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Abstract: Various stochastic models are proposed for transportation networks, by increasing order of complexity. The mathematical tools rely on queueing theory and asymptotic analysis. The models are mainly applied to service vehicle systems, like PRAXITÈLE, but the methods used are far more reaching. In particular, asymptotic independence of nodes is proved in thermodynamical limit, i.e. when the volume of the system increases.

(Résumé : tsvp)

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Modèles pour des Réseaux de Transport

Résumé : On propose divers modèles stochastiques de réseaux de transport, par ordre croissant de complexité. Les outils mathématiques mis en jeu font appel à la théorie des réseaux de files d'attente et à l'analyse asymptotique. Les applications visées sont principalement les systèmes de véhicules en libre service (type PRAXITÈLE), mais les méthodes d'analyse ont une portée générale. On montre notamment l'indépendance asymptotique des nœuds en *limite thermodynamique*, c'est à dire lorsque la taille du système augmente.

1 Introduction

Transportation problems have been for a long time a source of interesting mathematical problems. New transportation problems now arise in the context of demand driven systems (e.g. flexible route bus systems, flexible delivery, rental and self service cars, etc.). In these systems, the demand cannot be modelled by simple deterministic methods and stochastic models seem suitable to carry out analysis of performance. This paper is a continuation of the preliminary study [8] and provides some understanding in the behaviour of so-called *service vehicle networks* (SVN). It can be viewed as a creation of models. Some of them are classical and rely on so-called product form networks; others can hardly be solved analytically, but asymptotics when the size of the system increases can be obtained.

Basically, a customer arrives at a station (*node, station, parking lot*), where he takes a car, if any available one, to reach some other station (destination), where he leaves the car. Section 2 is devoted to the simplest case, when the number of available cars is unbounded. Here time dependent parameters are allowed. The model of section 3 assumes a finite number V of cars, no capacity constraints and no waiting room for customers, who are lost if there is no car when they arrive. Section 4 deals with a more difficult case: at each station, the capacity for parking places is limited and there is also a finite waiting room for clients at each station. An asymptotic analysis is presented for a symmetrical network, in which the number of cars and the numder of nodes simultaneously increase. In section 5, generalizations are proposed, some of them being the object of ongoing works.

2 Non time-homogeneous input process and infinite-server queues

Consider an open queueing network with N stations. At time t, customers arrive at node i, $i = 1, \ldots, N$ according to a non time-homogeneous Poisson process, with deterministic parameter $(\lambda_i(t), t \geq 0)$. These N processes are supposed to be independent. A customer arriving at node i is provided with a car and goes to node j with some probability p_{ij}, so that the matrix $P = (p_{ij})$

be stochastic. In this model, the number of available cars is supposed to be unlimited. After having reached his destination, a customer leaves the network. Let us introduce the following random variables, for all $i, j = 1, \ldots, N$:

- τ_{ij}, the time to go from node i to node j, the corresponding distribution function being $B_{ij}(x)$;
- $x_{ij}(t)$, the number of cars which, at time t, are on their way from i to j.

Now to each link (origin-destination pair) (i, j) with $p_{ij} > 0$, we associate a fictitious node having an infinite number of servers, the service-time of each server being equal to τ_{ij}.

As an immediate corollary of standard results, which likely can be traced to Palm [2], we have the following:

Theorem 2.1 *Assume that all service times, routing and arrival processes are mutually independent and that $\lambda_i(t), t \geq 0, i, j = \overline{1, N}$ are integrable functions. Then, for each t, $x_{ij}(t), i, j = \overline{1, N}$ are independent random variables having a Poisson distribution, with finite mean*

$$m_{ij}(t) \;=\; E x_{ij}(t) \;=\; E \int_{t-\tau_{ij}}^{t} p_{ij} \lambda_i(s) ds.$$

Remark 1 *These results are valid under more general assumptions, including time-dependent routing matrix. The reader is referred to [3] for an extensive survey and some new developments on these problems.*

Example 1 Let $B_{ij}(x) = B(x) = P\{\tau < x\}$ for all $i, j = \overline{1, N}$. Then the total number of moving cars at time t is

$$M(t) \;=\; E \int_{t-\tau}^{t} \lambda(s) ds,$$

where $\lambda(s) = \sum_{i=1}^{N} \lambda_i(s)$. The extreme values of $M(t)$ occur at all time instants t_0, satisfying the equation

$$\lambda(t_0) = \int_0^{t_0} \lambda(t_0 - u) dB(u).$$

3 Finite number of servers and no waiting room

3.1 Model description

Here we consider a more realistic situation, where the total number of cars in the system is finite, say V. Customers who do not find available cars at the station where they have arrived are lost. Moreover, a car arriving at a parking lot waits until the arrival of the next customer (no *empty displacements*); then both leave to reach their destination. Here we shall assume time-independent arrival rates $\lambda_i, i, j = \overline{1, N}$. All other parameters and notations in this network are the same as above.
As in the first model, we introduce virtual nodes denoted by (i, j). A car will be said to *be*

- at node (i, j), if he is moving from station i to station j;
- at station i, if it is waiting for a customer in parking lot i.

Then the system can be viewed as a *closed* network with $N(N+1)$ nodes, in which V cars are moving around and there are two kinds of nodes:

1. stations of type i, which are single-server queues with exponentially distributed service-times, with parameter $\lambda_i, i = \overline{1, N}$;
2. nodes denoted by pairs (i, j), which are depicted as infinite-server queues, with service-time distribution function $B_{ij}(x)$.

Let $x_i(t)$, $i = \overline{1, N}$, be the number of cars parked at station i at time t and $x_{ij}(t)$, $i, j = \overline{1, N}$, – the number of cars moving between i and j.
When $\tilde{\tau}_{ij} = E\tau_{ij} < \infty$ and matrix $\mathrm{P} = [p_{ij}]$ is ergodic, its invariant measure being denoted by $\pi = (\pi_1, \ldots, \pi_N)$, it is well known that the vector-process

$$X(t) = (x_i(t), i = \overline{1, N};\ x_{ij}(t), i, j = \overline{1, N})$$

has an equilibrium distribution. Indeed, we have the following

Theorem 3.1 *Under the above assumptions, the stationary distribution of $X(t)$ has a product form given by*

$$\lim_{t\to\infty} P\left(x_i(t) = n_i, i = \overline{1, N}; x_{ij}(t) = n_{ij}, i, j = \overline{1, N}\right) =$$

$$P\left(n_i, i=\overline{1,N}; n_{ij}, i,j=\overline{1,N}\right) = C\prod_{k=1}^{N}\left(\frac{\pi_k}{\lambda_k}\right)^{n_k}\prod_{i=1}^{N}\prod_{j=1}^{N}\frac{\left(\pi_i p_{ij}\tilde{\tau}_{ij}\right)^{n_{ij}}}{n_{ij}!}, \quad (3.1)$$

where

$$\sum_{i=1}^{N} n_i + \sum_{i=1}^{N}\sum_{j=1}^{N} n_{ij} = V,$$

and C is a normalizing constant.

Proof The statement of the theorem is an immediate corollary of classical results (see e.g. [4]). It suffices to remark that the transition probabilities through the network can be written as $p^{\star}_{\alpha,\beta}$, where $\alpha,\beta = i$ or (i,j) and

$$p^{\star}_{ij} = 0, \quad p^{\star}_{i,(ij)} = p_{ij}, \quad p^{\star}_{(ij),j} = 1.$$

Then

$$P(n_i, i=\overline{1,N}; n_{ij}, i,j=\overline{1,N}) = C\prod_{i=1}^{N} g_i(n_i)\prod_{i=1}^{N}\prod_{j=1}^{N} g_{ij}(n_{ij}),$$

where

$$g_i(n_i) = \left(\frac{l_i}{\lambda_i}\right)^{n_i}, \; g_{ij}(n_{ij}) = \frac{(l_{ij}\tilde{\tau}_{ij})^{n_{ij}}}{n_{ij}!},$$

and

$$l_i = \sum_{j=1}^{N} l_{ji} p^{\star}_{(ji),i} = \sum_{j=1}^{N} l_{ji}, \; l_{ji} = l_j p_{ji}.$$

Thus $l_i = \pi_i$ and $l_{ij} = \pi_i p_{ij}, \forall i,j = 1,\ldots,N$

The probability that a customer arriving at queue i be not served (*loss probability*) is exactly

$$P^{(i)}_{loss} = \sum_{\sum_{j\neq i} k_j + \sum_{m,j} k_{mj} = V} P(k_1,\ldots,k_{i-1},0,k_{i+1},\ldots,k_N;\, k_{mj}, m,j=\overline{1,N}).$$

3.2 Symmetrical network

When the network is symmetric, we have

$$p_{ij} = \frac{1}{N},\ \lambda_i = \lambda,\ \tau_{ij} = \tau,\ i,j = \overline{1,N}.$$

Let $g(t)$ be the number of moving cars at time t. Now the probability that a car goes to station j is equal to $\frac{1}{N}$, no matter which station the car was coming from: thus it suffices to introduce only one global virtual node, which will be assigned number 0 and contains all moving cars. The transition probabilities are then simply

$$p^\star_{ij} = 0,\ i,j = \overline{1,N};\ p^\star_{i0} = 1;\ p^\star_{0i} = \frac{1}{N},\ i = \overline{1,N};\ p^\star_{00} = 0.$$

The equilibrium state distribution is obtained as in the previous theorem:

$$\lim_{t\to\infty} P\left(x_0(t) = k_0, \ldots, x_N(t) = k_N\right) = P(k_0, \ldots, k_N) \ = \ C\lambda^{-V+k_0},$$

with $\sum_{i=0}^N k_i = V$ and

$$C = \lambda^V \left(\sum_{m=0}^{V} C^m_{N-1+m} \frac{\rho^{V-m}}{(V-m)!} \right)^{-1}.$$

Setting $\rho \ = \ N\lambda\tau$, the loss probability is given by

$$\begin{aligned} P_{loss} \ &= \ \sum_{k_0+\cdots+k_{N-1}=V} p(k_0, \ldots, k_{N-1}, 0) \ = \ C\lambda^V \sum_{m=0}^{V} C^m_{N-2+m} \frac{\rho^{V-m}}{(V-m)!} \\ &= \ \frac{\displaystyle\sum_{m=0}^{V} C^m_{N-2+m} \frac{\rho^{V-m}}{(V-m)!}}{\displaystyle\sum_{m=0}^{V} C^m_{N-1+m} \frac{\rho^{V-m}}{(V-m)!}}. \end{aligned} \tag{3.2}$$

3.3 Asymptotic estimate of the loss probability in the symmetrical case

It is of practical interest to compute P_{loss} in (3.2), when the number of cars increases with the number of stations. We shall take $V = rN$.

Let us introduce

$$u(s,l;x) = \sum_{m=0}^{s} C_{l+m}^{m} \frac{x^{s-m}}{(s-m)!}\,.$$

Then

$$P_{loss} \stackrel{\text{def}}{=} \varphi(N) = \frac{u(V,N-2;\rho)}{u(V,N-1;\rho)} = \frac{u(rN,N-2;N\lambda\tau)}{u(rN,N-1;N\lambda\tau)}.$$

One can check that $u(s,l;x)$ admits the following integral representation

$$u(s,l;x) = \frac{1}{2i\pi}\int_{\mathcal{C}} \frac{e^{xt}\,dt}{t^s(1-t)^{l+1}}\,,$$

where $\mathcal{C}$ stands for the unit circle. From this representation, exact asymptotic expansions can be obtained by using the classical *saddle-point* method in the complex plane. Simply quoting the main steps, we can write

$$u(rN,N-2;N\lambda\tau) = \frac{1}{2i\pi}\int_{\mathcal{C}} \frac{e^{N\lambda\tau t}\,dt}{t^{rN}(1-t)^{N-1}}\,.$$

Analogously,

$$u(rN,N-1;N\lambda\tau) = \frac{1}{2i\pi}\int_{\mathcal{C}} \frac{e^{N\lambda\tau t}\,dt}{t^{rN}(1-t)^{N}}\,.$$

It follows (skipping all intermediate derivations – see for instance [5]) that

$$\varphi(N) = 1 - \frac{2r}{\lambda\tau + r + 1 + \sqrt{(\lambda\tau + r + 1)^2 - 4\lambda\tau r}} + O(\frac{1}{N})\,. \tag{3.3}$$

4 Finite capacities for cars and customers

4.1 General model description

This last model tries to cope with still more realistic assumptions. As a consequence, there is a tremendous jump in terms of analytic complexity and closed-form solutions can hardly be expected. Nonetheless, it is possible to get interesting information on the behaviour of the system when its size becomes large.

There are N parking lots. Arrivals of customers at the i-th station form a Poisson process, with rate $\lambda_i(t)$, $i = 1, \dots, N$, not necessarily time-homogeneous. At station i, let m_i be the capacity of the i-th parking lot and k_i be the size of the waiting room for customers. Matrix $P = \{p_{ij}\}$, as before, denotes the routing probabilities of customers and τ_{ij} is the time to travel from node i to node j. A client arriving at a given node waits for a car, if there are free waiting places; otherwise he leaves the network and is lost. Similarly, a car arriving at a parking lot with free parking slots stays there (if there are no waiting customers); otherwise, if there is no free space, then the car immediately goes to node j with probability $\widetilde{p}_{ij}$.

The basic characteristics of this network are listed thereafter:

L - average number of cars waiting in parking lots;
L_0 - average number of cars moving without clients;
θ - average time cars are waiting in parking lots;
γ - average time necessary to look for a free parking place at the end of a trip;
P_{loss} - probability of loosing customers;
W - average waiting time of a customer.

The main results presented in the next sections concern the case where the number of nodes in the above network tends to infinity. Then it is proved, for a fully symmetrical network, that any fixed number of nodes become asymptotically independent, each of them being described by a birth and death process.

4.2 Fully symmetrical network

Here

$$k_i = k, \quad m_i = m, \quad p_{ij} = \widetilde{p}_{ij} = 1/N, \lambda_i(t) = \lambda(t)\ , \forall i, j = 1, \dots, N.$$

In addition all journeys are exponentially distributed with mean $1/\mu$. Now,

We first give a formal (and somehow artificial) construction of a network $\mathcal{N}$ with $2N$ nodes, the behaviour of which is clearly equivalent to the original

network.
$\mathcal{N}$ consist of N stations where the customers arrive and of N virtual nodes, which describe the cars in movement (infinite server queues). In the original network, all clients choose their destination with the same uniform probability $1/N$. A client arriving at a given station waits for a car, if there are free waiting places; otherwise he leaves the network and is lost. Similarly, a car arriving at a parking lot with free parking slots waits, if there are no customers; otherwise, when there is no free space, the car goes to node j with probability $1/N$.
In the network $\mathcal{N}$, the movement of a car is modelled as follows: it chooses a virtual node j with probability $1/N$, stays there for an exponential time with parameter μ and then chooses a new station with probability $1/N$.

Take an arbitrary station i in the network. Let $x_i(t)$ (resp. $y_i(t)$) be the number of customers (resp. cars) at this node at time t. Define the process

$$z_i(t) = \begin{cases} -x_i(t)\,, & \text{if } x(t) > 0,\, y(t) = 0, \\ y_i(t)\,, & \text{if } y(t) > 0,\, x(t) = 0, \\ 0\,, & \text{if } x(t) = 0,\, y(t) = 0. \end{cases} \tag{4.1}$$

Thus, $-k \leq z_i(t) \leq m, \, , \forall i \,, 1 \leq i \leq N$.
Similarly, the nodes representing the cars in movement are describded by means of the random variables $z'_i(t)\,, \forall i\,, 1 \leq i \leq N$, where $z'_i(t) \in \{0, 1, 2, \ldots\}$ denotes the number of cars in the i-th virtual node. We will analyze the behaviour of the Markov process

$$\mathbf{Z}^{(N)} = \{(z_1, \ldots, z_N; z'_1, \ldots, z'_N)\},$$

when $N \to \infty$.

4.2.1 Asymptotic independence

Here it will be proved that, for any fixed time t, the random variables $z_1(t)$, $z_2(t)$ describing the state of two arbitrary stations, are asymptotically independent. This is equivalent to say that, as $N \to \infty$,

$$\mathsf{P}^{(N)}\{z_1(t) = k_1, z_2(t) = k_2\} - \mathsf{P}^{(N)}\{z_1(t) = k_1\}\mathsf{P}^{(N)}\{z_2(t) = k_2\} \to 0. \tag{4.2}$$

Analogously, it is not difficult to show that $z_1(t)$ and $z_1'(t)$ are asymptotically independent, as well as $z_1'(t)$ and $z_2'(t)$. Indeed any fixed number of nodes and queues will be asymptotically independent.

Notation In the sequel, for the sake of shortness, the superscript (N) will be omitted and, whenever non ambiguous, we shall often write z_1, z_1', λ, instead of $z_1(t)$, $z_1'(t)$, $\lambda(t)$.

The proof of (4.2) is similar to that of asymptotic independence in [7], so we do not describe it in full details. For t fixed, consider a station i. In this section, all cars are assumed to be "different" and they "remember", which nodes they have visited up to time t. Then one can define *dependency set* $D_i(t)$ of the node i as follows.

1. If no cars have arrived to node i up to time t, and no cars from node i went to other nodes, then $D_i(t) = \{i\}$;

2. otherwise, let $t_1 < t$ be the last moment when one of the above events occurred (for example, a car arrived from node j). Then we put

$$D_i(t) = D_i(t_1 - 0) \cup D_j(t_1 - 0).$$

The number of elements in $D_i(t)$ will be denoted by $\varphi(t) \stackrel{\text{def}}{=} |D_1(t)|$. Majorizing $\varphi(t)$ by a branching process in continuous time, we get

$$\mathsf{E}|D_1(t)| \leq C\, e^{(\lambda+\mu r)t}. \tag{4.3}$$

Then we have

$$\begin{aligned}
&\mathsf{P}_t\{z_1 = k_1, z_2 = k_2\} = \\
&\qquad \mathsf{P}_t\{z_1 = k_1, z_2 = k_2 | D_1(t) \cap D_2(t) \neq \emptyset\}\mathsf{P}\{D_1(t) \cap D_2(t) \neq \emptyset\} + \\
&\qquad \mathsf{P}_t\{z_1 = k_1, z_2 = k_2 | D_1(t) \cap D_2(t) = \emptyset\}\mathsf{P}\{D_1(t) \cap D_2(t) = \emptyset\}.
\end{aligned}$$

Note that, if

$$D_1(t) \cap D_2(t) = \emptyset,$$

then $z_1(t)$ and $z_2(t)$ are conditionally independent. Thus if one can show

$$\mathsf{P}\{D_1(t) \cap D_2(t) \neq \emptyset\} \to 0, \quad \text{as } N \to \infty,$$

then (4.2) will hold. But

$$\begin{aligned}
&\mathsf{P}\{D_1(t)\cap D_2(t)\neq\emptyset\}\leq\\
&\quad\mathsf{P}\{D_1(t)\cap D_2(t)\neq\emptyset\,|\,|D_1(t)|\leq M,|D_2(t)|\leq M\}+\\
&\quad 2\mathsf{P}\{|D_1(t)|>M\}\stackrel{\text{def}}{=}F+G.
\end{aligned}$$

It is not difficult to see that $F < C_1M^2/N$ and, using (4.3) and Chebyshev inequality, we get $G \leq C_2/M$. Now choosing $M = N^{1/3}$ completes the proof of (4.2).

4.2.2 Kolmogorov's equations

Let us write now the *forward* Kolmogorov's equations (see [6]) for the process described in paragrah 4.2. Note that, since the state space, will be denoted by $\mathbf{X}$ is finite, these equations hold without any problem.

Let $P(t,x,y)$ be the transition function for the process,

$$x=(x_1,\ldots,x_N;x'_1,\ldots,x'_N),\quad y=(y_1,\ldots,y_N;y'_1,\ldots,y'_N).$$

The generator is denoted by

$$\begin{aligned}
a(x,y) &= \lim_{t\to 0}\frac{P(t,x,y)}{t},\quad x\neq y,\\
a(x) &= \lim_{t\to 0}\frac{1-P(t,x,x)}{t},\\
a(x,B) &= \sum_{y\in B:y\neq x} a(x,y),\quad \text{for any } B\subset\mathbf{X}.
\end{aligned}$$

Then, one can write

$$\frac{\partial P(t,x,B)}{\partial t} = -\sum_{y\in B}P(t,x,y)a(y)+\sum_{z\in X}P(t,x,z)a(z,B). \tag{4.4}$$

Moreover, it is not difficult to check the equality

$$a(z)=\mu\Big(1-\frac{1}{N^2}\sum_{i=1}^{N}I_{\{z_i=m\}}\Big)\sum_{i=1}^{N}z'_i+\lambda\sum_{i=1}^{N}I_{\{z_i\neq -k\}}.$$

4.2.3 Limiting equations for one node

In this section we derive the limiting equations describing the evolution of a given node, say node 1.
Set $p_l(t) = \mathsf{P}(z_1(t) = l) = P(t, x_0, B)$, where $x_0 = (r, \ldots, r; 0, \ldots, 0)$ is the initial state of the system and let

$$B = B^{(N)} = \{l\} \times \{-k, \ldots, m\}^{N-1} \times \{0, 1, 2, \ldots\}^N.$$

Then

$$a(z, B) = \begin{cases} 0\,, & \text{if } |z_1 - l| \geq 2, \\ \frac{\mu}{N} \sum_{i=1}^N z_i'\,, & \text{if } z_1 = l-1; \\ \lambda\,, & \text{if } z_1 = l+1; \\ \lambda \sum_{i=2}^N I_{\{z_i \neq -k\}} + \mu \frac{N-1}{N}\Big(1 - \frac{1}{N^2} \sum_{i=1}^N I_{\{z_i = m\}}\Big) \sum_{i=1}^N z_i'\,, & \text{if } z_1 = l. \end{cases}$$

Setting

$$\begin{aligned} \overline{B} &= \{l+1\} \times \{-k, \ldots, m\}^{N-1} \times \{0, 1, 2, \ldots\}^N, \\ \underline{B} &= \{l-1\} \times \{-k, \ldots, m\}^{N-1} \times \{0, 1, 2, \ldots\}^N, \\ w_N &= 1 - \frac{1}{N^2} \sum_{i=1}^N I_{\{z_i = m\}}, \end{aligned}$$

we get

$$\begin{aligned} \frac{dp_l(t)}{dt} &= \mu w_N \sum_{y \in B} \sum_{i=1}^N P(t, x_0, y) y_i' - \lambda \sum_{y \in B} \sum_{i=1}^N P(t, x_0, y) I_{\{y_i \neq -k\}} \\ &+ \frac{\mu}{N} \sum_{z \in \underline{B}} \sum_{i=1}^N P(t, x_0, z) z_i' + \lambda \sum_{z \in \overline{B}} P(t, x_0, z) \\ &+ \mu \frac{N-1}{N} w_N \sum_{z \in B} \sum_{i=1}^N P(t, x_0, z) z_i' + \lambda \sum_{z \in B} \sum_{i=2}^N P(t, x_0, z) I_{\{z_i \neq -k\}} \\ &\stackrel{\text{def}}{=} U_1 + U_2 + U_3 + U_4 + U_5 + U_6. \end{aligned}$$

Clearly,

$$U_1 + U_5 = -\frac{\mu}{N} w_N \sum_{z \in B} \sum_{i=1}^N P(t, x_0, z) z_i'$$

$$= \quad -\mu w_N \sum_{z\in B} P(t,x_0,z)z_1',$$

using the symmetry of the network. Also, when $l \neq -k$, we have

$$\begin{aligned} U_2 + U_6 &= -\lambda \sum_{z\in B} P(t,x_0,z) I_{\{z_1\neq -k\}} \\ &= -\lambda \sum_{z\in B} P(t,x_0,z) = -\lambda p_l(t). \end{aligned}$$

and

$$\begin{aligned} U_3 &= \mu \sum_{z\in \underline{B}} P(t,x_0,z)z_1', \\ U_4 &= \lambda p_{l+1}(t). \end{aligned}$$

Since $z_1(t)$ and $z_1'(t)$ are asymptotically independent, as $N\to\infty$, and $w_N \to 1$, we can write

$$\begin{aligned} \mu w_N \sum_{z\in B} P(t,x_0,z)z_1' &= \mu w_N \sum_{z\in B} P(t,x_0,z) \sum_{j=1}^{\infty} j I_{\{z_1'=j\}} \\ &= \mu w_N \sum_{j=1}^{\infty} j \sum_{z\in B} I_{\{z_1'=j\}} P(t,x_0,z) \\ &= \mu w_N \sum_{j=1}^{\infty} j \mathsf{P}_t\{z_1 = l, z_1' = j\} \\ &= \mu w_N p_l(t) \mathsf{E}(z_1'(t) \mid z_1(t) = l) \\ &\stackrel{N\to\infty}{\longrightarrow} \mu M(t) p_l(t), \end{aligned}$$

where $M(t)$ denotes the mean number of cars in the virtual nodes. Finally we obtain the following differential equation for $p_l(t)$, valid when $N\to\infty$,

$$\frac{dp_l(t)}{dt} = \mu M(t) p_{l-1}(t) - (\lambda + \mu M(t)) p_l(t) + \lambda p_{l+1}(t). \tag{4.5}$$

Remark It is not difficult to prove, still using asymptotic independence , that

$$\sum_{i=1}^{N} I_{\{z_i'=l\}} \xrightarrow{\mathsf{P}} p_l(t), \quad \text{as } N\to\infty.$$

The next step is to derive equations for $M(t)$. To this end it is possible to use the same procedure: write down Kolmogorov's equations for $q_l(t) = \mathsf{P}_t\{z_1' = l\}$, then take the summation $M(t) = \sum_{i=1}^{\infty} i\, q_i(t)$, pass to the limit as $N \to \infty$, and use the asymptotic independence. In this way the following equation could be obtained:

$$\frac{dM(t)}{dt} = \lambda(p_1(t) + \cdots + p_m(t)) - \mu M(t)(p_0(t) + \cdots + p_{m-1}(t)). \tag{4.6}$$

Then, using (4.5) and (4.6), it is possible to prove that

$$M(t) + \sum_{i=1}^{m} i\, p_i(t) = r. \tag{4.7}$$

But the easiest way to get (4.7) is simply to notice that, since the system is symmetrical, the mean number of cars in the first station plus the mean number of cars in the first "displacement" node is exactly r, and this is exactly (4.7).

The last question is whether the functions $p_l^{(N)}(t)$ and $M^{(N)}(t)$ converge to the solutions of (4.5) and (4.7), i.e. whether the convergence of the equations entails that their solutions also converge. But here the number of equations is finite (with continuous right-hand sides), so that the question of convergence should be answered positively. In fact, we just have proved the following

Theorem 4.1 *Quantities $p_l^{(N)}(t)$ and $M^{(N)}(t)$ converge, as $N \to \infty$, to the solutions of (4.5) and (4.7).*

4.2.4 Stationary case

Here we consider the stationary regime, i.e. the left-hand side of (4.5) is assumed to be equal 0, $M(t) \equiv M$ and $p_l(t) \equiv p_l$.
Note that for all finite N this regime exists and $p_l^{(N)} \stackrel{N\to\infty}{\longrightarrow} p_l$, $M^{(N)} \stackrel{N\to\infty}{\longrightarrow} M$, where p_l and M can be determined from

$$\mu M p_{l-1} - (\lambda + \mu M) p_l + \lambda p_{l+1} = 0\,, \tag{4.8}$$

and

$$M + \sum_{i=1}^{m} i\, p_i = r. \tag{4.9}$$

Let $\rho = \mu M/\lambda$. Using (4.8), after some elementary computations, we get

$$p_l = \frac{\rho^{l+k}(\rho - 1)}{\rho^{m+k+1} - 1}.$$

The following characteristics can be obtained:

- The average number of cars waiting in parking lots

$$L(\rho) = \frac{\rho^{k+1}(m\rho^{m+1} - (m+1)\rho^m + 1)}{(\rho - 1)(\rho^{m+k+1} - 1)}.$$

- The average number of waiting customers

$$K = \frac{(k+1)(1-\rho) - 1 + \rho^{k+1}}{(\rho - 1)(\rho^{m+k+1} - 1)}.$$

- The probability of loosing customers

$$P_{loss} \equiv p_{-k} = \frac{1-\rho}{1-\rho^{k+m+1}}.$$

- The probability of empty displacement

$$S_m \equiv p_m = \frac{\rho^{m+k}(1-\rho)}{1-\rho^{k+m+1}}.$$

Now all the basic characteristics of the network can be obtained from the last relations, provided that one can compute the intensity $\alpha = \mu M$ of the car process. Using (4.9), one gets that $\rho = \alpha/\lambda$ is the non-negative solution ρ_0 of the following equation

$$r - \frac{\lambda}{\mu}\rho = \frac{\rho^{k+1}(m\rho^{m+1} - (m+1)\rho^m + 1)}{(\rho - 1)(\rho^{m+k+1} - 1)} = L(\rho). \tag{4.10}$$

The function $L(\rho)$ is monotone increasing and

$$L(0) = 0, \quad L(1) = \lim_{\rho \to 1} L(\rho) = \frac{m(m+1)}{2(k+m+1)}, \quad L(\infty) = m,$$

the function $\psi(\rho) = r - \frac{\lambda}{\mu}\rho$ monotonically decreases and

$$\psi(0) = r, \quad \psi(\rho) > 0 \text{ if } \rho < \frac{\mu r}{\lambda}, \quad \psi(\infty) = -\infty.$$

Several situations can occur:

1. If $r \le \frac{\lambda}{\mu}$, then $\alpha_0 = \lambda\rho_0 < \lambda$.

2. If $\frac{\lambda}{\mu} < r < \frac{m(m+1)}{2(k+m+1)} + \frac{\lambda}{\mu}$, then again $\alpha_0 < \lambda$.

3. If $r > \frac{m(m+1)}{2(k+m+1)} + \frac{\lambda}{\mu}$, then the root of (4.10) is more than 1. This means that $\alpha_0 > \lambda$, i.e. cars arrive at a higher rate than customers, so that they will be frequently free, thus causing a bad utilization of the resources in the network.

It is also now not too difficult to determine network parameters in order to optimize the following natural efficiency criterion. Per unit of time, let c_1 (resp. c_2) be the cost of loosing a customer (resp. a car) and c_3, the profit coming from the transportation of a customer. The total equivalent cost per unit of time is

$$W(k, m, r, \lambda, \mu) = \lambda c_1 p_{-k} + c_2 p_m - c_3\lambda(1 - p_{-k}). \tag{4.11}$$

This function can be optimized in terms of k, r, m.

4.2.5 Speed of convergence and simulation results

A still open question concerns the speed of convergence w.r.t. N of the quantities $p_l^{(N)}(t)$ to the solutions of (4.5) and (4.7). Although we do not have yet analytical results on this problem, simulations tend to show that this convergence is pretty fast. To compare solutions of (4.5) and (4.7) with simulation curves, see Fig. 1–3.

In conclusion, we claim that equations (4.5) and (4.7) provide an accurate description of the evolution of the system.

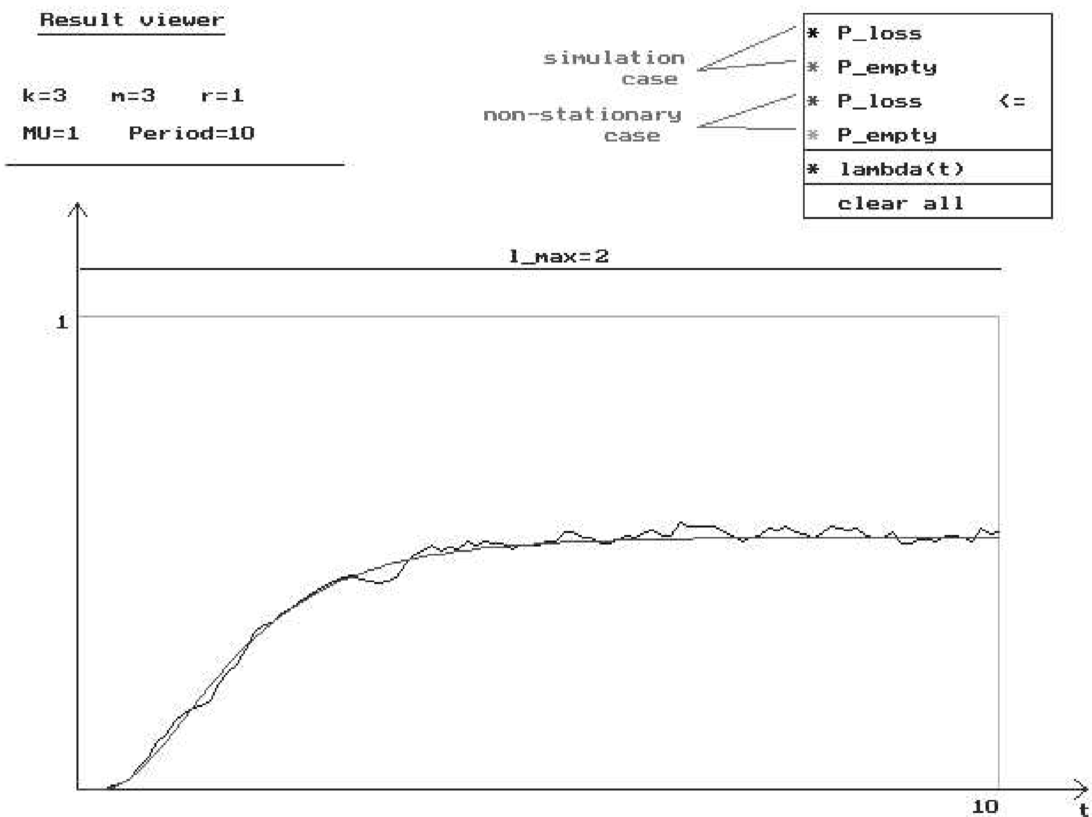
Result viewer
k=3 m=3 r=1
MU=1 Period=10
simulation case
non-stationary case
* P_loss
* P_empty
* P_loss <=
* P_empty
* lambda(t)
clear all
l_max=2
1
10
t

Figure 1: $N = 6$

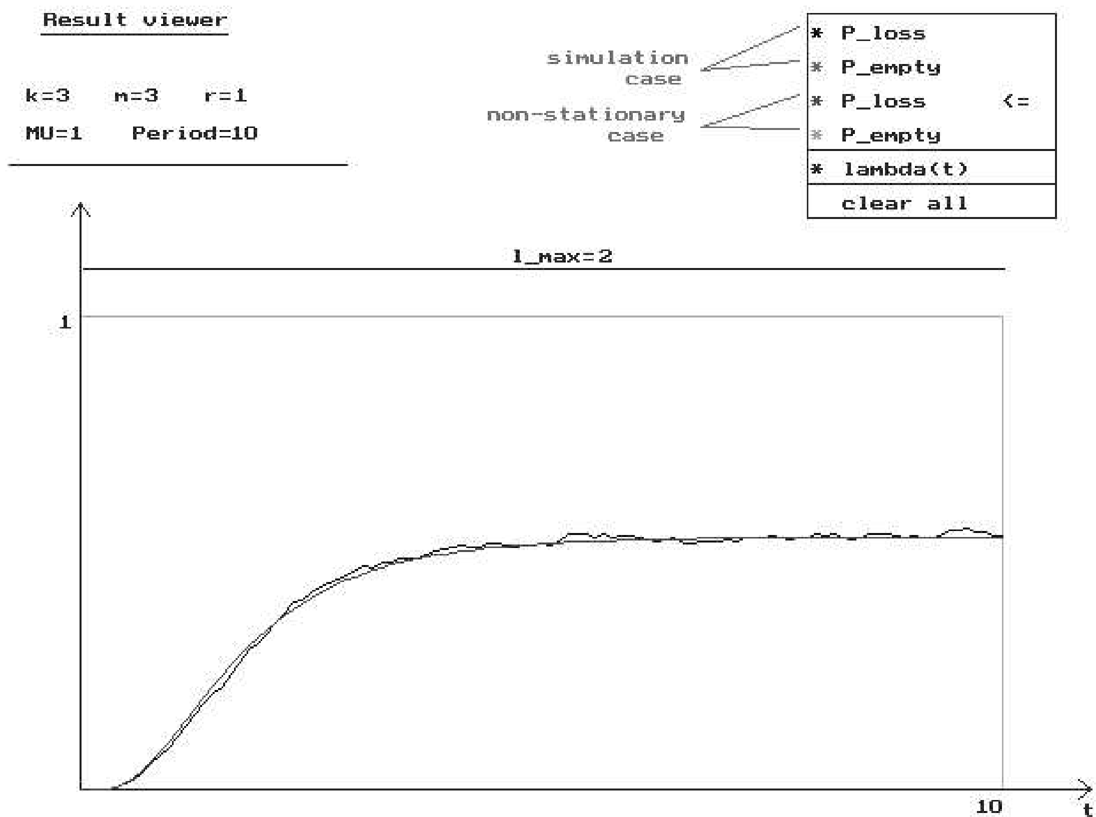
Result viewer
k=3 m=3 r=1
MU=1 Period=10
simulation case
non-stationary case
* P_loss
* P_empty
* P_loss <=
* P_empty
* lambda(t)
clear all
l_max=2
1
10
t

Figure 2: $N = 50$

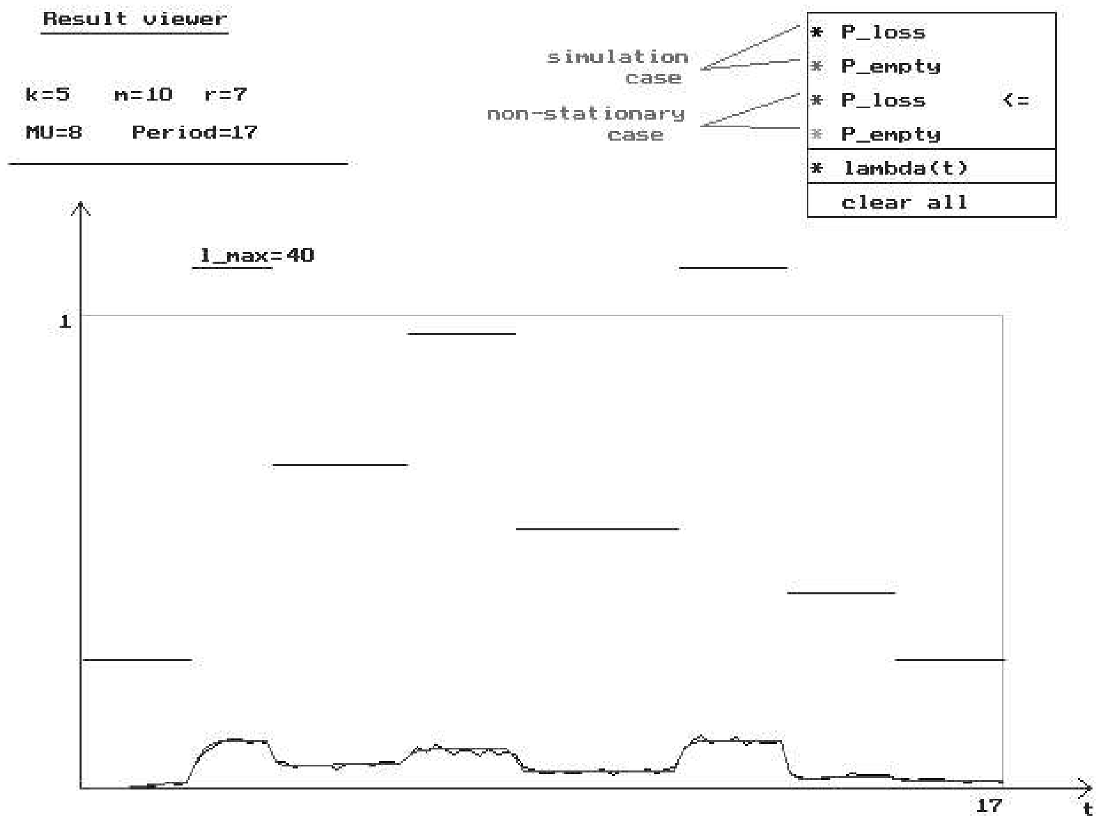

Figure 3: $N = 40$, $\lambda(t)$ is a step function

5 Generalizations

In this section we present some generalizations of the symmetrical model, without proofs since all of them are technically similar to those given in Section 4.2.

5.1 Model with impatience

Consider a fully symmetrical network, which differs from the previous one in only one respect: waiting customers may become impatient: if they do not get a car after some (random) time, then they leave the system.

Let, as before, k be the number of waiting places for customers. The laws of impatience times are exponentially distributed: if the customer is at the i-th position in his queue, then he leaves the system with intensity ν_i, i.e. his "rate of impatience" is ν_i. Introduce the quantities

$$\theta_l = \sum_{i=1}^{l} \nu_i\,, \qquad \text{with } \theta_0 = 0.$$

As $N \to \infty$, one can prove the following equations hold:

$$\begin{aligned}
\frac{dp_{-k}(t)}{dt} &= -(\theta_k + \mu M(t))p_{-k}(t) + \lambda(t)p_{-k+1}(t)\\
\frac{dp_l(t)}{dt} &= (\theta_{-l+1} + \mu M(t))p_{l-1}(t) - (\lambda(t) + \theta_{-l} + \mu M(t))p_l(t) + \lambda(t)p_{l+1}(t),\\
&\qquad\qquad -k < l \le 0;\\
\frac{dp_l(t)}{dt} &= \mu M(t)p_{l-1}(t) - (\lambda(t) + \mu M(t))p_l(t) + \lambda(t)p_{l+1}(t),\\
&\qquad\qquad 0 < l < m;\\
\frac{dp_m(t)}{dt} &= \mu M(t)p_{m-1}(t) - \lambda(t)p_m(t)\\
M(t) &= r - \sum_{i=1}^{m} ip_i(t).
\end{aligned} \tag{5.1}$$

As before, we can consider the stationary regime. The only question here is how to compute the probability of loosing the customer, i.e. the probability that an arbitrary customer coming to the system never gets a car. It is not difficult to prove that, when $p_l(t) \equiv p_l$ and $M(t) \equiv M$,

$$P_{loss} = p_{-k} + \sum_{i=0}^{k-1} p_{-i}\Big(1 - \frac{\mu M}{\mu M + \theta_{i+1}}\Big). \tag{5.2}$$

5.2 Model with different capacities of parking lots and waiting rooms

Here we allow the nodes to have different numbers of waiting places for customers and cars. Namely, let for $i \geq 0$ and $j \geq 1$ β_{ij} be the proportion of the nodes, having i places for customers and j places for cars, i.e.

$$\frac{\#\text{ of nodes with } i \text{ (resp. } j\text{) places for customers (resp. cars)}}{N} \stackrel{N\to\infty}{\longrightarrow} \beta_{ij}.$$

If k stands for the *maximal* size of a waiting room for customers, and m is a maximal number of parking slots for cars, then the elements of the matrix

$$\begin{pmatrix} \beta_{01} & \dots & \beta_{0m} \\ \dots & & \dots \\ \beta_{k1} & \dots & \beta_{km} \end{pmatrix}$$

are nonnegative and

$$\sum_{i,j} \beta_{ij} = 1.$$

All other characteristics of the network are the same as in Section 4.2. Here there are $(k+1)m$ types of nodes and for each type we can introduce the probabilities $p_l^{(ij)}(t)$. It is not difficult to prove that, as $N \to \infty$, there will be $(k+1)m$ systems of equations of the form (4.5) and (4.7) for $p_l^{(ij)}(t)$, together with one flow balance equation for $M(t)$:

$$M(t) + \sum_{i,j} \Big(\beta_{ij} \sum_{n=1}^{j} n p_n^{(ij)}(t)\Big) = r. \tag{5.3}$$

5.3 Models on a lattice

The next model involves some geometrical aspects. Let the stations be arranged on the square lattice, with size of the mesh h (see Fig. 4), and assume the existence of two non-negative functions $f(\rho)$ and $g(\rho)$, such that

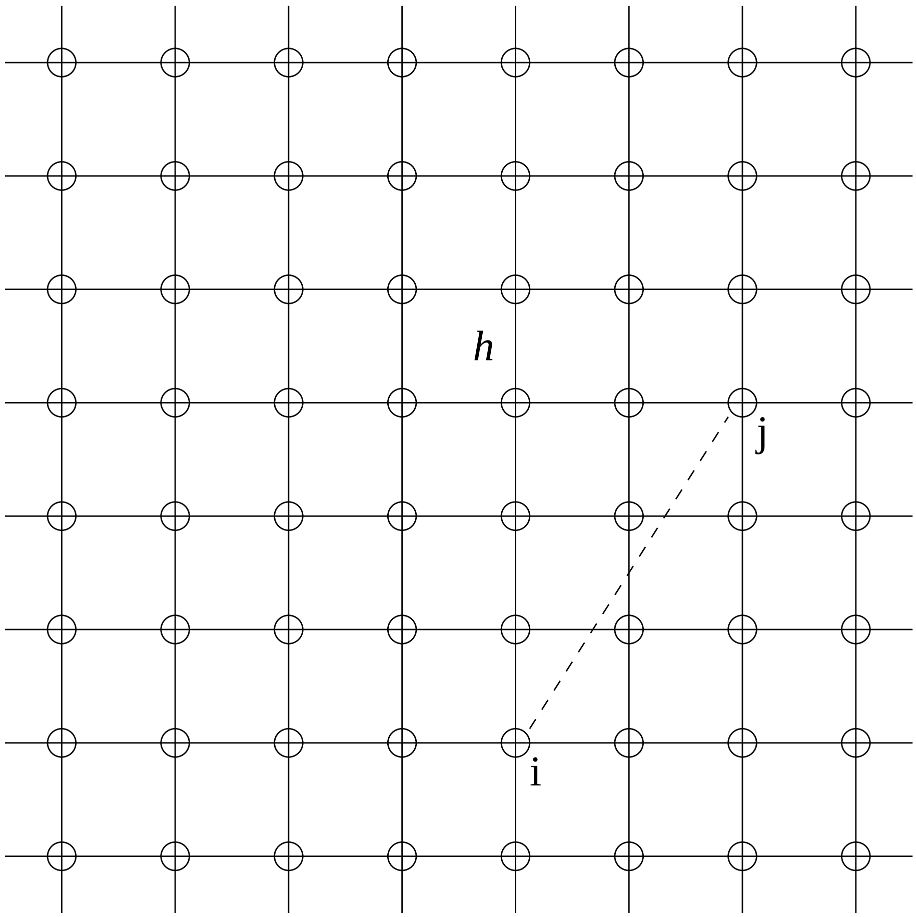

Figure 4: taxi on a square lattice

$$\int_0^{+\infty} f(\rho)\, d\rho < \infty.$$

The transition probabilities P_{ij} can be defined as $P_{ij} \sim f(\mathrm{dist}[i,j])$, that is, more exactly,

$$P_{ij} = \frac{f(\mathrm{dist}[i,j])}{\sum_k f(\mathrm{dist}[i,k])} ,$$

where $\mathrm{dist}[i,j]$ is the Euclidian distance between two nodes i and j. The travel time between nodes i and j is exponentially distributed with parameter $\mu_{ij} = g(\mathrm{dist}[i,j])$.

Let now $h \to 0$. As before, all nodes become asymptotically independent, and equations (4.5) and (4.7) hold, with μ substituted by

$$\frac{\int_0^{+\infty} \rho g(\rho) f(\rho)\, d\rho}{\int_0^{+\infty} \rho f(\rho)\, d\rho} . \tag{5.4}$$

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