as follows. Choose the smallest ν_k, such that

$$\text{pred}_k(d_k) \ge \rho \nu_k \, \text{vpred}_k(v_k). \tag{2.39}$$

If $\nu_k \le \nu_{k-1}$, set $\nu_k \leftarrow \nu_{k-1}$; otherwise increase ν_k if necessary so that $\nu_k \ge 1.5\, \nu_{k-1}$.

4. If

$$\phi(x_k + d_x, s_k + d_s; \nu_k) > \phi(x_k, s_k; \nu_k) - \eta \, \text{pred}_k(d_k)$$

or

$$d_s^{(i)} < -\tau s_k^{(i)}, \quad \text{for some } i,$$

decrease Δ_k by a constant factor and go to 1.

5. Set $x_{k+1} = x_k + d_x$, $s_{k+1} = \max(s_k + d_s, -g_{k+1})$, compute a new multiplier λ_{k+1}, update B_k, choose a new value $\Delta_{k+1} \ge \Delta_k$, increase k by 1 and go to 1.

Plantenga [20] describes an algorithm that has many common features with Algorithm I, but his approach has also important differences with ours.

Steps 3, 4 and 5 need some clarification. Writing $d_x = h_x + v_x$ and $d_s = h_s + v_s$, the total predicted reduction (2.7) becomes

$$\begin{aligned} &\text{pred}_k(d_k) = \\ &\quad -\nabla f_k^\top v_x - \frac{1}{2} v_x^\top B_k v_x - (\nabla f_k + B_k v_x)^\top h_x - \frac{1}{2} h_x^\top B_k h_x \\ &\quad + \nu_k \left(\|g_k + s_k\| - \|g_k + s_k + A_k^\top d_x + d_s\| \right) \\ &\quad + \mu \left(e^\top S_k^{-1} v_s - \frac{1}{2} v_s^\top S_k^{-2} v_s \right) + \mu \left(e^\top S_k^{-1} h_s - v_s^\top S_k^{-2} h_s - \frac{1}{2} h_s^\top S_k^{-2} h_s \right). \end{aligned}$$

Recalling the definitions (2.17) and (2.28) of the vertical and horizontal predicted reductions, we obtain

$$\text{pred}_k(d_k) = \nu_k \, \text{vpred}_k(v_k) + \text{hpred}_k(h_k) + \chi_k, \tag{2.40}$$

where

$$\chi_k = -\nabla f_k^\top v_x - \frac{1}{2} v_x^\top B_k v_x + \mu \left(e^\top S_k^{-1} v_s - \frac{1}{2} v_s^\top S_k^{-2} v_s \right). \tag{2.41}$$

We have noted ((2.20), (2.35)) that $\mathrm{vpred}_k(v_k)$ and $\mathrm{hpred}_k(d_k)$ are both nonnegative, but (2.41), which gives the change in the objective of (2.38) due to the vertical step v_k, can be of any sign. Condition (2.39) in Step 3 compensates for the possible negativity of this term by choosing a sufficiently large value of ν_k, so that $\mathrm{pred}_k(d_k)$ is at least a fraction ρ of $\nu_k \, \mathrm{vpred}_k(v_k)$. More precisely, from (2.41) we see that if $\mathrm{vpred}_k(v_k) > 0$, (2.39) holds when

$$\nu_k \geq \frac{-\chi_k}{(1-\rho)\,\mathrm{vpred}_k(v_k)}.$$

On the other hand, if $\mathrm{vpred}_k(v_k) = 0$, then by (2.24), it must be the case that the gradient of the squared objective in (2.9) is zero. In that case $v = 0$ is a solution to (2.9) and by the range space condition v_k is in the range of $(A_k^\top \; S_k^2)^\top$. Since $s_k > 0$ the squared objective of (2.9) is a positive definite quadratic on that subspace, so $v = 0$ is the unique minimizer in that space. This uniqueness implies that $v_k = 0$, since otherwise $\mathrm{vpred}_k(v_k)$ would be negative. In that case $\chi_k = 0$ and (2.39) is satisfied for any value of ν_k.

Observe also that the first inequality in Step 4 is satisfied when $s_k + d_s \not> 0$, so that the reduction of Δ_k that this implies, and the form of the trust region, guarantee that when Step 5 is reached, $s + d_s$ is positive. However, in our analysis requiring $s + d_s > 0$ is not sufficient; we need to keep the slacks sufficiently away from zero so that the logarithm can be considered Lipschitz continuous (see (3.2) below). This motivates the introduction of the parameter τ and the second inequality of Step 4.

Note that in step 5 we do not always set $s_{k+1} = s_k + d_s$, because when $g_{k+1}^{(i)} < 0$, the i-th constraint is feasible and we have more freedom in choosing the corresponding slack, $s_{k+1}^{(i)}$. In this case our rule ensures that the new slack is not unnecessarily small.

Finally note that we have left the strategy for computing the Lagrange multipliers and B_k unspecified. The treatment in this paper allows B_k to be any uniformly bounded approximation to $\nabla_{xx}^2 L(x_k, s_k, \lambda_k)$, and allows λ_k to be any multiplier estimate consistent with this boundedness. The important question of what choices of B_k and λ_k are most effective is not addressed here.

3 Well-posedness of Algorithm I

The purpose of this section is to show that, if an iterate (x_k, s_k) is not a stationary point of the barrier problem, then the trust region radius cannot shrink to zero and prevent the algorithm from moving away from that point. We begin by showing that m_k is an accurate local model of the merit function ϕ. To analyze this accuracy we define the *actual reduction* in the merit function ϕ from (x_k, s_k) to $(x_k + d_x, s_k + d_s)$ as

$$\mathrm{ared}_k(d) = \phi(x_k, s_k; \nu_k) - \phi(x_k + d_x, s_k + d_s; \nu_k). \tag{3.1}$$

Lemma 3.1. *Suppose that ∇f and A are Lipschitz continuous on an open convex set X containing all the iterates $\{x_k\}$ generated by Algorithm I, and assume that $\{B_k\}$ is bounded. Then there is a positive constant γ_L such that for any iterate (x_k, s_k) and step (d_x, d_s) such that the segment $[x_k, x_k + d_x]$ is in X, $s_k > 0$, and $d_s \geq -\tau s_k$,*

$$|\,\mathrm{pred}_k(d) - \mathrm{ared}_k(d)| \leq \gamma_L \Big((1 + \nu_k)\|d_x\|^2 + \|S_k^{-1} d_s\|^2 \Big).$$

Proof. Using the Lipschitz continuity of A, we have for some positive constant γ',

$$\begin{aligned}
&\Big| \|g(x_k + d_x) + s_k + d_s\| - \|g_k + s_k + A_k^\top d_x + d_s\| \Big| \\
&\quad \le \|g(x_k + d_x) - g_k - A_k^\top d_x\| \\
&\quad \le \sup_{\xi \in [x_k, x_k + d_x]} \|A(\xi) - A_k\| \, \|d_x\| \\
&\quad \le \gamma' \|d_x\|^2.
\end{aligned}$$

Similarly, for any scalars σ and σ' satisfying $\sigma > 0$ and $\sigma' \ge -\tau\sigma$,

$$\left| \ln(\sigma + \sigma') - \ln \sigma - \frac{\sigma'}{\sigma} \right| \le \sup_{t \in [\sigma, \sigma + \sigma']} \left| \frac{\sigma'}{t} - \frac{\sigma'}{\sigma} \right| = \frac{\sigma}{\sigma + \sigma'} \left(\frac{\sigma'}{\sigma} \right)^2 \le \frac{1}{1 - \tau} \left(\frac{\sigma'}{\sigma} \right)^2. \tag{3.2}$$

Using these two inequalities, the definitions (3.1), (2.7) of $\operatorname{ared}_k(d)$ and $\operatorname{pred}_k(d)$, the Lipschitz continuity of ∇f, and the boundedness of $\{B_k\}$, we have

$$\begin{aligned}
&|\operatorname{pred}_k(d) - \operatorname{ared}_k(d)| \\
&= \Bigg| f(x_k + d_x) - f_k - \nabla f_k^\top d_x - \frac{1}{2} d_x^\top B_k d_x \\
&\qquad + \nu_k \Big(\|g(x_k + d_x) + s_k + d_s\| - \|g_k + s_k + A_k^\top d_x + d_s\| \Big) \\
&\qquad - \mu \sum_{i=1}^m \left(\ln(s_k + d_s)^{(i)} - \ln s_k^{(i)} - \frac{d_s^{(i)}}{s_k^{(i)}} + \frac{1}{2} \left(\frac{d_s^{(i)}}{s_k^{(i)}} \right)^2 \right) \Bigg| \\
&\le \gamma''(1 + \nu_k) \|d_x\|^2 + \mu \left(\frac{1}{1 - \tau} + \frac{1}{2} \right) \|S_k^{-1} d_s\|^2,
\end{aligned}$$

for some positive constant γ''. □

In the next proposition, we show that Algorithm I determines an acceptable step with a finite number of reductions of Δ_k, i.e., that there can be no infinite cycling between steps 1 and 4 of Algorithm I. For this it is important that we ensure that, by decreasing the trust region radius, we are able to make the displacement in s arbitrarily small. As discussed in the paragraph following (1.19), defining $\delta_k = \max(\beta, \Delta_k)$ instead of $\delta_k = \Delta_k$ accomplishes this goal.

Proposition 3.2. *Suppose that $s_k > 0$ and that (x_k, s_k) is not a stationary point of the barrier problem (2.2). Then there exists $\Delta_k^0 > 0$, such that if $\Delta_k \in (0, \Delta_k^0)$, the inequality $\operatorname{ared}_k(d_k) \ge \eta \operatorname{pred}_k(d_k)$ holds.*

Proof. We proceed by contradiction, supposing that there is a subsequence (indexed by i, the iteration counter k is fixed here) of trust region radii $\Delta_{k,i}$ converging to zero, and corresponding steps $d_{k,i} = v_{k,i} + h_{k,i}$ and penalty parameters $\nu_{k,i}$, such that $\operatorname{ared}_{k,i}(d_{k,i}) < \eta \operatorname{pred}_{k,i}(d_{k,i})$ for all i.

The inequality $\text{ared}_{k,i}(d_{k,i}) < \eta\, \text{pred}_{k,i}(d_{k,i})$ and the assumption $\eta \in (0,1)$ imply that $|\,\text{pred}_{k,i}(d_{k,i}) - \text{ared}_{k,i}(d_{k,i})| > (1-\eta)\,\text{pred}_{k,i}(d_{k,i})$. This together with the limits $d_x^{k,i} \to 0$, $d_s^{k,i} \to 0$, and Lemma 3.1 gives

$$\text{pred}_{k,i}(d_{k,i}) = (1+\nu_{k,i})o(\|d_x^{k,i}\|) + o(\|d_s^{k,i}\|). \tag{3.3}$$

We will show that this equation leads to a contradiction, which will prove the proposition. For the rest of the proof γ_1', γ_2', ..., denote positive constants (independent of i but not of k), and to simplify the notation, we omit the arguments in $\text{vpred}_{k,i}(v_{k,i})$, $\text{hpred}_{k,i}(h_{k,i})$, and $\text{pred}_{k,i}(d_{k,i})$.

Consider first the case when $g_k + s_k = 0$. From (2.16), (2.18) and (2.17) we see that $\text{vpred}_{k,i} = 0$. Also, since $g_k + s_k = 0$, (2.9) has a solution ($v = 0$) in the range space of $(A_k^\top \;\; S_k^2)^\top$, so that the range space condition (2.14) implies that $v_{k,i}$ is of the form (2.14), for some vector $w_{k,i}$. Therefore $0 = \text{vpred}_{k,i} = \|(A_k^\top A_k + S_k^2)w_{k,i}\|$, which implies that $w_{k,i} = v_{k,i} = 0$ because the matrix inside the parenthesis is nonsingular. Given that $\text{vpred}_{k,i} = v_{k,i} = 0$, we have from (2.40), (2.41) and (2.35) that $\text{pred}_{k,i} = \text{hpred}_{k,i} \ge 0$. Hence, inequality (2.39) holds independently of the value of $\nu_{k,i}$, implying that $\{\nu_{k,i}\}_{i\ge 1}$ is bounded. Therefore, (3.3) gives

$$\text{pred}_{k,i} = o(\|d_x^{k,i}\|) + o(\|d_s^{k,i}\|). \tag{3.4}$$

On the other hand, from (2.32) and $v_{k,i} = 0$ we see that $p_k^c = -Z_x^\top \nabla f_k + \mu Z_s^\top S_k^{-1} e$. This vector is nonzero; otherwise the KKT conditions of the barrier problem (2.2) and the definition (2.29) of Z_k, would imply that (x_k, s_k) is a stationary point of the problem. Then (2.36), the trust region in (2.38), and the fact that $\delta_{k,i} = \beta$ for large i, and $h_{k,i} = d_{k,i}$ give

$$\text{pred}_{k,i} = \text{hpred}_{k,i} \ge \gamma_1' \hat{\Delta}_{k,i} \ge \gamma_1' \|(d_x^{k,i}, D_{k,i} d_s^{k,i})\|_T \ge \gamma_2'(\|d_x^{k,i}\| + \|d_s^{k,i}\|).$$

This contradicts (3.4).

Consider now the case when $g_k + s_k \ne 0$. Since the matrix $(A_k^\top \;\; S_k)$ has full rank, and by $\tilde{\Delta}_{k,i} \to 0$, we deduce from (2.24) that for i large

$$\text{vpred}_{k,i} \ge \gamma_3' \tilde{\Delta}_{k,i}. \tag{3.5}$$

Then, from Step 3 of the algorithm, (3.5), and the fact that $\|d_x^{k,i}\| + \|d_s^{k,i}\| \le (\gamma_4')^{-1}\tilde{\Delta}_{k,i}$, we obtain

$$\begin{aligned} \text{pred}_{k,i} &\ge \rho\nu_{k,i}\,\text{vpred}_{k,i} \\ &\ge \rho\nu_{k,i}\gamma_3'\tilde{\Delta}_{k,i} \\ &\ge \rho\nu_{k,i}\gamma_3'\gamma_4'\left(\|d_x^{k,i}\| + \|d_s^{k,i}\|\right). \end{aligned}$$

Since, $\nu_{k,i} \ge \nu_{-1} > 0$ this contradicts (3.3), concluding the proof. □

4 Global analysis of Algorithm I

We now analyze the global behavior of Algorithm I when applied to the barrier problem (2.2) for a fixed value of μ. To establish the main result of this section we make the following assumptions about the problem and the iterates generated by the algorithm.

Assumptions 4.1. (a) The functions f and g are differentiable on an open convex set X containing all the iterates and ∇f, g, and A are Lipschitz continuous on X. (b) The sequence $\{f_k\}$ is bounded below and the sequences $\{\nabla f_k\}$, $\{g_k\}$, $\{A_k\}$ and $\{B_k\}$ are bounded.

Note that we have not assumed that the matrices of constraint gradients A_k have full rank (a commonly made assumption) because we want to explore how the algorithm behaves in the presence of dependent constraint gradients. Our most restrictive assumption is (b), which could be violated if the iterates diverge. The practical value of our analysis, as we will show, is that the situations under which Algorithm I can fail represent problem characteristics that are of interest to a user and that can be characterized in simple mathematical terms. As we proceed with the analysis, we will point out how it makes specific demands on some of the more subtle aspects of Algorithm I whose role may not be apparent to the reader at this point. Therefore the analysis that follows provides a justification for the design of our algorithm.

We adopt the notation $\alpha^+ = \max(0, \alpha)$, for a scalar α, while for a vector, u^+ is defined componentwise by $(u^+)^{(i)} = (u^{(i)})^+$. We also make use of the measure of infeasibility $x \mapsto \|g(x)^+\|$, which vanishes if and only if x is feasible for the original problem (2.1). Note that $\|g(\cdot)^+\|^2$ is differentiable and has for gradient

$$\nabla \|g(x)^+\|^2 = 2A(x)g(x)^+.$$

We make use of the following definitions; here $A^{(i)}$ denotes the i-th column of A.

Definitions 4.2. A sequence $\{x_k\}$ is *asymptotically feasible* if $g(x_k)^+ \to 0$. We say that the sequence $\{(g_k, A_k)\}$ has a limit point $(\bar{g}, \bar{A})$ *failing the linear independence constraint qualification*, if the set $\{\bar{A}^{(i)} : \bar{g}^{(i)} = 0\}$ is rank deficient.

Note that the concept of constraint qualification usually applies to a point x, but that we extend it to characterize limit points of the sequence $\{(g_k, A_k)\}$, and thus our definition is not standard. The main result we will establish for Algorithm I is the following.

Theorem 4.3. *Suppose that Algorithm I is applied to the barrier problem* (2.2) *and that Assumptions* 4.1 *hold. Then,*

1) *the sequence of slack variables* $\{s_k\}$ *is bounded,*
2) $A_k(g_k + s_k) \to 0$ *and* $S_k(g_k + s_k) \to 0$.

Furthermore, one of the following three situations occurs.

(i) *The sequence* $\{x_k\}$ *is not asymptotically feasible. In this situation, the iterates approach stationarity of the measure of infeasibility* $x \mapsto \|g(x)^+\|$, *meaning that* $A_k g_k^+ \to 0$, *and the penalty parameters* ν_k *tend to infinity.*

(ii) *The sequence $\{x_k\}$ is asymptotically feasible, but the sequence $\{(g_k, A_k)\}$ has a limit point $(\bar{g}, \bar{A})$ failing the linear independence constraint qualification. In this situation also, the penalty parameters ν_k tend to infinity.*

(iii) *The sequence $\{x_k\}$ is asymptotically feasible and all limit points of the sequence $\{(g_k, A_k)\}$ satisfy the linear independence constraint qualification. In this situation, $\{s_k\}$ is bounded away from zero, the penalty parameter ν_k is constant and g_k is negative for all large indices k, and stationarity of problem (2.2) is obtained, i.e., $\nabla f_k + A_k \lambda_k \to 0$, where the multipliers are defined by $\lambda_k = \mu S_k^{-1} e$.*

Note that this theorem isolates two situations where the KKT conditions may not be satisfied in the limit, both of which are of interest. Outcome (i) is a situation where in the limit there is no direction improving feasibility to first order. This indicates that finding a feasible point is a problem that a local method cannot always solve without a good starting point. In considering outcome (ii) we must keep in mind that in some cases the solution to problem (2.2) is a point where the linear independence constraint qualification fails, and which is not a KKT point. Thus outcome (ii) may be just as relevant to the problem as satisfying the KKT conditions.

The rest of the section is devoted to the proof of this theorem, which will be presented in a sequence of lemmas addressing in order all the statements in the theorem. It is convenient to work with the following multiple of the merit function ϕ

$$\tilde{\phi}(x, s; \nu) \equiv \frac{1}{\nu}\phi(x, s; \nu) = \frac{1}{\nu}\left(f(x) - \mu \sum_{i=1}^{m} \ln s^{(i)}\right) + \|g(x) + s\| \qquad (s > 0).$$

Since step 4 of Algorithm I requires that $\tilde{\phi}$ be reduced sufficiently at every new iterate, we have that

$$\tilde{\phi}(x_k, s_k; \nu_{k-1}) \leq \tilde{\phi}(x_{k-1}, s_{k-1}; \nu_{k-1}) - \frac{\eta \operatorname{pred}_{k-1}}{\nu_{k-1}},$$

and therefore

$$\tilde{\phi}(x_k, s_k; \nu_k) \leq \tilde{\phi}(x_{k-1}, s_{k-1}; \nu_{k-1}) + \left(\frac{1}{\nu_k} - \frac{1}{\nu_{k-1}}\right)\left(f_k - \mu \sum_{i=1}^{m} \ln s_k^{(i)}\right) - \frac{\eta \operatorname{pred}_{k-1}}{\nu_{k-1}}. \quad (4.1)$$

This indicates that the sequence $\{\tilde{\phi}(x_k, s_k; \nu_k)\}$ is not necessarily monotone when ν_k is updated. To deal with this difficulty, we first establish that, under mild assumptions, the slack variables are bounded above.

Lemma 4.4. *Assume that $\{f_k\}$ is bounded below and that $\{g_k\}$ is bounded. Then the sequence $\{s_k\}$ is bounded, which implies that $\{\phi(x_k, s_k; \nu_k)\}$ is bounded below.*

Proof. Let γ be an upper bound for $-f_k$ and for $\|g_k\|$. Since

$$\sum_{i=1}^{m} \ln s_k^{(i)} \leq m \ln \|s_k\|_\infty \leq m \ln \|s_k\|, \qquad (4.2)$$

equation (4.1), the fact that the sequence $\{\nu_k\}$ is monotone non-decreasing, and the non-negativity of pred_k give

$$\tilde{\phi}(x_k, s_k; \nu_k) \leq \tilde{\phi}(x_1, s_1; \nu_1) + \left(\frac{1}{\nu_1} - \frac{1}{\nu_k}\right)(\gamma + \mu m \max_{2 \leq j \leq k} \ln \|s_j\|). \tag{4.3}$$

On the other hand, from the definition of $\tilde{\phi}$ and (4.2) we have that for any k,

$$\tilde{\phi}(x_k, s_k; \nu_k) \geq -\frac{1}{\nu_k}(\gamma + \mu m \ln \|s_k\|) + \|s_k\| - \|g_k\|. \tag{4.4}$$

Now, consider the indices l_j such that $\|s_{l_j}\| = \max_{k \leq l_j} \|s_k\|$. Then combining (4.3)–(4.4) for k given by any such l_j we obtain

$$-\frac{1}{\nu_{l_j}}(\gamma + \mu m \ln \|s_{l_j}\|) + \|s_{l_j}\| - \|g_{l_j}\| \leq \tilde{\phi}(x_1, s_1; \nu_1) + \left(\frac{1}{\nu_1} - \frac{1}{\nu_{l_j}}\right)(\gamma + \mu m \ln \|s_{l_j}\|),$$

and thus

$$\|s_{l_j}\| \leq \tilde{\phi}(x_1, s_1; \nu_1) + \gamma + \frac{1}{\nu_1}(\gamma + \mu m \ln \|s_{l_j}\|). \tag{4.5}$$

Since the ratio $(\ln \|s\|)/\|s\|$ tends to 0 when $\|s\| \to \infty$, relation (4.5) implies that $\{s_{l_j}\}$ must be bounded. By definition of the indices l_j we conclude that the whole sequence $\{s_k\}$ is bounded. $\square$

Given that the slack variables are bounded above and that f_k is bounded below, it is clear that we may redefine the objective function f – by adding a constant to it – so that

$$f_k - \mu \sum_{i=1}^{m} \ln s_k^{(i)} > 0$$

at all iterates, and that this change does not affect the problem or the algorithm in any way. This positivity, the fact that ν_k is nondecreasing and (4.1) imply that

$$\tilde{\phi}(x_k, s_k; \nu_k) \leq \tilde{\phi}(x_{k-1}, s_{k-1}; \nu_{k-1}) - \frac{\eta \,\text{pred}_{k-1}}{\nu_{k-1}} \tag{4.6}$$

for all k. This fact will be useful in proving the next result, which deals with the function $(x, s) \mapsto \|g(x) + s\|^2$, which is another measure of feasibility for the barrier problem (2.2). Note that the gradient of this function is

$$2 \begin{pmatrix} A(x) \\ S \end{pmatrix} (g(x) + s).$$

We now show that the iterates generated by the algorithm approach stationarity of this feasibility function $\|g(x) + s\|^2$.

Lemma 4.5. *Assume that the sequences $\{g_k\}$, $\{A_k\}$, and $\{B_k\}$ are bounded, that $\{f_k\}$ is bounded below, and that g, A, and ∇f are Lipschitz continuous on an open set containing all the iterates. Then*

$$\lim_{k\to\infty} \begin{pmatrix} A_k \\ S_k \end{pmatrix} (g_k + s_k) = 0.$$

Proof. By the assumptions on A and g, we have that the function

$$\theta(x,s) \equiv \left\| \begin{pmatrix} A(x) \\ S \end{pmatrix} (g(x) + s) \right\|$$

is Lipschitz continuous on an open set Ω containing all the iterates, i.e., there is a constant $\gamma_L' > 0$ such that

$$|\theta(x,s) - \theta(x_l, s_l)| \le \gamma_L' \|(x,s) - (x_l, s_l)\|, \tag{4.7}$$

for any two points (x,s) and (x_l, s_l) in Ω.

Now consider an arbitrary iterate (x_l, s_l) such that $\theta_l \equiv \theta(x_l, s_l) \neq 0$. We first want to show that in a neighborhood of this iterate all sufficiently small steps are accepted by Algorithm I. To do this define the ball

$$\mathcal{B}_l \equiv \{(x,s) : \|(x,s) - (x_l, s_l)\| < \theta_l/(2\gamma_L')\}.$$

By (4.7), for any $(x,s) \in \mathcal{B}_l$ we have that

$$\theta(x,s) \ge \frac{1}{2}\theta_l,$$

which implies that $g(x) + s \neq 0$. We also know that the vertical step satisfies (2.24), and have shown in Lemma 4.4 that $\{s_k\}$ is bounded. Using this, (2.39) and the boundedness assumptions on $\{A_k\}$ and $\{g_k + s_k\}$, we see that there is a constant γ_1' (independent of k and l), such that for any such iterate (x_l, s_l) and any iterate $(x_k, s_k) \in \mathcal{B}_l$

$$\text{pred}_k \ge \rho\nu_k \,\text{vpred}_k \ge \nu_k \gamma_1' \theta_l \,\min\left(\xi\beta, \tilde{\Delta}_k, \theta_l\right). \tag{4.8}$$

Therefore, if Δ_k is sufficiently small we have

$$\text{pred}_k \ge \nu_k \gamma_1' \theta_l \,\tilde{\Delta}_k. \tag{4.9}$$

Now for sufficiently small Δ_k, we have from (2.22) and the definition of D_k (see step 2 of Algorithm I) that

$$\|d_x\|^2 + \|S_k^{-1} d_s\|^2 \le \max(1, \frac{1}{\beta^2})\gamma_{\scriptscriptstyle T}^{-2} \|(d_x, D_k d_s)\|_{\scriptscriptstyle T}^2 \le \gamma_2' \Delta_k^2, \tag{4.10}$$

for some constant γ_2'. Using this in Lemma 3.1, recalling that $\tilde{\Delta}_k = \xi\Delta_k$, and using (4.9) we obtain

$$\frac{|\,\text{ared}_k - \text{pred}_k\,|}{\text{pred}_k} \le \frac{(1+\nu_k)\gamma_L \gamma_2' \Delta_k^2}{\nu_k \gamma_1' \theta_l \xi \Delta_k}.$$

By making Δ_k sufficiently small we can ensure that the last term is less than or equal to $1-\eta$, and therefore

$$\text{ared}_k \geq \eta\, \text{pred}_k, \tag{4.11}$$

implying (by (3.1)) acceptance of the step in Algorithm I.

Next we want to show that the rest of the iterates $\{x_k\}_{k>l}$ cannot remain in $\mathcal{B}_l$. We proceed by contradiction and assume that for all $k>l$, $x_k \in \mathcal{B}_l$ and therefore (4.11) holds for sufficiently small Δ_k. This implies that there exists $\Delta^0 > 0$ such that $\Delta_k > \Delta^0$ for all $k > l$. This, together with (4.6) and (4.8) gives

$$\tilde{\phi}_{k+1} \leq \tilde{\phi}_k - \frac{\eta}{\nu_k}\, \text{pred}_k \leq \tilde{\phi}_k - \eta\gamma_1'\theta_l \min\left(\xi\beta, \xi\Delta^0, \theta_l\right),$$

where $\tilde{\phi}_k \equiv \tilde{\phi}(x_k, s_k; \nu_k)$. Since the last term in the right hand side is constant, this relation implies that $\tilde{\phi}_k \to -\infty$, contradicting the conclusion of Lemma 4.4 that $\{\tilde{\phi}_k\}$ is bounded below. Therefore the sequence of iterates must leave $\mathcal{B}_l$ for some $k > l$.

Now let (x_{k+1}, s_{k+1}) be the first iterate after (x_l, s_l) that is not contained in $\mathcal{B}_l$. We must consider two possibilities. First, if there exists some $j \in [l,k]$ such that $\tilde{\Delta}_j > \min(\xi\beta, \theta_l)$, then we have from (4.6) and (4.8) that

$$\begin{aligned}
\tilde{\phi}_{k+1} &\leq \tilde{\phi}_{j+1} \\
&\leq \tilde{\phi}_j - \frac{\eta}{\nu_j}\, \text{pred}_j \\
&\leq \tilde{\phi}_j - \eta\gamma_1'\theta_l \min(\xi\beta, \theta_l) \\
&\leq \tilde{\phi}_l - \eta\gamma_1'\theta_l \min(\xi\beta, \theta_l).
\end{aligned} \tag{4.12}$$

The other possibility is that for all $j \in [l,k]$, $\tilde{\Delta}_j \leq \min(\xi\beta, \theta_l)$. In that case it follows from (4.6) and (4.8) that

$$\begin{aligned}
\tilde{\phi}_{k+1} &\leq \tilde{\phi}_l - \sum_{j=l}^{k} \frac{\eta}{\nu_j}\, \text{pred}_j \\
&\leq \tilde{\phi}_l - \sum_{j=l}^{k} \eta\gamma_1'\theta_l\xi\Delta_j.
\end{aligned} \tag{4.13}$$

Let $(\gamma_3')^{-1}$ be an upper bound on $\{s_k\}$; then since (x_{k+1}, s_{k+1}) has left the ball $\mathcal{B}_l$, whose radius is $\theta_l/(2\gamma_L')$, and $\Delta_j \leq \beta$

$$\begin{aligned}
\sum_{j=l}^{k} \Delta_j &\geq \sum_{j=l}^{k} \|(d_x^j, D_j d_s^j)\|_T \\
&\geq \gamma_T \min(1, \beta\gamma_3') \sum_{j=l}^{k} \|d_j\| \\
&\geq \gamma_T \min(1, \beta\gamma_3') \|(x_{k+1}, s_{k+1}) - (x_l, s_l)\| \\
&\geq \gamma_T \min(1, \beta\gamma_3')\theta_l/(2\gamma_L') \\
&= \gamma_4'\theta_l,
\end{aligned}$$

for some constant γ_4'. Substituting in (4.13) we obtain

$$\tilde{\phi}_{k+1} \leq \tilde{\phi}_l - \eta\gamma_1'\gamma_4'\xi\theta_l^2. \tag{4.14}$$

To conclude the proof note that since $\{\tilde{\phi}_k\}$ is decreasing and bounded below, we have that $\tilde{\phi}_l \to \tilde{\phi}_*$ for some infimum value $\tilde{\phi}_*$. Since l was chosen arbitrarily, the fact that either (4.12) or (4.14) must hold at (x_l, s_l) implies that $\theta_l \to 0$. $\square$

This result shows that $A_k(g_k+s_k) \to 0$ and $S_k(g_k+s_k) \to 0$. This is of course satisfied when $g_k + s_k \to 0$, that is when feasibility is attained asymptotically. However it can also occur when $g_k + s_k \not\to 0$ and the matrices A_k and S_k lose rank, a possibility we now investigate.

The procedure for updating the slack variables in step 5 of Algorithm I becomes important now. It ensures that

$$g_k + s_k \geq g_k^+ \geq 0 \tag{4.15}$$

holds at every iteration. Lemma 4.6 first uses this relation to show that the gradient $A_k g_k^+$ of the measure of infeasibility $x \mapsto \frac{1}{2}\|g(x)^+\|^2$ converges to zero. Then Lemma 4.6 shows that the case $g_k^+ \not\to 0$ implies that the penalty parameters tend to infinity.

Lemma 4.6. *Under the conditions of Lemma 4.5, $A_k g_k^+ \to 0$. Moreover, if the sequence of iterates is not asymptotically feasible, i.e., if $g_k^+ \not\to 0$, then the penalty parameters ν_k tend to infinity.*

Proof. Let $\hat{A}$, $\hat{g}$, and $\hat{s}$ be limit points of the sequences $\{A_k\}$, $\{g_k\}$, and $\{s_k\}$. Since these sequences are bounded, we only have to show that $\hat{A}\hat{g}^+ = 0$.

If $\hat{g}^{(i)} \geq 0$, the conditions $\hat{s} \geq 0$ and $\hat{S}(\hat{g}+\hat{s}) = 0$ (from Lemma 4.5) imply that $\hat{s}^{(i)} = 0$. If $\hat{g}^{(i)} < 0$, then from (4.15), $\hat{s}^{(i)} \neq 0$, which together with the equation $\hat{S}(\hat{g}+\hat{s}) = 0$ implies that $\hat{s}^{(i)} = -\hat{g}^{(i)}$. This shows that $\hat{g} + \hat{s} = \hat{g}^+$. Using the equation $\hat{A}(\hat{g}+\hat{s}) = 0$ (from Lemma 4.5), we obtain that $\hat{A}\hat{g}^+ = 0$, which proves the first part of the lemma.

If $g_k^+ \not\to 0$, (4.15) implies that there is an index i such that $(g_k+s_k)^{(i)} \not\to 0$. Since $S_k(g_k+s_k) \to 0$, there is a subsequence of indices k such that $s_k^{(i)} \to 0$ and $\ln s_k^{(i)} \to -\infty$. Since $\{f_k\}$ is bounded below, this is incompatible with the decrease of $\phi(x_k, s_k; \nu)$ for a fixed value of the penalty parameter $\nu > 0$. Therefore ν_k is increased infinitely often, and because this is always at least by a constant factor, $\{\nu_k\}$ is unbounded. $\square$

This completes our discussion of the case when the sequence $\{x_k\}$ is not asymptotically feasible (item (i) of Theorem 4.3).

To continue the analysis we consider from now on only the case when feasibility is approached asymptotically. We will divide the analysis in two cases depending on whether the matrices $(A_k^\top \; S_k)$ lose rank or not. We use the notation $\sigma_{\min}(C)$ to denote the smallest singular value of a matrix C, and recall that in Definitions 4.2 we describe our notion of linear independence constraint qualification.

Lemma 4.7. *Suppose that the sequences* $\{g_k\}$ *and* $\{A_k\}$ *are bounded, and that* $g_k + s_k \to 0$*. Then, either there is some bound* $\hat{\sigma} > 0$ *such that*

$$\sigma_{\min}((A_k^\top \;\; S_k)) \geq \hat{\sigma}$$

for all k*, or the sequence* $\{(g_k, A_k)\}$ *has a limit point* $(\bar{g}, \bar{A})$ *failing the linear independence constraint qualification. In the latter case, the penalty parameter* ν_k *goes to infinity.*

Proof. If $\liminf \sigma_{\min}((A_k^\top \;\; S_k)) = 0$, there is a subsequence of iterates for which the smallest singular value of $(A_k^\top \;\; S_k)$ converges to 0. Thus, since the sequence $\{(A_k, g_k, s_k)\}$ is bounded (by the assumptions and Lemma 4.4), it has a limit point $(\bar{A}, \bar{g}, \bar{s})$ such that the matrix $(\bar{A}^\top \;\; \bar{S})$ is rank deficient. Now $\bar{S}$ is diagonal, so that the set $\mathcal{I} = \{i : \bar{s}^{(i)} = 0\}$ cannot be empty and the columns of $\bar{A}$ with index in $\mathcal{I}$ must be linearly dependent. Since we assume $g_k + s_k \to 0$, we have that $\bar{g}^{(i)} = 0$ if and only if $i \in \mathcal{I}$, and it follows that the set $\{\bar{A}^{(i)} : \bar{g}^{(i)} = 0\}$ is rank deficient.

Since for $i \in \mathcal{I}$, a subsequence of $\{s_k^{(i)}\}$ tends to zero, a subsequence of $\{-\ln s_k^{(i)}\}$ goes to infinity. This is incompatible with the decrease of $\phi(x_k, s_k; \nu)$, which would occur if ν_k were eventually constant. By the update rule for the penalty parameter, if ν_k is changed infinitely often then $\{\nu_k\}$ is unbounded. □

For the rest of this section we will focus on the case where $\sigma_{\min}((A_k^\top \;\; S_k)) \geq \hat{\sigma} > 0$ for all k. First we will use this condition to bound the length of the vertical step $v = (v_x, v_s)$ by a constant multiple of vpred_k (Lemma 4.8); then we can use this relation to show that the sequence of merit function parameters ν_k is bounded (Lemma 4.9). Finally we will be able to show that the stationarity conditions for problem (2.2) are asymptotically satisfied.

Lemma 4.8. *Suppose that Assumptions* 4.1 *hold and that for some* $\hat{\sigma} > 0$*,*

$$\sigma_{\min}((A_k^\top \;\; S_k)) \geq \hat{\sigma} > 0, \tag{4.16}$$

for all k*. Then, there are positive constants* γ_3 *and* γ_4 *such that if* $\|g_k + s_k\| \leq \gamma_3$*,*

$$\left\| \begin{pmatrix} v_x \\ S_k^{-1} v_s \end{pmatrix} \right\| \leq \gamma_4 \, \text{vpred}_k \,. \tag{4.17}$$

Proof. Recall that, by Lemma 2.2, the vertical step must satisfy

$$\|g_k + s_k\| \, \text{vpred}_k \geq \frac{\gamma_1}{2} \left\| \begin{pmatrix} A_k \\ S_k \end{pmatrix} (g_k + s_k) \right\| \min \left(\gamma_T, \frac{\gamma_T \tilde{\Delta}_k}{\hat{\beta}}, \frac{\left\| \begin{pmatrix} A_k \\ S_k \end{pmatrix} (g_k + s_k) \right\|}{\|(A_k^\top \;\; S_k)\|^2} \right).$$

(We may assume that $\|g_k + s_k\| \neq 0$, for otherwise $v_k = \text{vpred}_k = 0$ and (4.17) is trivially satisfied.) Using (4.16) and letting $\bar{\sigma}_1 = \sup_k \|(A_k^\top \;\; S_k)\|$, this implies

$$\text{vpred}_k \geq \frac{\gamma_1 \hat{\sigma}}{2} \min \left(\gamma_T, \frac{\gamma_T \tilde{\Delta}_k}{\hat{\beta}}, \frac{\hat{\sigma} \|g_k + s_k\|}{\bar{\sigma}_1^2} \right). \tag{4.18}$$

Let us now assume that $\|g_k + s_k\|$ is strictly smaller than the constant $\gamma_T \bar{\sigma}_1^2 / \hat{\sigma}$; then the minimum in (4.18) cannot occur at γ_T. Then (4.18) becomes

$$\text{vpred}_k \geq \frac{\gamma_1 \hat{\sigma}}{2} \min \left(\frac{\gamma_T \tilde{\Delta}_k}{\hat{\beta}}, \frac{\hat{\sigma} \|g_k + s_k\|}{\bar{\sigma}_1^2} \right). \tag{4.19}$$

We now consider two cases:

Case 1. Suppose $\|g_k + s_k\| \geq \frac{\gamma_T \hat{\sigma}}{2\hat{\beta}} \tilde{\Delta}_k$. Then, using $\hat{\sigma} \leq \bar{\sigma}_1$,

$$\text{vpred}_k \geq \frac{\gamma_1 \hat{\sigma}}{2} \min \left(\frac{\gamma_T}{\hat{\beta}}, \frac{\gamma_T \hat{\sigma}^2}{2\hat{\beta} \bar{\sigma}_1^2} \right) \tilde{\Delta}_k \geq \frac{\gamma_1 \gamma_T \hat{\sigma}^3}{4\hat{\beta} \bar{\sigma}_1^2} \tilde{\Delta}_k. \tag{4.20}$$

From (2.10) noting that $\tilde{\delta}_k \geq \beta$, and by (4.20),

$$\begin{aligned} \left\| \begin{pmatrix} v_x \\ S_k^{-1} v_s \end{pmatrix} \right\| &\leq \frac{1}{\min(1, \tilde{\delta}_k) \gamma_T} \|(v_x, \tilde{D}_k v_s)\|_T \\ &\leq \frac{1}{\min(1, \beta) \gamma_T} \tilde{\Delta}_k \\ &\leq \frac{4\hat{\beta} \bar{\sigma}_1^2}{\min(1, \beta) \gamma_T^2 \gamma_1 \hat{\sigma}^3} \text{vpred}_k, \end{aligned}$$

from which (4.17) follows.

Case 2. Suppose

$$\|g_k + s_k\| \leq \frac{\gamma_T \hat{\sigma}}{2\hat{\beta}} \tilde{\Delta}_k. \tag{4.21}$$

Consider an arbitrary $n+m$ vector $\bar{v}$ in the range of $(A_k^\top \; S_k^2)^\top$ that decreases the objective function of the vertical subproblem (2.9). We claim such a vector is within the trust region if $\|g_k + s_k\|$ is sufficiently small. Since $\bar{v} = (A_k^\top \; S_k^2)^\top w$ for some vector $w \in \mathbf{R}^m$,

$$\|g_k + s_k\|^2 \geq \|g_k + s_k + (A_k^\top \; S_k) \begin{pmatrix} A_k \\ S_k \end{pmatrix} w\|^2$$

or

$$\|(A_k^\top A_k + S_k^2) w\|^2 \leq -2(g_k + s_k)^\top (A_k^\top A_k + S_k^2) w.$$

Using the Cauchy-Schwarz inequality, this implies that

$$\|(A_k^\top A_k + S_k^2) w\| \leq 2\|g_k + s_k\|$$

and by (4.16), it follows that

$$\left\| \begin{pmatrix} \bar{v}_x \\ S_k^{-1} \bar{v}_s \end{pmatrix} \right\| = \left\| \begin{pmatrix} A_k \\ S_k \end{pmatrix} w \right\| \leq \frac{2}{\hat{\sigma}} \|g_k + s_k\|. \tag{4.22}$$

Then by (2.10), (4.22) and (4.21)

$$
\begin{aligned}
\|(\bar{v}_x, \tilde{D}_k \bar{v}_s)\|_T &\le \frac{1}{\gamma_T} \max(\hat{\beta}, \tilde{\Delta}_k) \|(\bar{v}_x, S_k^{-1} \bar{v}_s)\| \\
&\le \frac{2}{\gamma_T \hat{\sigma}} \max(\hat{\beta}, \tilde{\Delta}_k) \|g_k + s_k\| \\
&= \frac{2}{\gamma_T \hat{\sigma}} \max(\hat{\beta} \|g_k + s_k\|, \tilde{\Delta}_k \|g_k + s_k\|) \\
&\le \max\left(\tilde{\Delta}_k, \frac{2\tilde{\Delta}_k \|g_k + s_k\|}{\gamma_T \hat{\sigma}} \right) \\
&\le \tilde{\Delta}_k, \qquad (4.23)
\end{aligned}
$$

provided that $\|g_k + s_k\| \le \gamma_T \hat{\sigma}/2$.

Now consider the problem (2.9) and its transformed equivalent (2.12). Since $(A_k^\top \; S_k)$ is of full rank there is a solution $\bar{u}$ to the equation $g_k + s_k + A_k^\top u_x + S_k u_s = 0$, of minimum Euclidean norm, which is known to lie in the range of $(A_k^\top \; S_k)^\top$. Thus $\bar{v} = (\bar{u}_x, S_k \bar{u}_s)$ lies in the range of $(A_k^\top \; S_k^2)^\top$, and gives a value of zero for the objective of (2.9). By the above argument, if $\|g_k + s_k\|$ is sufficiently small, $\bar{v}$ must satisfy (4.23), and is therefore a solution to (2.9). Since $\bar{v}$ is a solution to (2.9) lying in the range of $(A_k^\top \; S_k^2)^\top$, the range space condition implies that the vertical step v_k must also lie in the range of $(A_k^\top \; S_k^2)^\top$. This implies that since $\mathrm{vpred}_k(v_k) \ge 0$, v_k satisfies (4.22), so that

$$
\left\| \begin{pmatrix} v_x \\ S_k^{-1} v_s \end{pmatrix} \right\| \le \frac{2}{\hat{\sigma}} \|g_k + s_k\|. \qquad (4.24)
$$

Now recall that by (4.19) and (4.21),

$$
\mathrm{vpred}_k \ge \frac{\gamma_1 \hat{\sigma}}{2} \min\left(\frac{2}{\hat{\sigma}}, \frac{\hat{\sigma}}{\bar{\sigma}_1^2} \right) \|g_k + s_k\|, \qquad (4.25)
$$

which together with (4.24) implies (4.17).

$\square$

We should note that if the Lagrange multipliers λ_k are defined as the least squares solution to

$$
\begin{pmatrix} \nabla f(x_k) + A_k \lambda \\ S_k \lambda - \mu e \end{pmatrix} = 0,
$$

then (4.16) implies that the sequence $\{\lambda_k\}$ is bounded. The boundedness assumption on B_k is now easy to enforce in this case, particularly if B_k is defined as $\nabla^2_{xx} L(x_k, s_k, \lambda_k)$.

With the bound (4.17) on the vertical step, in the case where $g_k + s_k \to 0$, we can show that the parameter ν_k eventually becomes fixed.

Lemma 4.9. *Suppose that Assumptions* 4.1 *are satisfied, and that* (4.17) *holds for* k *sufficiently large. Then, the sequence of penalty parameters* $\{\nu_k\}$ *is bounded. In addition, there exists an index* k_1 *and positive scalars* $\bar{\nu}$ *and* γ_5*, such that for all* $k \geq k_1$*,*

$$\nu_k = \bar{\nu}$$

and

$$\text{pred}_k(d_k) \geq \gamma_5 \, \text{hpred}_k \, . \tag{4.26}$$

Proof. In Step 3 of Algorithm I, ν_k is chosen to be sufficiently large such that

$$\text{pred}_k(d_k) \geq \rho \nu_k \, \text{vpred}_k, \tag{4.27}$$

where, as in (2.40)-(2.41)

$$\begin{aligned} \text{pred}_k(d_k) \;\; &= \;\; \nu_k \, \text{vpred}_k + \text{hpred}_k \\ &\quad - \nabla f_k^\top v_x - \frac{1}{2} v_x^\top B_k v_x + \mu \left(e^\top S_k^{-1} v_s - \frac{1}{2} v_s^\top S_k^{-2} v_s \right) . \end{aligned} \tag{4.28}$$

We consider the terms in the second line of the above equation. By Assumptions 4.1, $\{\nabla f_k\}$, $\{A_k\}$, and $\{B_k\}$ are all bounded. Note also that $\{\text{vpred}_k\}$ is bounded, since by (2.17), $\text{vpred}_k \leq \|g_k + s_k\|$, and this quantity is bounded as a consequence of Assumption 4.1 and Lemma 4.4. Therefore, using (4.17), there is a constant $\gamma_1' > 0$ such that

$$-\nabla f_k^\top v_x - \frac{1}{2} v_x^\top B_k v_x + \mu \left(e^\top S_k^{-1} v_s - \frac{1}{2} v_s^\top S_k^{-2} v_s \right) \geq - \gamma_1' \; \text{vpred}_k \, .$$

Hence from (4.28) the predicted decrease satisfies

$$\text{pred}_k(d_k) \geq \nu_k \, \text{vpred}_k + \text{hpred}_k - \gamma_1' \, \text{vpred}_k \, . \tag{4.29}$$

Since vpred_k and hpred_k are nonnegative, we deduce from this inequality that condition (4.27) is satisfied if $\nu_k \geq \gamma_1'/(1-\rho)$. Therefore, if ν_k becomes larger than $\gamma_1'/(1-\rho)$, it will never be increased. This, together with the fact that whenever Algorithm I increases ν_k it does so by a constant factor, implies that after some iterate, k_1 say, ν_k will remain unchanged at some value $\bar{\nu}$.

Now consider (4.29) when $k > k_1$ and $\nu_k = \bar{\nu}$. If $\text{hpred}_k > -2(\bar{\nu} - \gamma_1') \, \text{vpred}_k$ then $\text{pred}_k(d_k) > \frac{1}{2} \text{hpred}_k$. Otherwise, $\text{hpred}_k \leq -2(\bar{\nu} - \gamma_1') \, \text{vpred}_k$ and it must be the case that $\bar{\nu} - \gamma_1' < 0$, in which case, by (4.27),

$$\text{pred}_k(d_k) \geq \frac{\rho \bar{\nu}}{2(\gamma_1' - \bar{\nu})} \, \text{hpred}_k \, .$$

So (4.26) holds in either case. □

Theorem 4.10. *Suppose that Assumptions* 4.1 *hold and that the singular values of the matrices* $(A_k^\top \; S_k)$ *are bounded away from zero. Then,*
(*i*) s_k *is bounded away from zero and* g_k *is negative for all large* k*,*

(ii) $\nabla f_k + \mu A_k S_k^{-1} e \to 0$.

Proof. By Lemma 4.5, $g_k + s_k \to 0$, and thus (4.17) eventually holds at all iterates. So, by Lemma 4.9, we have that $\nu_k = \bar{\nu}$ for all $k \geq k_1$. Since Algorithm I decreases the merit function at every iteration we have

$$\phi(x_k, s_k; \bar{\nu}) \leq \phi(x_{k_1}, s_{k_1}; \bar{\nu}), \quad \text{for } k \geq k_1.$$

Thus

$$-\mu \sum_{i=1}^{m} \ln s_k^{(i)} \leq \phi(x_{k_1}, s_{k_1}; \bar{\nu}) - f_k - \bar{\nu}\|g_k + s_k\|.$$

Since we assume that $\{f_k\}$ is bounded below, this implies that there is a vector $\bar{s} > 0$ such that

$$s_k \geq \bar{s}, \quad \text{for } k \geq 1.$$

Thus, because $g_k + s_k \to 0$, we have that $g_k < 0$ for large k, proving (i).

Next, recall that, by Lemma 2.3, (h_x, h_s) satisfies

$$\text{hpred}_k \geq \frac{\gamma_2}{2}\|p_k^c\| \min\left(\frac{\gamma_T \hat{\Delta}_k}{\|Z_x^\top Z_x + Z_s^\top D_k^2 Z_s\|^{1/2}}, \frac{\|p_k^c\|}{\|Z_x^\top B_k Z_x + \mu Z_s^\top S_k^{-2} Z_s\|} \right), \tag{4.30}$$

where

$$p_k^c = -Z_x^\top(\nabla f_k + B_k v_x) + \mu Z_s^\top (S_k^{-1} e - S_k^{-2} v_s),$$

and where the null space basis matrix $Z_k = \begin{pmatrix} Z_x^\top & Z_s^\top \end{pmatrix}^\top$ is assumed to have singular values that are both bounded above and bounded away from zero. Since $\{B_k\}$ is bounded, and since we have shown that all components of s_k are bounded away from zero, it follows that $Z_x^\top B_k Z_x + \mu Z_s^\top S_k^{-2} Z_s$ is bounded. Similarly, $Z_x^\top Z_x + Z_s^\top D_k^2 Z_s$ is clearly bounded if $\Delta_k \leq \beta$, since in that case $D_k = \beta S_k^{-1}$. If $\Delta_k > \beta$, then $D_k = \Delta_k S_k^{-1}$ and, because by (2.27), $\hat{\Delta}_k \geq (1-\xi)\Delta_k$, there is a similar bound for $\|Z_x^\top Z_x + Z_s^\top D_k^2 Z_s\|^{1/2}/\hat{\Delta}_k$. Hence, inequality (4.30) becomes

$$\text{hpred}_k \geq \gamma_1' \, \|p_k^c\| \min\left(1, \hat{\Delta}_k, \|p_k^c\|\right), \tag{4.31}$$

for some positive constant γ_1'.

To show that $\nabla f_k + \mu A_k S_k^{-1} e \to 0$, we relate this quantity to p_k^c. Note that the matrix $\begin{pmatrix} I & -A_k \end{pmatrix}^\top$ is a null space basis (see (2.29)), and that using the equivalence of null space bases we get

$$\begin{aligned} \nabla f_k + \mu A_k S_k^{-1} e &= \begin{pmatrix} I & -A_k \end{pmatrix} \begin{pmatrix} \nabla f(x_k) \\ -\mu S_k^{-1} e \end{pmatrix} \\ &= \begin{pmatrix} I & -A_k \end{pmatrix} Z_k (Z_k^\top Z_k)^{-1} Z_k^\top \begin{pmatrix} \nabla f(x_k) \\ -\mu S_k^{-1} e \end{pmatrix}, \end{aligned} \tag{4.32}$$

for the chosen null space basis Z_k. By the boundedness of A_k and of the singular values of Z_k it follows from (4.32) that for some constant γ_2'

$$\|p_k^c\| \geq \gamma_2' \left(\|q_k\| - \|v_k\|\right)$$

for all k, where $q_k = \nabla f_k + \mu A_k S_k^{-1} e$.

We use a similar argument to that used in the proof of Lemma 4.5. To obtain a contradiction, suppose that $\theta \equiv \frac{1}{4} \limsup_{k\to\infty} \|q_k\|$ is nonzero. Since $v_k \to 0$, we can find an iterate (x_l, s_l) with arbitrarily large l such that $\|q_l\| > 3\theta$ and such that $\|v_k\| < \theta$ for all $k \geq l$. Let $\bar{\gamma}_L$ be the Lipschitz constant for $q(x,s) = \nabla f(x) + \mu A(x) S^{-1} e$. Then any iterate (x_k, s_k), with $k \geq l$, in the ball $\mathcal{B} \equiv \{(x,s) : \|(x,s) - (x_l, s_l)\| < \theta/\bar{\gamma}_L\}$, satisfies

$$\|p_k^c\| \geq \gamma_2' \left(\|q_l\| - \|q_l - q_k\| - \|v_k\|\right) \geq \gamma_2'(3\theta - \theta - \theta) = \gamma_2'\theta.$$

By Lemma 4.9 and (4.31),

$$\mathrm{pred}_k \geq \gamma_5 \, \mathrm{hpred}_k \geq \gamma_5 \gamma_1' \, \gamma_2' \, \theta \min\left(1, \hat{\Delta}_k, \gamma_2' \, \theta\right). \tag{4.33}$$

Therefore, for any iterate (x_k, s_k) in $\mathcal{B}$, with $k \geq l$, if $\hat{\Delta}_k$ is sufficiently small we have

$$\mathrm{pred}_k \geq \gamma_5 \gamma_1' \, \gamma_2' \theta \hat{\Delta}_k.$$

Then by Lemma 3.1,

$$\frac{|\,\mathrm{ared}_k - \mathrm{pred}_k\,|}{\mathrm{pred}_k} \leq \frac{\gamma_L (1+\bar{\nu}) \Delta_k^2}{\gamma_T^2 \min(1, \beta^2) \gamma_5 \gamma_1' \, \gamma_2' \theta \hat{\Delta}_k} \leq 1 - \eta \tag{4.34}$$

for Δ_k sufficiently small, implying acceptance of the step.

Next we show that the rest of the iterates $\{(x_k, s_k)\}_{k \geq l}$ cannot remain in $\mathcal{B}$. To prove this by contradiction we assume that for all $k > l$, $(x_k, s_k) \in \mathcal{B}$ and therefore (4.34) holds for sufficiently small Δ_k. This implies that there exists $\Delta^0 > 0$ such that $\hat{\Delta}_k > \Delta^0$ for all $k > l$. This, together with (4.33) and step 4 of Algorithm I, gives

$$\phi_{k+1} \leq \phi_k - \eta \, \mathrm{pred}_k \leq \phi_k - \eta \gamma_5 \gamma_1' \, \gamma_2' \theta \min\left(1, \Delta^0, \gamma_2' \theta\right).$$

Since the right hand side is constant, this relation implies that $\phi_k \to -\infty$, which gives a contradiction because Lemma 4.4 shows that $\{\phi_k\}$ is bounded below. Therefore the sequence of iterates must leave $\mathcal{B}$ for some $k > l$.

Now let (x_{k+1}, s_{k+1}) be the first iterate after (x_l, s_l) that is not contained in $\mathcal{B}$. We must consider two possibilities. First, if there exists some $j \in [l, k]$ such that $\hat{\Delta}_j > \min(1, \gamma_2' \theta)$, then we have from (4.33) that

$$\begin{aligned} \phi_{k+1} &\leq \phi_{j+1} \\ &\leq \phi_j - \eta \mathrm{pred}_j \\ &\leq \phi_j - \eta \gamma_5 \gamma_1' \, \gamma_2' \theta \min\left(1, \gamma_2' \theta\right) \\ &\leq \phi_l - \eta \gamma_5 \gamma_1' \, \gamma_2' \theta \min\left(1, \gamma_2' \theta\right). \end{aligned} \tag{4.35}$$

The other possibility is that for all $j \in [k,l]$, $\hat{\Delta}_j \le \min(1, \gamma_2'\theta)$. In that case, it follows from (4.33) and (2.27) that

$$\begin{aligned}
\phi_{k+1} &\le \phi_l - \eta \sum_{j=l}^{k} \mathrm{pred}_j \\
&\le \phi_l - \eta\gamma_5\gamma_1'\gamma_2'\theta(1-\xi)\sum_{j=l}^{k}\Delta_j \\
&\le \phi_l - \eta\gamma_5\gamma_1'\gamma_2'\gamma_3'(1-\xi)\theta^2. \qquad (4.36)
\end{aligned}$$

The last inequality follows from the fact that (x_{k+1}, s_{k+1}) has left the ball $\mathcal{B}$, whose radius is $\theta/\bar{\gamma}_L$, so that, as at the end of Lemma 4.5, $\sum_{j=l}^{k}\Delta_j \ge \gamma_3'\theta$, for some constant γ_3'.

Since the sequence $\{\phi_k\}$ is decreasing and bounded below, it converges. This is in contradiction with the fact that l may be chosen arbitrarily large in (4.35) or (4.36), and the fact that $\theta \ne 0$. Therefore $q_k \to 0$. $\square$

Now we have essentially established all points of our main convergence result, Theorem 4.3, which we restate and whose proof we now summarize.

Theorem 4.11. *Suppose that Algorithm I is applied to the barrier problem* (2.2) *and that Assumptions* 4.1 *hold. Then,*

1) *the sequence of slack variables $\{s_k\}$ is bounded,*
2) *$A_k(g_k + s_k) \to 0$ and $S_k(g_k + s_k) \to 0$.*

Furthermore, one of the following three situations occurs.

(*i*) *The sequence $\{x_k\}$ is not asymptotically feasible. In this situation, the sequence $\{x_k\}$ approaches stationarity of the measure of infeasibility $x \mapsto \|g(x)^+\|$, meaning that $A_k g_k^+ \to 0$, and the penalty parameter ν_k goes to infinity.*

(*ii*) *The sequence $\{x_k\}$ is asymptotically feasible, but the sequence $\{(g_k, A_k)\}$ has a limit point $(\bar{g}, \bar{A})$ failing the linear independence constraint qualification. In this situation also, the penalty parameter ν_k goes to infinity.*

(*iii*) *The sequence $\{x_k\}$ is asymptotically feasible and all limit points of the sequence $\{(g_k, A_k)\}$ satisfy the linear independence constraint qualification. In this situation, s_k is bounded away from zero, the penalty parameter ν_k is constant for all large indices k, g_k is negative for all large indices k, and stationarity of problem* (2.2) *is obtained, i.e., $\nabla f_k + A_k\lambda_k \to 0$, where the multipliers are defined by $\lambda_k = \mu S_k^{-1} e$.*

Proof. Conclusion (1) was established in Lemma 4.4, and conclusion (2) in Lemma 4.5. In the case that $\{x_k\}$ is not asymptotically feasible ($g_k^+ \not\to 0$), it was shown in Lemma 4.6 that situation (*i*) occurs. If $g_k^+ \to 0$, it was shown in Lemma 4.7, Lemma 4.9, and Theorem 4.10 that either (*ii*) or (*iii*) must hold. $\square$

5 Overall algorithm

In this section we consider the overall algorithm, in which Algorithm I is run for decreasing values of the barrier parameter μ. We are not concerned here with conditions assuring a good rate of convergence, but consider only the global convergence properties of this algorithm.

Algorithm II. Choose an initial value $\mu_1 > 0$ for the barrier parameter, a reduction factor $a \in (0, 1)$, and a sequence of stopping tolerances $\{\epsilon_l\}_{l \geq 1}$ that tends to zero. Choose an initial iterate (x_1, s_1) and set $l = 1$ and $k_0 = 1$.

1. Apply Algorithm I from the point $(x_{k_{l-1}}, s_{k_{l-1}})$ until it finds a point (x_{k_l}, s_{k_l}) satisfying
$$\|g_{k_l} + s_{k_l}\| \leq \epsilon_l, \tag{5.1}$$
$$\|\nabla f_{k_l} + A_{k_l}\lambda_{k_l}\| \leq \epsilon_l, \tag{5.2}$$
where $\lambda_{k_l} = \mu_l S_{k_l}^{-1} e$.

2. Choose $\mu_{l+1} \in (0, a\mu_l)$.

3. Increase l by 1, and go to step 1.

All the iterates generated by this algorithm form a single sequence $\{(x_k, s_k)\}_{k \geq 1}$. The index k_{l-1} $(l \geq 1)$ labels the starting point of the lth outer iteration, which ends at the point (x_{k_l}, s_{k_l}).

Theorem 5.1. *Suppose that $\{(x_k, s_k)\}$ is generated by Algorithm II and that, for each barrier problem, Assumptions* 4.1 *hold. Then, one of the following two possible outcomes can occur.*

(*A*) *For some parameter μ_l, either inequality* (5.1) *is never satisfied, in which case the stationarity condition for minimizing $x \mapsto \|g(x)^+\|$ is satisfied in the limit, i.e., $A(x_k)g(x_k)^+ \to 0$, or else $g_k + s_k \to 0$ but inequality* (5.2) *is never satisfied, in which case the sequence $\{(g_k, A_k)\}$ has a limit point $(\bar{g}, \bar{A})$ failing the linear independence constraint qualification.*

(*B*) *At each outer iteration l of Algorithm II, the inner algorithm succeeds in finding a pair (x_{k_l}, s_{k_l}) satisfying* (5.1)–(5.2)*. All limit points $\hat{x}$ of $\{x_{k_l}\}$ are feasible. Furthermore, if any limit point $\hat{x}$ of $\{x_{k_l}\}$ satisfies the linear independence constraint qualification, then the first order optimality conditions of the problem*

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & g(x) \leq 0 \end{array}$$

hold at $\hat{x}$: there exists $\hat{\lambda} \in \mathbf{R}^m$ such that

$$\nabla \hat{f} + \hat{A}\hat{\lambda} = 0, \quad \hat{g} \leq 0, \quad \hat{\lambda} \geq 0, \quad \hat{g}^\top \hat{\lambda} = 0.$$

Proof. Suppose that, for some value of μ_l, Algorithm II fails to find a point satisfying (5.1) and (5.2). This implies that Algorithm I generates an infinite sequence for problem (2.2) with $\mu = \mu_l$, but that outcome (*iii*) of Theorem 4.11 does not occur. Since Assumptions 4.1

hold, this implies for that value of μ that either outcome (i) or (ii) of Theorem 4.11 occurs, which leads to conclusion (A).

The only other possibility is that Algorithm II satisfies (5.1)–(5.2) for all $l \geq 1$. Let $\mathcal{L}$ be a subsequence of indices l, such that $x_{k_l} \to \hat{x}$ when $l \to \infty$ in $\mathcal{L}$. Since $0 \leq g_{k_l}^+ \leq g_{k_l} + s_{k_l}$ and $g_{k_l} + s_{k_l} \to 0$, one has $\hat{g} = g(\hat{x}) \leq 0$ ($\hat{x}$ is feasible) and $s_{k_l} \to \hat{s} = -\hat{g}$ when $l \to \infty$ in $\mathcal{L}$.

Now suppose that the linear independence constraint qualification holds at $\hat{x}$ and consider the set of indices

$$\mathcal{I} = \{i : \hat{g}^{(i)} = 0\}.$$

For $i \notin \mathcal{I}$, $\hat{g}^{(i)} < 0$ and $\hat{s}^{(i)} > 0$, so that $\lambda_{k_l}^{(i)} = \mu_l / s_{k_l}^{(i)} \to 0$ when $l \to \infty$ in $\mathcal{L}$. From this and $\nabla f_{k_l} + A_{k_l} \lambda_{k_l} \to 0$, we deduce that

$$\nabla f_{k_l} + \sum_{i \in \mathcal{I}} \lambda_{k_l}^{(i)} \nabla g_{k_l}^{(i)} \to 0. \tag{5.3}$$

By the constraint qualification hypothesis, the vectors $\{\nabla \hat{g}^{(i)} : i \in \mathcal{I}\}$ are linearly independent, so that, by (5.3), the positive sequence $\{\lambda_{k_l}\}_{l \in \mathcal{L}}$ converges to some value $\hat{\lambda} \geq 0$. Now, it remains to take the limit in $\nabla f_{k_l} + A_{k_l} \lambda_{k_l}$ when $l \to \infty$ in $\mathcal{L}$ and to observe that $\hat{g}^\top \hat{\lambda} = 0$. Therefore conclusion (B) holds. □

6 Final Remarks

In this paper we have presented and analyzed a trust region method for solving the barrier problem (1.3). This is an optimization problem with nonlinear *equality* constraints, plus the implicit constraint $s > 0$. Our strategy has been to use a well-developed algorithm for equality constrained optimization and enforce the constraint $s > 0$ by means of the trust region and the barrier term. Another benefit of using a trust region is the ability of the method to deal with indefiniteness of the Hessian and rank deficiency of the constraints.

Some specific ideas on how to define the trust region have been described in §1, but many other possibilities exist, and this remains an open area of investigation. In practice it may be advantageous to handle the steps in x and s separately and to use non-symmetric trust region shapes; see for example [15].

The algorithmic framework given in §1 can be used to implement primal or primal-dual interior point methods. In this paper we have focused on primal methods because they are easier to implement, and we have devoted much attention to their global convergence properties because the analysis provides important clues on how to design the algorithms. Computational experience with the primal interior point method is given in [15]; that paper also provides some preliminary computational results with primal-dual methods.

Another question to be dealt with is how to ensure that a good rate of convergence is obtained. This requires, among other things, a careful strategy for updating the barrier parameter μ and deciding how accurately to solve the barrier subproblems. We should also mention that since our merit function is non-differentiable, getting fast convergence may necessitate use of a second-order correction or a watch-dog strategy to avoid the Maratos effect. Our computational experience with equality constrained problems [17] indicates that use of a second-order correction can be an efficient strategy for this purpose.

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