# Weighted $H^2$ Approximation of Transfer Functions

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# Weighted $H^2$ approximation of transfer functions

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**Abstract:** The aim of this work is to generalize to the case of weighted $L^2$ spaces some results about $L^2$ approximation by analytic and rational functions which are useful to perform the identification of unknown transfer functions of stable (linear causal time–invariant) systems from incomplete frequency data.

**Key-words:** frequency domain identification, weighted Hardy spaces, extremal problems, rational approximation, orthogonal polynomials

*(Résumé : tsvp)*

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# Approximation $H^2$ pondérée de fonctions de transfert

**Résumé :** L'objet de ce rapport est de généraliser au cas d'une norme $L^2$ pondérée certains résultats obtenus pour l'approximation rationnelle et l'approximation analytique en norme $L^2$. Ces résultats s'appliquent à l'identification des fonctions de transfert de systèmes linéaires (stationnaires et causaux) stables, à partir de données sur une bande de fréquence.

**Mots-clé :** identification fréquentielle, espaces de Hardy pondérés, problèmes extrémaux, approximation rationnelle, polynômes orthogonaux

# 1 Introduction

Stability properties of linear stationary causal systems are related to the fact that their transfer functions belong to some Hardy spaces. For discrete time single–input / single–output strictly causal systems, these Hardy space are to be defined in the framework of the disk as follows. The space $\bar{H}_0^p$ is the space of functions analytic outside the closed unit disk, vanishing at infinity, and satisfying the growth condition:

$$\sup_{r>1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(r\, e^{i\theta})|^p d\theta < \infty\,, \quad \text{for } p < \infty\,, \tag{1}$$

or:

$$\sup_{r>1} \sup_{\theta \in [-\pi,\pi]} \text{ess} |f(r\, e^{i\theta})| < \infty\,, \quad \text{for } p = \infty\,.$$

The value of $p$ is linked to the stability properties satisfied by the system under consideration, the choices $p = 2$ and $p = \infty$ being of particular interest.
We are concerned here with the Hilbert case where $p = 2$ for which the $\bar{H}_0^2$ norm of the transfer function is equal to the input–output gain either from $l^2(\mathbb{N})$ into $l^\infty(\mathbb{N})$ or from $l^1(\mathbb{N})$ into $l^2(\mathbb{N})$ ; on the stochastic hand–side, it is also equal to the output variance for a white noise input (see e.g. [17]).

The approximation issues approached in this paper are mainly motivated by the following typical frequency domain identification problem for systems as above.
For an unknown $\bar{H}_0^2$ transfer function, assume that we are only given some of its (possibly noisy) pointwise values at frequencies belonging to a symmetric subset $K$ of $\mathbb{T}$, corresponding to the bandwidth of the associated system. Such measurements may be obtained using harmonic identification procedures. In order to identify the unknown system, we want to find a rational $\bar{H}_0^2$ function of bounded Mac–Millan degree accounting well enough for these data.

A preliminary interpolation step must be performed which consists in getting an interpolant $\phi \in L^2(K)$ for the given experimental pointwise data. If further information is available concerning the behaviour of the system outside the bandwidth, it must also be given by a function $\kappa \in L^2(J)$, $J = \mathbb{T} \setminus K$ (if nothing is known, one can take $\kappa = 0$). Then, our identification problem

can be approached by two consecutive stages consisting in solving $(i)$ and $(ii)$ below.
$(i)$ A bounded extremal problem (analytic approximation stage):
*given $\phi \in L^2(K)$, $\kappa \in L^2(J)$, and $M > 0$, find a function $f_0 \in \bar{H}_0^2$ which minimizes the $L^2(K)$ distance to $\phi$ under the constraint that its $L^2(J)$ distance to $\kappa$ is bounded by $M$.*
The function $f_0$ provides the best $\bar{H}_0^2$ approximant for the interpolating function $\phi$ on $K$ under the given gauge constraint on $J$. It provides an infinite dimensional model of the unkown system.
$(ii)$ A rational approximation problem (model order reduction stage) related to the solution of $(i)$:
*given an integer $n > 0$, find a rational function $r_0$ in $\bar{H}_0^2$ of Mac–Millan degree at most $n$ which minimizes the $L^2(\mathbb{T})$ distance to $f_0$.*

Whenever the circle is endowed with the Lebesgue measure, these issues have already been deeply studied and algorithms are available in order to build their solutions, see [2], [5] for $(i)$ and [4], [7] for $(ii)$. Our aim here is to study analogous weighted problems and mainly to generalize the corresponding resolution algorithms. In other terms, we want to set up a similar identification procedure that can be used when the unit circle is endowed with some finite positive measure $\mu$. This makes sense for measures $\mu$ such that the spaces of square–summable functions with respect to $\mu$ and to the Lebesgue measure coincide, space equality for which we will provide a necessary and sufficient condition in section 3, before solving for $(i)$ and $(ii)$ in sections 4 and 5, respectively.

In the classical stochastic framework, this type of weighted approximation problems come up when minimizing the variance of the output error between the searched model and the "true system", the quantity $d\mu/d\lambda$ being the spectral density of the noisy input (when this input is a white noise, then $\mu = \lambda$, as already mentionned). Such a weighting $\mu$ may also be present in the criterion when one pursues an identification procedure with the purpose of designing a controller, in which case $\mu$ represents the control performance criterion. Moreover, these weighted $L^2$ criteria may arise in control problems, when computing stable optimal controllers under some parametrization, for example. It may simply be used to weight some frequencies more than the others and to represent the confidence one has in the available measurements for either identification, filtering or control issues.

Note finally that a frequency domain identification problem for continuous time system can also be translated into this unit disk framework by means of a Möbius transform.

# 2 Notation and statement of the problems

Let $\mu$ be any positive finite measure on the unit circle $\mathbb{T}$ satisfying $d\mu(-\theta) = -d\mu(\theta)$ and let $L^2(\mu)$ be the *real* Hilbert space of functions on the unit circle $\mathbb{T}$ that are square–summable w.r.t. $\mu$ and that satisfy the conjugate–symmetry property $f(e^{-i\theta}) = \overline{f(e^{i\theta})}$ (such functions possess real Fourier coefficients); $L^2(\mu)$ is endowed with the inner product defined by:

$$< f\, ,\, g >_\mu = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\, g(e^{-i\theta})\, d\mu(\theta)\,, \tag{2}$$

and with the associated norm $\|\ \|_\mu$. For any symmetric subset $\Gamma$ of $\mathbb{T}$, $L^2(\Gamma,\mu)$ stands for the *real* Lebesgue space of square summable functions on $\Gamma$ w.r.t. $\mu$ that are conjugate–symmetric. If $\chi_\Gamma$ denotes the characteristic function of $\Gamma$, the norm on $L^2(\Gamma,\mu)$ is defined by: $\|f\|_{\Gamma,\mu} = \|\chi_\Gamma\, f\|_\mu$.

The disk algebra, i.e. the collection of functions which are continuous on the closed unit disk and analytic in the interior $\mathbb{D}$, will be denoted by $\mathcal{A}$, while $\mathcal{A}_0 \subset \mathcal{A}$ will stand for the set of functions in $\mathcal{A}$ which vanish at the origin. The family $\{z^k,\ k \geq 0\}$ is uniformly dense in $\mathcal{A}$, see e.g. [10] or [11] for proofs or details about the considerations of this section. We also denote by $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}_0$ the corresponding spaces for the complementary of the unit disk. The family $\{1/z^k,\ k > 0\}$ is uniformly dense in $\bar{\mathcal{A}}_0$. Each function in $\mathcal{A}$ or $\bar{\mathcal{A}}$ is the Poisson integral of its restriction to $\mathbb{T}$, so that we may identify these functions with their traces on $\mathbb{T}$, considering $\mathcal{A}$ and $\bar{\mathcal{A}}$ as subspaces of $L^2(\mu)$. This allows to define the *real* weighted Hardy spaces $H^2(\mu)$ and $\bar{H}^2_0(\mu)$ as the $L^2(\mu)$ closures of $\mathcal{A}$ and $\bar{\mathcal{A}}_0$, respectively. The two spaces $H^2(\mu)$ and $\bar{H}^2_0(\mu)$ are isometrical under the map:

$$\begin{array}{rcl} L^2(\mu) & \to & L^2(\mu) \\ f(z) & \mapsto & \dfrac{f(1/z)}{z} = \check{f}(z)\,. \end{array} \tag{3}$$

Whenever $\mu = \lambda$, the Lebesgue measure for which we write $d\lambda(\theta) = d\theta$, we use $L^2(\lambda) = L^2(\mathbb{T})$ to denote the classical *real* Hilbert space of square–summable conjugate–symmetric functions on the unit circle $\mathbb{T}$. Still for the sake of simplicity, we note $<\ ,\ >_\lambda = <\ ,\ >$ and $\|\ \|_\lambda = \|\ \|$ for its inner product and associated norm. Also, we write $H^2(\lambda) = H^2$ for the classical *real* Hardy space of the unit disk $\mathbb{D}$; it coincides with the closed subspace of $L^2(\mathbb{T})$ of functions whose Fourier coefficients of negative index are zero. These functions admit an analytic extension in $\mathbb{D}$ and $H^2$ is isometric to the space of analytic functions in $\mathbb{D}$ satisfying the growth condition (1) for $p = 2$ and $r < 1$ which provides a norm on this space (see e.g. [10, II.3]). Furthermore, $\bar{H}_0^2(\lambda) = \bar{H}_0^2$ is the orthogonal complement of $H^2$ in $L^2(\mathbb{T})$ w.r.t. $\lambda$. It consists in $L^2(\mathbb{T})$ functions which possess Fourier coefficients of non–negative index equal to zero or, equivalently, in functions analytic outside the closed unit disk and vanishing at infinity satisfying the growth condition (1) for $p = 2$.
Let $L^\infty(\mathbb{T})$ be the *real* Banach space of essentially bounded conjugate–symmetric functions. The Hardy space $H^\infty$ is defined to be the $L^\infty(\mathbb{T})$ closure of polynomials; it can also be characterized by ([10, II.4]):

$$H^\infty = H^2 \cap L^\infty(\mathbb{T})\,. \tag{4}$$

Considering now the distance induced on $\mathbb{T}$ by the $L^2(\mu)$ norm, we come back to our approximation problems $(i)$ and $(ii)$. Problem $(i)$ can be formulated as follows.

$(P_1)$ *Given* $\phi \in L^2(K)$, $\kappa \in L^2(J)$, *and* $M > 0$, *find a function* $f_0 \in \bar{H}_0^2$ *which minimizes* $\|\phi - f\|_{K,\mu}$ *among the functions* $f \in \bar{H}_0^2$ *which satisfy the constraint* $\|\kappa - f\|_{J,\mu} \leq M$.

As an approximation issue, problem $(ii)$ must be set up for an arbitrary function $f \in \bar{H}_0^2$, even if $f = f_0$, solution to $(P_1)$, in our global identification scheme.

$(P_2)$ *Given $f \in \bar{H}_0^2$ and an integer $n > 0$, find a rational function $r_0$ which minimizes $\|f - r\|_\mu$, where $r$ ranges over the rational functions in $\bar{H}_0^2$ of Mac–Millan degree at most $n$.*

For these two problems to make sense, we must have the following topological Hilbert spaces inclusion:

$$L^2(\mathbb{T}) \subset L^2(\mu),$$

which requires any square–summable function for the Lebesgue measure to be square–summable for $\mu$ and also the identitity map from $L^2(\mathbb{T})$ to $L^2(\mu)$ to be continuous. It implies in particular that *$\mu$ is absolutely continuous with respect to the Lebesgue measure* and that the $L^2(\mathbb{T})$ closure of a set of functions is contained in its $L^2(\mu)$ closure. In this case, $H^2(\mu)$ and $\bar{H}_0^2(\mu)$ are the $L^2(\mu)$ closures of $H^2$ and $\bar{H}_0^2$ respectively. Moreover, for our best approximant to be reached, $\bar{H}_0^2$ must be a closed subspace of $L^2(\mu)$, and we shall have:

$$\bar{H}_0^2 = \bar{H}_0^2(\mu)\,. \tag{5}$$

*Remark 1* Note that the following equalities as between topological spaces are equivalent:
*(a)* $L^2(\mathbb{T}) = L^2(\mu)\,,$
*(b)* $\bar{H}_0^2 = \bar{H}_0^2(\mu)\,,$
*(c)* $H^2 = H^2(\mu)\,.$
Indeed, if *(a)* holds, the $L^2(\mathbb{T})$ closure and the $L^2(\mu)$ closure agree, and we have *(b)* and *(c)*. On another hand, *(b)* and *(c)* are equivalent by (3). Finally, if *(b)* and *(c)* hold, since trigonometric polynomials are dense in $L^2(\mu)$, we get:

$$L^2(\mu) = H^2(\mu) \oplus \bar{H}_0^2(\mu) = H^2 \oplus \bar{H}_0^2 = L^2(\mathbb{T})\,,$$

where $\oplus$ stands for the direct sum.

# 3 Weighted $H^2$ spaces

In view of remark 1, the space equality (5) can be ensured by the following result.

**Theorem 1** *Let $\mu$ be a finite positive measure such that $d\mu(-\theta) = -d\mu(\theta)$. Then*

$$L^2(\mathbb{T}) = L^2(\mu) \tag{6}$$

*if and only if $\mu$ is of the form*

$$d\mu(\theta) = |\nu(e^{i\theta})|^2 \, d\theta \tag{7}$$

*for a function $\nu$ which belongs to $H^\infty$ and is invertible in $H^\infty$.*

*Proof:* the space equality (6) is obviously satisfied whenever (7) holds. Conversely, assume that $\mu$ is such that (6) holds. As already mentionned, this space equality implies that $\mu$ and $\lambda$ are mutually absolutely continuous. Moreover, it implies that 1 does not lie in the $L^2(\mu)$ closure of $\mathcal{A}_0$ since it does not lie in the $L^2(\mathbb{T})$ closure of $\mathcal{A}_0$ (see remark 1). Let $F$ denotes the orthogonal projection of 1 on the $L^2(\mu)$ closure of $\mathcal{A}_0$, and let $\nu$ be given by:

$$\nu = \nu(0)\,(1-F)^{-1}\,,$$

with

$$|\nu(0)|^2 = \frac{1}{2\pi} \inf_{f\in\mathcal{A}_0} \int_0^{2\pi} |1-f|^2 \, d\mu = \frac{1}{2\pi} \int_0^{2\pi} |1-F|^2 \, d\mu\,.$$

It is then a consequence of Szego–Kolmogoroff–Krein theorem [11, ch.4] that $\nu$ is an outer $H^2$ function such that

$$d\mu(\theta) = |\nu(e^{i\theta})|^2 \, d\theta\,.$$

Now, we have that $f \in L^2(\mu)$ if and only if $\nu f \in L^2(\mathbb{T})$, which in view of (6) and [19, thm.13.14] implies that $\nu$ and $1/\nu$ belong to $L^\infty(\mathbb{T})$. It then follows from (4) that $\nu \in H^\infty$. Furthermore, by definition, a function $f$ belongs to $H^2(\mu)$ if and only if $\nu\, f$ belongs to the $L^2(\mathbb{T})$ closure of $\nu\,\mathcal{A}$; but, $\nu$ being outer, it is a consequence of Beurling's theorem [10, cor.II.7.3] that the latter space is equal to $H^2$. Therefore:

$$H^2(\mu) = H^2/\nu\,,$$

which is also equal to $H^2$. Thus, $1/\nu$ belongs to $H^2$ and, still using (4), we are done. $\square$

$(H)$ *In the remaining of this paper, we make the standing assumption that $\mu$ satisfies (7) for a function $\nu$ belonging to $H^\infty$ and invertible in $H^\infty$.*

*Remark 2* As another consequence of hypothesis (7),

$$L^2(\Gamma,\mu) = L^2(\Gamma,\lambda) = L^2(\Gamma)\,,$$

for any symmetric subset $\Gamma$ of $\mathbb{T}$, whence we also get, for $f \in L^2(\Gamma)$:

$$\|f\|_{\Gamma,\mu} = \|\nu\, f\|_{\Gamma}\,. \tag{8}$$

Meanwhile, for all $f\,,\ g \in L^2(\mathbb{T})$, using notation (3), the scalar product (2) can be written as:

$$< f\,,\, g >_\mu = \frac{1}{2i\pi}\int_{\mathbb{T}} \check{f}(z)\, g(z)\, |\nu(z)|^2\, dz\,. \tag{9}$$

# 4 Weighted $H^2$ approximation

In this section, we get a solution to problem $(P_1)$ by solving the following analogous problem $(P_1')$ in $H^2$.

$(P_1')$ *Given $\varphi \in L^2(K)$, $h \in L^2(J)$, and $M > 0$, find a function $g_0 \in H^2$ which minimizes $\|\varphi - g\|_{K,\mu}$ among the functions $g \in H^2$ which satisfy the constraint $\|h - g\|_{J,\mu} \leq M$.*

Thanks to isometry (3), the bounded extremal problems $(P_1)$ and $(P_1')$ are equivalent whenever $K$ is a symmetric subset of $\mathbb{T}$: if we take $\varphi = \check{\phi}$, $h = \check{\kappa}$ and if $g_0$ is the associated solution to $(P_1')$ for some $M > 0$, then $f_0 = \check{g}_0$ solves $(P_1)$ with $\phi$, $\kappa$, and $M$.

For $\mu = \lambda$, problem $(P_1')$ has been solved in [2] when $h = 0$ and in [12] when $\phi = 0$; since then, it has been approached in the general $H^p$ setting, $1 \leq p < \infty$, in [5], and in $H^\infty$, in [6]. From these results, we deduce below the existence and a characterization of a solution to problem $(P_1')$ for measures $\mu$ satisfying $(H)$. Let

$$B_M^h(\mu) = \{g \in H^2 \text{ s.t. } \|h - g\|_{J,\mu} \leq M\} \text{ and } C_M^h(\mu) = \{g_{|K}\,,\ g \in B_M^h(\mu)\}\,.$$

Denote by $P_{H^2}$ the $\lambda$–orthogonal projection from $L^2(\mathbb{T})$ onto $H^2$ and let $T : H^2 \to H^2$ be the Toeplitz operator with symbol $\chi_J$:

$$T(g) = P_{H^2}(\chi_J \, g) \ , \ \forall g \in H^2 \, .$$

**Theorem 2** *Let $K$ be a symmetric subset of $\mathbb{T}$ such that both $K$ and its complementary $J$ are of positive measure $\mu$, where $\mu$ satisfies $(H)$. Then, there exists a unique solution $g_0 \in H^2$ to problem $(P_1')$. Moreover, $\|h - g_0\|_{J,\mu} = M$ whenever $\varphi \notin C_M^h(\mu)$ and $g_0$ is given in this case by the implicit equation:*

$$g_0 = \nu^{-1} \, (1 + l \, T)^{-1} \, P_{H^2} \, \left(\nu(\varphi \vee (l+1) \, h)\right) \ , \tag{10}$$

*where $l \in (-1, +\infty)$ is the unique real number such that $\|g_0 - h\|_{J,\mu} = M$.*

*Proof:* first of all, observe that $(P_1')$ can be formulated shortly as follows.
*Find $g_0 \in H^2$ such that*

$$\|\varphi - g_0\|_{K,\mu} = \min_{g \in B_M^h(\mu)} \|\varphi - g\|_{K,\mu} \, .$$

Under hypothesis $(H)$ and since (8) holds, $g_0$ provides a solution to $(P_1')$ if and only if $\gamma_0 = \nu \, g_0$ satisfies the unweighted bounded extremal problem:

$$\|\nu \, \varphi - \gamma_0\|_K = \min_{\gamma \in B_M^{\nu h}(\lambda)} \|\nu \, \varphi - \gamma\|_K \, . \tag{11}$$

That (11) has a unique solution $\gamma_0$ which verifies

$$\|\nu \, h - \gamma_0\|_J = M$$

whenever

$$\nu \, \varphi \notin C_M^{\nu \, h}(\lambda) \, , \tag{12}$$

is precisely the content of [5, thm.2]. Of course, (12) is equivalent to the assumption that $\varphi \notin C_M^h(\mu)$, and this establishes the first part of the theorem. Towards the characterization of $g_0$, it is stated in [5, thm.4] that, if (12) holds, then the solution $\gamma_0$ to (11) is given by the implicit equation:

$$\gamma_0 = (1 + l \, T)^{-1} \, P_{H^2} \, \left(\nu \, \varphi \vee (l+1) \, \nu \, h\right) \ ,$$

where $l \in (-1, +\infty)$ is the unique real number such that $\|\gamma_0 - \nu\, h\|_J = M$. Taking now $g_0 = \nu^{-1}\, \gamma_0$ achieves the proof. $\square$

Note that the Toeplitz operator $T$ is bounded, self–adjoint, positive, with norm 1 and spectrum equal to $[0,1]$ (see e.g. [15, chap.3]); in particular, $(1 + l\,T)$ is invertible for $l > -1$.

Observe that without norm constraint on $g_0$ outside $K$, problem $(P_1')$ becomes ill–posed in general, as is shown below. However, this is not the case if $\varphi$ is already the trace on $K$ of an $H^2$ function. If this holds, $(P_1')$ can be interpreted when $M \to \infty$ as a recovery issue of the $H^2$ function $\varphi$ from its values on $K$.

**Proposition 1**
*(i) The space $H^2_{|_K}$, consisting in traces on $K$ of $H^2$ functions, is dense in $L^2(K)$ for $\mu$ whenever $\mu(J) > 0$.*
*(ii) Let $\varphi \in L^2(K)$ and let $(g_n)$ be a sequence of $H^2$ functions such that $\|\varphi - g_n\|_{K,\mu}$ tends to 0. If $\varphi \notin H^2_{|_K}$, then* $\lim_{n\to\infty} \|g_n\|_{J,\mu} = \infty$.

*Proof:* it is established in [6, prop.1] that the space $H^2_{|_K}$ is dense in $L^2(K)$ for $\lambda$ whenever $\lambda(J) > 0$. In view of theorem 1 and of remark 1, this is amounts to assertion $(i)$. Assertion $(ii)$ directly follows from [5, prop.3]. However, if $\varphi \in H^2_{|K}$, one can deduce from recovery formulae in [14] that $\|\varphi - g_n\|_\mu$ tends to 0. $\square$

In order to compute $g_0$, we have to get through the implicit character in $M$ of equation (10). To this end, it can be shown by using analogous properties in the unweighted case [5, prop.4] that $M$ is a smoothly decreasing function of the Lagrange parameter $l$ from $(-1, \infty)$ onto $(0, \infty)$, if $\varphi \notin H^2_{|K}$. Hence, $M$ being given, $g_0$ can be numerically computed using a dichotomy procedure on $l$. Furthermore, as $l \to -1$, then the error $e_\mu = \|\varphi - g_0\|_{K,\mu}$ goes to zero while $M \to \infty$, if $\varphi \notin H^2_{|K}$. Differential equations can be written down in the same vein as in [2], showing that $M$ and $e_\mu$ cannot behave arbitrarily when $l \to -1$, though asymptotic formulae are not yet available.

It may however be useful towards these asymptotic issues to get another characterization of $g_0$ in the framework of Carleman formulae, see [1], [14]. Let

$\varrho > 1$ and let $\zeta$ be the outer $H^\infty$ function of modulus equal to $\varrho$ a.e. on $K$ and to 1 a.e. on $J$:

$$\zeta(z) = \exp\left\{\frac{\log \varrho}{2\,\pi} \int_K \frac{e^{it}+z}{e^{it}-z}\, dt\right\}\,,\ z \in \mathbb{D}\,. \tag{13}$$

If $\varphi \notin C_M^h(\mu)$, it follows from [5, cor.1] that (10) equivalently rewrites as

$$g_0(z) = \frac{1}{2\,i\,\pi} \int_{\mathbb{T}} \frac{\nu(\xi)}{\nu(z)} \left(\frac{\zeta(\xi)}{\zeta(z)}\right)^{\alpha} (\varphi \vee h)(\xi)\, \frac{d\xi}{\xi - z}\,,\ \forall z \in \mathbb{D}\,, \tag{14}$$

where

$$\alpha = -\frac{\log(l+1)}{2\,\log \varrho}\,,$$

$l \in (-1, \infty)$ being the unique real number such that $\|g_0 - h\|_{J,\mu} = M$.

# 5 Weighted rational approximation

## 5.1 The function $\Psi_n$.

In this section, we consider the rational approximation problem $(P_2)$. Observe first that a rational function $p/q$ belongs to $\bar{H}_0^2$ if and only if $q$ has its roots inside the unit disk $\mathbb{D}$ ($p/q$ is *stable*) and the degree of $p$ is less than the degree of $q$ ($p/q$ is *strictly proper*). The Mac–Millan degree of such a rational $\bar{H}_0^2$ function is thus the degree of $q$. Recall that the problem $(P_2)$ consists in minimizing a $\mu$–weighted criterion

$$\|f - \frac{p}{q}\|_\mu \tag{15}$$

over the set of rational functions $p/q$ in $\bar{H}_0^2$ of Mac–Millan degree at most $n$, for some fixed $n$. We first establish a normality result which generalizes [7, prop.2.1].

**Proposition 2** *If $f \in \bar{H}_0^2$ is not a rational function of degree less than $n$, then the argument of any local minimum of* (15) *is an irreducible fraction whose degree is equal to $n$.*

*Proof:* assume that there exists a local minimum $p_0/q_0$ of (15) where $p_0$ and $q_0$ are coprime polynomials and $\mathrm{d}^o\, q_0 < n$. In this case, there exists a neighbourhood $U$ of 0 in $\mathbb{R}$ such that

$$\forall a \in U\,,\ \forall b \in (-1,+1)\,,\ \left\| f - \left(\frac{p_0}{q_0} + \frac{a}{z-b}\right)\right\|_\mu \geq \left\| f - \frac{p_0}{q_0}\right\|_\mu .$$

By expansion, this yields

$$a^2 \left\|\frac{1}{z-b}\right\|_\mu^2 - 2\,a \left\langle f - \frac{p_0}{q_0}, \frac{1}{z-b}\right\rangle_\mu \geq 0\,,$$

which holds for any $a \in U$, if and only if

$$\left\langle f - \frac{p_0}{q_0}, \frac{1}{z-b}\right\rangle_\mu = 0\ ,\ \forall b \in (-1,+1)\,.$$

Since the family $\{1/(z-b)\,,\ b \in (-1,+1)\}$, is dense in $\bar{H}_0^2$ for the Lebesgue norm, it follows from remark 1 that it remains dense in $\bar{H}_0^2$ w.r.t. the $L^2(\mu)$ norm; thus, we must have

$$\left\langle f - \frac{p_0}{q_0}, \bar{H}_0^2\right\rangle_\mu = 0\ .$$

Hence, $f = p_0/q_0$ and $f$ is rational of degree $n$. □

*We assume in the following that $f$ is not rational of degree less than $n$.*

As a consequence of proposition 2, we are going to look for a solution $p/q$ to problem $(P_2)$ over the set of $\bar{H}_0^2$ rational functions of Mac–Millan degree $n$.

*We shall therefore assume $q$ to be monic.*

The first step is to eliminate the numerator $p$. Let $V_q$ be the $n$–dimensional linear subspace of $\bar{H}_0^2$ generated by $\{z^i/q\},\ i = 0, \cdots, n-1$. Any local minimum $p/q$ of (15) must then be the orthogonal projection of $f$ onto $V_q$ with respect to $\mu$, and thus $p$ satisfies the linear system

$$< f - \frac{p}{q}, \frac{z^i}{q} >_\mu = 0\ ,\ i = 0, \cdots, n-1\,.$$

In this way, $p$ becomes a linear function of $q$ denoted by $L_n(q)$.

Let $\Delta_n$ be the set of real monic polynomials of degree $n$ whose roots belong to $\mathbb{D}$. Note that an element $q$ of $\Delta_n$, which writes $q(z) = z^n + q_{n-1}z^{n-1} + \ldots + q_0$, can be identified with the $\mathbb{R}^n$ vector of its coefficients, $(q_{n-1}, q_{n-2}, \ldots, q_0)$; this allows us to consider $\Delta_n$ as an open subset of $\mathbb{R}^n$. We finally have:

**Proposition 3** *Problem $(P_2)$ can be solved by minimizing the function $\Psi_n$ defined on $\Delta_n$ by:*

$$\begin{array}{rccl} \Psi_n : & \Delta_n & \longrightarrow & \mathbb{R} \\ & q & \longmapsto & \|f - \dfrac{L_n(q)}{q}\|_\mu^2 , \end{array} \tag{16}$$

*where $L_n(q)/q$ is the orthogonal projection of $f$ onto $V_q$ in $L^2(\mu)$.*

In the case of the Lebesgue measure ($\mu = \lambda$), $L_n(q)$ can be easily computed as the remainder of some division in $H^2$ (see [7]). In general, the situation is more complicated. However, we propose below an integral representation formula for $L_n(q)$.

Fix $q$ for a while and let $\{\Phi_j\}$, $j \geq 0$, be the system of orthonormal polynomials on $\mathbb{T}$ for the measure $d\mu/|q|^2$ (see [16, XI] and also [8]). The orthogonal polynomial $\Phi_j$ has precisely degree $j$ and its roots lie in $\mathbb{D}$.

Define the reciprocal polynomial $\tilde{P}$ of a real polynomial $P$ of degree $k$ by

$$\tilde{P}(z) = z^k\, P(1/z)\,.$$

This operation also apply to a polynomial $P$ whose degree is unknown but bounded by $k$, for example to the remainder in a division by a polynomial of degree $k+1$. In this case, $P$ has exact degree $k$ if and only if $\tilde{P}(0) \neq 0$. Note also that the polynomials $P$ and $\tilde{P}$ always have the same roots on $\mathbb{T}$.

**Proposition 4** *The polynomial $L_n(q)$ is given by*

$$L_n(q)(z) = \frac{1}{2i\pi} \int_{\mathbb{T}} \frac{\check{f}}{q}(\xi) \frac{\widetilde{\Phi_n}(\xi)\widetilde{\Phi_n}(z) - \Phi_n(\xi)\Phi_n(z)}{1 - \xi z}\, |\nu(\xi)|^2\, d\,\xi\,. \tag{17}$$

*Proof:* by choosing $\{\Phi_j/q\}$, $j = 0, \cdots, n-1$, as a basis of $V_q$, we get that:

$$L_n(q) = \sum_{j=0}^{n-1} < f, \frac{\Phi_j}{q} >_\mu \Phi_j \, ,$$

or else, in view of (9),

$$L_n(q)(z) = \frac{1}{2i\pi} \int_{\mathbb{T}} \frac{\check{f}}{q}(\xi) \sum_{j=0}^{n-1} \Phi_j(\xi)\Phi_j(z) \, |\nu(\xi)|^2 \, d\xi \, .$$

Using the Christoffel–Darboux formula [16, XI] for the Szegö kernel:

$$\sum_{j=0}^{n-1} \Phi_j(\xi) \, \Phi_j(z) = \frac{\widetilde{\Phi_n}(\xi) \, \widetilde{\Phi_n}(z) - \Phi_n(\xi) \, \Phi_n(z)}{1 - \xi \, z} \, , \tag{18}$$

we obtain (17). □

Following [4] we prove in the next sections that the function $\Psi_n$ defined by (16) does extend smoothly to an open neighbourhood of $\Delta_n$.

## 5.2 Extension of the domain of $\phi_n$.

*We shall restrict ourselves to the case where*

$$\nu \;=\; \frac{1}{\tilde{w}}, \tag{19}$$

*$w$ being a monic polynomial of degree $d$ whose roots lie inside the unit disk $\mathbb{D}$ and which does not vanish at zero.* This is an important special case treated in [16] in which the system $\{\Phi_j\}$ can be computed, explicitely for $j \geq n + d$:

$$\Phi_j(z) = \; z^{j-n-d} q(z) w(z), \quad j \geq n + d \, , \tag{20}$$

and by the recurrence formulae:

$$\widetilde{\Phi_j}(0) \, \widetilde{\Phi_j}(z) \;=\; \widetilde{\Phi_{j+1}}(0) \, \widetilde{\Phi_{j+1}}(z) \;-\; \Phi_{j+1}(0)\Phi_{j+1}(z) \, , \; 0 \leq j \leq n + d - 1 \, , \tag{21}$$

which would also hold for any system of orthonormal polynomials on $\mathbb{T}$, their leading coefficients being given by

$$\widetilde{\Phi_j}(0)^2 \;=\; \widetilde{\Phi_{j+1}}(0)^2 \;-\; \Phi_{j+1}(0)^2 = \sum_{k=0}^{j} \Phi_k^2(0)\,. \tag{22}$$

Formulas (20), (21), and (22) allow to compute the polynomials $\Phi_j$, for every $j \geq 0$. Let us introduce the family of maps $\{\phi_j\}$ defined on $\Delta_n$ by

$$\phi_j \;:\; q \mapsto \Phi_j, \tag{23}$$

where $\Phi_j$ is the $j$-th orthogonal polynomial for the measure $d\mu/|q|^2$.

**Proposition 5** *The map $\phi_n \;:\; q \mapsto \Phi_n$, smoothly extends to an open neighbourhood $\mathcal{V}$ of $\Delta_n$. Moreover, if $q = uq'$ where $u$ is monic of degree $m$ and has all its roots of modulus 1 while $q'$ belongs to $\Delta_{n-m}$, then $\phi_n(q) = u\,\phi_{n-m}(q')$, where $\phi_n$ now denotes the extended map.*

*Proof:* first, for $j \geq n+d$, formula (20) defines polynomials $\phi_j(q)$ of degree $j$ for each polynomial $q$ of degree $n$ that are clearly smooth functions of $q$ on any open neighbourhood of $\Delta_n$. Let us then proceed by induction in order to show that this holds in fact for all $j \geq n$. To this end, let $k \geq n$ and assume that for $j \geq k+1$, $\phi_j$ has been extended to some neighbourhood $\mathcal{V}_j$ of $\Delta_n$. Let us prove that $\phi_k$ can in turn be extended to some open $\mathcal{V}_k$ such that $\overline{\Delta}_n \subset \mathcal{V}_k \subset \mathcal{V}_{k+1}$, where $\overline{\Delta}_n$ is the closure of $\Delta_n$.

Formula (21) asserts that the coefficients of $\Phi_k$ are algebraic functions of the coefficients of $\Phi_{k+1}$, whose denominator is given by (22); hence, if for all $q \in \mathcal{V}_k$,

$$\widetilde{\phi_{k+1}(q)}(0)^2 - \phi_{k+1}(q)(0)^2 \neq 0, \tag{24}$$

then $\phi_k$ does smoothly extend to $\mathcal{V}_k$. In this case, since $\widetilde{\phi_k(q)}(0) \neq 0$, the degree of $\phi_k(q)$ is $k$ on the whole $\mathcal{V}_k$.

Assume first that $q$ belongs to $\overline{\Delta}_n$. Then $\Phi_{k+1}$ has all its roots in $\bar{\mathbb{D}}$, as the limit of orthonormal polynomials whose roots lie inside $\mathbb{D}$ (see [16, thm.11.4.1])).

Suppose that (24) is false. Then $\Phi_{k+1}(0)/\tilde{\Phi}_{k+1}(0)$, which is equal to the product of the roots of $\Phi_{k+1}$, has modulus 1. It follows that each root of $\Phi_{k+1}$ belongs to $\mathbb{T}$. However, this cannot occur since $\Phi_{k+1}$ has at most $n < k+1$ roots on $\mathbb{T}$, which we prove now.

More precisely, we establish that, for $j \geq k+1$, $\Phi_j$ and $q$ have the same roots on $\mathbb{T}$. Indeed, by (21), the roots of $\Phi_{j+1}$ on the unit circle are roots of $\Phi_j$, since every roots of $\Phi_{j+1}$ which has modulus 1 is also a root of $\widetilde{\Phi_{j+1}}$. Conversely, combining (21) and the reciprocal formula

$$\widetilde{\Phi_j}(0)\, z\Phi_j(z) \;=\; \widetilde{\Phi_{j+1}}(0)\,\Phi_{j+1}(z) \;-\; \Phi_{j+1}(0)\widetilde{\Phi_{j+1}}(z)$$

gives:

$$\widetilde{\Phi_j}(0)\,\Phi_{j+1}(z) \;=\; \Phi_{j+1}(0)\,\widetilde{\Phi_j}(z) \;+\; \widetilde{\Phi_{j+1}}(0)\, z\Phi_j(z),$$

which in turn proves that the roots of $\Phi_j$ on $\mathbb{T}$ are also roots of $\Phi_{j+1}$.

Thus, equation (24) is satisfied on $\overline{\Delta}_n$. Now, since the map $\phi_{k+1}$ is continuous on $\mathcal{V}_{k+1}$, equation (24) is also satisfied on a neighbourhood $\mathcal{V}_k \subset \mathcal{V}_{k+1}$ of $\Delta_n$; thus, $\phi_k$ does smoothly extend to $\mathcal{V}_k$.

To prove the second assertion of proposition 5, let $q = uq'$ as in the statement, and let $\{\Phi_j\}$ and $\{\Phi'_j\}$ be the systems of orthonormal polynomials for the measures $d\mu/q$ and $d\mu/q'$, respectively. Formula (20) implies that $\Phi_{n+d} = u\Phi'_{n+d-m}$. Moreover, using that $u(0)^2 = \tilde{u}(0)^2 = 1$ and $u(z) = u(0)\tilde{u}(z)$, it can be proved by induction from (21) and (22) that $\Phi_j = u\Phi'_{j-m}$ for $j \geq n$. $\square$

## 5.3 Extension of the domain of $\Psi_n$.

**Proposition 6** *Assume that $\check{f}$ and $\nu$ are analytic in a disk $D_r = \{\, z, |z| < r\,\}$ for some $r > 1$. Then, the map $\Psi_n$ smoothly extends to a neighbourhood $\mathcal{V}$ of $\Delta_n$.*

*Proof:* introduce

$$f_w \;=\; \frac{f}{w} \quad \text{which gives} \quad \check{f}_w(z) \;=\; \frac{\check{f}(z)\, z^d}{\tilde{w}(z)}\,.$$

Since we get from (19) that

$$|\nu(z)|^2 = \frac{z^d}{w(z)\,\tilde{w}(z)}\,,\ z \in \mathbb{T}\,, \tag{25}$$

the integral representation (17) can be rewritten as:

$$L_n(q)(z) = \frac{1}{2i\pi}\int_\Gamma \frac{\check{f}_w}{qw}(\xi)\frac{\widetilde{\phi_n(q)}\,(\xi)\widetilde{\phi_n(q)}\,(z) - \phi_n(q)\,(\xi)\phi_n(q)\,(z)}{1-\xi z}\,d\xi\,, \tag{26}$$

where $\Gamma$ is any contour contained in the domain of holomorphy of $\check{f}_w$ that encompasses the zeroes of $q$.
By assumption, the function $\check{f}_w$ is analytic in the disk $D_r$ (since $\nu$ is analytic in $D_r$, $\tilde{w}$ has no zero inside $D_r$). Moreover, it follows from proposition 5 that the map $\phi_n$ smoothly extends to a neighbourhood $\mathcal{V}$ of $\Delta_n$. Choose $\mathcal{V}$ in order for $q$ to have its roots in $D_r$. The contour $\Gamma$ may thus be deformed within $D_r$ thereby allowing (26) to remain defined for $q \in \mathcal{V}$. This integral representation yields a smooth extension of $L_n$.
Furthermore, if $q \in \Delta_n$, the properties of the orthogonal projection show that

$$\Psi_n(q) \;=\; \|f - \frac{L_n(q)}{q}\|_\mu^2 \;=\; \|f\|_\mu^2 - < f, \frac{L_n(q)}{q} >_\mu\,. \tag{27}$$

It is now sufficient to smoothly extend the map $q \mapsto < f, z^j/q >_\mu$ for every $j$. This can be done similarly by noting that from (9) and (25),

$$< f, z^j/q >_\mu \;=\; \frac{1}{2i\pi}\int_\Gamma \check{f}_w(\xi)\frac{\xi^j}{q(\xi)w(\xi)}d\xi.$$

□

We are then in position to establish two results concerning the recursive properties of $\Psi_n$.

**Lemma 1** *Let $q \in \partial\Delta_n$, and suppose that $q = uq'$ where $u$ is monic of degree $m$ and has all its roots of modulus 1 while $q'$ belongs to $\Delta_{n-m}$. Then $\Psi_n(q) = \Psi_{n-m}(q')$.*

*Proof:* in view of (27), it is sufficient to prove that $L_n(q) = uL_{n-m}(q')$. This follows from (26) and proposition 5. $\square$

We can thus apply differential tools as in the case of the Lebesgue measure $\lambda$ for which problem $(P_2)$ has been solved in [4] and [7]. Let us denote by $\nabla_n(q)$ the gradient vector of $\Psi_n$ at the point $q$.

**Corollary 1** *Assume that $q$ belongs to some smooth part of $\partial\Delta_n$ and let $q = u\,q'$ as in lemma 1 with $m = 1$ such that $q'$ is a critical point of $\Psi_{n-1}$. Then $\nabla_n(q)$ is orthogonal to $\partial\Delta_n$ and points outwards.*

*Proof:* As in [7], we can deduce from lemma 1 that the projection of $\nabla_n(q)$ onto $\partial\Delta_n$ is just $\nabla_{n-1}(q')$ which is 0 by hypothesis, so that $\nabla_n(q)$ is orthogonal to $\partial\Delta_n$. Moreover, it cannot point inwards because this would imply that $L_{n-1}(q')/q'$, which is rational of degree $n-1$, is locally a best approximant to $f$ among rational functions of degree $n$, hence by proposition 2 that $f$ itself is rational of degree $< n$. $\square$

## 5.4 An algorithm to find a local minimum.

We shall assume in this section that, for $k = 1 \dots n$, $\nabla_k$ does not vanish on $\partial\Delta_k$ and that all the critical points of $\Psi_k$ on $\Delta_k$ are non degenerate. Note that for the Lebesgue measure, these two properties hold in an open dense subset of the space of $\bar{H}_0^2$ functions that are analytic ouside a disk $D_r$ for $r < 1$, see [3]. Although this has not been established yet in the weighted case, it seems reasonable to expect that this kind of “genericity” result still holds. Anyway, whenever these assumptions are satisfied, $\Psi_k$ has a finite number of critical points in $\Delta_k$ and an algorithm can be described following the same scheme than in [4].

The function $\Psi_n$ is smooth and its local minima belong to $\Delta_n$ by proposition 2. Therefore, local minima are critical points of $\Psi_n$ and can be found by a gradient algorithm. Such a procedure leads either to a critical point or to the boundary of the domain. More precisely, it goes as follows: we integrate the vector field $-\nabla_n$ from an initial point. If we meet the boundary of $\Delta_n$

then, by lemma 1, we are led to solve a problem of lower order. Conversely, by corollary 1, a local minimum of $\Psi_k$, $k < n$, provides a suitable boundary initial point to integrate $-\nabla_{k+1}$. The procedure can thus continue through different orders (stricly positive, since $\Psi_0$ is a constant function whose value $\|f\|_\mu^2$ is an upper bound for $\Psi_n$ on $\Delta_n$) whereas the value of the criterion (which is $\Psi_k$ while integrating $-\nabla_k$) decreases. Thus, a local minimum cannot be met twice; since local minima are finite in number, the procedure converges.

## 5.5 More about $L_n(q)$.

In this section, we establish different expressions of the polynomial $L_n(q)$ when $\nu$ is given by (19). In this case, we already know how to compute $L_n(q)$ using (26) together with the recurrence formulae (21), (22) initialized by (20) for $j = n+d$. However, for the above algorithm to work out efficiently, it might be interesting to provide and compare various schemes to get $L_n(q)$.

1) From formula (17).

It is easily seen from (18) that

$$L_n(q)(z) = A(z) - \sum_{j=n}^{n+d-1} < f, \Phi_j/q >_\mu \ \Phi_j(z)\,,$$

where $A(z)$ is a polynomial of degree $n+d-1$ given by

$$A(z) = \frac{1}{2i\pi} \int_{\mathbb{T}} \frac{\check{f}}{q}(\xi) \frac{\widetilde{\Phi_{n+d}}(\xi)\widetilde{\Phi_{n+d}}(z) - \Phi_{n+d}(\xi)\Phi_{n+d}(z)}{1-\xi z} \, |\nu(\xi)|^2 \, d\xi\,,$$

or else, in view of (20) and (25),

$$A(z) = \frac{1}{2i\pi} \int_{\mathbb{T}} \frac{\check{f}_w}{qw}(\xi) \frac{\widetilde{qw}(\xi)\widetilde{qw}(z) - qw(\xi)qw(z)}{1-\xi z} \, d\xi\,.$$

In view of proposition 4, we recognise in this integral expression for $A$ the best numerator associated to the denominator $\Phi_{n+d} = qw$ in the unweighted approximation of $f_w$ at degree $n+d$. Using the integral representations of quotient and remainder in the division of analytic functions [18], it can also be interpreted as follows [7, prop.2.3.]:

$$\tilde{A}(z) = \mathcal{R}_{qw}(\check{f}_w\, \widetilde{qw}) = \mathcal{R}_{qw}(\check{f}(z)\tilde{q}(z)z^d),$$

where $\mathcal{R}_{qw}(.)$ denotes the remainder in the division of some analytic function by $qw$. Finally $L_n(q)$ can be writen:

$$L_n(q) = \widetilde{\mathcal{R}_{qw}}(\check{f}(z)\tilde{q}(z)z^d) - \sum_{j=n}^{n+d-1} < f, \Phi_j/q >_\mu \ \Phi_j \,. \tag{28}$$

2) Using divided differences.

We come back to the early definition of $L_n(q)$; it satisfies the linear system:

$$< f - \frac{L_n(q)}{q}, \frac{z^i}{q} >_\mu = 0 \ , \ i = 0, \cdots, n-1 \,,$$

which can be written using (25)

$$\frac{1}{2i\pi} \int_{\mathbb{T}} G(\xi) \ \frac{\xi^i}{q(\xi)w(\xi)} d\,\xi = 0 \ , \ i = 0, \cdots, n-1 \,, \tag{29}$$

where $G$ is given by

$$G(z) \ = \ \left(\check{f} - \frac{\widetilde{L_n(q)}}{\tilde{q}}\right)(z) \ \frac{z^d}{\tilde{w}(z)}.$$

Let $z_1, z_2, \ldots, z_{n+d}$ be the roots of $q\,w$, and write down the division in $H^2$ of $G$ by $q\,w$ as:

$$G \ = \ B\; q\, w \ + \ R. \tag{30}$$

The Newton formula [9, 2.6.6] asserts that:

$$R(z) \ = \ \sum_{k=1}^{n+d} [G(z_1) \ G(z_2) \ldots G(z_k)] \ (z - z_1)(z - z_2) \ldots (z - z_{k-1}),$$

where $[G(z_1)\, G(z_2) \dots G(z_k)]$ denotes the divided difference of $G(z_1), G(z_2), \dots, G(z_k)$ with respect to the roots $z_1, \dots, z_k$ of $q\,w$; it is given by (see [9, 3.6.3]):

$$[G(z_1)\; G(z_2) \dots G(z_k)] \;=\; \frac{1}{2i\pi} \int_{\mathbb{T}} \frac{G(\xi)}{(\xi - z_1)(\xi - z_2) \dots (\xi - z_k)} d\,\xi.$$

Then, (29) implies that $[G(z_1)\; G(z_2) \dots G(z_k)] = 0$, for $k \geq d+1$; hence, *$R$ is a polynomial of degree $d$.*

Now, (30) yields

$$\check{f}(z)\tilde{q}(z)z^d \;=\; B(z)\,\tilde{q}(z)\tilde{w}(z)\,q(z)w(z) \;+\; R(z)\,\tilde{q}(z)\tilde{w}(z) \;+\; z^d\,\widetilde{L_n(q)}(z),$$

or else

$$\mathcal{R}_{qw}(\check{f}(z)\tilde{q}(z)z^d) \;=\; \mathcal{R}_{qw}(R\,\tilde{q}\tilde{w}) \;+\; z^d\,\widetilde{L_n(q)}, \tag{31}$$

where $R$ is a polynomial of degree $d$. Note that (31) can be written as a linear system of $n+d$ equations in the coefficients of $R$ and $\widetilde{L_n(q)}$ (that is $n+d$ unknowns) and thus allows to compute $L_n(q)$.

Of course, formulas (28) and (31) are deeply linked. Indeed, it can be easily established that

$$\widetilde{\mathcal{R}_{qw}}(R\tilde{q}\tilde{w}) \;=\; \sum_{j=n}^{n+d-1} < f, \frac{\Phi_j}{q} >_\mu \Phi_j.$$

# 6 Conclusion

For the family of measures considered in this work, the solution of the weighted bounded extremal problem $(P_1)$ can be deduced from the solution of the unweighted one by using an explicit change of variable. Concerning the weighted rational approximation issue $(P_2)$, orthogonal polynomials on $\mathbb{T}$ for $d\mu/|q|^2$ are used to express the best numerator in the criterion (16) and to establish its smoothness property. The natural idea to appeal to a basis of orthogonal polynomials on $\mathbb{T}$ has been used in [8]. In this note, an unweighted rational approximation problem is studied, which can be expressed as $(P_2)$ for $\mu = \lambda$ with the additional and difficult constraint that the degree of the numerator should be less or equal to some fixed $m < n$. Excepted for $m = n-1$, this constraint prevents from smoothly extending the criterion. This relies on the fact that, if $q$ has more than $m$ roots on $\mathbb{T}$, then $\phi_m$ does not extend smoothly to $\overline{\Delta}_n$. It would be interesting to study possible generalizations of our present work and of results in [8] to an analogous $(m, n)$ weighted rational approximation problem.

Further theoretical questions could also be considered, such as the consistency problem: if $f$ is already rational of degree $n$, is it the *single* critical point of the problem? Once again, the answer does not come straightfully as in the unweighted scalar case and depends on the measure $\mu$. This is a relevant question when studying identification schemes, which is classically handled in a stochastic framework, see [13]. For arbitrary $f$ in $\bar{H}_0^2$, the criterion $\Psi_n$ generally has several local minima and despite our algorithm will converge to one of these, we cannot get sure to find them all. This is an additional motivation for introducing a weight in the rational approximation problem $(P_2)$ since it allows to consider the following uniqueness issue: given $f$ in $\bar{H}_0^2$, is it possible to find a measure $\mu$ which ensures uniqueness of the critical points of $\Psi_n$?

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