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▶ **To cite this version:**

Kaitai Li, Aixiang Huang, Qinghuai Huang. The Detection of Multiple Singular Point in PC-Continuation Method. [Research Report] RR-2867, INRIA. 1996. <inria-00073824>

HAL Id: inria-00073824

https://hal.inria.fr/inria-00073824

Submitted on 24 May 2006

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INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

The Detection of Multiple Singular Point in PC-Continuation Method

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N° 2867

Avril 1996

THÈME 4

Rapport de recherche

ISSN 0249-6399

The Detection of Multiple Singular Point in PC-Continuation Method *

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Thème 4 — Simulation
et optimisation
de systèmes complexes
Projet Estime

Rapport de recherche n° 2867 — Avril 1996 — 25 pages

Abstract: In this paper we show how to use the detection method for multiple singular points, including multiple limit point and multiple bifurcation point, in order to solve numerically nonlinear systems of equations by Predictor-Corrector continuation methods.

Key-words: Detection, Multiple singular point, PC-continuation method
Mos(AMS)Subject classiflcation:58E07

(Résumé : tsvp)

*Supported by State Key Major Project of Basic Research and NSFC

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La Détection de Points Singuliers Multiples dans la Méthode de Continuation PC

Résumé : Dans cet article, on montre comment utiliser la méthode de détection des points singuliers multiples – points limites multiples et points de bifurcation multiples – en vue de la résolution numérique de systèmes d'équations non linéaires par des méthodes de continuation de type Prédicteur-Correcteur.

Mots-clé : Détection, Points singuliers multiples, Méthode de continuation PC

1 Introduction

The PC-continuation method is mostly used to numerically compute bifurcation branches at present (see [EA90]). The detection of singular point is necessary during PC-continuation computation. Recently, the detection for simple singular point has been discussed, see for example [EA90]. In [Kat76], the case of odd multiple singular points was studied. We here,give a general detection for both cases, odd and even.

Assume that E is a Banach space,for the sake of simplicity, $E = R^N$ (for infinite dimension,the conclusion is still valid). We consider a smooth enough nonlinear functional equation with a bifurcation parameter $\lambda \in R$:

$$G(u, \lambda) = 0, E \times R \to E. \tag{1.1}$$

A point (u_0, λ_0) is called m a multiple singular point if it satisfies:

(H1) $G(u_0, \lambda_0) = 0$.

(H2) $D_u G(u_0, \lambda_0)$ is a Fredholm operator with index zero,zero is one of its eigenvalues,both the algebraic and geometric multiplicity is $m \geq 1$.

Here $D_u G(u, \lambda)$ denotes the Frechet derivative with respect to the variable u, and $D_u G_0$ will denote $D_u G(u_0, \lambda_0)$, the Frechet derivative evaluated at the point $(u, \lambda) = (u_0, \lambda_0)$.

Furthermore, if

(H3) $D_\lambda G_0 \in \mathrm{Rang}(D_u G_0)$ or $D_\lambda G_0 \notin \mathrm{Rang}(D_u G_0)$

is satisfied too, then (u_0, λ_0) is called a m-multiple regular bifurcation point (RBP) or an m-multiple limit point (LP).

The Fredholm operator theory shows that there exist eigenfunctions $\varphi_i, i = 1, 2, \cdots, m$ and adjoint eigenfunctions $\psi_i, i = 1, 2, \cdots, m$ such that

$$\begin{aligned}
E_0 &\equiv \ker(D_uG_0) = \text{span}\{\varphi_i, i = 1, 2, \cdots, m\}, \\
E_0^* &\equiv \ker(D_uG_0^*) = \text{span}\{\psi_i, i = 1, 2, \cdots, m\}, \\
\hat{E} &\equiv \text{Rang } (D_uG_0) = \{v \in E, < \psi_i, v >= 0, \forall i = 1, 2, \cdots, m\}, \\
\hat{E}^* &\equiv \text{Rang } (D_uG_0^*) = \{v \in E^*, < \varphi_i, v >= 0, \forall_i = 1, 2, \cdots, m\}, \\
< \varphi_i, \varphi_j > &= \delta_{ij}, < \psi_i, \varphi_j >= \delta_{ij}, i, j = 1, 2, \cdots m, \\
E &= \ker(D_uG_0) \oplus \text{Rang}(D_uG_0) = E_0 \oplus \hat{E} \\
E^* &= \ker(D_uG_0^*) \oplus \text{Rang}(D_uG_0^*) = E_0^* \oplus \hat{E}^*.
\end{aligned}$$

Here E^* denotes the dual space to E.

In addition, there exists $\varphi_0 \in \text{Rang}(D_uG_0)$ such that

$$\begin{cases} D_uG_0\varphi_0 + D_\lambda G_0 = 0 & \text{if } (u_0, \lambda_0) \text{is a RBP} \\ \varphi_0 = 0 & \text{if } (u_0, \lambda_0) \text{is a LP} \\ < \varphi_0, \psi_i >= 0, i = 1, 2, \cdots, m. & \end{cases} \tag{1.2}$$

Let

$$E = \ker(D_uG_0) \oplus \text{span}\{\varphi_0\}.$$

It is obvious that $E = \ker(D_uG_0)$ for the case of an LP.

Let π be a linear bounded functional such that

$$\forall v \in E, \quad ((v - \pi v)v) \in \ker(D_uG_0).$$

In the sequel we introduce the notation

$$\begin{aligned}
q_{ij} &= D_{uu}G_0\varphi_i\varphi_j, \\
q_{i0} &= D_uDG_0(\varphi_0, 1)\varphi_i = \begin{cases} D_{uu}G_0\varphi_0\varphi_i + D_{u\lambda}G_0\varphi_i & \text{for RBP} \\ D_{u\lambda}G_0\varphi_i & \text{for LP,} \end{cases} \\
q_{00} &= D^2G_0(\varphi_0, 1)(\varphi_0, 1) = \begin{cases} D_{uu}G_0\varphi_0\varphi_0 + 2D_{u\lambda}G_0\varphi_0 + D_{\lambda\lambda}G_0 & \text{for RBP} \\ D_{\lambda\lambda}G_0 & \text{for LP,} \end{cases}
\end{aligned}$$

$$\begin{aligned}
q_{ijk} &= D_{uuu}G_0\varphi_i\varphi_j\varphi_k, & q_{ij0} &= D_{uu}DG_0(\varphi_0, 1)\varphi_i\varphi_j, \\
q_{i00} &= D_uD^2G_0(\varphi_0, 1)(\varphi_0, 1)\varphi_i, & q_{000} &= D^3G_0(\varphi_0, 1)(\varphi_0, 1)(\varphi_0, 1),
\end{aligned}$$

$$z_{ij} = \begin{cases} 0 & \text{if } q_{ij} \notin \text{Rang } (D_u G_0) \\ z_{ij} \in \text{Rang } (D_u G_0), D_u G_0 z_{ij} + q_{ij} = 0 & \text{if } q_{ij} \in \text{Rang } (D_u G_0), \end{cases}$$
$$z_{i0} = \begin{cases} 0 & \text{if } q_{i0} \notin \text{Rang } (D_u G_0) \\ z_{i0} \in \text{Rang}(D_u G_0), D_u G_0 z_{i0} + q_{i0} = 0 & \text{if } q_{i0} \in \text{Rang } (D_u G_0), \end{cases}$$
$$z_{00} = \begin{cases} 0 & \text{if } q_{00} \notin \text{Rang } (D_u G_0) \\ z_{00} \in \text{Rang } (D_u G_0), D_u G_0 z_{00} + q_{00} = 0 & \text{if } q_{00} \in \text{Rang } (D_u G_0), \end{cases}$$

$$\begin{array}{lll} a^k_{ijl} =< \psi_k, q_{ijl} >, & a^k_{ij0} =< \psi_k, q_{ij0} >, & a^k_{i00} =< \psi_k, q_{i00} >, \\ a^k_{000} =< \psi_k, q_{000} >, & d^k =< \psi_k, D_\lambda G_0 >, & \\ a^k_{ij} =< \psi_k, q_{ij} >, & a^k_{i0} =< \psi_k, q_{i0} >, & a^k_{00} =< \psi_k, q_{00} >, \\ b^k_{ij} =< \psi_k, z_{ij} >, & b^k_{i0} =< \psi_k, z_{i0} >, & b^k_{00} =< \psi_k, z_{00} > \end{array}$$

It is clear that $q_{ij}, q_{ijk}, a^k_{ij}, a^k_{ijl}$ are symmetric with respect to the subindex. We also need to introduce the matrices and vectors:

$$\begin{array}{ll} A_k = (a^k_{ij}), A^0_k = \{a^k_{i0}\}, & A^{00} = \{a^k_{00}\} \quad i, j, k = 1, 2, \cdots, m \\ A^0 = \{A^0_1, A^0_2, \cdots, A^0_m\}^T, & A = \{A_1, A_2, \cdots, A_m\}^T, \\ M_k = \begin{pmatrix} A_k & A^0_k \\ A^{0T}_k & a^k_{00} \end{pmatrix}, & M = \{M_1, M_2, \cdots, M_m\}^T. \end{array}$$

2 Preliminary

In this section we will give some Lemmas which will be used frequently.

Lemma 1 Rank$(M) = m + 1$ iff $\forall 0 \neq v \in E$
there exists a $w \in E, w \neq 0$ such that

$$D^2 G_0(v, \pi v)(w, \pi w) \notin \text{Rang}(D_u G_0) \tag{2.1}$$

Proof It is sufficient that if (2.1) is not valid then Rank $(M) < m + 1$; if Rank$(M) < m + 1$ then (2.1) is not valid too.

Assume that (2.1) is not true, then there exists $v_0 \in E, \forall w_0 \in E$ and we have

$$D^2 G_0(v_0, \pi v_0)(w_0, \pi w_0) \in \text{Rang } (D_u G_0). \tag{2.2}$$

Let $\rho^T = \{\alpha^1, \alpha^2, \cdots, \alpha^m, \alpha^0\} \in R^{m+1}$ such that

$$v_0 = \alpha^i \varphi_i + \alpha^0 \varphi_0, \quad \sum_{i=1}^{m} (\alpha^i)^2 + (\alpha^0)^2 \neq 0$$

and $\forall \nu^T = \{\beta^1, \beta^2, \cdots \beta^m, \beta^0\} \in R^{m+1}$ such that

$$w_0 = \beta^i \varphi_i + \beta^0 \varphi_0.$$

Then, (2.2) yields

$$< \psi_k, D^2 G_0(v_0, \pi v_0)(w_0, \pi w_0) >= 0, k = 1, 2, \cdots, m. \tag{2.3}$$

A simple calculation shows that

$$\rho^T M_k \nu = 0, \quad \forall \nu \in R^{m+1}. \tag{2.4}$$

Hence

$$\rho^T M_k = 0$$

So that ρ is orthogonal to every row vector of M_k. Because of dim $\{\text{span}(\rho^\perp)\} = m$ hence Rank $(M) < m + 1$.

Conversely,if Rank $(M) < m + 1$. Let Y be a spanning subspace by all row vectors of M and $\dim(Y) < m + 1$. Therefore there exists a nontrivial vector $\sigma^T = \{\sigma^1, \sigma^2, \cdots, \sigma^m, \sigma^0\} \in Y^\perp$ such that

$$\begin{aligned} \sigma^T M_k &= 0 \quad \forall 1 \leq k \leq m \\ \sigma^T M_k \mu &= 0 \quad \forall \mu \in R^{m+1}, \mu^T = \{\mu^1, \mu^2, \cdots \mu^m, \mu^0\}. \end{aligned}$$

Set $v = \sigma^i \varphi_i + \sigma^0 \varphi_0, w = \mu^i \varphi_i + \mu^0 \varphi_0$. From

$$D^2 G_0(v, \pi v)(w, \pi w) \in \text{Rang } (D_u G_0), \quad \forall w \in E,$$

we obtain that (2.1) is not valid. The proof ends. □

Remark 1 Indeed,

$$\forall v = \alpha^i \varphi_i + \alpha^0 \varphi_0, \quad w = \mu^i \varphi_i + \mu^0 \varphi_0,$$

$$D^2G_0(v,\pi v)(w,\pi w) = q_{ij}\alpha^i\mu^i + q_{i0}(\alpha^i\mu^0 + \mu^i\alpha^0) + q_{00}\alpha^0\mu^0, \tag{2.5}$$

$$< \psi_k, D^2G_0(v,\pi v)(w,\pi w) >= a^k_{ij}\alpha^i\mu^j + a^k_{i0}(\alpha^i\mu^0 + \mu^i\alpha^0) + a^k_{00}\alpha^0\mu^0, \quad k = 1,\cdots,m. \tag{2.6}$$

Lemma 1 is equivalent to that Rank $(M) = m+1$ iff $\forall(\alpha,\alpha^0) \in R^{m+1}$, equation

$$a^k_{ij}\alpha^i\mu^j + a^k_{i0}(\alpha^i\mu^0 + \mu^i\alpha^0) + a^k_{00}\alpha^0\mu^0 = 0$$

has no nontrivial solution (μ,μ^0).

Denote

$$A(\alpha) = (a^k_{ij}\alpha^i, k,j = 1,2,\cdots m)_{m\times m} \tag{2.7}$$

$$A_0(\alpha) = \{a^k_{i0}\alpha^i, k = 1,2,\cdots m\}^T_{m\times 1}. \tag{2.8}$$

Then the system (2.3) can be rewritten

$$(A(\alpha) + \alpha_0A^0, A_0(\alpha) + \alpha^0A^{00}) \begin{pmatrix} \mu \\ \mu^0 \end{pmatrix} = 0. \tag{2.9}$$

When (u_0,λ_0) is a RBP then $\varphi_0 \neq 0$. In this case, (2.9) has a nontrivial solution iff

$$\text{Rank } (A(\alpha) + \alpha_0A^0, A_0(\alpha) + \alpha^0A^{00}) < m+1, \forall(\alpha,\alpha^0) \in R^m \times R.$$

If (u_0,λ_0) is a LP then $\varphi_0 = 0$.(2.9) is a system with m equations and m unknowns. Therefore, (2.9) has a nontrivial solution iff

$$\text{Rank}A(\alpha) < m,$$

$$\text{Rank}(M) < m+1 \rightleftharpoons D^2G_0(v,\pi w) \in \text{Rank}(D_uG_0), \forall v \in E, \forall w \in E. \tag{2.10}$$

Lemma 2 For any matrix $T \in R^{m\times m}$, any $l, r \in R^m, a \in R$, we have

$$\det \begin{pmatrix} T & r \\ l^T & a \end{pmatrix} = a \det(T) - l^T T^* r, m \geq 2 \tag{2.11}$$

where T^* is a cofactor matrix of T.

Proof See in [YW].

Lemma 3 Assume $T \in R^{m\times m}$,. Then

$$T^*T = TT^* = \det(T)I, \tag{2.12}$$

where I is an identity matrix,T^*.
Furthermore, if $\mathrm{Rank}(T) = m-1$, then

$$T^* = \varphi\psi_0^T, \tag{2.13}$$

where φ_0, ψ_0 are the right and left null vector of T respectively.
If Rank $(T) < m-1$, then

$$T^* = 0. \tag{2.14}$$

(for the proof see in [YW]).

Remark 2 From (2.12) it follows that

$$\det(T^*T) = \det(TT^*) = (\det T)^m.$$

Therefore

$$\det(T^*) = \det(T)^{m-1} \tag{2.15}$$

Remark 3 By (2.11) and (1.1) the determinant of M_k is given by

$$\det(M_k) = \alpha_{00}^k \det(A_k) - A_k^{0T} A_k^* A_k^0 \tag{2.16}$$

3 Bifurcation Equations and Branching Solutions

Assume that there exist a solution arc $c(s) = (u(s), \lambda(s))$ of (1.1) in the neighborhood at (u_0, λ_0) depending smoothly on some parameter s,, for example the arclength such that

$$G(u(s), \lambda(s)) = 0, u(0) = u_0, \lambda(0) = \lambda_0. \tag{3.1}$$

Differentiating (3.1) twice with respect to s and evaluating at $s = 0$, we obtain

$$D_u G_0 \dot{u}(0) + D_\lambda G_0 \dot{\lambda}(0) = 0, \tag{3.2}$$

$$D_u G_0 \ddot{u}(0) + D_\lambda G_0 \ddot{\lambda}(0) = -D^2 G_0(\dot{u}(0), \dot{\lambda}(0))(\dot{u}(0), \dot{\lambda}(0)). \tag{3.3}$$

(3.2) yields that

$$\xi^0 \equiv \dot{\lambda}(0) \begin{cases} \neq 0 & \text{for RBP,} \\ = 0 & \text{for LP.} \end{cases} \tag{3.4}$$

Combining(3.2) and (1.2),we can write

$$D_u G_0(\dot{u}(0) - \xi^0 \varphi_0) = 0.$$

Hence

$$\dot{u}(0) \in E, \dot{u}(0) = \xi^i \varphi_i + \xi^0 \varphi_0 \tag{3.5}$$

where $\xi = (\xi^1, \xi^2, \cdots \xi^m) \in R^m$. Substituting (3.5) into (3.3) leads to

$$D_u G_0 \ddot{u}(0) = -D_\lambda G_0 \eta - q_{ij} \xi^i \xi^j - 2 q_{i0} \xi^i \xi^0 - q_{00} \xi^0 \xi^0 \tag{3.6}$$

with

$$\eta = \ddot{\lambda}(0) \tag{3.7}$$

It is clear that there exists a nontrivial solution $\ddot{u}(0)$ iff

$$D_\lambda G_0\eta + q_{ij}\xi^i\xi^j + 2q_{i0}\xi^i\xi^0 + 2q_{00}\xi^0\xi^0 \in \mathrm{Rang}(D_\lambda G_0). \tag{3.8}$$

Then (3.8) implies

$$f_k(\xi, \xi^0, \eta) \equiv a^k_{ij}\xi^i\xi^j + 2a^k_{i0}\xi^i\xi^0 + a^k_{00}\xi^0\xi^0 + d^k\eta = 0, k = 1, \cdots, m. \tag{3.9}$$

or

$$f_k = \xi^T A_k\xi + 2A^0_k\xi\xi^0 + a^k_{00}\xi^0\xi^0 + d^k\eta = 0, \tag{3.10}$$

$$(\xi^T, \xi^0)M_k \begin{pmatrix} \xi \\ \xi^0 \end{pmatrix} + d^k\eta = 0, k = 1, 2, \cdots m. \tag{3.11}$$

To avoid indetermination, we add to (3.9) the normalization equations

$$\begin{cases} (\xi^T, \xi^0)M_k \begin{pmatrix} \xi \\ \xi^0 \end{pmatrix} = 0 \\ \xi^T\xi + \xi^0\xi^0 = 1 \end{cases} \quad \text{for RBP}, \tag{3.12}$$

$$\begin{cases} \xi^T A_k\xi + d^k\eta = 0 \\ \xi^T\xi = 1 \end{cases} \quad \text{for LP}, \tag{3.13}$$

where we use $d^k = 0$ for BRP, $\xi^0 = 0$ for LP.

(3.12) is called the algebraic bifurcation equations (ABE), and (3.13) is called the limit point bifurcation equations (LPBE)(see [DWD82]).

Equations (3.12) (3.13) can be rewritten as

$$\mathrm{ABE}: \begin{cases} A(\xi)\xi + 2\xi^0 A^0\xi + \xi^0\xi^0 A^{00} = 0 \\ \xi^T\xi + \xi^0\xi^0 = 1 \end{cases} \quad \text{for RBP}, \tag{3.14}$$

$$\mathrm{LPBE}: \begin{cases} A(\xi)\xi + d\eta = 0 \\ \xi^T\xi = 1 \end{cases} \quad \text{for LP}. \tag{3.15}$$

It is wellknown [DWD82] that not all roots of bifurcation equations can be used to generate solution branches of (1.1). Only all isolated roots of bifurcation equations can be used to generate solution branches. However,a root of bifurcation equation is isolated iff the corresponding Jacobian matrix evaluated at the root is nonsingular.

The Jacobian matrices of RBP,LP respectively,are

$$J_B(\xi,\xi^0) = 2\begin{bmatrix} A(\xi)+\xi^0 A^0 & A^0\xi + A^{00}\xi^0 \\ \xi^T & \xi^0 \end{bmatrix} \quad \text{for ABE} \tag{3.16}$$

$$J_L(\xi,\eta) = 2\begin{bmatrix} A(\xi) & d \\ \xi^T & 0 \end{bmatrix} \quad \text{for LPBE} \tag{3.17}$$

By applying lemma 2,we can evaluate the determinant of Jacobian J_B, J_L as follows

$$\begin{aligned} &\det(J_B(\xi,\xi^0)) = 2[\xi^0 \det(A(\xi)+\xi^0 A^0) \\ &-\xi^T(A(\xi)+\xi^0A^0)^*(A^0\xi+\xi^0A^{00})], \end{aligned} \tag{3.18}$$

$$\det(J_L(\xi,\eta)) = -2\xi^T A(\xi)^* d. \tag{3.19}$$

In the following we discuss above ABE and LPBE separately.

1. We consider the case of ABE.

Firstly, assume that $(\xi,0)$ is a root of ABE, here $\xi^0 = \dot{\lambda}(0) = 0$. In this case (3.14) becomes

$$\begin{cases} A(\xi)\xi & = 0 \\ \xi^T\xi & = 1 \end{cases} \tag{3.20}$$

Therefore,

(A) $\xi \neq 0$ iff $A(\xi)$ is singular,det $A(\xi) = 0$;

(B) ξ is a eigenvector cooresponding to zero eigenvalue of $A(\xi)$.

From (3.18) we have

$$\det(J_B(\xi,0)) = -\xi^T A(\xi)^*(A^0\xi) \tag{3.21}$$

By applying Lemma 3, if $\mathrm{Rank}(A(\xi)) < m-1$ then $A(\xi)^* = 0$; if Rank $(A(\xi)) = m-1$, then by (2.13) it follows that

$$-\xi^T A(\xi)^*(A^0\xi) = -\xi^T\xi\xi^T(A^0\xi) = -\xi^T A^0\xi.$$

Consequently,we obtain

$$\det(J_B(\xi,0)) = \begin{cases} 0 & \text{if } \mathrm{Rank}(A(\xi)) < m-1 \\ -\xi^T A^0\xi & \text{if } \mathrm{Rank}(A(\xi)) = m-1 \end{cases} \tag{3.22}$$

Secondly,assume $\xi^0 \neq 0$, from (3.14) it follows that

$$A^0\xi + \xi^0 A^{00} = -(\xi^0)^{-1}(A(\xi) + \xi^0 A^0)\xi. \tag{3.23}$$

Substituting (3.23) into (3.18) leads to

$$\begin{aligned} &\det J_B(\xi,\xi^0) = 2[\xi^0\det(A(\xi)+\xi^0A^0) \\ &+(\xi^0)^{-1}\xi^T(A(\xi)+\xi^0A^0)^*(A(\xi)+\xi^0A^0)\xi] \end{aligned} \tag{3.24}$$

Using (2.12) implies that

$$\begin{aligned} \det J_B(\xi,\xi^0) &= 2[\xi^0\det(A(\xi)+\xi^0A^0) + (\xi^0)^{-1}\xi^T\det(A(\xi)+\xi^0A^0)I\xi] \\ &= 2(\xi^0)^{-1}\det(A(\xi)+\xi^0A^0)(|\xi|^2+(\xi^0)^2) \end{aligned}$$

i.e.

$$\det J_B(\xi,\xi^0) = 2(\xi^0)^{-1}\det(A(\xi)+\xi^0A^0) \tag{3.25}$$

Finally,we conclude

Theorem 1 Assume that (ξ,ξ^0) is a root of ABE,

(A) If $\xi^0 = 0$ then $J_B(\xi,\xi^0)$ is nonsingular iff
(i) $A(\xi)$ is singular with Rank $(A(\xi)) = m-1$
(ii) ξ is an eigenvector of $A(\xi)$ corresponding to zero eigenvalue but not an eigenvector of A^0 corresponding to the zero eigenvalue.

Furthermore, $\det J_B(\xi,0)$ can be evaluated by

$$\det J_B(\xi,0) = -\xi^T A^0\xi \tag{3.26}$$

(B) If $\xi^0 \neq 0$ then $J_B(\xi, \xi^0)$ is nonsingular iff $A(\xi) + \xi^0 A^0$ is nonsingular. Furthermore, $\det J_B(\xi, \xi^0)$ can be evaluated by

$$\det J_B(\xi, \xi^0) = 2(\xi^0)^{-1} \det(A(\xi) + \xi^0 A^0). \tag{3.27}$$

2. We consider the case of LPBE.

Firstly, we assume $\eta = \ddot{\lambda}(0) = 0,$. From (3.16) we obtain

$$\begin{cases} A(\xi)\xi = 0 \\ \xi^T \xi = 1. \end{cases} \tag{3.28}$$

$A(\xi)$ is singular, ξ is a eigenvector corresponding to zero eigenvalue of $A(\xi)$. By applying Lemma 3,we have

$$A(\xi)^* = \begin{cases} 0 & \text{if Rank } A(\xi) < m-1 \\ \xi\xi^T & \text{if Rank } A(\xi) = m-1. \end{cases}$$

From (3.19) we have

$$\det J_L(\xi, 0) = \begin{cases} 0 & \text{if Rank } A(\xi) < m-1 \\ -2\xi^T d & \text{if Rank } A(\xi) = m-1. \end{cases} \tag{3.29}$$

Secondly, we assume $\eta \neq 0$. From (3.15) it follows that

$$d = -\eta^{-1} A(\xi)\xi.$$

Substituting it into (3.19) leads to

$$\det J_L(\xi, \eta) = 2\eta^{-1}\xi^T A(\xi)^* A(\xi)\xi$$

In view of Lemma 3,we obtain

$$\det J_L(\xi, \eta) = 2\eta^{-1}\xi^T \det(A(\xi)) I\xi = 2\eta^{-1} \det(A(\xi)) \tag{3.30}$$

Finally, we conclude

Theorem 2 Assume that (ξ, η) is a root of LPBE.

(A) If $\eta = 0$ then $J_L(\xi, 0)$ is nonsingular iff

(i) $A(\xi)$ is singular with Rank $A(\xi) = m - 1$,

(ii) $d \notin$ Rang $(A(\xi))$.

Furthermore det $J_L(\xi, 0)$ can be computed by

$$\det J_L(\xi, 0)) = -2\xi^T d. \tag{3.31}$$

(B) If $\eta \neq 0$ then $J_L(\xi, \eta)$ is nonsingular iff $A(\xi)$ is nonsingular too. Furthermore,det $J_L(\xi, \eta)$ can be evaluated by

$$\det J_L(\xi, \eta) = 2\eta^{-1} \det(A(\xi)) \tag{3.32}$$

The number of isolated root of bifurcation equation (ABE,LPBE) is limited. Equations (3.14) and (3.15) form a system with m equations and $(m+1)$, so Bezout's theorem (see [EA90]) allows a maximum of 2^m isolated roots in the m dimensional complex projective plane. Because $(\xi^1, \xi^2, \cdots, \xi^m, \xi^0)$ or $(\xi^1, \xi^2, \cdots, \xi^m, \eta)$ is a root of ABE or LPBE respectively, then so is $(-\xi^1, -\xi^2, \cdots, -\xi^m, -\xi^0)$ or $(-\xi^1, -\xi^2, \cdots, -\xi^m, \eta)$, which is distinct in complex projective plane but generates a branch with the two tangent vectors

$$t_\pm = \pm(\dot{u}(0), \dot{\lambda}(0)) = (\pm\xi^i\varphi_i, \pm\xi^0) \quad \text{or } t_\pm = (\pm\xi^i\varphi_i, 0)$$

determined by the direction of approach to (u_0, λ_0), both roots relate to the same arc, hence they will not be considered distinct.Thus we can have at most 2^{m-1} isolated real roots of bifurcation equations.

Assume that (ξ_*^i, ξ_*^0) or (ξ_*^i, η_*) is an isolated root of ABE or LPBE respectively, then we have

$$\dot{u}(0) = \xi_*^i\varphi_i + \xi_*^0\varphi_0, \quad \xi_*^0 = \dot{\lambda}(0) \quad \text{for ABE}, \tag{3.33}$$

or

$$\dot{u}(0) = \xi_*^i\varphi_i, \quad \dot{\lambda}(0) = 0 \qquad\qquad \text{for LPBE}. \tag{3.34}$$

Consequently (3.6) has unique solution,i.e.

$$D_u G_0(\ddot{u}(0) - \ddot{\lambda}(0)\varphi_0) = -(q_{ij}\xi_*^i\xi_*^j + 2q_{i0}\xi_*^i\xi_*^0 + q_{00}\xi_*^0\xi_*^0) \quad \text{for ABE}, \tag{3.35}$$

or

$$D_u G_0 \ddot{u}(0) = -\eta_* D_\lambda G_0 - (q_{ij}\xi_*^i \xi_*^j) \quad \text{for LPBE,} \tag{3.36}$$

with

$$< \ddot{u}(0) - \ddot{\lambda}(0)\varphi_0, \psi_k >= 0,$$

or

$$< \ddot{u}(0), \psi_k >= 0, k = 1, 2, \cdots, m.$$

Let us denote this unique solution by

$$v_0 = \begin{cases} \ddot{u}(0) - \ddot{\lambda}(0)\varphi_0 & \text{for ABE} \\ \ddot{u}(0) & \text{for LPBE} \end{cases} \tag{3.37}$$

It is well known [Kat76] that there exists a unique solution corresponding to (ξ^*, ξ_*^0) or (ξ^*, η^*) and starting at (u_0, λ_0) along direction $\dot{u}(0)$ defined by (3.33) or (3.34) which can be expressed as

$$\begin{cases} u(s) = u_0 + s(\xi^i(s)\varphi_i + \xi^0(s)\varphi_0) + \frac{1}{2}s^2 v(s) \\ \lambda(s) = \lambda_0 + s\xi^0(s) \quad \text{for RBP} \end{cases} \tag{3.38}$$

$$\begin{cases} u(s) = u_0 + s(\xi^i(s)\varphi_i) + \frac{1}{2}s^2 v(s) \quad \text{for LP} \\ \lambda(s) = \lambda_0 + \frac{1}{2}s^2 \eta(s) \end{cases} \tag{3.39}$$

where functions $\xi^i(s), \xi^0(s), \eta(s)$ and $v(s)$ are unique with initial conditions

$$\xi^i(0) = \xi_*^i, \xi^0(0) = \xi_*^0, \eta(0) = \eta^*, v(0) = v_0. \tag{3.40}$$

4 Example

Taking $m=2$, let

$$\begin{aligned}
\alpha_{11} &= \begin{vmatrix} a_{11}^1 & a_{12}^1 \\ a_{11}^2 & a_{12}^2 \end{vmatrix}, \alpha_{12} = \begin{vmatrix} a_{11}^1 & a_{22}^1 \\ a_{11}^2 & a_{22}^2 \end{vmatrix}, \alpha_{22} = \begin{vmatrix} a_{12}^1 & a_{22}^1 \\ a_{12}^2 & a_{22}^2 \end{vmatrix} \\
\alpha_{10} &= \begin{vmatrix} a_{10}^1 & a_{20}^1 \\ a_{11}^2 & a_{12}^2 \end{vmatrix} + \begin{vmatrix} a_{11}^1 & a_{12}^1 \\ a_{10}^2 & a_{20} \end{vmatrix} \\
\alpha_{20} &= \begin{vmatrix} a_{10}^1 & a_{20}^1 \\ a_{12}^2 & a_{22}^2 \end{vmatrix} + \begin{vmatrix} a_{12}^1 & a_{22}^1 \\ a_{10}^2 & a_{20}^2 \end{vmatrix}, \alpha_{00} = \begin{vmatrix} a_{10}^1 & a_{20}^1 \\ a_{10}^2 & a_{20}^2 \end{vmatrix} \\
\sigma^2 &= -\begin{vmatrix} a_{11}^1 & a_{12}^1 \\ a_{21}^1 & a_{22}^1 \end{vmatrix} \\
\beta_{11} &= \begin{vmatrix} a_{11}^1 & d^1 \\ a_{11}^2 & d^2 \end{vmatrix}, \beta_{12} = \begin{vmatrix} a_{12}^1 & d^1 \\ a_{12}^2 & d^2 \end{vmatrix}, \beta_{22} = \begin{vmatrix} a_{22}^1 & d^1 \\ a_{22}^2 & d^2 \end{vmatrix} \\
\gamma_{11} &= \begin{vmatrix} a_{11}^1 & a_{00}^1 \\ a_{11}^2 & a_{00}^2 \end{vmatrix}, \gamma_{12} = \begin{vmatrix} a_{12}^1 & a_{00}^1 \\ a_{12}^2 & a_{00}^2 \end{vmatrix}, \gamma_{22} = \begin{vmatrix} a_{22}^1 & a_{00}^1 \\ a_{22}^2 & a_{00}^2 \end{vmatrix} \\
\gamma_{10} &= \begin{vmatrix} a_{10}^1 & a_{00}^1 \\ a_{10}^2 & a_{00}^2 \end{vmatrix}, \gamma_{20} = \begin{vmatrix} a_{20}^1 & a_{00}^1 \\ a_{20}^2 & a_{00}^2 \end{vmatrix}
\end{aligned}$$

$$-\rho^2 = \gamma_{12}^2 - \gamma_{11}\gamma_{22}$$

Elementary calculation shows that

$$\det A(\xi) = \sum_{i,j=1}^{2} \alpha_{ij}\xi^i\xi^j, \tag{4.1}$$

$$\det(A(\xi) + \xi^0 A^0) = \sum_{i,j=0}^{2} \alpha_{ij}\xi^i\xi^j \tag{4.2}$$

For LPBE, we introduce

$$\xi^1 = \sin\theta, \xi^2 = \cos\theta,$$

then LPBE can be rewritten as

$$\begin{aligned} a_{11}^1 \sin^2\theta + 2a_{12}^1 \cos\theta\sin\theta + a_{22}^1\cos^2\theta + \eta d^1 = 0, \\ a_{11}^2 \sin^2\theta + 2a_{12}^2 \cos\theta\sin\theta + a_{22}^2\cos^2\theta + \eta d^2 = 0. \end{aligned} \tag{4.3}$$

For ABE, we introduce

$$\xi^1 = \sin\varphi\sin\theta, \xi^2 = \cos\varphi\sin\theta, \xi^0 = \cos\theta,$$

then ABE can be rewritten as

$$\begin{aligned} &a^1_{11}\sin^2\varphi\sin^2\theta + 2a^1_{12}\sin\varphi\cos\varphi\sin^2\theta + a^1_{22}\cos^2\varphi\sin^2\theta \\ &+2a^1_{10}\sin^2\varphi\sin\theta\cos\theta + 2a^1_{20}\cos\varphi\sin\theta\cos\theta + a^1_{00}\cos^2\theta = 0, \end{aligned} \tag{4.4}$$

$$\begin{aligned} &a^2_{11}\sin^2\varphi\sin^2\theta + 2a^2_{12}\sin\varphi\cos\varphi\sin^2\theta + a^2_{22}\cos^2\varphi\sin^2\theta \\ &+2a^2_{10}\sin^2\varphi\sin\theta\cos\theta + 2a^2_{20}\cos\varphi\sin\theta\cos\theta + a^2_{00}\cos^2\theta = 0. \end{aligned} \tag{4.5}$$

(i) Assume that $(\xi, 0)$ is a root of LPBE.
There is a nontrivial solution to (4.3) iff

$$\sigma^2 > 0, \alpha^2_{ij} = \alpha a^1_{ij} \tag{4.6}$$

and

$$\xi^T D \propto \begin{cases} d^1 a^1_{22} + d^2 a^1_{12} & \text{if } a^1_{11} = 0 \\ d^2 a^1_{11} + d^2 a^1_{22} \pm \sigma d^1 & \text{if } a^1_{11} \neq 0 \end{cases} \tag{4.7}$$

(ii) Assume that (ξ, η) is a root of LPBE there is a nontrivial solution to (4.3) iff

$$\beta^2_{12} - \beta_{11}\beta_{22} > 0 \tag{4.8}$$

because of

$$\beta_{11}\sin^2\theta + 2\beta_{12}\sin\theta\cos\theta + \beta_{22}\cos^2\theta = 0$$

which can be obtained from (4.3). In view of

$$\det A(\xi) = a_{11}\sin^2\theta + 2a_{12}\sin\theta\cos\theta + a_{22}\cos^2\theta, \tag{4.9}$$

we can conclude (4.8) and $\alpha_{ij} \neq \alpha\beta_{ij}$ garanties that $A(\xi)$ is nonsingular.

(iii) Assume that $(\xi, 0)$ is a root of ABE then (4.4) and (4.5) become

$$a_{11}^1 \sin^2\varphi + 2a_{12}^1 \sin\varphi\cos\varphi + a_{22}^1\cos^2\varphi = 0,$$

$$a_{11}^2 \sin^2\varphi + 2a_{12}^2 \sin\varphi\cos\varphi + a_{22}^2\cos^2\varphi = 0. \tag{4.10}$$

It is obvious that (4.10) has a nontrivial solution iff

$$\sigma^2 = -\begin{vmatrix} a_{11}^1 & a_{12}^1 \\ a_{21}^1 & a_{22}^1 \end{vmatrix} \geq 0, a_{ij}^2 = aa_{ij}^1. \tag{4.11}$$

In this case

$$\xi^T A^0 \xi = a_{10}^1 \sin^2\varphi + (a_{20}^1 + a_{10}^2)\sin\varphi\cos\varphi + a_{20}^2\cos^2\varphi \tag{4.12}$$

It is clear that if $a_{11}^1 = 0$ then $J_B(\xi, 0)$ is nonsingular iff

$$a_{10}^1(a_{22}^1)^2 + (a_{20}^1 + a_{10}^2)a_{22}^1 a_{12}^1 + a_{20}^2(a_{12}^1)^2 \neq 0 \tag{4.13}$$

if $a_{11}^1 \neq 0$ then $J_B(\xi, 0)$ is nonsingular iff

$$a_{10}^1(a_{12}^1 \pm \sigma)^2 + (a_{20}^1 + a_{20}^2)a_{11}^1(a_{12}^1 \pm \sigma) + a_{22}^1(a_{11}^1)^2 \neq 0 \tag{4.14}$$

(iv) Assume that $(\xi, \xi^0 \neq 0)$ is a root of ABE. From ABE we can obtain

$$(\gamma_{11}\sin^2\theta + 2\gamma_{12}\sin\varphi\cos\varphi + \gamma_{22}\cos^2\varphi)\sin^2\theta$$
$$+2(\gamma_{20}\sin\varphi + \gamma_{20}\cos\varphi)\sin\theta\cos\theta = 0. \tag{4.15}$$

If

$$\rho^2 = -(\gamma_{12}^2 - \gamma_{11}\gamma_{22}) > 0, \tag{4.16}$$

combining(4.15),(4.4),(4.5) we obtain

$$p_0\sin^4\varphi + p_1\sin^3\varphi\cos\varphi + p_2\sin^2\varphi\cos^2\varphi +$$
$$p_3\sin\varphi\cos^3\varphi + p_4\cos^4\varphi = 0 \tag{4.17}$$

where

$$\begin{aligned}
p_0 = &\ 4a_{11}^1\gamma_{12}^2 + a_{00}^1\gamma_{11} - 4\gamma_{10}\gamma_{11}a_{10}^1 \\
p_1 = &\ 8a_{12}^1\gamma_{10}^2 + 8a_{11}^1\gamma_{10}\gamma_{20} - 8a_{10}^1\gamma_{10}\gamma_{12} - 4\gamma_{12}(a_{20}^1\gamma_{10} + a_{10}^1\gamma_{20}) + 4\gamma_{11}\gamma_{12}a_{00}^1 \\
p_2 = &\ 16a_{12}^1\gamma_{10}\gamma_{20} + 4a_{11}^1\gamma_{20}^2 + 4a_{22}^1\gamma_{10}^2 - 4\gamma_{12}a_{10}^1\gamma_{12} \\
&-4a_{20}^1\gamma_{20}\gamma_{11} - 8(\gamma_{10}a_{20}^1 + \gamma_{20}a_{10}^1)\gamma_{12} + 4a_{00}^1\gamma_{12}^2 + 2a_{00}^1\gamma_{11}\gamma_{12} \\
p_3 = &\ 8a_{12}^1\gamma_{20}^2 + 8\gamma_{10}\gamma_{20}a_{12}^1 - 4(\gamma_{10}a_{20}^1 + \gamma_{20}a_{10}^1)\gamma_{22} - 8\gamma_{12}a_{20}^1\gamma_{20} + 4a_{00}^1\gamma_{12}\gamma_{22} \\
p_4 = &\ 4a_{22}^1\gamma_{20}^2 - 4\gamma_{20}\gamma_{22}a_{20}^1 + a_{00}^1\gamma_{22}^2
\end{aligned}$$

If $p_0 \neq 0$ then (4.17) has four roots.

5 The Case of Degeneration

In this section, we consider the degenerate case. We need the information from a high order derivative of $G(u,\lambda)$ at (u_0,λ_0). To do that, differentiating (3.1) three times with respect to s and evaluate at $s=0$, we obtain

$$\begin{aligned} D_uG_0 \dddot{u}(0) + D_\lambda G_0 \dddot{\lambda}(0) = -3(q_{ij}\xi^i\eta^j + q_{i0}\xi^i\eta^0 + q_{0i}\xi^0\eta^i + q_{00}\xi^0\eta^0) \\ -q_{ijk}\xi^i\xi^j\xi^k - 3q_{ij0}\xi^i\xi^j\xi^0 - 3q_{i00}\xi^i\xi^0\xi^0 - q_{000}\xi^0\xi^0\xi^0 \end{aligned} \tag{5.1}$$

where

$$\dot{u}(0) = \xi^i\varphi_i + \xi^0\varphi_0, \ddot{u}(0) = \eta^i\varphi_i + \eta^0\varphi_0 \tag{5.2}$$

and $\xi^0 = \dot{\lambda}(0), \eta^0 = \ddot{\lambda}(0)$. Let $\varsigma = \dddot{\lambda}(0)$. Taking the inner product of (5.1) with ψ_k we get

$$\begin{aligned} a^k_{ijl}\xi^i\xi^j\xi^l + 3a^k_{ij0}\xi^i\xi^j\xi^0 + 3a^k_{i00}\xi^i\xi^0\xi^0 + a^k_{000}\xi^0\xi^0\xi^0 \\ +3(a^k_{ij}\xi^i\eta^j + a^k_{i0}(\xi^i\eta^0 + \eta^i\xi^0) + a^k_{00}\xi^0\eta^0) + d^k\varsigma = 0 \end{aligned} \tag{5.3}$$

$$\eta^T\eta + \eta^0\eta^0 = 1 \quad \text{for RBP} \tag{5.4}$$

$$\eta^T\eta = 1 \quad \text{for LP} \tag{5.5}$$

Equations (5.3)-(5.5) with ABE or LPBE are the bifurcation equations of the degenerate case.

Set

$$\begin{aligned} &\mathcal{A}(\xi) = (a^k_{ijl}\xi^j\xi^l), \mathcal{A}^0(\xi) = (a^k_{ij0}\xi^j) \\ &\mathcal{A}^{00} = (a^k_{i00}), \mathcal{A}^{000} = \{a^k_{000}\}, \qquad \forall\xi\in R^m, \xi^0\in R. \end{aligned}$$

Then (5.3)-(5.5) can be rewritten as

$$\begin{cases} \mathcal{A}(\xi)\xi + 3\xi^0\mathcal{A}^0(\xi)\xi + 3\xi^0\xi^0\mathcal{A}^{00}\xi + \xi^0\xi^0\xi^0\mathcal{A}^{000} \\ 3(A(\xi)\eta + \eta^0A^0\xi + \xi^0A^0\eta + A^{00}\xi^0\eta^0) = 0 \quad \text{for RBP} \\ \xi^T\xi + \xi^0\xi^0 = 1, \eta^T\eta + \eta^0\eta^0 = 1 \end{cases} \tag{5.6}$$

$$\begin{cases} \mathcal{A}(\xi)\xi + 3A(\xi)\eta + d\varsigma = 0 \\ \xi^T\xi = 1, \eta^T\eta = 1 \end{cases} \quad \text{for LP} \tag{5.7}$$

The Jacobian matrices of (5.6) and (5.8) are denoted by

$$\mathcal{J}_\beta = [J_{ij}]_{i=\bar{1,4},j=\bar{1,3}} \mathcal{J}_L = [\hat{J}_{ij}]_{i,j=\bar{1,3}} \tag{5.8}$$

where

$$\begin{aligned} &J_{11} = 3(\mathcal{A}(\xi) + 2\xi^0\mathcal{A}^0(\xi) + \xi^0\xi^0\mathcal{A}^{00} + A(\eta) + \eta^0 A^0), \\ &J_{12} = 3(A(\xi) + \xi^0 A^0), \\ &J_{13} = 3(\mathcal{A}^0(\xi)\xi + 2\xi^0\mathcal{A}^{00}\xi + \xi^0\xi^0\mathcal{A}^{000} + A^0\eta + \eta^0 A^{00}), \\ &J_{14} = 3A^0\xi + 3\xi^0 A^{00}, \end{aligned} \tag{5.9}$$

$$\begin{aligned} &J_{21} = 2\xi^T, J_{22} = 0, J_{23} = 2\xi^0, J_{24} = 0, \\ &J_{31} = 0, J_{32} = 2\eta^T, J_{33} = 0, J_{34} = 2\eta^0; \\ &\hat{J}_{11} = 3(\mathcal{A}(\xi) + A(\eta)), \hat{J}_{12} = 3A(\xi), \hat{J}_{13} = d, \\ &\hat{J}_{21} = 2\xi^T, \hat{J}_{22} = 0, \hat{J}_{23} = 0, \\ &\hat{J}_{31} = 0, \hat{J}_{32} = 2\eta^T, \hat{J}_{33} = 0. \end{aligned} \tag{5.10}$$

For example, consider the nonlinear elliptic boundary value problem

$$\begin{cases} \Delta u + \lambda f(u) = 0 & \text{in } \Omega = [0,1] \times [0,1] \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{5.11}$$

Assume that $f(u)$ satisfies

$$f'(0) = 1, f'''(0) < 0. \tag{5.12}$$

Let $X = H_0^1(\Omega), a(u,v) = (\nabla u, \nabla v) \quad \forall u,v \in X$. The linear operator $T : g \in L^2(\Omega) \to Tg \in X$ is defined by

$$a(Tg, v) = -(g, v) \quad \forall v \in X \tag{5.13}$$

Problem (5.11) can be expressed by

$$G(u, \lambda) \equiv u + \lambda T f(u) = 0. \tag{5.14}$$

The mapping $G : X \times R \to X$ is defined well and smooth.
The singular set S_0 of (5.14) on $\{(0, \lambda); \lambda \in R\}$ is

$$S_0 = \{(0, \lambda_0), \lambda_0 = (p^2 + q^2)\pi, (p, q) \in N \times N\}.$$

Simple calculation yields

$$\begin{array}{l} D_u G_0 = I + \lambda_0 I, D_\lambda G_0 = 0, D_{\lambda\lambda} G_0 = 0, \\ D_{uu} G_0 = 0, D_{u\lambda} G_0 = T, D_{\lambda\lambda\lambda} G_0 = 0, \\ D_{uuu} G_0 = \lambda_0 f'''(0) T, D_{uu\lambda} G_0 = 0, D_{u\lambda\lambda} G_0 = 0. \end{array} \tag{5.15}$$

The eigenfunctions corresponding to λ_0 are given

$$\begin{array}{ll} \varphi(x, y) = 2 \sin p\pi x \sin q\pi y & \\ \hat{\varphi}(x, y) = 2 \sin q\pi x \sin p\pi y & \end{array} \quad \forall (p, q) \in N \times N$$

We take $\lambda_0 = 65\pi$ as an example, then

$$\begin{array}{rclc} (p_1, q_1) & = & (8, 1), & (p_3, q_3) = (7, 4) \\ (p_2, g_2) & = & (1, 8), & (p_4, q_4) = (4, 7). \end{array}$$

. The four eigenfunctions are

$$\begin{array}{ll} \varphi_1(x, y) = 2 \sin 8\pi x \sin \pi y, & \varphi_3(x, y) = 2 \sin 7\pi x \sin 4\pi y, \\ \varphi_2(x, y) = 2 \sin \pi x \sin 8\pi y, & \varphi_4(x, y) = 2 \sin 4\pi x \sin 7\pi y. \end{array} \tag{5.16}$$

Elementary calculation shows that

$$\begin{array}{l} a^k_{ij} = 0, a^k_{00} = 0, a^k_{i0} = -\frac{2}{\lambda_0}\delta_{ik}, \quad i, j, k = 1, 2, 3, 4 \\ a^k_{ijl} = f'''(0) < \varphi_i \varphi_j \varphi_l, \varphi_k > \\ a^1_{111} = a^2_{222} = a^2_{333} = a^4_{444} = \frac{9}{4} f'''(0) \\ a^k_{kij} = a^k_{ikj} = a^k_{ijk} = a^i_{kkj} = a^i_{kjk} = a^i_{jkk} = f'''(0) \quad k \neq i \neq j \\ a^k_{ijl} = 0 \quad \text{otherwise.} \end{array} \tag{5.17}$$

Bifurcation equations become

$$\begin{array}{l} 3\delta_{ki} \xi^i \eta^0 + a^k_{ijl} \xi^i \xi^j \xi^l = 0 \\ (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 + (\xi^4)^2 = 1 \end{array} \tag{5.18}$$

$((0),\lambda_0)$ is LP from (5.17),(5.18) and we have

$$\begin{array}{l}
\xi^1[3\eta^0+(\frac{9}{4}(\xi^1)^2+2\xi^2\xi^3+2\xi^3\xi^4+2\xi^2\xi^4)f'''(0)]=0,\\
\xi^2[3\eta^0+(\frac{9}{4}(\xi^2)^2+2\xi^1\xi^3+2\xi^1\xi^4+2\xi^3\xi^4)f'''(0)]=0,\\
\xi^3[3\eta^0+(\frac{9}{4}(\xi^3)^2+2\xi^2\xi^1+2\xi^1\xi^4+2\xi^2\xi^4)f'''(0)]=0,\\
\xi^4[3\eta^0+(\frac{9}{4}(\xi^4)^2+2\xi^1\xi^2+2\xi^1\xi^3+2\xi^2\xi^3)f'''(0)]=0,\\
(\xi^1)^2+(\xi^2)^2+(\xi^3)^2+(\xi^4)^2=1.
\end{array}$$

6 Detection of a singular point

We consider

$$\sigma(s)=\det\begin{pmatrix} D_uG(s) & D_\lambda G(s)\\ \dot u^T(s) & \dot\lambda(s)\end{pmatrix}. \tag{6.1}$$

By using lemma 3 we obtain

$$\sigma(s)=\dot\lambda(s)\det(D_uG(s))-\dot u^T(s)(D_uG(s))^*D_\lambda G(s), \tag{6.2}$$

where

$$D_uG(s)=D_uG(u(s),\lambda(s)).$$

$(D_uG(s))^*$ is the adjoint matrix of $D_uG(s)$. On the other hand,

$$D_uG(s)\dot u(s)+D_\lambda G(s)\dot\lambda(s)=0. \tag{6.3}$$

If $\dot\lambda(s)\neq 0$, multiplying both sides of (6.2) with $\dot\lambda(s)$ and combining (6.3) we obtain

$$\dot\lambda(s)\sigma(s)=|\dot\lambda(s)|^2\det(D_uG(s))+\dot u^T(D_uG(s))^*D_uG(s).\dot u(s).$$

Taking lemma 3 into account we derive

$$\dot\lambda(s)\sigma(s)=(|\dot\lambda(s)|^2+|\dot u(s)|^2)\det(D_uG(s)). \tag{6.4}$$

It is obvious that $\dot\lambda(s)\sigma(s)$ and $\det(D_uG(s))$ have the same sign. On the other hand,

$$\lambda(s)=\lambda_0+\begin{cases}\frac{1}{2}\xi^0 s+o(s^2) & \text{for RBP}\\ \frac{1}{2}\eta s^2+o(s^3) & \text{for LP}\end{cases}$$

$$\dot\lambda(s)=\begin{cases}\frac{1}{2}\xi^0 & \text{for RBP}\\ \eta s+o(s^2) & \text{for LP}\end{cases}$$

This means that $\dot{\lambda}(s)$ will change sign passing through $s = 0$ for LP and do not so for RBP. Consequently, we conclude

Theorem 3 Suppose (u_0, λ_0) be a singular point of (1.1).$(\xi, \xi^0), (\xi, \eta)$ are the root of ABE and LPBE respectively.
If $\xi^0 = \dot{\lambda}(0) \neq 0$ for RBP, then $\sigma(s)$ and $\det D_u G(s)$ will change or do not change their sign passing through $s = 0$ simultaneously.
If $\eta \neq 0$ for LP,then one of $\sigma(s)$ and $\det(D_u G(s))$ will change sign passing through $s = 0$ and another one do not so.

By applying Kato's perturbation theory of eigenvalues, $D_u G(s)$ on the solution branch in the neighborhood at (u_0, λ_0) has an eigenvalue $\rho(s)$ with m multiplicity (algebraic and geometric) continuously depending on s without changing multiplicity (there exist a exceptional point with respect to s for a special kind of $D_u G_0$ such that multiplicity will be change (see [Kat76],Chapter 2),we do not consider this case).

Let $\varphi_i(s)$ be associate eigenfunctions

$$D_u G(s)\varphi_i(s) = \rho(s)\varphi_i(s) \quad i = 1, 2, \cdots, m, \tag{6.5}$$

with

$$\rho(0) = 0, \varphi_i(0) = \varphi_i \quad i = 1, 2, \cdots, m. \tag{6.6}$$

Differentiating (6.5) with respect to s and evaluating at $s = 0$ we obtain

$$\begin{aligned} D_{uu}G(s)\dot{u}(s)\varphi_i(s) + D_{u\lambda}G(s)\dot{\lambda}(s)\varphi_i(s) + D_u G(s)\dot{\varphi}_i(s) \\ = \dot{\rho}(s)\varphi_i(s) + \rho(s)\dot{\varphi}_i(s). \end{aligned}$$

Hence we have

$$D_u G_0 \dot{\varphi}_i(0) = \begin{cases} -D_{uu}G_0\dot{u}(0)\varphi_i - D_u G_0 \varphi_i \xi^0 + \dot{\rho}(0)\varphi_i & \text{for RBP} \\ -D_{uu}G_0\dot{u}(0)\varphi_i + \dot{\rho}(0)\varphi_i & \text{for LP.} \end{cases} \tag{6.7}$$

Taking the inner product with ψ_k we derive that

$$\begin{array}{ll} a_{ij}^k \xi^j + a_{i0}^k \xi^0 = \dot{\rho}(0)\delta_{ki} & \text{for RBP,} \\ a_{ij}^k \xi^j = \dot{\rho}(0)\delta_{ki} & \text{for LP} \end{array}$$

i.e.

$$A(\xi) + \xi^0 A^0 = \dot{\rho}(0) I \quad \text{for RBP} \tag{6.8}$$

$$A(\xi) = \dot{\rho}(0) I \qquad \text{for LP} \tag{6.9}$$

where (ξ, ξ^0) and (ξ, η) are the root of ABE and LPBE respectively.
It follows that

$$\det(A(\xi) + \xi^0 A^0) = \dot{\rho}(0)^m \quad \text{for RBP} \tag{6.10}$$

$$\det A(\xi) = \dot{\rho}(0)^m \qquad \text{for LP} \tag{6.11}$$

Hence,we can conclude that $A(\xi) + \xi^0 A^0$(for RBP) and $A(\xi)$(for LP) are non-singular iff $\dot{\rho}(0) \neq 0$. Since $\rho(0) = 0, \dot{\rho}(0) \neq 0$ means $\rho(s)$ will be change sign passing through $s = 0$.

Theorem 4 On any solution branch of (1.1) generated by an isolated root (ξ, ξ^0) or (ξ, η) of ABE or LPBE with:

$$\begin{array}{ll} \xi^0 \neq 0 & \text{for RBP} \\ \eta \neq 0 & \text{for LP} \end{array}$$

we can conclude that if m is odd $\rho(s) \det(D_u G(s))$ will change its signs passing through (u_0, λ_0), if m is even then sign of $\det(D_u G(s))$ will not change.

Proof Since $(\xi, \xi^0), (\xi, \eta)$ are isolated root of ABE or LPBE respectively. $A(\xi) + \xi^0 A^0, A(\xi)$ are nonsingular according to theorems 1 and 2 . Therefore (6.10), (6.11) show that $\rho(s)$ will change sign. Because other eigenvalue of $D_u G_0$ do not change its sign passing through $s = 0$, the sign of $\det(D_u G(s))$ depends on $\rho^m(s)$ only. If m is odd, the sign of $\det(D_u G(s))$ changes passing through (u_0, λ_0), if m is even it does not change. The proof ends. □

Combining Theorem 3 and 4 we conclude

(1) For RBP,if $\xi^0 \neq 0, \sigma(s)$ change sign passing through $s = 0$ if m is odd, it does not change sign passing through $s = 0$ if m is even.

(2) For LP, if $\eta \neq 0$,then $\sigma(s)$ does not change sign passing through $s = 0$ when m is odd, $\sigma(s)$ changes sign through s=0 when m is even.

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Éditeur
INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
ISSN 0249-6399