The Mortality of a Pair of 2x2 Matrices is Decidable

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Abstract: A pair of matrices is said to be mortal if there is a serie of these matrices for which the product is the null matrix. A recent result have established that the general problem of the mortality of a pair of integral matrices is undecidable. In this article, we prove by using only linear algebra that the mortality of a pair of 2×2 integral matrices is decidable.

Key-words: Mortality; Decidability; Linear algebra

(Résumé : tsvp)

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La mortalité d'une paire de matrices 2×2 est décidable

Résumé : Une paire de matrices est dite mortelle si il existe une suite de ces deux matrices dont le produit soit la matrice nulle. Un résultat récent montre que le problème général de la mortalité d'une paire de matrices entières est indécidable. Dans cet article, nous prouvons en utilisant seulement l'algèbre linéaire que la mortalité d'une paire de matrices entières 2×2 est décidable.

Mots-clé : Mortalité; Décidabilité; Algèbre linéaire

1 Introduction

In the article [1], the authors recently investigates the problem of the mortality of a pair of matrix. They obtain essentially two results: (i) the mortality of a pair of integral 33×33-matrix is undecidable in general, (ii) the mortality of a pair of integral positive matrix can be solved in exponential time. In the article [3], the author consider as for him the general problem of 2×2 matrices and exhibit a criterion for a set of such matrices to be mortal. Unfortunately, this criterion is a theoretic response to this problem and does not give an algorithm. In this article we focus on the problem of the mortality of a single pair of integral 2×2 matrices and we prove that this problem is decidable.

2 Definitions and theorems

At first, let us define the mortality.

Definition 2.1 (mortality) *Let $\mathcal{R}$ be a commutative ring and $A_1, A_2, ..., A_m$ a finite set of non-null matrices with size $n \times n$ and with coefficients in $\mathcal{R}$. This set is called* mortal *if there exists two finite sets of numbers i_l and k_l, with $1 \leq l \leq M$ such that:*

$$A_{i_1}^{k_1}.A_{i_2}^{k_2}. \ ... \ A_{i_M}^{k_M} = \mathcal{O}$$

where $\mathcal{O}$ designates the null matrix.

It may be noted that if a set of matrices is mortal and if the ring do not contain divisors of zero, then necessarily at least one of the matrices of the set have to be singular (i.e. with rank not maximal). In the following study, since we limit ourselves to the set $\mathcal{Z}$ of natural integers, it will be the case.

A theorem will be of great use in the following:

Theorem 2.1 (Cayley-Hamilton theorem) *Let M be a matrix with coefficients over a ring $\mathcal{R}$. Its characteristic polynomial C_M, belongs to the set $\mathcal{R}[X]$ of polynomials with coefficients in $\mathcal{R}$ and is defined by $C_M = det(M - X.I)$, where det is the determinant, X is the formal unknown variable and I is the identity matrix of the same size than M. Let f_M be the morphism of rings defined by $f_M(1_\mathcal{R}) = I$ and $f_M(X) = M$ where $1_\mathcal{R}$ is the unity of the ring $\mathcal{R}$. We have then $f_M(C_M) = \mathcal{O}$.*

Moreover if M is a 2×2-matrix, then $C_M = X^2 - tr(M).X + det(M)$ and so $M^2 = tr(M).M - det(M).I$ where tr is the linear application trace.

We will also use properties of diagonalisation and trigonalisation. In order to lighten the theorems we only exposed them for 2×2-matrices:

Theorem 2.2 *Let M be a 2×2-matrix over. A matrix N is said to be* semblable *to M if there exists an inversible matrix P such that $M = P^{-1}NP$. There is then two cases:*

- *if the characteristic polynomial $X^2 - tr(M).X + det(M)$ has two distinct roots α_1 and α_2, then the matrix M is semblable to the diagonal matrix $\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$. The matrix M is then said to be* diagonalisable.
- *if the characteristic polynomial $X^2 - tr(M).X + det(M)$ has a double root α, then the matrix M is semblable to one upper triangular matrix of the form $\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$. The matrix M is then said to be* trigonalisable *if $\beta \neq 0$ and again* diagonalisable *on the case of the contrary.*

Moreover in both cases, the matrix P of change of basis is known.

This easy theorem generalize into the Jordan theorem [2].

3 Mortality of a pair of 2×2-matrices

In this case, we have only two matrices A_1 and A_2 that we can multiply as many as desired in order to obtain the null matrix. As we have said previously, at least one of these two matrices have to be singular. Since our matrices are supposed to be non-null, this implies that either A_1 or A_2 has a rank equal to 1. In what follows we suppose that at least A_1 is of rank 1. There is then three different cases.

Case when $tr(A_1) = 0$ Since A_1 is a singular matrix, then $det(A_1) = 0$. In virtue of the Cayley-Hamilton theorem, we have then $A_1^2 = 0$ and thus the pair of matrices is mortal.

Case when both A_1 and A_2 are singular Here in virtue of the preceding case, we can suppose that $tr(A_1) \neq 0$ and $tr(A_2) \neq 0$. Let us suppose that we have two sets of numbers i_l with values in $\{1, 2\}$ and k_l with strictly positive integral values such that:

$$A_{i_1}^{k_1}.A_{i_2}^{k_2}. \dots A_{i_M}^{k_M} = \mathcal{O}$$

In this case we have $det(A_1) = det(A_2) = 0$ since both matrices are singular, thus the Cayley-Hamilton theorem gives use the equations $A_1^2 = tr(A_1).A_1$ and $A_2^2 = tr(A_2).A_2$. We have then $A_{i_l}^{k_l} = tr(A_{i_l})^{k_l \ div \ 2}.A_{i_l}^{k_l \% 2}$ where div is the integer quotient and % is the integer remainder. Since the indices i_l can be supposed successively alterning (in the case of not you can group successive terms into a single one), the mortality of a pair of matrices, in this case is equivalent to a system of either the form:

$$A_1.A_2. \dots A_{i_M} = \mathcal{O}$$

or:

$$A_2.A_1. \dots A_{i_M} = \mathcal{O}$$

Now just consider the first case. If $i_M = 2$, this equation means that the matrix $(A_1.A_2)$ is nilpotent. On the contrary, if $i_M = 1$, you may add a A_2 at the end of the left-hand side of the equation and again the matrix $(A_1.A_2)$ reveals to be nilpotent.

For the second case, you add a A_1 term at the beginning and using the same argument as just above you deduce again that $(A_1.A_2)$ is nilpotent. So in this case if the pair of matrix is mortal you can deduce that $(A_1.A_2)$ is nilpotent. Reciprocally if $(A_1.A_2)$ is nilpotent then the pair of matrix is mortal.

But if a 2×2-matrix M is nilpotent then necessarily we have $M^2 = \mathcal{O}$. We may use the variant of Jordan theorem to see this. As we have seen, the matrix M is semblable to a matrix N of the form $\begin{pmatrix} \alpha_1 & \beta \\ 0 & \alpha_2 \end{pmatrix}$, with here α_1 and α_2 eventually equal. Since $M = P^{-1}.N.P$ for some inversible matrix P then $M^k = P^{-1}.N^k.P$ and thus M is k-nilpotent if and only if N is also k-nilpotent. But by recurrence, we have that N^k is of the form $\begin{pmatrix} \alpha_1^k & \beta_k \\ 0 & \alpha_2^k \end{pmatrix}$ for some β_k. Thus a necessary condition for N to be k-nilpotent is that $\alpha_1 = \alpha_2 = 0$. We have then that N is equal to $\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$, and thus $N^2 = \mathcal{O}$ and so $M^2 = \mathcal{O}$.

So in this case the pair of matrix A_1 and A_2 is mortal if and only if $(A_1.A_2)^2 = \mathcal{O}$.

Case when A_2 is non singular As previously, let us consider the following system:

$$A_{i_1}^{k_1}.A_{i_2}^{k_2}.\ ...\ A_{i_M}^{k_M} \quad = \quad \mathcal{O}$$

We may still consider that the k_l values for which $i_l = 1$ are equal to 1. Moreover without limitation of the generalization, we can consider that $i_M = 1$. Indeed, if $i_M = 2$ since A_2 is inversible, we can multiply both side by $A_2^{-k_M}$ and thus obtain the equation:

$$A_{i_1}^{k_1}.A_{i_2}^{k_2}.\ ...\ A_{i_{M-1}}^{k_{M-1}} \quad = \quad \mathcal{O}$$

We remind here that a matrix M is null if and only if for all vector $\vec{v}$, we have $M\vec{v} = \vec{0}$. Let us apply this criterion to our system. We have then for all vector $\vec{v}$:

$$A_{i_1}^{k_1}.A_{i_2}^{k_2}.\ ...\ A_1\vec{v} \quad = \quad \vec{0}$$

We have then necessarily $M \geq 2$. Indeed if $M = 1$ this system implies that $A_1 = \mathcal{O}$ which is excluded. So our system rewrites as:

$$A_{i_1}^{k_1}.A_{i_2}^{k_2}.\ ...\ .A_2^{k_{M-1}}.A_1\vec{v} \quad = \quad \vec{0}$$

Again since A_2 is inversible, if we suppose that $M = 2$ we are led again to $A_1 = \mathcal{O}$. Thus we have $M \geq 3$ and the system is:

$$A_{i_1}^{k_1}.A_{i_2}^{k_2}.\ ...\ .A_1.A_2^{k_{M-1}}.A_1\vec{v} \quad = \quad \vec{0}$$

But now without loss of generality, we can consider that in fact $M = 3$. Indeed the space image of A_1 is of dimension 1, i.e. it is a line sustained by a vector, say $\vec{w}$. As a consequence the image of a subspace by A_1 is either the empty space or the entire space image of A_1. So either the space image of $A_1.A_2^{k_{M-1}}.A_1$ is the empty space and the matrix $A_1.A_2^{k_{M-1}}.A_1$ is equal to $\mathcal{O}$, or it is the space image of A_1. But in this latter case, this space is also generated simply by A_1 and thus you have necessarily:

$$A_{i_1}^{k_1}.A_{i_2}^{k_2}.\ ...\ .A_1.A_2^{k_{M-3}}.A_1\vec{v} \quad = \quad \vec{0}$$

Thus by recurrence you can easily deduce that we can limit us to the case where $M = 3$ and so the pair of matrices A_1 and A_2 is mortal if and only if there exist an integer k such that:

$$A_1.A_2^k.A_1 \quad = \quad \mathcal{O}$$

In order to solve this problem, we will trigonalize the matrix A_2. Let then be P a non singular matrix and T_2 a trigonal matrix such that $A_2 = P^{-1}.T_2.P$. We have then:

$$A_1.P^{-1}.T_2^k.P.A_1 \quad = \quad \mathcal{O}$$

Both matrices $A_1.P^{-1}$ and $P.A_1$ have a rank equal to one. Let respectively be $\vec{u}$ and $\vec{v}$ the directions of the kernel of $A_1.P^{-1}$ and of the image of $P.A_1$. The latter equation is then equivalent to the existence of a value k such that $T_2^k(\vec{v})$ is colinear to $\vec{u}$. We will pose $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$.

Let us suppose at first that T_2 is diagonal with $T_2 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$. We have then $T_2^k(\vec{v})$ colinear to $\vec{u}$ if and only if the determinant $det(T_2^k(\vec{v}), \vec{u})$ is null i.e.:

$$\begin{aligned} \begin{vmatrix} \alpha_1^k.v_1 & u_1 \\ \alpha_2^k.v_2 & u_2 \end{vmatrix} &= 0 \\ &\Longleftrightarrow \\ \alpha_1^k.v_1.u_2 - \alpha_2^k.v_2.u_1 &= 0 \end{aligned}$$

If $\alpha_1 = \alpha_2$, then the condition is just $v_1.u_2 - v_2.u_1 = 0$. Thus the value of k has no importance and so in this case, the pair of matrix is mortal if and only if $A_1.A_2.A_1 = \mathcal{O}$.

Now suppose that $\alpha_1 \neq \alpha_2$. Since the matrix A_2 is non singular, both these values cannot be null. Suppose now that either v_1 or u_2 is null, then to satisfy the condition necessarily we have either v_2 or u_1 null. Again if it is the case, the value of k is indifferent and the pair of matrix is mortal if and only if $A_1.A_2.A_1 = \mathcal{O}$. Now we suppose that none of the values u_1, u_2, v_1 and v_2 is null. Then the condition is equivalent to:

$$\begin{array}{rcl} k.log(\alpha_1) + log(v_1.u_2) & = & k.log(\alpha_2) + log(v_2.u_1) \\ & \Longleftrightarrow & \\ k & = & \dfrac{log(v_2.u_1) - log(v_1.u_2)}{log(\alpha_1) - log(\alpha_2)} \end{array}$$

Thus if we have an estimation of the righthand side precise enough (say at an accuracy of less than $1/2$), then a single value of k is possible and thus it is possible to test the mortality.

Now we suppose that T_2 is diagonal of the form $T_2 = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$ with $\beta \neq 0$. Again since T_2 is not singular then $\alpha \neq 0$. Again we have to decide whether there exists an index k for which $T_2^k(\vec{v})$ is colinear to $\vec{u}$. In this case we prove easily by recurrence that $T_2^k = \begin{pmatrix} \alpha^k & k\beta\alpha^{k-1} \\ 0 & \alpha^k \end{pmatrix}$ for all $k \geq 1$. The condition $det(T_2^k(\vec{v}), \vec{u})$ rewrites then in:

$$\begin{array}{rcl} \begin{vmatrix} \alpha^k v_1 + k\beta\alpha^{k-1} v_2 & u_1 \\ \alpha^k v_2 & u_2 \end{vmatrix} & = & 0 \\ & \Longleftrightarrow & \\ \alpha^k v_1.u_2 + k\beta\alpha^{k-1} v_2.u_2 - \alpha^k v_2.u_1 & = & 0 \\ \alpha(v_1.u_2 - v_2.u_1) + k\beta v_2.u_2 & = & 0 \end{array}$$

At this point if $v_2.u_2 = 0$, then the condition is met if and only if $v_1.u_2 - v_2.u_1 = 0$. In this case the value of k is again indifferent and thus the pair of matrices is mortal if and only if $A_1.A_2.A_1 = \mathcal{O}$. Now if $v_2.u_2 \neq 0$, we have:

$$\begin{array}{rcl} \alpha(v_1.u_2 - v_2.u_1) + k\beta v_2.u_2 & = & 0 \\ & \Longleftrightarrow & \\ k & = & \dfrac{\alpha(v_2.u_1 - v_1.u_2)}{\beta v_2.u_2} \end{array}$$

Again, if we obtain an approximation acurate enough of the right-hand side we obtain a candidate value for k that may be verified on the initial matrices A_1 and A_2.

But every values which arise in the previous expression are real values but with computable decimal expansions. As a consequence they can be computed as acurate as desired as well as any expression involving them with elementary functions. This latter remark finishes the proof.

Conclusion

In this article we have proven that we can decide of the mortality of a pair of 2×2-matrix. The major characteristic of this demonstration is that it only requires linear algebra. As a consequence it generalizes to other commutative rings (gaussian integers e.g.). This problem poses also the problem of the exact size of matrices from which the problem becomes undecidable. This problem is certainly a hard open problem. The reader will also note that the general problem of reachability for 2×2 matrices is still undecided.

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