Bishop-Phelps Cones and Convexity: Applications to Stability of Vector Optimization Problems

Ewa M. Bednarczuk

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Bishop-Phelps cones and convexity: applications to stability of vector optimization problems.

Ewa M.Bednarczuk

N° 2806

Février 1996

PROGRAMME 6

Rapport de recherche

ISSN 0249-6399

Bishop-Phelps cones and convexity: applications to stability of vector optimization problems.

Ewa M.Bednarczuk

Programme 6 — Calcul scientifique, modélisation et logiciel numérique
Projet NUMATH

Rapport de recherche n2806 — Février 1996 — 11 pages

Abstract: In this report we study the stability of cone support points $Min(A|\mathcal{K})$ of a given set A in a topological vector space Y, equipped with a closed convex cone $\mathcal{K} \subset Y$. We prove sufficient conditions for the lower continuity of $Min(A|\mathcal{K})$ when A is subjected to perturbations (Theorem 2.2, Theorem 2.3). The crucial assumption is that the set $Min(A|\mathcal{K})$ is dense in the set of strict cone support points (Definition 2.1). In normed vector spaces Y the set of strict support points contains the set of super efficient points in the sense of Borwein and Zhuang. By making use of the density result for super efficient points Theorem 4.2 gives sufficient conditions for the lower continuity of cone support points for cones with weakly compact bases and the original set A being closed and convex.

When $\mathcal{K}$ is a Bishop-Phelps cone in a Banach space Y we give a simple characterisation of strict support points (Theorem 3.2) which allows us to give a variant of the result of Attouch and Riahi (Theorem 3.4) without any compactness assumption (Theorem 3.5.

Key-words: stability of minimal points, lower continuity, super efficiency, Bishop-Phelps cones, convexity

(Résumé : tsvp)

Cônes de Bishop-Phelps et convexité: applications à la stabilité de problèmes d'optimisation vectorielle

Résumé : Dans cet article nous étudions la stabilité de l'ensemble des points de support $Min(A|\mathcal{K})$ pour des parties quelconques A d'un espace topologique vectoriel Y muni d'un cône convexe fermé $\mathcal{K}$. Dans l'optimisation vectorielle $Min(A|\mathcal{K})$ est nommé l'ensemble des points minimaux de A par rapport à $\mathcal{K}$. Nous démontrons des conditions suffisantes pour la continuité inférieure de $Min(A|\mathcal{K})$ (Théorème 2.2, 2.3). Dans ces deux résultats l'hypothèse essentielle est que l'ensemble $Min(A|\mathcal{K})$ est dense dans l'ensemble de points de support stricte (Definition 2.1). Dans les espaces normés l'ensemble de points de support stricte contient l'ensemble de points super efficaces au sens de Borwein Zhuang (Proposition 3.1). En utilisant les conditions suffisantes pour la densité des points super efficaces on obtient la continuité inférieure des points minimaux pour cône $\mathcal{K}$ possedant un base faiblement compact et partie A convexe fermé (Théorème 3.1, Théorème 4.2).

Dans le cas où $\mathcal{K}$ est le cône de Bishop-Phelps dans l'espace de Banach nous donnons une caractérisation simple de points de support stricte (Théorème 3.2) qui nous permets de formuler un resultat proche de celui de Attouch et Riahi (Théorème 3.5) sans aucune hypothèse de compacité (Théorème 3.4).

Mots-clé : la stabilité de points minimaux, la continuité inférieure, super efficacité au sens de Borwein et Zhuang, cônes de Bishop-Phelps, convexité

1 Introduction

Let Y be a topological vector space ordered by a closed convex pointed cone $\mathcal{K}$. The cone $\mathcal{K}$ **supports** a subset A of Y at $a_0 \in A$ if $(A - a_0) \cap (-\mathcal{K}) = \{a_0\}$. The point a_0 is called a **cone support point** of A. In vector optimization cone support points are called **minimal** and the set of all cone support points is denoted by $Min(A|\mathcal{K})$.

In this note we investigate lower continuity of cone support points when A is subjected to perturbations.

Let U be a topological space. A multivalued mapping $\Gamma : U \to Y$ is **lower continuous** (l.c.) at (y_0, u_0) if for each $0-$neighbourhood W in Y there exists a neighbourhood U_0 of u_0 such that $(y_0 + W) \cap \Gamma(u) \neq \emptyset$ for all $u \in U_0$. Γ is l.c. at u_0 if it is l.c. at every point $y_0 \in \Gamma(u_0)$. We say that Γ is **upper Hausdorff continuous** (u.H.c.) at u_0 if for each $0-$neighbourhood W of Y there exists a neighbourhood U_0 of u_0 such that $\Gamma(u) \subset \Gamma(u_0) + W$.

One of crucial properties ensuring lower continuity of support points is the domination property. We say that the **domination property** (DP) holds for A if $A \subset Min(A|\mathcal{K}) + \mathcal{K}$.

To derive our basic continuity results (Theorem 2.2, Theorem 2.3) we need also to distinguish some subsets of cone support points. We say that a cone support point $a_0 \in Min(A|\mathcal{K})$ is a **strong proper cone support point**, or a **strongly properly minimal point**, and we write $a_0 \in SPMin(A|\mathcal{K})$, if there exists a closed convex pointed cone $\mathcal{K}_0$, $\mathrm{int}\mathcal{K}_0 \neq \emptyset$, such that $\mathcal{K} \setminus \{0\} \subset \mathrm{int}\mathcal{K}_0$, and for each $0-$neighbourhood W there exists a $0-$neighbourhood O

$$(\mathcal{K} \setminus W) + O \subset \mathcal{K}_0\,, \qquad (*)$$

and $a_0 \in Min(A|\mathcal{K}_0)$.

The following auxiliary lemma will be of use in the sequel.

Lemma 1.1 *If $a_0 \in SPMin(A|\mathcal{K})$, then for any $0-$neighbourhood W there exists a $0-$neighbourhood O such that for all $z \notin a_0 + W$ and $z \notin a_0 - \mathcal{K}_0$ we have $(z + O) \cap (a_0 - \mathcal{K}) = \emptyset$.*

Proof. We start by showing that if the condition $(*)$ holds, then for any $0-$neighbourhood W there exists a $0-$neighbourhood O such that for any $z \notin W$, and $z \notin \mathcal{K}_0$ we have

$$(z + O) \cap (-\mathcal{K}) = \emptyset\,.$$

Let us first note that for any $0-$neighbourhood W one can choose $0-$neighbourhoods W_1, W_2 such that $[W^c + W_1] \cap W_2 = \emptyset$, where W^c stands for the complement of W. Moreover, by $(*)$, there exists a $0-$neighbourhood O_2 such that

$$(\mathcal{K} \setminus W_2) + O_2 \subset \mathcal{K}_0\,,$$

and hence

$$(\mathcal{K} \setminus W_2) + O_2 \cap W_1 \subset \mathcal{K}_0\,. \quad (1)$$

Now, suppose on the contrary that there exists $\bar{W}$ such that for any O one can find $z \notin \bar{W}$, and $z \notin (-\mathcal{K}_0)$ such that

$$(z + O) \cap (-\mathcal{K}) \neq \emptyset\,. \qquad (2)$$

Let us take $O = O_2 \cap W_1$. By (2), there exists $z \notin \bar{W}$, $z \notin (-\mathcal{K}_0)$ such that $z + w = -k$, where $w \in O_2 \cap W_1$, $k \in \mathcal{K}$, and $k \notin W_2$. Hence, by (1), $k + O_2 \cap W_1 \subset \mathcal{K}_0$, and consequently $k - w = -z \in \mathcal{K}_0$, contrary to the fact that $z \notin (-\mathcal{K}_0)$.

[1]This work was supported by the INRIA project NUMATH during the author's stay at the Institut Elie Cartan, Universite de Nancy 1, Departement de Mathematiques. On leave from the Systems Research Institute, Polish Academy of Sciences, 01-447 Warszawa, Newelska 6

Now, if we take $z-a_0 \notin W$ and $z-a_0 \notin (-\mathcal{K}_0)$ we have $[(z-a_0)+O]\cap(-\mathcal{K})=\emptyset$, which means that $(z+O)\cap(a_0-\mathcal{K})=\emptyset$.

$\square$

2 Basic results

In [2] we have proved the following stability result.

Theorem 2.1 *Suppose that* $\Gamma(u_0)\neq\emptyset$, *and*

$$Min(\Gamma(u_0)|\mathcal{K}) = SPMin(\Gamma(u_0)|\mathcal{K}))\,,$$

and (DP) *holds for all* $\Gamma(u)$ *in a certain neighbourhood* U_0. *If* Γ *is l.c. and u.H.c. at* u_0, *then* $\mathcal{M}:U\to Y$, $\mathcal{M}(u)=Min(\Gamma(u)|\mathcal{K})$ *is l.c. at* u_0.

Here we prove the following refinement of this result.

Theorem 2.2 *Let* $\Gamma(u_0)\neq\emptyset$, *and* (DP) *holds for all* $\Gamma(u)$ *in a certain neighbourhood* U_0 *of* u_0. *Assume that*

$$Min(\Gamma(u_0)|\mathcal{K}) \subset cl(SPMin(\Gamma(u_0)|\mathcal{K}))\,. \qquad (i)$$

If Γ *is l.c. and u.H.c. at* u_0, *then* $\mathcal{M}$ *is l.c. at* u_0.

Proof. Since $\Gamma(u_0)$ is nonempty, Γ is l.c. at u_0, and (DP) holds for all $\Gamma(u)$, $u\in U_0$, we have $Min(\Gamma(u)|\mathcal{K})\neq\emptyset$ for $u\in U_0$.

Let W be a $0-$neighbourhood and let $y_0\in\mathcal{M}(u_0)$. Let W_1, W_2 be $0-$neighbourhoods such that $W_1+W_1\subset W$ and $W_2+W_2\subset W_1$. By (i), there exists a strongly properly minimal element y_1 such that $y_1\in y_0+W_2$.

Since $y_1\in SPMin(\Gamma(u_0)|\mathcal{K})$, by Lemma 1.1, there exists a $0-$neighbourhood O such that

$$(z+O)\cap(y_1-\mathcal{K})=\emptyset$$

for all $z\in(y_1-\mathcal{K}_0)^c$, $z\notin y_1+W_2$. This means that

$$[[(y_1-\mathcal{K}_0)^c\setminus(y_1+W_2)]+O]\cap(y_1-\mathcal{K})=\emptyset\,,$$

and consequently,

$$[[(y_1-\mathcal{K}_0)^c\setminus(y_1+W_2)]+O_1]\cap[(y_1+O_1)-\mathcal{K}]=\emptyset\,, \qquad (1)$$

for any $0-$neighbourhood O_1 such that $O_1+O_1\subset O$.

On the other hand, since $y_1\in Min(\Gamma(u_0)|\mathcal{K}_0)$,

$$\Gamma(u_0)\subset(y_1-\mathcal{K}_0)^c\cup\{y_1\}\,.$$

Therefore,

$$\Gamma(u_0)\subset[(y_1-\mathcal{K}_0)^c\setminus(y_1+W_2)]\cup(y_1+W_2)\,,$$

and

$$\Gamma(u_0)+O_1\cap W_2\subset[[(y_1-\mathcal{K}_0)^c\setminus(y_1+W_2)]+O_1\cap W_2]\cup(y_1+W_1)\,.$$

There exists a neighbourhood U_1 of u_0 such that for all $u\in U_1$ we have

$$\Gamma(u)\subset\Gamma(u_0)+O_1\cap W_2\subset[[(y_1-\mathcal{K}_0)^c\setminus(y_1+W_2]+O_1\cap W_2]\cup(y_1+W_1)\,. \qquad (2)$$

Moreover, there exists a neighbourhood U_2 of u_0 such that

$$(y_1 + O_1 \cap W_2) \cap \Gamma(u) \neq \emptyset\,,$$

for $u \in U_2$, i.e., there exist y_u,

$$y_u \in \Gamma(u) \cap (y_1 + O_1 \cap W_2)\,,$$

and consequently,

$$y_u - \mathcal{K} \subset y_1 + O_1 \cap W_2 - \mathcal{K}\,.$$

By (1) ,

$$(y_u - \mathcal{K}) \cap [[(y_0 - \mathcal{K}_0]^c \setminus (y_1 + W_2)] + O_1 \cap W_2] = \emptyset\,,$$

and, by (2) , for $u \in U_1 \cap U_2$ we have

$$(y_u - \mathcal{K}) \cap \Gamma(u) \subset y_1 + W_1 \subset y_0 + W\,.$$

Now, by (DP), for each $u \in U_1 \cap U_2 \cap U_0$ there exists $\eta_u \in Min(\Gamma(u)|\mathcal{K}) = \mathcal{M}(u)$ such that

$$\eta_u \in (y_u - \mathcal{K}) \cap \Gamma(u) \subset y_0 + W\,.$$

This completes the proof.

□

Lemma 1.1 allows us to make an important, though elementary observation.

Remark 2.1 *If $a_0 \in SPMin(A|\mathcal{K})$, then $A \cap (a_0 - \mathcal{K}_0) = \{a_0\}$. Hence, for any $a \in A$ and any $0-$nieghbourhood W such that $a \notin a_0 + W$ we have $a \notin a_0 - \mathcal{K}_0$. By Lemma 1.1, there exists a $0-$neighbourhood O such that*

$$[(A \setminus (a_0 + W)) + O] \cap (a_0 - \mathcal{K}) = \emptyset\,.$$

This leads us to the following definition.

Definition 2.1 *An element $a_0 \in A$ is a* **strict cone support point***, or a* **strictly minimal element***, $a_0 \in SMin(A|\mathcal{K})$, if for any $0-$neighbourhood W there exists a $0-$neighbourhood O such that*

$$[(A \setminus (a_0 + W)) + O] \cap (a_0 - \mathcal{K}) = \emptyset\,. \qquad (**)$$

Note that each strict cone support point is a cone support point. Suppose on the contrary that there exists $a_1 \in A$, $a_1 \neq a_0$ such that $a_1 \in A \cap (a_0 - \mathcal{K})$. Then there exists a $0-$neighbourhood W such that $a_1 \in A \setminus (a_0 + W)$ but, for each $0-$neighbourhood O , $a_1 \in [(A \setminus (a_0 + W)) + O] \cap (a_0 - \mathcal{K})$ contrary to $(**)$.

Moreover, by Remark 2.1, each strong proper cone support point is a strict cone support point.

Let us note that $(**)$ can be rephrased as

$$[((A - a_0) \setminus W) + O] \cap [-\mathcal{K}] = \emptyset\,, \qquad (**)'$$

or

$$[((A - a_0) \setminus W] \cap [O - \mathcal{K}] = \emptyset\,. \qquad (**)''$$

With this definition we can prove a stronger version of Theorem 2.2.

Theorem 2.3 *Let $\Gamma(u_0) \neq \emptyset$ and (DP) holds for $\Gamma(u)$ for all u in a certain neighbourhood U_0 of u_0.*

Assume that

$$y_0 \in cl(SMin(\Gamma(u_0)|\mathcal{K})) \qquad (ii)$$

If Γ is l.c. and u.H.c. at u_0, then $\mathcal{M}$ is l.c. at (y_0, u_0).

Proof. By our assumptions we have $Min(\Gamma(u_0)|\mathcal{K}) \neq \emptyset$.

Let W be a $0-$neighbourhood and let $y_0 \in \mathcal{M}(u_0)$. Let W_1, W_2 be $0-$neighbourhoods such that $W_1 + W_1 \subset W$ and $W_2 + W_2 \subset W_1$. By (ii), there exists $y_1 \in SMin(\Gamma(u_0)|\mathcal{K})$ such that that $y_1 \in y_0 + W_2$.

By definition,

$$[(\Gamma(u_0) \setminus (y_1 + W_2)) + O] \cap (y_1 - \mathcal{K}) = \emptyset,$$

and consequently

$$[(\Gamma(u_0) \setminus (y_1 + W_2)) + O_1] \cap (y_1 + O_1 - \mathcal{K}) = \emptyset, \qquad (1)$$

for any $0-$neighbourhood O_1 such that $O_1 + O_1 \subset O$.

On the other hand,

$$\Gamma(u_0) \subset (\Gamma(u_0) \setminus (y_1 + W_2)) \cup (y_1 + W_2).$$

Therefore,

$$\Gamma(u_0) + O_1 \cap W_2 \subset [(\Gamma(u_0) \setminus (y_1 + W_2)) + O_1 \cap W_2] \cup (y_1 + W_1).$$

Now, there exists a neighbourhood U_1 of u_0 such that for all $u \in U_1$

$$\Gamma(u) \subset \Gamma(u_0) + O_1 \cap W_2 \subset [(\Gamma(u_0) \setminus (y_1 + W_2)) + O_1 \cap W_2] \cup (y_1 + W_1). \quad (2)$$

Moreover, there exists a neighbourhood U_2 of u_0 such that

$$(y_1 + O_1 \cap W_2) \cap \Gamma(u) \neq \emptyset,$$

i.e., for each $u \in U_2$ there exists y_u,

$$y_u \in \Gamma(u) \cap (y_1 + O_1 \cap W_2),$$

and consequently

$$y_u - \mathcal{K} \subset y_1 + O_1 \cap W_2 - \mathcal{K}.$$

Now, by (1),

$$(y_u - \mathcal{K}) \cap [(\Gamma(u_0) \setminus (y_1 + W_2)) + O_1 \cap W_2] = \emptyset.$$

By (2), for $u \in U_1 \cap U_2$

$$(y_u - \mathcal{K}) \cap \Gamma(u) \subset y_1 + W_1 \subset y_0 + W.$$

By (DP), for each $u \in U_0 \cap U_1 \cap U_2$ there exists $\eta_u \in Min(\Gamma(u)|\mathcal{K}) = \mathcal{M}(u)$ such that

$$\eta_u \in (y_u - \mathcal{K}) \cap \Gamma(u) \subset y_0 + W,$$

which completes the proof.

□

Clearly, to get lower continuity of $\mathcal{M}$ at u_0 it is enough to replace (ii) by

$$Min(\Gamma(u_0)|\mathcal{K}) \subset \mathrm{cl}(SMin(\Gamma(u_0)|\mathcal{K})). \qquad (ii)'$$

We close this section with a characterization of strong proper cone support points for cones with bases.

Recall that cone $\mathcal{K}$ has a base Θ if Θ is convex, $0 \notin \mathrm{cl}(\Theta)$, and $\mathcal{K} = \mathrm{cone}(B)$ If cone $\mathcal{K}$ is based, it is necessarily convex and pointed.

Proposition 2.1 *Suppose that cone $\mathcal{K}$ posseses a base Θ, then*

$$\mathcal{K}^O = cone(\Theta + O) \subset \mathcal{K}_0 \qquad (*)'$$

for some $0-$neighbourhood O.

Moreover, if base Θ is topologically bounded, then $(*)'$ implies $(*)$.

Proof Since $0 \notin \Theta$ there exists a $0-$neighbourhood W such that $\Theta \cap W = \emptyset$. Thus, by $(*)$ there exists a $0-$neighbourhood O such that $\Theta + O \subset \mathcal{K}_0$, or $cone(\Theta + O) \subset \mathcal{K}_0$.

On the other hand, if $(*)$ is not satisfied, then there exists a $0-$neighbourhood W such that for each $0-$neighbourhood O one can find an element $k_o \in \mathcal{K}$, $k_o \notin W$, such that $k_o + O \notin \mathcal{K}_0$, ie., $k_o + o \notin \mathcal{K}_0$. Now, $k_o = \lambda_o \theta_o$ and there exists λ_0 such that for $0 \leq \lambda \leq \lambda_0$ we have $\lambda\Theta \subset W$. Thus, $\lambda_o > \lambda_0$, and

$$\lambda_o/\lambda_0(\lambda_0\theta_o + \lambda_0/\lambda_o o) \notin \mathcal{K}_0 .$$

This means that

$$\lambda_0\theta_o + \lambda_0/\lambda_o o \notin \mathcal{K}_0 .$$

Since $\Theta_0 = \lambda_0\Theta$ is also a base we get

$$\Theta_0 + O \not\subset \mathcal{K}_0 ,$$

which contradicts $(*)'$.

□

3 Applications to normed spaces

Let Y be a normed space with the unit ball B. We start with the following proposition.

Definition 3.1 *A point $a_0 \in A$ is said to be* **super efficient in the sense of Borwein and Zhuang** *[3], $a_0 \in SE(A|\mathcal{K})$, if there exists a number M such that*

$$cl(cone(A - a_0)) \cap (B - \mathcal{K}) \subset MB .$$

Each super efficient point is efficient. Moreover, we show that each super efficient point is a strict support point.

Proposition 3.1 *For any subset A of Y we have*

$$SE(A|\mathcal{K}) \subset SMin(A|\mathcal{K}) .$$

Proof.

In normed spaces strict minimality can be rephrased as follows: for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$[(A \setminus (a_0 + \varepsilon B)) + \delta B] \cap (a_0 - \mathcal{K}) = \emptyset ,$$

or equivalently

$$[(A - a_0) \setminus \varepsilon B] \cap [\delta B - \mathcal{K}] = \emptyset .$$

Thus, if $a_0 \notin SMin(A|\mathcal{K})$ there exists $\varepsilon_0 > 0$ such that for each $\delta > 0$

$$[(A - a_0) \setminus \varepsilon_0 B] \cap [\delta B - \mathcal{K}] \neq \emptyset .$$

Hence, there exists $a_n \in A$, $\|a_n - a_0\| \geq \varepsilon_0$, such that

$$a_n - a_0 = 1/n(b_n - k_n)\,.$$

This implies that $n(a_n - a_0) = b_n - \bar{k}_n$ and $\|n(a_n - a_0)\| \to +\infty$. This, however, means that $a_0 \notin SE(A|\mathcal{K})$.

□

Proposition 3.2 *Suppose that $\mathcal{K}$ has a bounded base Θ. Then*

$$SPMin(A|\mathcal{K}) \subset SE(A|\mathcal{K})\,.$$

Proof . If $a_0 \in SPMin(A|\mathcal{K})$, then, by Proposition 2.1, there exists $\varepsilon > 0$ such that

$$(A - a_0) \cap \text{cone}(-\mathcal{K}_\varepsilon) = \{0\}\,,$$

where $\mathcal{K}_\varepsilon = \text{cone}(\Theta + \varepsilon B)$. Thus,

$$\text{cone}(A - a_0) \cap (\varepsilon B - \Theta) = \emptyset\,.$$

For any $y \in Y$ such that $y \in (a - a_0) - \mathcal{K}$ we have

$$a - a_0 = y - \lambda\theta\,,$$

for some $\lambda \geq 0$ and $\theta \in \Theta$. The rest of the proof is the same as the proof of Proposition 3.4 of [3]. If $\lambda = 0$, then $\|a - a_0\| = \|y\|$. If $\lambda > 0$,

$$\lambda^{-1}(a - a_0) = (\lambda^{-1}y - \theta)\,,$$

Since $\lambda^{-1} \in \text{cone}(A - a_0)$,

$$\|\lambda^{-1}y\| \geq \varepsilon\,.$$

Now

$$\begin{aligned} \|a - a_0\| &\leq \|y\| + \lambda \sup\theta \\ &\leq \|y\| + \lambda m = \|y\|(1 + \tfrac{\lambda m}{\|y\|}) \\ &\leq \|y\|(1 + \tfrac{m}{\varepsilon}) = M\|y\|\,, \end{aligned}$$

where, by assumption, $m = \sup\|\theta\| < +\infty$. This means that $a_0 \in SE(A|\mathcal{K})$.

□

In view of the above results, we have the following variant of Theorem 2.2.

Theorem 3.1 *Let $\Gamma(u_0) \neq \emptyset$. Suppose that*

$$y_0 \in cl(SE(\Gamma(u_0)|\mathcal{K}))\,, \qquad (iii)$$

and (DP) holds for all $\Gamma(u)$ is a certain neighbourhood U_0 of u_0. If Γ is l.c. at (y_0, u_0) and u.H.c. at u_0, then $\mathcal{M}$ is l.c. at (y_0, u_0).

Proof. By Proposition 3.1, each super efficient point is a strict support, hence $y_0 \in \text{cl}SMin(\Gamma(u_0)|\mathcal{K})$, and by applying Theorem 2.3 we obtain the assertion.

□

As previously, to get lower continuity of $\mathcal{M}$ at u_0, it is enough to replace (iii) by

$$Min(\Gamma(u_0)|\mathcal{K}) \subset \text{cl}(SE(\Gamma(u_0)|\mathcal{K}))\,. \qquad (iii)'$$

3.1 Bishop-Phelps cones

Let Y be a Banach space and $\mathcal{K}$ be a Bishop-Phelps cone, i.e.,

$$\mathcal{K}_\alpha = \{y \in Y \mid f(y) \geq \alpha \|y\| \|f\|\},$$

where f is a linear continuous functional on Y and $0 < \alpha < 1$. This is a closed convex pointed cone. If it is nontrivial, then $\mathcal{K}_\alpha$ has a bounded base Θ

$$\Theta = \{z \in \mathcal{K} \mid f(z) = 1\}.$$

The following characterisation holds.

Theorem 3.2 *Let Y be a Banach space, A a nonempty subset of Y and $a_0 \in Min(A|\mathcal{K}_\alpha)$. If there exists $\beta < \alpha$ such that $a_0 \in Min(A|\mathcal{K}_\beta)$, then $a_0 \in SPMin(A|\mathcal{K}_\alpha)$.*

Proof. We show that cone $\mathcal{K}_\beta$ satisfies the condition $(*)$ of the definition of strong proper minimality.

Let us take any $\varepsilon > 0$ and $z \in \mathcal{K}_\alpha$, $\|z\| \geq \varepsilon$. We have

$$\begin{aligned} f(z+o) = f(z) + f(o) &\geq \alpha \|f\| \|z\| + f(o) \\ &\geq \alpha \|z+o\| \|f\| - \alpha \|f\| \|o\| - \|f\| \|o\| \\ &\geq \|f\| \|z+o\| [\alpha - \tfrac{(\alpha+1)\|o\|}{\|z+o\|}] \\ &\geq \|f\| \|z+o\| [\alpha - \tfrac{(\alpha+1)\|o\|}{\varepsilon - \|o\|}]. \end{aligned}$$

To have $\alpha - \frac{(\alpha+1)\|o\|}{\varepsilon - \|o\|} \geq \beta$ we choose

$$\|o\| < \frac{(\alpha - \beta)\varepsilon}{2\alpha + 1 - \beta}.$$

□

In [7] Phelps proved the following result.

Theorem 3.3 *Let Y be a Banach space and A a nonempty closed subset of Y. If $\inf f(A) > -\infty$, then for any $a \in A$ there exists $a_0 \in A$ such that $a_0 \in (a - \mathcal{K}_\alpha)$ and a_0 is a cone support point, i.e., $A \cap (a_0 - \mathcal{K}_\alpha) = \{a_0\}$.*

Proof. Let $a = a_1 \in A$ and

$$A_1 = A \cap (a_1 - \mathcal{K}).$$

Having defined $a_1, \ldots, a_n$, we choose $a_{n+1} \in A_n$ such that

$$\inf f(A_n) > f(x_{n+1}) + 1/n.$$

Since $a_{n+1} \in a_n - \mathcal{K}_\alpha$ we have $a_{n+1} - \mathcal{K}_\alpha \subset a_n - \mathcal{K}_\alpha$ and $A_{n+1} \subset A_n$. Moreover, for $a \in A_{n+1}$ we have

$$f(a) \geq \inf f(A_{n+1}) \geq \inf f(A_n)$$

Now

$$\alpha \|f\| \|a_{n+1} - a\| \leq f(a_{n+1}) - f(a) \leq f(a_{n+1}) - \inf f(A_n \leq 1/n.$$

This means that $\operatorname{diam}(A_{n+1}) \leq \frac{2}{n\|f\|\alpha}$. By the completeness, $\bigcap A_n = \{a_0\}$. Moreover, for all n,

$$A \cap (a_0 - \mathcal{K}_\alpha) \subset A_n,$$

hence $A \cap (a_0 - \mathcal{K}_\alpha) = \{a_0\}$.

□

Theorem 3.4 *Let Y be a Banach space. Assume that*

(i) there exists a neighbourhood U_0 of u_0 such that all the sets $\Gamma(u)$ are closed and $\inf_{y\in\Gamma(u)} f(y) > -\infty$,

(ii)

$$y_0 \in cl(\bigcup_{\beta<\alpha} Min(\Gamma(u_0)|\mathcal{K}_\beta)) . \qquad (iv)$$

If Γ is l.c. at (y_0, u_0) and u.H.c. at u_0, then $\mathcal{M}$ is l.c. at (y_0, u_0) .

Proof. By Theorem 3.3, (DP) holds for all $\Gamma(u)$ in U_0 . By Theorem 3.2 we have

$$\mathrm{cl}(\bigcup_{\beta<\alpha} Min(\Gamma(u_0)|\mathcal{K}_\beta)) \subset SPMin(\Gamma(u)|\mathcal{K}_\alpha) .$$

Now, the assertion follows from Theorem 2.2 .

□

Theorem 3.4 is a variant of the following result proved by Attouch and Riahi [1].

Theorem 3.5 *Let Y be a Banach space, $\{D_n ;\, n \in N\}$ a sequence of closed nonempty subset which Painleve-Kuratowski converges to $D \subset Y$, and $\mathcal{K}$ is a closed pointed convex cone in Y , $\mathcal{K} \subset \{y \in Y\, ;\, l(x) + \varepsilon\|x\| \le 0\}$ for some $l \in Y^*$ and $\varepsilon > 0$.*

Suppose that the following conditions are satisfied:

(i) $\inf_{n\in N} \inf_{D_n} l > -\infty$,

(ii)for every $\rho > 0$ there exists a compact subset $K_\rho \subset Y$ such that for every $n \in N$

$$Min(D_n|\mathcal{K}) \cap \rho B \subset K_\rho .$$

Then $Min(D|\mathcal{K}) \neq \emptyset$ and

$$Min(D|\mathcal{K}) \subset \lim\inf_{n\to\infty} (Min(D_n|\mathcal{K}) .$$

We see that in Theorem 3.4 the condition (ii) of Theorem 3.5 is replaced by a weaker condition (iv), but a stronger type of convergence is used.

4 Density problems

The conditions (i) , (ii) , (iii) , and (iv) are density type assumptions. They express the property that $Min(A|\mathcal{K})$ is dense in the set of certain kinds of proper cone support points. This property has been investigated in many different setting and for different notions of properness (e.g., [3], [4], [6], [8]).

Here we cite the result of Borwein and Zhuang [3] which is particularily useful to our purposes.

We say that a subset A of Y is $\mathcal{K}$-lower bounded if there is some constant $M > 0$ such that

$$A \subset MB + \mathcal{K} .$$

A subset A is $\mathcal{K}-$lower bounded if either it is topologically bounded, i.e., $A \subset MB$ for some positive constant $M > 0$, or there exists an element m such that $a - m \in \mathcal{K}$ for all $a \in A$.

Theorem 4.1 (Borwein, Zhuang [3]) *Let Y be a Banach space, $\mathcal{K}$ an ordering cone and A a nonempty subset of Y . Assume that $\mathcal{K}$ has a closed and bounded base Θ . If either of the following conditions is satisfied, then $SE(A|\mathcal{K})$ is norm-dense in the nonempty set $Min(A|\mathcal{K})$:*

(i) A is weakly compact;

(ii) A is weakly closed and $\mathcal{K}-$lower bounded while Θ is weakly compact.

For convex sets the condition (ii) can be rewritten in the form

(ii)' A is closed and $\mathcal{K}-$lower bounded while Θ is weakly compact.

In view of this result we can rewrite Theorem 3.1 in the following form.

Theorem 4.2 *Let Y by a Banach space. Suppose that $\mathcal{K}$ posseses a weakly compact base, $\Gamma(u_0)$ is nonempty, closed and convex, $Min(\Gamma(u_0)|\mathcal{K})$ is bounded, and (DP) holds for all $\Gamma(u)$ in a certain neighbourhood of u_0.*

If Γ is l.c. and u.H.c. at u_0, then $\mathcal{M}$ is l.c. at u_0.

Proof. It is enough to observe that if $Min(\Gamma(u_0)|\mathcal{K})$ is bounded and (DP) holds for $\Gamma(u_0)$, then $\Gamma(u_0)$ is $\mathcal{K}-$lower bounded. Thus, by Theorem 4.1, $Min(\Gamma(u_0)|\mathcal{K}) \subset \mathrm{cl}(SE(\Gamma(u_0)|\mathcal{K}))$. Now, the assertion follows from Theorem 3.1. □

5 Concluding remarks

The results presented here for problems in normed spaces concern mainly cones with bounded bases. It was shown by Petschke [6] that each cone with a bounded base can be represented as a Bishop-Phelps cone. On the other hand, there exist important classes of cones which do not have bounded bases. It was shown by Dauer and Gallagher [4] that nonnegative orthants in the spaces l^p, and L^p for $1 < p < +\infty$, do not have bounded bases, and hence, weakly compact bases.

References

[1] Attouch H., Riahi H., Stability results for Ekeland's ε-variational principle and cone extremal solutions, Mathematics of OR(1993), vol.18,173-201

[2] Bednarczuk E.M., Berge-type theorems for vector optimization problems, optimization(1995), vol.32, 373-384

[3] Borwein J.M.,Zhuang D., Super efficiency in vector optimization, Trans.AMS(1993), vol.338,105-122

[4] Dauer J.P.,Gallagher R.J., Positive proper efficient points and related cone results in vector optimization theory, SIAM J.Cont.Optim.(1990), vol.28, 158-172

[5] Guerraggio a., Molho E., Zaffaroni A., On the notion of proper efficiency in vector optimization, J.O.T.A.(1994), vol.82, 1-21

[6] Petschke M., On a theorem of Arrow, Barankin, and Blackwell, SIAM J.Cont.Optim.(1990), vol.28, 395-401

[7] Phelps R., Support cones in Banach spaces, Advances in Mathematics(1994), vol.13, 1-19

[8] Jahn J., A generalization of a theorem of Arrow-Barankin-Blackwell, SIAM J.Cont.Optim.(1988),vol.26,995-1005

Unité de recherche INRIA Lorraine, Technopôle de Nancy-Brabois, Campus scientifique,
615 rue du Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY
Unité de recherche INRIA Rennes, Irisa, Campus universitaire de Beaulieu, 35042 RENNES Cedex
Unité de recherche INRIA Rhône-Alpes, 46 avenue Félix Viallet, 38031 GRENOBLE Cedex 1
Unité de recherche INRIA Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex
Unité de recherche INRIA Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex

Éditeur
INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
ISSN 0249-6399