

# The Mortality of a Pair of 2x2 Matrices is Decidable

Yannick Saouter

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*The mortality of a pair of  $2 \times 2$  matrices is  
decidable*

Yannick Saouter

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## The mortality of a pair of $2 \times 2$ matrices is decidable

Yannick Saouter<sup>†</sup>

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**Abstract:** A pair of matrices is said to be mortal if there is a serie of these matrices for which the product is the null matrix. A recent result have established that the general problem of the mortality of a pair of integral matrices is undecidable. In this article, we prove by using only linear algebra that the mortality of a pair of  $2 \times 2$  integral matrices is decidable.

**Key-words:** Mortality; Decidability; Linear algebra

*(Résumé : *tsvp*)*

<sup>†</sup> email: [saouter@irisa.fr](mailto:saouter@irisa.fr)

Unité de recherche INRIA Rennes  
IRISA, Campus universitaire de Beaulieu, 35042 RENNES Cedex (France)  
Téléphone : (33) 99 84 71 00 – Télécopie : (33) 99 84 71 71

## La mortalité d'une paire de matrices $2 \times 2$ est décidable

**Résumé :** Une paire de matrices est dite mortelle si il existe une suite de ces deux matrices dont le produit soit la matrice nulle. Un résultat récent montre que le problème général de la mortalité d'une paire de matrices entières est indécidable. Dans cet article, nous prouvons en utilisant seulement l'algèbre linéaire que la mortalité d'une paire de matrices entières  $2 \times 2$  est décidable.

**Mots-clé :** Mortalité; Décidabilité; Algèbre linéaire

## 1 Introduction

In the article [1], the authors recently investigate the problem of the mortality of a pair of matrix. They obtain essentially two results: (i) the mortality of a pair of integral  $33 \times 33$ -matrix is undecidable in general, (ii) the mortality of a pair of integral positive matrix can be solved in exponential time. In the article [3], the author consider as for him the general problem of  $2 \times 2$  matrices and exhibit a criterion for a set of such matrices to be mortal. Unfortunately, this criterion is a theoretic response to this problem and does not give an algorithm. In this article we focus on the problem of the mortality of a single pair of integral  $2 \times 2$  matrices and we prove that this problem is decidable.

## 2 Definitions and theorems

At first, let us define the mortality.

**Definition 2.1** (mortality) *Let  $\mathcal{R}$  be a commutative ring and  $A_1, A_2, \dots, A_m$  a finite set of non-null matrices with size  $n \times n$  and with coefficients in  $\mathcal{R}$ . This set is called mortal if there exists two finite sets of numbers  $i_l$  and  $k_l$ , with  $1 \leq l \leq M$  such that:*

$$A_{i_1}^{k_1} \cdot A_{i_2}^{k_2} \cdot \dots \cdot A_{i_M}^{k_M} = \mathcal{O}$$

where  $\mathcal{O}$  designates the null matrix.

It may be noted that if a set of matrices is mortal and if the ring do not contain divisors of zero, then necessarily at least one of the matrices of the set have to be singular (i.e. with rank not maximal). In the following study, since we limit ourselves to the set  $\mathcal{Z}$  of natural integers, it will be the case.

A theorem will be of great use in the following:

**Theorem 2.1** (Cayley-Hamilton theorem) *Let  $M$  be a matrix with coefficients over a ring  $\mathcal{R}$ . Its characteristic polynomial  $C_M$ , belongs to the set  $\mathcal{R}[X]$  of polynomials with coefficients in  $\mathcal{R}$  and is defined by  $C_M = \det(M - X.I)$ , where  $\det$  is the determinant,  $X$  is the formal unknown variable and  $I$  is the identity matrix of the same size than  $M$ . Let  $f_M$  be the morphism of rings defined by  $f_M(1_{\mathcal{R}}) = I$  and  $f_M(X) = M$  where  $1_{\mathcal{R}}$  is the unity of the ring  $\mathcal{R}$ . We have then  $f_M(C_M) = \mathcal{O}$ .*

*Moreover if  $M$  is a  $2 \times 2$ -matrix, then  $C_M = X^2 - \text{tr}(M).X + \det(M)$  and so  $M^2 = \text{tr}(M).M - \det(M).I$  where  $\text{tr}$  is the linear application trace.*

We will also use properties of diagonalisation and trigonalisation. In order to lighten the theorems we only exposed them for  $2 \times 2$ -matrices:

**Theorem 2.2** *Let  $M$  be a  $2 \times 2$ -matrix over. A matrix  $N$  is said to be semblable to  $M$  if there exists an inversible matrix  $P$  such that  $M = P^{-1}NP$ . There is then two cases:*

- *if the characteristic polynomial  $X^2 - \text{tr}(M).X + \det(M)$  has two distinct roots  $\alpha_1$  and  $\alpha_2$ , then the matrix  $M$  is semblable to the diagonal matrix  $\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$ . The matrix  $M$  is then said to be diagonalisable.*
- *if the characteristic polynomial  $X^2 - \text{tr}(M).X + \det(M)$  has a double root  $\alpha$ , then the matrix  $M$  is semblable to one upper triangular matrix of the form  $\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$ . The matrix  $M$  is then said to be trigonalisable if  $\beta \neq 0$  and again diagonalisable on the case of the contrary.*

Moreover in both cases, the matrix  $P$  of change of basis is known.

This easy theorem generalize into the Jordan theorem [2].

### 3 Mortality of a pair of $2 \times 2$ -matrices

In this case, we have only two matrices  $A_1$  and  $A_2$  that we can multiply as many as desired in order to obtain the null matrix. As we have said previously, at least one of these two matrices have to be singular. Since our matrices are supposed to be non-null, this implies that either  $A_1$  or  $A_2$  has a rank equal to 1. In what follows we suppose that at least  $A_1$  is of rank 1. There is then three different cases.

**Case when  $tr(A_1) = 0$**  Since  $A_1$  is a singular matrix, then  $det(A_1) = 0$ . In virtue of the Cayley-Hamilton theorem, we have then  $A_1^2 = 0$  and thus the pair of matrices is mortal.

**Case when both  $A_1$  and  $A_2$  are singular** Here in virtue of the preceding case, we can suppose that  $tr(A_1) \neq 0$  and  $tr(A_2) \neq 0$ . Let us suppose that we have two sets of numbers  $i_l$  with values in  $\{1, 2\}$  and  $k_l$  with strictly positive integral values such that:

$$A_{i_1}^{k_1} \cdot A_{i_2}^{k_2} \cdot \dots \cdot A_{i_M}^{k_M} = \mathcal{O}$$

In this case we have  $det(A_1) = det(A_2) = 0$  since both matrices are singular, thus the Cayley-Hamilton theorem gives use the equations  $A_1^2 = tr(A_1) \cdot A_1$  and  $A_2^2 = tr(A_2) \cdot A_2$ . We have then  $A_{i_l}^{k_l} = tr(A_{i_l})^{k_l \text{ div } 2} \cdot A_{i_l}^{k_l \% 2}$  where  $div$  is the integer quotient and  $\%$  is the integer remainder. Since the indices  $i_l$  can be supposed successively alternating (in the case of not you can group successive terms into a single one), the mortality of a pair of matrices, in this case is equivalent to a system of either the form:

$$A_1 \cdot A_2 \cdot \dots \cdot A_{i_M} = \mathcal{O}$$

or:

$$A_2 \cdot A_1 \cdot \dots \cdot A_{i_M} = \mathcal{O}$$

Now just consider the first case. If  $i_M = 2$ , this equation means that the matrix  $(A_1 \cdot A_2)$  is nilpotent. On the contrary, if  $i_M = 1$ , you may add a  $A_2$  at the end of the left-hand side of the equation and again the matrix  $(A_1 \cdot A_2)$  reveals to be nilpotent.

For the second case, you add a  $A_1$  term at the beginning and using the same argument as just above you deduce again that  $(A_1 \cdot A_2)$  is nilpotent. So in this case if the pair of matrix is mortal you can deduce that  $(A_1 \cdot A_2)$  is nilpotent. Reciprocally if  $(A_1 \cdot A_2)$  is nilpotent then the pair of matrix is mortal.

But if a  $2 \times 2$ -matrix  $M$  is nilpotent then necessarily we have  $M^2 = \mathcal{O}$ . We may use the variant of Jordan theorem to see this. As we have seen, the matrix  $M$  is semblable to a matrix  $N$  of the form  $\begin{pmatrix} \alpha_1 & \beta \\ 0 & \alpha_2 \end{pmatrix}$ , with here  $\alpha_1$  and  $\alpha_2$  eventually equal. Since  $M = P^{-1} \cdot N \cdot P$  for some inversible matrix  $P$  then  $M^k = P^{-1} \cdot N^k \cdot P$  and thus  $M$  is  $k$ -nilpotent if and only if  $N$  is also  $k$ -nilpotent. But by recurrence, we have that  $N^k$  is of the form  $\begin{pmatrix} \alpha_1^k & \beta_k \\ 0 & \alpha_2^k \end{pmatrix}$  for some  $\beta_k$ . Thus a necessary condition for  $N$  to be  $k$ -nilpotent is that  $\alpha_1 = \alpha_2 = 0$ . We have then that  $N$  is equal to  $\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$ , and thus  $N^2 = \mathcal{O}$  and so  $M^2 = \mathcal{O}$ .

So in this case the pair of matrix  $A_1$  and  $A_2$  is mortal if and only if  $(A_1 \cdot A_2)^2 = \mathcal{O}$ .

**Case when  $A_2$  is non singular** As previously, let us consider the following system:

$$A_{i_1}^{k_1} . A_{i_2}^{k_2} . \dots . A_{i_M}^{k_M} = \mathcal{O}$$

We may still consider that the  $k_i$  values for which  $i_i = 1$  are equal to 1. Moreover without limitation of the generalization, we can consider that  $i_M = 1$ . Indeed, if  $i_M = 2$  since  $A_2$  is inversible, we can multiply both side by  $A_2^{-k_M}$  and thus obtain the equation:

$$A_{i_1}^{k_1} . A_{i_2}^{k_2} . \dots . A_{i_{M-1}}^{k_{M-1}} = \mathcal{O}$$

We remind here that a matrix  $M$  is null if and only if for all vector  $\vec{v}$ , we have  $M\vec{v} = \vec{0}$ . Let us apply this criterion to our system. We have then for all vector  $\vec{v}$ :

$$A_{i_1}^{k_1} . A_{i_2}^{k_2} . \dots . A_1 \vec{v} = \vec{0}$$

We have then necessarily  $M \geq 2$ . Indeed if  $M = 1$  this system implies that  $A_1 = \mathcal{O}$  which is excluded. So our system rewrites as:

$$A_{i_1}^{k_1} . A_{i_2}^{k_2} . \dots . A_2^{k_{M-1}} . A_1 \vec{v} = \vec{0}$$

Again since  $A_2$  is inversible, if we suppose that  $M = 2$  we are led again to  $A_1 = \mathcal{O}$ . Thus we have  $M \geq 3$  and the system is:

$$A_{i_1}^{k_1} . A_{i_2}^{k_2} . \dots . A_1 . A_2^{k_{M-1}} . A_1 \vec{v} = \vec{0}$$

But now without loss of generality, we can consider that in fact  $M = 3$ . Indeed the space image of  $A_1$  is of dimension 1, i.e. it is a line sustained by a vector, say  $\vec{w}$ . As a consequence the image of a subspace by  $A_1$  is either the empty space or the entire space image of  $A_1$ . So either the space image of  $A_1 . A_2^{k_{M-1}} . A_1$  is the empty space and the matrix  $A_1 . A_2^{k_{M-1}} . A_1$  is equal to  $\mathcal{O}$ , or it is the space image of  $A_1$ . But in this latter case, this space is also generated simply by  $A_1$  and thus you have necessarily:

$$A_{i_1}^{k_1} . A_{i_2}^{k_2} . \dots . A_1 . A_2^{k_{M-3}} . A_1 \vec{v} = \vec{0}$$

Thus by recurrence you can easily deduce that we can limit us to the case where  $M = 3$  and so the pair of matrices  $A_1$  and  $A_2$  is mortal if and only if there exist an integer  $k$  such that:

$$A_1 . A_2^k . A_1 = \mathcal{O}$$

In order to solve this problem, we will trigonalize the matrix  $A_2$ . Let then be  $P$  a non singular matrix and  $T_2$  a trigonal matrix such that  $A_2 = P^{-1} . T_2 . P$ . We have then:

$$A_1 . P^{-1} . T_2^k . P . A_1 = \mathcal{O}$$

Both matrices  $A_1 . P^{-1}$  and  $P . A_1$  have a rank equal to one. Let respectively be  $\vec{u}$  and  $\vec{v}$  the directions of the kernel of  $A_1 . P^{-1}$  and of the image of  $P . A_1$ . The latter equation is then equivalent to the existence of a value  $k$  such that  $T_2^k(\vec{v})$  is colinear to  $\vec{u}$ . We will pose  $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$  and  $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ .

Let us suppose at first that  $T_2$  is diagonal with  $T_2 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$ . We have then  $T_2^k(\vec{v})$  colinear to  $\vec{u}$  if and only if the determinant  $\det(T_2^k(\vec{v}), \vec{u})$  is null i.e.:

$$\begin{aligned} \begin{vmatrix} \alpha_1^k . v_1 & u_1 \\ \alpha_2^k . v_2 & u_2 \end{vmatrix} &= 0 \\ \iff \\ \alpha_1^k . v_1 . u_2 - \alpha_2^k . v_2 . u_1 &= 0 \end{aligned}$$



If  $\alpha_1 = \alpha_2$ , then the condition is just  $v_1.u_2 - v_2.u_1 = 0$ . Thus the value of  $k$  has no importance and so in this case, the pair of matrix is mortal if and only if  $A_1.A_2.A_1 = \mathcal{O}$ .

Now suppose that  $\alpha_1 \neq \alpha_2$ . Since the matrix  $A_2$  is non singular, both these values cannot be null. Suppose now that either  $v_1$  or  $u_2$  is null, then to satisfy the condition necessarily we have either  $v_2$  or  $u_1$  null. Again if it is the case, the value of  $k$  is indifferent and the pair of matrix is mortal if and only if  $A_1.A_2.A_1 = \mathcal{O}$ . Now we suppose that none of the values  $u_1, u_2, v_1$  and  $v_2$  is null. Then the condition is equivalent to:

$$\begin{aligned} k.\log(\alpha_1) + \log(v_1.u_2) &= k.\log(\alpha_2) + \log(v_2.u_1) \\ &\iff \\ k &= \frac{\log(v_2.u_1) - \log(v_1.u_2)}{\log(\alpha_1) - \log(\alpha_2)} \end{aligned}$$

Thus if we have an estimation of the righthand side precise enough (say at an accuracy of less than  $1/2$ ), then a single value of  $k$  is possible and thus it is possible to test the mortality.

Now we suppose that  $T_2$  is diagonal of the form  $T_2 = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$  with  $\beta \neq 0$ . Again since  $T_2$  is not singular then  $\alpha \neq 0$ . Again we have to decide whether there exists an index  $k$  for which  $T_2^k(\vec{v})$  is colinear to  $\vec{u}$ . In this case we prove easily by recurrence that  $T_2^k = \begin{pmatrix} \alpha^k & k\beta\alpha^{k-1} \\ 0 & \alpha^k \end{pmatrix}$  for all  $k \geq 1$ . The condition  $\det(T_2^k(\vec{v}), \vec{u})$  rewrites then in:

$$\begin{aligned} \begin{vmatrix} \alpha^k v_1 + k\beta\alpha^{k-1} v_2 & u_1 \\ \alpha^k v_2 & u_2 \end{vmatrix} &= 0 \\ &\iff \\ \alpha^k v_1.u_2 + k\beta\alpha^{k-1} v_2.u_2 - \alpha^k v_2.u_1 &= 0 \\ \alpha(v_1.u_2 - v_2.u_1) + k\beta v_2.u_2 &= 0 \end{aligned}$$

At this point if  $v_2.u_2 = 0$ , then the condition is met if and only if  $v_1.u_2 - v_2.u_1 = 0$ . In this case the value of  $k$  is again indifferent and thus the pair of matrices is mortal if and only if  $A_1.A_2.A_1 = \mathcal{O}$ . Now if  $v_2.u_2 \neq 0$ , we have:

$$\begin{aligned} \alpha(v_1.u_2 - v_2.u_1) + k\beta v_2.u_2 &= 0 \\ &\iff \\ k &= \frac{\alpha(v_2.u_1 - v_1.u_2)}{\beta v_2.u_2} \end{aligned}$$

Again, if we obtain an approximation accurate enough of the right-hand side we obtain a candidate value for  $k$  that may be verified on the initial matrices  $A_1$  and  $A_2$ .

But every values which arise in the previous expression are real values but with computable decimal expansions. As a consequence they can be computed as accurate as desired as well as any expression involving them with elementary functions. This latter remark finishes the proof.

## Conclusion

In this article we have proven that we can decide of the mortality of a pair of  $2 \times 2$ -matrix. The major characteristic of this demonstration is that it only requires linear algebra. As a consequence it generalizes to other commutative rings (gaussian integers e.g.). This problem poses also the problem of the exact size of matrices from which the problem becomes undecidable. This problem is certainly a hard open problem. The reader will also note that the general problem of reachability for  $2 \times 2$  matrices is still undecided.

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Unité de recherche INRIA Lorraine, Technopôle de Nancy-Brabois, Campus scientifique,  
615 rue du Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY  
Unité de recherche INRIA Rennes, Irisa, Campus universitaire de Beaulieu, 35042 RENNES Cedex  
Unité de recherche INRIA Rhône-Alpes, 46 avenue Félix Viallet, 38031 GRENOBLE Cedex 1  
Unité de recherche INRIA Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex  
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