Higher Order Multifractal Analysis

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Abstract: We extend the multifractal analysis to the description of the distribution of the n-th order correlation of singularities. For this purpose, new multifractal spectra are defined, and some general relations between them are given. We finally perform explicit computations in the case of deterministic multinomial measures, and show that the multifractal formalism does not hold for correlations on this class of measures.

Key-words: Hausdorff dimension, multifractal spectrum, multifractal correlation, multinomial measures.

(Résumé : tsvp)

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Analyse multifractale d'ordre supérieur

Résumé : Nous étendons l'analyse multifractale à la description de la distribution des corrélations d'ordre n des singularités. Dans cette optique, de nouveaux spectres multifractals sont définis, et nous donnons quelques résultats généraux liant ces spectres. Nous donnons également des expressions explicites de ces spectres dans le cas des mesures multinomiales déterministes, et montrons que le formalisme multifractal du deuxième ordre n'a pas lieu pour cette classe de mesures.

Mots-clé : dimension de Hausdorff, spectre multifractal, corrélations multifractales, mesures multinomiales.

1. Introduction

Multifractal analysis has recently drawn much attention as a tool for studying the structure of singular measures, both in theory and in applications. Efforts have mainly been focused on special cases such as self-similar and self-affine measures, both in the deterministic case [1, 2, 3, 4, 5, 6, 7] and in the random case [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. Multifractal analysis has also been extended to a larger class of set functions, namely the class of Choquet capacities [21]. Recent efforts have also been made to extend the analysis to point functions [22, 23, 24].

In the multifractal scheme, the pointwise structure of a singular measure is analyzed through the so-called "multifractal spectrum", which gives either geometrical or probabilistic information about the distribution of points having the same singularity. Several definitions of a multifractal spectrum (of a measure) appear in the litterature. Some are related to measure theory (Hausdorff and packing measures), other to large deviation theory and to probability (Rényi exponents and Legendre spectrum). The so-called "multifractal formalism" assesses that, in some situations, all the spectra coincide. One motivation for studying this multifractal formalism stems from the fact that both Hausdorff and packing dimensions are known to be very awkward to compute. The other definitions are more suited to applications, as they are easier to evaluate. The multifractal formalism is known to hold for the class of multiplicative measures [7] but it fails in general [7, 21].

Our concern in this work is different. The full characterization of the singularity structure of a sequence of Choquet capacities (and of measures) goes beyond the information contained in the multifractal spectra [21]. Other tools are needed if one wants to describe more precisely this singularity structure. A natural extension is to recognize that the usual spectra correspond to one point statistics, and to generalize the analysis to higher order statistics.

In this view, we introduce here the notion of *multifractal correlation*. We extend the definitions of the aforementionned spectra (section 2), and give general results relating them (sections 4 and 3). In section 5, we perform a two point statistics on a class of deterministic multiplicative measures (namely the multinomial measures) and show that the multifractal formalism does not hold in higher order statistics. The random case will be treated in a forthcoming paper.

2. Definitions

In what follows, we give definitions of quantities measuring the multifractal correlations between singularities of a given probability measure. The parameters we consider are just extensions of the Rényi exponents and the various multifractal spectra.

2.1. Notations. Let μ be a probability measure with support in $[0,1)$. For $n \in \mathbb{N}^\star$, let $\mathcal{P}_n := \{I_k^n\,;\ 0 \leq k < \nu_n\}$ be a partition of $[0,1)$, $\mathcal{P} := \bigcup_{n\in\mathbb{N}^\star} \mathcal{P}_n$, and $x_k^n := \inf I_k^n$ for all k.

Throughout the paper, we assume that the conditions (C1) through (C4) are met:

(C1) $\lim_{n\to\infty} \max_{0\leq j<\nu_n} |I_j^n| = 0$,

(C2) $\forall n, k,\quad I_k^n$ is an interval, semi-open to the right.

(C3) $\forall\ n,\ \forall\ j,\ 0 \leq j < \nu_n\ \ \exists\ k$ such that $I_j^n \subsetneqq I_k^{n-1}$, where $I_0^0 := E$.

(C4) $\forall\ \alpha > 0$,

$$\limsup_{I\in\mathcal{P},|I|\to 0} |I|^\alpha\, k(I) \leq 1$$

$$\text{where } k(I_j^n) := \sup\left\{\frac{|I_j^n|}{|I_k^{n+1}|}\,;\ I_k^{n+1} \subset I_j^n\right\}.$$

We stress the fact that, in our case, a multifractal analysis is relative to a fixed sequence of partitions $\mathcal{P}$. In particular, if $\mathcal{P}$ changes, all the quantities defined below (i.e. α, f_h, f_g, τ and f_l) may vary.

For $x \in [0,1)$ and $n \in \mathbb{N}$, let $I^n(x)$ be the interval I_k^n containing x.

Let $p \in \mathbb{N}$, $\varepsilon > 0$, $r := (r_n)_{n\in\mathbb{N}}$ a sequence of $(0,1)^p$, with $r_n := (r_n^1, r_n^2, \ldots, r_n^p)$ for all n. The sequence r is such that either $\lim_n r_n^i = 0$ for all i, or the sequence is constant.

Let $D := \liminf_n D^n$, where $D^n := [0, 1 - \max_{1\leq i\leq p} r_n^i)$.
For all $x \in D^n$, set

$$\alpha_n(x) := \frac{\log \mu(I^n(x))}{\log |I^n(x)|}$$

which is defined when $\mu(I^n(x)) \neq 0$, and

$$\forall\ n \in \mathbb{N}\quad \alpha_n(x,r)\ := (\alpha_n(x), \alpha_n(x + r_n^1), \ldots, \alpha_n(x + r_n^p))$$

$$\forall\ x \in D\quad \alpha(x,r)\ := \left(\lim_n \alpha_n(x), \lim_n \alpha_n(x + r_n^1), \ldots, \lim_n \alpha_n(x + r_n^p)\right)$$

when the considered limits exist.

Let $r^\infty := (r_n)_{n\in\mathbb{N}}$ be a sequence of $(0,1)^\infty$, with $r_n := (r_n^1, r_n^2, \ldots, r_n^p, \ldots)$, every sequence $(r_n^i)_n$ converging towards 0 for all i.

We define

$$\alpha_n(x, r^\infty)\ := (\alpha_n(x), \alpha_n(x + r_n^1), \ldots, \alpha_n(x + r_n^p), \ldots)\qquad \text{for all } n$$

$$\alpha(x, r^\infty)\ := \left(\lim_n \alpha_n(x), \lim_n \alpha_n(x + r_n^1), \ldots, \lim_n \alpha_n(x + r_n^p), \ldots\right)$$

where x belongs to $\liminf_n [0, 1 - \sup_i r_n^i)$.
Throughout the rest of the paper, we shall always refer to the case $p = 0$ by replacing the sequence r with the symbol $\emptyset$.
Thus,

$$\begin{aligned} \alpha_n(x, \emptyset) &:= \alpha_n(x) \\ \alpha(x, \emptyset) &:= \lim_n \alpha_n(x) \end{aligned}$$

2.2. Hausdorff spectrum of order p (or Hausdorff p-spectrum). We will use the following definition of the dimension of a set E, $\dim E$, which is similar to that of the classical Hausdorff dimension, except the fact that the coverings are restricted to the elements of $\mathcal{P}$.
Let

$$\begin{aligned} \mathcal{M}_\delta^s(E) &:= \inf\left\{\sum_{i=1}^{+\infty} \mathrm{diam}(E_i)^s \;/ E \subset \bigcup_i E_i, \quad \mathrm{diam}(E_i) \leq \delta, \quad E_i \in \mathcal{P} \;\; \forall i\right\} \\ \mathcal{M}^s(E) &:= \lim_{\delta \to 0} \mathcal{M}_\delta^s(E) \\ \dim E &:= \inf\{s \;/\; \mathcal{M}^s(E) = 0\} = \sup\{s \;/\; \mathcal{M}^s(E) = +\infty\} \end{aligned}$$

Since the elements of $\mathcal{P}$ satisfy conditions (C1) through (C4), the definition of dim coincides with that of the classical Hausdorff dimension [25].

Let $\alpha := (\alpha_0, \alpha_1, \ldots, \alpha_p) \in (\mathbb{R}^+)^{p+1}$.
Set

$$E_{\alpha,r} := \{x \in [0,1) \;/\; \alpha(x,r) = \alpha\}$$

for all $\alpha \in (\mathbb{R}^+)^{p+1}$.
The *Hausdorff p-spectrum* of μ is defined as

$$f_h(\alpha, r) := \dim E_{\alpha,r}$$

2.3. Large deviation p-spectrum. Let $n \in \mathbb{N}$ and

$$D^n_{k_0,\ldots,k_p} := I^n_{k_0} \bigcap \left(\bigcap_{i=1}^{p} (I^n_{k_i} - r^i_n)\right)$$

Notice that the $D^n_{k_0,\ldots,k_p}$ form a partition of $D^n := [0, 1 - \max_{1\leq i\leq p} r^i_n)$, and that $D^n = [0,1)$ if $p = 0$.
Hence, for all $x \in D^n$, we note $D^n(x)$ the (unique) interval $D^n_{k_0,\ldots,k_p}$ containing x.
Let $U^n := \{x \in D^n \,;\; \mu(I^n(x)) \prod_{i=1}^{p} \mu(I^n(x + r^i_n)) \neq 0\}$.

Set

$$G_\varepsilon^n(\alpha,r) := \left\{x \in U^n \;/\; \alpha_n(x,r) \in \prod_{i=1}^{p} [\alpha_i-\varepsilon, \alpha_i+\varepsilon]\right\}$$
$$K_\varepsilon^n(\alpha,r) := \{(k_n(x), k_n(x+r_n^1), \dots, k_n(x+r_n^p))\,;\; x \in G_\varepsilon^n(\alpha,r)\}$$

where $k_n(x)$ is the (unique) integer k such that $x \in I_k^n$.
Define, for all $\varepsilon, \beta > 0$,

$$S_\varepsilon^n(\alpha,\beta,r) := \sum_{(k_0,\dots,k_p)\in K_\varepsilon^n(\alpha,r)} |D_{k_0,\dots,k_p}^n|^\beta$$
$$S_\varepsilon(\alpha,\beta,r) := \limsup_n S_\varepsilon^n(\alpha,\beta,r)$$

(with the convention $\sum_\emptyset = 0$).
Using (C1), it is easy to show, by analogy with the Hausdorff dimension, that there exists a real number $f_g^\varepsilon(\alpha,r)$ such that

$$\beta < f_g^\varepsilon(\alpha,r) \implies S_\varepsilon(\alpha,\beta,r) = +\infty$$
$$\beta > f_g^\varepsilon(\alpha,r) \implies S_\varepsilon(\alpha,\beta,r) = 0$$

$f_g^\varepsilon(\alpha,r)$ is non decreasing in ε, and we note

$$f_g(\alpha,r) := \lim_{\varepsilon\downarrow 0} f_g^\varepsilon(\alpha,r) = \inf_{\varepsilon>0} f_g^\varepsilon(\alpha,r)$$

the *large deviation p-spectrum* of μ.

Remark: when $p = 0$ (no correlation), and when all the intervals have the same size ν_n^{-1}, it is straightforward (lemma 4) that, for all $\alpha \in \mathbb{R}^+$,

$$f_g(\alpha,\emptyset) = \lim_{\varepsilon\downarrow 0}\limsup_n \frac{\log \text{card } \{I_k^n\;;\; |I_k^n|^{\alpha+\varepsilon} \le \mu(I_k^n) \le |I_k^n|^{\alpha-\varepsilon}\}}{\log \nu_n}$$

which is the usual quantity considered in the litterature [26][14][3].

2.4. Legendre p-spectrum. Here we extend the definitions given in [7].
Let $(\lambda_n)_{n\ge 1}$ be a sequence of positive integers such that

$$\sum_{n>0} \exp(-\eta\lambda_n) < \infty \text{ for all } \eta > 0$$

Recall that $U^n := \{x \in D^n\,;\; \mu(I^n(x)) \prod_{i=1}^p \mu(I^n(x+r_n^i)) \neq 0\}$. For all $x \in D^n$,

$$\mu(I^n(x)) \prod_{i=1}^{p} \mu(I^n(x+r_n^i)) = \sum_{k_0,k_1,\dots,k_p=0}^{\nu_n-1} \mu(I_{k_0}^n) \prod_{i=1}^{p} \mu(I_{k_i}^p) \mathbb{1}_{D_{k_0,\dots,k_p}^n}(x)$$

which shows that the left-hand side quantity is a step function of x.
This proves that U^n is a finite union of semi-open intervals, and hence a Borel set.
For $q := (q_0,\dots,q_p) \in \mathbb{R}^{p+1}$ and $\tau \in \mathbb{R}$, we can now define

$$C_n(q,\tau,r) := \int_{U^n} \mu(I^n(x))^{q_0 \frac{\log|D^n(x)|}{\log|I^n(x)|}} \prod_{i=1}^{p} \mu(I^n(x+r_n^i))^{q_i \frac{\log|D^n(x)|}{\log|I^n(x+r_n^i)|}} \; |D^n(x)|^{-(\tau+1)} dx$$

and

$$C(q,\tau,r) = \limsup_n \lambda_n^{-1} \log C_n(q,\tau,r)$$

We suppose that $C(q,\tau,r)$ is not constantly equal to 0 or ∞ (this imposes the growth of the sequence $(\lambda_n)_n$).
Set

$$\Omega := \{(q,\tau) \in \mathbb{R}^{p+2} \ / \ C(q,\tau,r) < 0\}$$

One verifies that C is convex, non decreasing in τ, and non increasing in q_i for all i. A similar argument as the one found in [7] allows to show that there exists a concave map $\tau : q \mapsto \tau(q,r)$ such that

$$\mathring{\Omega} = \{(q,\tau) \in \mathbb{R}^{p+2} \ / \ \tau < \tau(q,r)\}$$

($\mathring{\Omega}$ is the interior of Ω).

We suppose that τ is finite on an open set $\mathcal{I}$ of $\mathbb{R}^{p+1}$ containing 0.
We define the *Legendre p-spectrum* f_l of μ as being the following Legendre transform of τ :

$$f_l(\alpha,r) := \inf_q [< q\,,\alpha > - \ \tau(q,r)]$$

where $< .\,,. >$ denotes the inner product in $\mathbb{R}^{p+1}$.

Remark 1: the definition of C_n can be also stated as follows : since the $D^n_{k_0,\dots,k_p}$ form a partition of D^n, we can write

$$\begin{aligned} C_n(q,\tau,r) &= \int_{U^n} |D^n(x)|^{q_0\alpha_n(x)+\sum_{i=1}^p q_i\alpha_n(x+r_n^i)-(\tau+1)} dx \\ &= \sum_{k_0,k_1,\dots,k_p=0}^{\nu_n-1}{}' |D^n_{k_0,\dots,k_p}|^{\sum_{i=0}^p q_i\alpha_n(x^n_{k_i})-\tau} \end{aligned}$$

where $'$ means that the summation runs through those indices $k_0,k_1,\dots,k_p$ such that $\prod_{i=0}^p \mu(I^n_{k_i}) \neq 0$.

Remark 2: when $p=0$, we have $D^n_k = I^n_k$, and we obtain the definitions considered in [7]. In particular, if all the intervals have the same size $\exp(-\lambda_n) = \nu_n^{-1}$, one can easily show (lemma 5) that we come up with the usual formulae

$$\begin{aligned} \tau_n(q,\emptyset) &= -\frac{1}{\log \nu_n} \log \sum_{k \ / \ \mu(I^n_k)\neq 0} \mu(I^n_k)^q \\ \tau(q,\emptyset) &= \liminf_n \tau_n(q,\emptyset) \end{aligned}$$

3. Particular cases

The definitions given above may be simplified when, for large n, all the intervals I^n_k have the same size $\exp(-\lambda_n) = \nu_n^{-1}$ for all k, and when $r^i_n \nu_n \in \mathbb{N}$.

Lemma 1.

$$D^n_{k_0,\dots,k_p} = \begin{cases} I^n_{k_0} & \textit{if } \; k_i = k_0 + r^i_n \nu_n \;\; \textit{for all } i \geq 1 \\ \emptyset & \textit{otherwise} \end{cases}$$

Proof. Assume $D^n_{k_0,\dots,k_p} \neq \emptyset$, and let $x \in D^n_{k_0,\dots,k_p}$. Then $k_n(x) = k_0$ and $k_n(x + r^i_n) = k_i$ for all $1 \leq i \leq p$.
Since $r^i_n \nu_n \in \mathbb{N}$, we have $k_i = k_n(x + r^i_n) = k_n(x) + r^i_n \nu_n = k_0 + r^i_n \nu_n$ (recall that $k_n(x) = [\nu_n \; x]$). If $k_i = k_0 + r^i_n \nu_n$, then $D^n_{k_0,\dots,k_p} = I^n_{k_0}$. Otherwise, $D^n_{k_0,\dots,k_p} = \emptyset$. □

Corollary 1.

$$\forall \; x \in D^n \qquad D^n_{k_n(x),k_n(x+r^1_n),\dots,k_n(x+r^p_n)} = I^n(x)$$

Proof. Since $k_n(x + r^i_n) = k_n(x) + r^i_n \nu_n$ for all $1 \leq i \leq p$, we deduce, using lemma 1,

$$D^n_{k_n(x),k_n(x+r^1_n),\dots,k_n(x+r^p_n)} = I^n_{k_n(x)} = I^n(x)$$

□

Set

$$\tilde{G}^n_\varepsilon(\alpha, r) := \{0 \leq k < \nu_n \, ; \alpha_n(x^n_k) \in [\alpha_0 - \varepsilon, \alpha_0 + \varepsilon], \alpha_n(x^n_k + r^i_n) \in [\alpha_i - \varepsilon, \alpha_i + \varepsilon], \; 1 \leq i \leq p\}$$

Lemma 2.

$$\tilde{G}^n_\varepsilon(\alpha, r) = \{k_n(x); \; x \in G^n_\varepsilon(\alpha, r)\}$$

Proof. Let $k \in \{k_n(x); \; x \in G^n_\varepsilon(\alpha, r)\}$. There exists x such that

$$\begin{aligned} \alpha_n(x) \quad &\in \quad [\alpha_0 - \varepsilon, \alpha_0 + \varepsilon] \\ \alpha_n(x + r^i_n) \quad &\in \quad [\alpha_i - \varepsilon, \alpha_i + \varepsilon], \qquad 1 \leq i \leq p \end{aligned}$$

with $k = k_n(x)$.
Since $r^i_n \nu_n \in \mathbb{N}$, we have $k_n(x + r^i_n) = k_n(x) + r^i_n \nu_n$ and

$$\begin{aligned} \alpha_n(x) \quad &= \quad \alpha_n(x^n_k) \\ \alpha_n(x + r^i_n) \quad &= \quad \alpha_n(x^n_k + r^i_n) \qquad 1 \leq i \leq p \end{aligned}$$

Thus, $k \in \tilde{G}^n_\varepsilon(\alpha, r)$.

Conversely, let $k \in \tilde{G}^n_\varepsilon(\alpha, r)$, and let x be any element of I^n_k (thus, $k_n(x) = k$). Then

$$\alpha_n(x) = \alpha_n(x^n_k)$$

and

$$\alpha_n(x + r^i_n) = \alpha_n(x^n_k + r^i_n) \qquad 1 \leq i \leq p$$

yielding

$$x \in G^n_\varepsilon(\alpha, r)$$

We conclude $k \in \{k_n(x); \; x \in G^n_\varepsilon(\alpha, r)\}$. □

Lemma 3.

$$K_\varepsilon^n(\alpha,r)=\left\{(k,k+r_n^1\nu_n,\dots,k+r_n^p\nu_n);\ k\in\tilde{G}_\varepsilon^n(\alpha,r)\right\}$$

Proof.

$$\begin{aligned}K_\varepsilon^n(\alpha,r) &:= \{(k_n(x),k_n(x+r_n^1),\dots,k_n(x+r_n^p))\,;\ x\in G_\varepsilon^n(\alpha,r)\}\\ &= \{(k_n(x),k_n(x)+r_n^1\nu_n,\dots,k_n(x)+r_n^p\nu_n)\,;\ x\in G_\varepsilon^n(\alpha,r)\}\\ &= \left\{(k,k+r_n^1\nu_n,\dots,k+r_n^p\nu_n)\,;\ k\in\tilde{G}_\varepsilon^n(\alpha,r)\right\} \qquad \text{(lemma 2)}\end{aligned}$$

□

Lemma 4.

$$f_g(\alpha,r)=\lim_{\varepsilon\downarrow 0}\limsup_n\frac{\log \mathit{card}\ \tilde{G}_\varepsilon^n(\alpha,r)}{\log\nu_n}$$

Proof.

$$\begin{aligned}S_\varepsilon(\alpha,\beta,r) &= \limsup_n\sum_{k\in\tilde{G}_\varepsilon^n(\alpha,r)}|I_k^n|^\beta \qquad \text{(lemmas 1 and 3)}\\ &= \limsup_n \nu_n^{-\beta}\text{card }\tilde{G}_\varepsilon^n(\alpha,r)\\ &= \limsup_n \exp\left(-\log\nu_n\left(\beta-\frac{\log\text{card }\tilde{G}_\varepsilon^n(\alpha,r)}{\log\nu_n}\right)\right)\end{aligned}$$

It is then straightforward that, for a given $\varepsilon>0$,

$$\begin{aligned}\beta>\limsup_n\frac{\log\text{card }\tilde{G}_\varepsilon^n(\alpha,r)}{\log\nu_n} &\implies S_\varepsilon(\alpha,\beta,r)=0\\ \beta<\limsup_n\frac{\log\text{card }\tilde{G}_\varepsilon^n(\alpha,r)}{\log\nu_n} &\implies S_\varepsilon(\alpha,\beta,r)=+\infty\end{aligned}$$

yielding

$$f_g^\varepsilon(\alpha,r)=\limsup_n\frac{\log\text{card }\tilde{G}_\varepsilon^n(\alpha,r)}{\log\nu_n}$$

□

Lemma 5. *$f_l(\alpha,r)=\inf_q\{<\alpha\,,q>-\tau(q,r)\}$ where*

$$\tau(q,r)=\liminf_n\frac{\log\sum_k{}'\mu(I_k^n)^{q_0}\prod_{i=1}^p\mu(I_k^n+r_n^i)^{q_i}}{\log\nu_n^{-1}}$$

$\sum'$ meaning that the summation runs through those indices k such that $\mu(I_k^n)\prod_{i=1}^p\mu(I_k^n+r_n^i)\neq 0$.

Proof. We have, using lemma 1,

$$\begin{aligned}C_n(q,\tau,r) &= \sum_k{}'|I_k^n|^{q_0\alpha_n(x_k^n)+\sum_{i=1}^p q_i\alpha_n(x_k^n+r_n^i)-\tau}\\ &= \nu_n^\tau\sum_k{}'\mu(I_k^n)^{q_0}\prod_{i=1}^p\mu(I_k^n+r_n^i)^{q_i}\end{aligned}$$

and

$$C(q,\tau,r) = \tau - \liminf_n \frac{\log \sum_k{}' \mu(I_k^n)^{q_0} \prod_{i=1}^p \mu(I_k^n + r_n^i)^{q_i}}{\log \nu_n^{-1}}$$

□

4. General properties

Let us recall some properties on the "classical" spectra $f_h(.,\emptyset)$, $f_g(.,\emptyset)$ and $f_l(.,\emptyset)$.

Theorem 1. [7, 21] *We have*

$$f_h(.,\emptyset) \leq f_g(.,\emptyset) \leq f_l(.,\emptyset)$$

Theorem 2. [21] *Let*

$$A_n := \left\{ \frac{\log \mu(I_k^n)}{\log |I_k^n|} ; 0 \leq k < \nu_n \right\}$$

If

$$\lim_n \frac{\log card\ A_n}{\lambda_n} = 0$$

then

$$f_l(.,\emptyset) = f_g^{\star\star}(.,\emptyset)$$

where * *denotes the Legendre transform. Therefore,* $f_l(.,\emptyset)$ *is the concave hull of* $f_g(.,\emptyset)$.

Our aim in this section is to extend these results to higher order statistics.

4.1. Comparison of f_h and f_g.

The following result holds:

Theorem 3.

$$f_h(.,r) \leq f_g(.,r)$$

for all sequence r *of* $(0,1)^p$, $p \in \mathbb{N}^*$.

Proof. Consider $p \geq 1$, $\alpha \in (\mathbb{R}^+)^{p+1}$, and assume that $E_{\alpha,r} \neq \emptyset$ (the case $E_{\alpha,r} = \emptyset$ is trivial).
Let $x \in E_{\alpha,r}$ and $\varepsilon > 0$. Then there exists $n_0 := n_0(x,\alpha,r)$ such that, for all $n \geq n_0$,

$$\begin{array}{rcl} \alpha_n(x) & \in & [\alpha_0 - \varepsilon, \alpha_0 + \varepsilon] \\ \alpha_n(x + r_n^i) & \in & [\alpha_i - \varepsilon, \alpha_i + \varepsilon] \qquad 1 \leq i \leq p \end{array}$$

Hence, $x \in G_\varepsilon^n(\alpha, r)$ and $(k_n(x), k_n(x+r_n^1), \ldots, k_n(x+r_n^p)) \in K_\varepsilon^n(\alpha,r)$. Since in addition $x \in D^n_{k_n(x),k_n(x+r_n^1),\ldots,k_n(x+r_n^p)}$, we obtain

$$\forall\, \varepsilon > 0 \quad x \in \bigcup_l \bigcap_{n \geq l} \bigcup_{(k_0,\ldots,k_p)\in K_\varepsilon^n(\alpha,r)} D^n_{k_0,\ldots,k_p}$$

Set $E^\varepsilon_{\alpha,l,r} := \bigcap_{n\geq l}\bigcup_{(k_0,\dots,k_p)\in K^n_\varepsilon(\alpha,r)} D^n_{k_0,\dots,k_p}$ and $s^\varepsilon_{\alpha,l,r} := \dim E^\varepsilon_{\alpha,l,r}$.
Clearly, for all $\varepsilon > 0$, $E_{\alpha,r} \subset \bigcup_l E^\varepsilon_{\alpha,l,r}$ and

$$\forall\ \varepsilon > 0 \qquad \dim E_{\alpha,r} \leq \sup_l s^\varepsilon_{\alpha,l,r}$$

Let $s,\delta \in \mathbb{R}^+$. For large n, we have $\max_{(k_0,\dots,k_p)} |D^n_{k_0,\dots,k_p}| \leq \delta$ (condition (C1)), and thus

$$\mathcal{H}^s_\delta\left(E^\varepsilon_{\alpha,l,r}\right) \leq \sum_{(k_0,\dots,k_p)\in K^n_\varepsilon(\alpha,r)} |D^n_{k_0,\dots,k_p}|^s =: S^n_\varepsilon(\alpha,s,r)$$

which yields

$$\mathcal{H}^s\left(E^\varepsilon_{\alpha,l,r}\right) \leq S_\varepsilon(\alpha,s,r)$$

and

$$\forall\ \varepsilon > 0 \quad \forall\ l \qquad s^\varepsilon_{\alpha,l,r} \leq f^\varepsilon_g(\alpha,r)$$

We conclude

$$f_h(\alpha,r) \leq f_g(\alpha,r)$$

□

4.2. Comparison of f_g and f_l.

Theorem 4. *Under conditions (C1) and (C2), we have*

$$f_g(.,r) \leq f_l(.,r)$$

for all sequence r of $(0,1)^p$, $p \in \mathbb{N}^*$.

Theorem 5. *Assume that conditions (C1) and (C2) are met. Let $p \in \mathbb{N}^\star$, and let r be a sequence of $(0,1)^p$.*
Set

$$A_n := \left\{\frac{\log \mu(I^n_k)}{\log |I^n_k|}\,; 0 \leq k < \nu_n\right\}$$

If

$$\lim_n \frac{\log \mathit{card}\ A_n}{\lambda_n} = 0$$

then

$$f_l(.,r) = f^{\star\star}_g(.,r)$$

where * denotes the Legendre transform. Therefore, f_l is the concave hull of f_g.

Proof of theorem 4. Let $\tau \in \mathbb{R}, q \in \mathbb{R}^{p+1}, \varepsilon > 0$ and α such that $f_g(\alpha, r) \neq -\infty$. For all $n \in \mathbb{N}$,

$$\begin{aligned}
C_n(q,\tau,r) &= \sum_{k_0,k_1,\dots,k_p=0}^{\nu_n-1} |D^n_{k_0,\dots,k_p}|^{q_0\alpha_n(x^n_{k_0})+\sum_{i=1}^p q_i\alpha_n(x^n_{k_i})-\tau} \\
&\geq \sum_{(k_0,\dots,k_p)\in K^n_\varepsilon(\alpha,r)} |D^n_{k_0,\dots,k_p}|^{q_0\alpha_n(x^n_{k_0})+\sum_{i=1}^p q_i\alpha_n(x^n_{k_i})-\tau} \\
&\geq \sum_{(k_0,\dots,k_p)\in K^n_\varepsilon(\alpha,r)} |D^n_{k_0,\dots,k_p}|^{\sum_{i=0}^p q_i(\alpha_i\pm\varepsilon)-\tau} \\
& \quad \text{(choose } + \text{ if } q_i \geq 0, - \text{ if } q_i < 0) \\
C_n(q,\tau,r) &\geq S^n_\varepsilon(\alpha, \sum_{i=0}^p q_i(\alpha_i \pm \varepsilon) - \tau, r)
\end{aligned}$$

Let $\tau < \tau(q,r)$. Then there exists $c > 0$ such that, for large n, $C_n(q,\tau,r) < \exp(-c\lambda_n)$, implying $S_\varepsilon(\sum_{i=0}^p q_i(\alpha_i \pm \varepsilon) - \tau, r) = 0$, and hence $\sum_{i=0}^p q_i(\alpha_i \pm \varepsilon) - \tau \geq f_g^\varepsilon(\alpha, r)$ for all $\varepsilon > 0$.
Letting ε tend to 0 gives

$$<\alpha\,,q> - \ \tau(q,r) \geq f_g(\alpha,r) \quad \forall q$$

We conclude

$$f_g(\alpha,r) \leq f_l(\alpha,r)$$

□

Proof of theorem 5. From theorem 4, we already have $f_g^{\star\star} \leq f_l$. Let us prove the opposite inequality.
Let $q := (q_0, q_1, \dots, q_p) \in \mathbb{R}^{p+1}$.
By assumption, there exists an integer $N(n)$ such that

$$\left\{(\alpha_n(x^n_{k_0}),\dots,\alpha_n(x^n_{k_p}))\,;\ k_0,\dots,k_p = 0,\dots,\nu_n - 1\right\} = \bigcup_{i=1}^{N(n)} \{\alpha_{n,i}\}$$

with

$$\lim_n \frac{\log N(n)}{\lambda_n} = 0$$

Let $K_{n,i} := \left\{(k_0,\dots,k_p)\,;\alpha_{n,i} = (\alpha_n(x^n_{k_0}),\dots,\alpha(x^n_{k_p}))\right\}$.
Then

$$C_n(q,\tau,r) = \sum_{i=1}^{N(n)} \sum_{(k_0,\dots,k_p)\in K_{n,i}} |D^n_{k_0,\dots,k_p}|^{<q\,,\alpha_{n,i}> \ -\tau}$$

Set

$$i(n) \ := \text{argmax}_{i=1,\dots,N(n)} \sum_{(k_0,\dots,k_p)\in K_{n,i}} |D^n_{k_0,\dots,k_p}|^{<q\,,\alpha_{n,i}> \ -\tau}$$

($i(n)$ may be not unique. In that case, take the smallest one, for instance).
This yields (set α_n for $\alpha_{n,i(n)}$ and K_n for $K_{n,i(n)}$)

$$C_n(q,\tau,r) \leq N(n) \sum_{(k_0,\dots,k_p)\in K_n} |D^n_{k_0,\dots,k_p}|^{<q\,,\alpha_n> \ -\tau} \qquad (1)$$

This inequality holds in particular for those indices n' such that $C(q,\tau,r) = \lim_{n'} \lambda_{n'}^{-1} \log C_{n'}(q,\tau,r)$. Furthermore, we can extract from $(\alpha_{n'})$ a sequence (α_j) converging towards a limit denoted $\bar{\alpha} := (\bar{\alpha}_0, \bar{\alpha}_1, \ldots, \bar{\alpha}_p)$.
More precisely,

$$\forall\, \varepsilon > 0 \;\; \exists\, j_0 \;\; \forall\, j \geq j_0 \qquad \alpha_j \in \prod_{i=0}^{p} [\bar{\alpha}_i - \varepsilon, \bar{\alpha}_i + \varepsilon]$$

Since $\alpha_j \in K_j$ for all j, we deduce

$$\forall\, \varepsilon > 0 \;\; \exists\, j_0 \;\; \forall\, j \geq j_0 \qquad K_j \subset K_\varepsilon^j(\bar{\alpha}, r)$$

Inequality (1) reads, for large j,

$$\forall\, q \qquad C_j(q,\tau,r) \;\leq\; N(j) \sum_{(k_0,\ldots,k_p)\in K_\varepsilon^j} |D^j_{k_0,\ldots,k_p}|^{\sum_{i=0}^{p} q_i(\bar{\alpha}_i \pm \varepsilon) - \tau}$$

$$C_j(q,\tau,r) \;\leq\; N(j)\; S_\varepsilon^j(\bar{\alpha}, \sum_{i=0}^{p} q_i(\bar{\alpha}_i \pm \varepsilon) - \tau, r)$$

This inequality, along with the hypothesis of the theorem, yields

$$\forall\, \varepsilon > 0 \qquad \tau(q,r) \geq \sum_{i=0}^{p} q_i(\bar{\alpha}_i \pm \varepsilon) - f_g^\varepsilon(\bar{\alpha}_i, r)$$

and

$$\tau(q,r) \geq \,<\bar{\alpha}\,, q> - f_g(\bar{\alpha}, r) \geq f_g^\star(q,r)$$

We conclude

$$f_g^{\star\star} \geq f_l$$

□

5. Two point statistics of multinomial measures

In this section, our aim is to compute the spectra defined in the section 2 on a specific class of measures, namely the multinomial measures for the case $p = 1$. We also show that the multifractal formalism does not hold in higher order statistics. We first recall the definitions of such measures and some of their properties. We close this section by giving the values of the spectra.

5.1. Multinomial measures : definitions and properties.

Let B be an integer, $B \geq 2$. Let $(m_0, m_1, \ldots, m_{B-1}) \in (0,1)^B$ such that $\sum_{i=0}^{B-1} m_i = 1$.
We construct a sequence $(\mu_n)_n$ of probability measures on $[0,1)$ as follows : start with μ_0 = Lebesgue measure. Split the interval $[0,1)$ in B intervals of equal size, say $\{I_{i_1}\,;\; 0 \leq i_1 < B\}$. Define a set function μ_1 on $[0,1)$ by $\mu_1(I_{i_1}) = m_{i_1}$. A well known theorem of Caratheodory's asserts that this set function defines a probability measure on $[0,1)$. Iterate the process on each interval I_{i_1}, so as to obtain B^2 intervals $\{I_{i_1,i_2}\,;\; i_1, i_2 = 0, \ldots, B-1\}$ of size B^{-2}. Then define a probability measure μ_2 such that $\mu_2(I_{i_1,i_2}) = m_{i_1} m_{i_2}$.

This iterative construction yields a sequence $(\mu_n)_n$ of probability measures defined on $[0,1)$, and a sequence $\{I_{i_1,i_2,\dots,i_n}\,;\; i_1,i_2,\dots,i_n = 0,\dots,B-1\}$ of partitions of $[0,1)$ in B-adic intervals such that $|I_{i_1,i_2,\dots,i_n}| = B^{-n}$, and $\mu_n(I_{i_1,i_2,\dots,i_n}) = m_{i_1}m_{i_2}\dots m_{i_n}$. This sequence possesses a weak* limit μ called *multinomial measure* of base B, with weights $(m_0, m_1, \dots, m_{B-1})$.

The multinomial measures possess the following well-known properties [27, 26, 7] :

Proposition 1.

Consider $x \in [0,1)$ and its B-adic expansion $x = \sum_{k\geq 1} x_k\, B^{-k}$ with $x_k \in \{0,\dots,B-1\}$ for all k (non-terminating expansion are not considered here). For all $i < B$, let $\varphi_i(x) := \lim\limits_{n\to\infty} \text{card}\,\{k \leq n\,;\; x_k = i\}/n$ be the proportion of i's in the expansion of x (when this limit exists).

The following results hold :

1. *$\alpha(x)$ exists almost everywhere, and equals $-\sum_{i=0}^{B-1} \varphi_i(x)\log_B m_i$.*
2. *For all $q \in \mathbb{R}$, $\tau(q,\emptyset) = -\log_B(\sum_{i=0}^{B-1} m_i^q)$.*
3. *$f_l(\alpha,\emptyset) = -\infty$ if and only if $\alpha \notin D := [\alpha_{\min}, \alpha_{\max}]$, with $\alpha_{\min} := -\log_B \max_i(m_i)$, $\alpha_{\max} := -\log_B \min_i(m_i)$.*
4. *For all $\alpha \in \overset{o}{D}$, $\exists !\; q_\alpha \in \mathbb{R}$ such that*

$$\alpha = \tau'(q_\alpha,\emptyset)$$

   and $f_l(\alpha,\emptyset) = \alpha q_\alpha - \tau(q_\alpha,\emptyset)$.
5. *$\lim\limits_{q\to+\infty} \tau'(q,\emptyset) = \alpha_{\min}$, $\lim\limits_{q\to-\infty} \tau'(q,\emptyset) = \alpha_{\max}$.*
6. *$f_l(\alpha,\emptyset) = -\sum_{i=0}^{B-1} \varphi_i(\alpha)\log_B \varphi_i(\alpha)$ where $(\varphi_0(\alpha),\dots,\varphi_{B-1}(\alpha))$ is a solution of the constrained problem*

$$\left\{\begin{array}{c} \min(\sum_{i=0}^{B-1} \varphi_i \log_B \varphi_i) \\ \sum_{i=0}^{B-1} \varphi_i = 1 \\ -\sum_{i=0}^{B-1} \varphi_i \log_B m_i = \alpha \end{array}\right.$$

   where α is known (with the convention $0\log 0 = 0$).
7. *$f_h(.,\emptyset) = f_g(.,\emptyset) = f_l(.,\emptyset)$.*

Proof. See [27, 26, 7] for instance. □

Let μ be a multinomial measure with weights $(m_0, m_1, \dots, m_{B-1})$, where $B \geq 2$. In what follows, we consider a sequence $r := (r_n)_n$ whose general term is $r_n := B^{-p(n)}$, $(p(n))_n$ being a non decreasing sequence of integers chosen such that $\beta := \lim\limits_{n\to+\infty} \dfrac{p(n)}{n}$ exists. In the following, we consider only the case $p(n) \leq n$, the case $p(n) > n$ leading to a trivial analysis.

For $(\alpha, \alpha') \in (\mathbb{R}^+)^2$, we shall write $f_h(\alpha, \alpha', r)$, $f_g(\alpha, \alpha', r)$ and $f_l(\alpha, \alpha', r)$ for $f_h((\alpha, \alpha'), r)$, $f_g((\alpha, \alpha'), r)$ and $f_l((\alpha, \alpha'), r)$ respectively. In addition, when $p = 0$, the symbol $\emptyset$ will be omitted.

Since the computations of these spectra require several lemmas, we state the results in this section and postpone all the proofs until section 6.

We have the following results:

Proposition 2. *(Hausdorff spectrum)*

$$f_h(\alpha, \alpha', r) = \begin{cases} f_h(\alpha, \emptyset) & \text{if } \alpha = \alpha' \\ -\infty & \text{if } \alpha \neq \alpha' \end{cases}$$

The computation of f_h is rather simple, and leads to a trivial result which does not give much new information. For this reason, we shall compute another function, also defined in terms of a Hausdorff dimension, which allows a finer analysis of the measure in terms of two point statistics.
Let

$$\tilde{E}_{\alpha,\alpha',r} := \left\{ t \in [0,1)\,;\ \lim_n \alpha_n(t) = \alpha \text{ and } \limsup_n \alpha_n(t + r_n) = \alpha' \right\}$$

This aforementionned function is defined as follows:

$$F_h(\alpha, \alpha', r) := \dim \tilde{E}_{\alpha,\alpha',r}$$

Set $\alpha_0 = -\log_B m_0$ and $\alpha_{B-1} = -\log_B m_{B-1}$.
If $m_0 \neq m_{B-1}$, set

$$\lambda := \frac{\alpha' - \alpha}{\alpha_0 - \alpha_{B-1}} \quad \text{and} \quad \alpha_c := \alpha_c(\alpha, \alpha') := \frac{\alpha' \alpha_{B-1} - \alpha \alpha_0}{(\alpha_{B-1} - \alpha_0) + (\alpha' - \alpha)}$$

Proposition 3. *(F_h spectrum)*

- *If* $m_0 \geq m_{B-1}$,

$$F_h(\alpha, \alpha', r) = \begin{cases} f_h(\alpha) & \text{if } \alpha' = \alpha \\ -\infty & \text{if } \alpha' \neq \alpha \end{cases}$$

- *If* $m_0 < m_{B-1}$,

$$F_h(\alpha, \alpha', r) = \begin{cases} f_h(\alpha) & \text{if } (\alpha' = \alpha) \\ 0 & \text{if } \alpha = \alpha_{B-1} \text{ and } \alpha' < \alpha(1-\beta) + \beta\alpha_0 \\ -\infty & \text{otherwise} \end{cases}$$

Thus, if for instance $\beta = 1$, all correlations between (α, α') occur when $\alpha = \alpha_{B-1}$ and $\alpha' \in [\alpha_{B-1}, \alpha_0]$.

REMARK 1: In the definition of $\tilde{E}_{\alpha,\alpha',r}$, we could as well consider a lim inf instead of a lim sup. The relation between $\alpha_n(t + r_n)$ and $\alpha_n(t)$ (see page 28) shows that this is equivalent to interchanging the role of the weitghts m_0 and m_{B-1} in the computation of all the three spectra.

Proposition 4. *(Legendre spectrum)*

- *If* $m_0 \neq m_{B-1}$,

$$f_l(\alpha, \alpha', r) = \begin{cases} (1-\lambda)\ f_l(\alpha_c) & \textit{if } \alpha_c \in\ (\alpha_{\min}, \alpha_{\max}) \ \textit{ and } \ \lambda \neq 1, \lambda \geq 0 \\ f_l(\alpha) & \textit{if } \alpha = \alpha' \in \{\alpha_{\min}\,;\alpha_{\max}\} \\ 0 & \textit{if } \alpha = \alpha_{B-1} \ \textit{ and } \ \alpha' = \alpha_0 \\ -\infty & \textit{otherwise} \end{cases}$$

- *If* $m_0 = m_{B-1}$,

$$f_l(\alpha, \alpha', r) = \begin{cases} f_l(\alpha) & \textit{if } \ \alpha = \alpha' \\ -\infty & \textit{if } \ \alpha \neq \alpha' \end{cases}$$

Proposition 5. *(Large deviation spectrum)*

$$f_g \equiv f_l$$

REMARK 2: In particular, when $m_0 > m_{B-1}$, we have

$$f_h(\alpha_{B-1}, \alpha_0, r) = F_h(\alpha_{B-1}, \alpha_0, r) = -\infty < 0 = f_l(\alpha_{B-1}, \alpha_0, r)$$

i.e. a second order analog of the "multifractal formalism" does not hold for multinomial measures.

REMARK 3: Let B be an integer greater than 1, μ be a multinomial measure with weights $(m_0, \ldots, m_{B-1})$, and η be a multinomial measure with weights $(p_0, \ldots, p_{B-1})$. μ and η have the same first order spectra if and only if $\{m_0, \ldots, m_{B-1}\} = \{p_0, \ldots, p_{B-1}\}$. Thus, for instance if $B = 2$, the first order spectra do not differentiate between μ_2 with weights $M = (m_0, m_1)$, and η_2 with weights $\tilde{M} = (m_1, m_0)$. It is straightforward to check that the second order spectra of μ_2 and ν_2 do differ. It is also easy to prove that the second order spectra allow to differentiate two trinomial measures as long as all the weights are different. More generally, we conjecture that for $2 \leq p \leq \infty$, the p-th order spectra are different for all p-nomial measures, even when the sets of weights are the same.

6. Proofs

In the computations of the spectra, we widely use B-adic expansions of real numbers of $[0,1)$. Throughout the paper, only terminating expansions are considered.

6.1. Computation of $\tau(q,q',r)$. Note $x_k^n := 0.a_1a_2\dots a_n$ the B-adic expansion of x_k^n, with $a_i \in \{0,1,\dots,B-1\}$. For sake of simplicity, let us write p for $p(n)$.
χ_n can be written

$$\begin{aligned}\chi_n(q,q',r) &= \sum_{x_k^n/a_p<B-1} \mu(I_k^n)^q\ \mu(I_k^n+B^{-p})^{q'} + \sum_{x_k^n/a_p=B-1} \mu(I_k^n)^q\ \mu(I_k^n+B^{-p})^{q'} \\ &:= S_0+S_1\end{aligned}$$

6.1.1. *Computation of S_0.* In this case, only the p-th digit of x_k^n and $x_k^n+B^{-p}$ are different (one can notice that $x_k^n+B^{-p}$ remains in $[0,1)$). Set $i := a_p$ and $|N| := n_0+n_1+\dots+n_{B-1}$ with $N := (n_0,n_1,\dots,n_{B-1})$, each n_i being the number of i's in the B-adic expansion of x_k^n. Clearly,

$$\mu(I_k^n+B^{-p}) = \frac{m_{i+1}}{m_i}\ \mu(I_k^n)$$

and thus

$$\begin{aligned}S_0 &= \sum_{i=0}^{B-2}\sum_{|N|=n-1}\binom{n-1}{n_0,n_1,\dots,n_{B-1}}(m_0^{n_0}m_1^{n_1}\dots m_{B-1}^{n_{B-1}}m_i)^{q+q'}\left(\frac{m_{i+1}}{m_i}\right)^{q'} \\ &= \left[\sum_{i=0}^{B-2}\left(\frac{m_{i+1}}{m_i}\right)^{q'} m_i{}^{q+q'}\right]\left[\sum_{|N|=n-1}\binom{n-1}{n_0,n_1,\dots,n_{B-1}}(m_0^{q+q'})^{n_0}(m_1^{q+q'})^{n_1}\dots(m_{B-1}^{q+q'})^{n_{B-1}}\right] \\ S_0 &= \left(\sum_{i=0}^{B-2} m_i^q m_{i+1}^{q'}\right)\left(\sum_{i=0}^{B-1} m_i{}^{q+q'}\right)^{n-1}\end{aligned}$$

6.1.2. *Computation of S_1.* Suppose that within the first $p-1$ first digits, the last l digits have value $B-1$, and that $a_{p-l-1}\le B-2$:

$$x_k^n = 0.\underbrace{a_1\dots a_{p-l-2}}\ a_{p-l-1}\ \ \underbrace{(B-1)\dots(B-1)}_{a_{p-l}\text{ through }a_{p-1}}\ \ (B-1)\underbrace{a_{p+1}\dots a_n}$$

When adding 1 to the p-th digit, a_p changes to 0, and so do the l preceding digits, whereas the $(p-l-1)$-th digit changes its value to $a_{p-l-1}+1\le B-1$:

$$x_k^n+B^{-p} = 0.\underbrace{a_1\dots a_{p-l-2}}\,(a_{p-l-1}+1)\ \underbrace{0\dots0}\ 0\ \underbrace{a_{p+1}\dots a_n}$$

We are led to the following conditions on l: since we assume that there are $l+1$ digits having value $B-1$, the first condition we get is $l+1\le n_{B-1}$. The second condition is also straightforward: $l+1\le p-1$. Thus, we must have $-1\le l\le \min(n_{B-1}-1,p-2)$

(by convention, the case $l = -1$ refers to the case $a_p \le B-2$, which leads to the computation of S_0). We then come up with the following result (set $i := a_{p-l-1}$):

$$\mu(I_k^n + B^{-p}) = \frac{m_0^{l+1} m_{i+1}}{m_{B-1}^{l+1} m_i}\ \mu(I_k^n)$$

For a fixed l, the number of x_k^n's having such an expansion is $\begin{pmatrix} n-l-2 \\ n_0, n_1, \dots, n_{B-1} \end{pmatrix}$ with $|N| = n-l-2$. We deduce that

$$\begin{aligned}
S_1 &= \sum_{i=0}^{B-2}\sum_{l=0}^{p-2}\sum_{|N|=n-l-2}\begin{pmatrix} n-l-2 \\ n_0, n_1, \dots, n_{B-1} \end{pmatrix}(m_0^{n_0} m_1^{n_1}\dots m_{B-1}^{n_{B-1}} m_{B-1}^{l+1} m_i)^{q+q'}\left(\frac{m_0^{l+1} m_{i+1}}{m_{B-1}^{l+1} m_i}\right)^{q'} \\
&= \sum_{i=0}^{B-2}\sum_{l=0}^{p-2}\sum_{|N|=n-l-2}\begin{pmatrix} n-l-2 \\ n_0, n_1, \dots, n_{B-1} \end{pmatrix}(m_0^{q+q'})^{n_0}(m_1^{q+q'})^{n_1}\dots(m_{B-1}^{q+q'})^{n_{B-1}} \times \\
&\qquad\qquad (m_0^{q'})^{l+1}(m_{B-1}^{q})^{l+1} m_i{}^q (m_{i+1})^{q'} \\
&= m_0^{q'} m_{B-1}^q \left[\sum_{i=0}^{B-2} m_i{}^q (m_{i+1})^{q'}\right] \times \\
&\quad \left\{\sum_{l=0}^{p-2}(m_0^{q'} m_{B-1}^q)^l \left[\sum_{|N|=n-l-2}\begin{pmatrix} n-l-2 \\ n_0, n_1, \dots, n_{B-1} \end{pmatrix}(m_0^{q+q'})^{n_0}(m_1^{q+q'})^{n_1}\dots(m_{B-1}^{q+q'})^{n_{B-1}}\right]\right\} \\
&= m_0^{q'} m_{B-1}^q \left[\sum_{i=0}^{B-2} m_i{}^q m_{i+1}^{q'}\right]\left[\sum_{l=0}^{p-2}(m_0^{q'} m_{B-1}^q)^l \left(\sum_{k=0}^{B-1} m_k^{q+q'}\right)^{n-l-2}\right] \\
&= m_0^{q'} m_{B-1}^q \left[\sum_{i=0}^{B-2} m_i{}^q m_{i+1}^{q'}\right]\left[\sum_{k=0}^{B-1} m_k^{q+q'}\right]^{n-2} \sum_{l=0}^{p-2}\left[\frac{m_0^{q'} m_{B-1}^q}{\sum_{k=0}^{B-1} m_k^{q+q'}}\right]^l \\
&= m_0^{q'} m_{B-1}^q \left[\sum_{i=0}^{B-2} m_i{}^q m_{i+1}^{q'}\right]\left[\sum_{k=0}^{B-1} m_k^{q+q'}\right]^{n-2} \quad \xi(p-1, M(q,q'))
\end{aligned}$$

with

$$M(q,q') := \frac{m_0^{q'} m_{B-1}^q}{\sum_{k=0}^{B-1} m_k^{q+q'}}$$

and

$$\xi(p,M) \quad = \quad \begin{cases} \dfrac{M^p - 1}{M-1} & \text{if } M \neq 1 \\ p & \text{if } M = 1 \end{cases}$$

Combining this result with S_0 gives

$$\chi_n(q,q',B^{-p(n)}) = \left[\sum_{i=0}^{B-2} m_i{}^q m_{i+1}^{q'}\right]\left[\sum_{k=0}^{B-1} m_k^{q+q'}\right]^{n-1} \xi(p(n), M(q,q'))$$

and

$$\begin{aligned}
\tau(q,q',r) &= -\lim_{n\to+\infty}\frac{1}{n}\log_B \chi_n(q,q',B^{-p(n)}) \\
&= \tau(q+q') - \lim_{n\to+\infty}\frac{1}{n}\log_B \xi(p(n),M(q,q')) \\
\tau(q,q',r) &= \tau(q+q') - \beta\max(0,L(q,q',r))
\end{aligned}$$

with $L(q,q',r) := \tau(q+q') + q\log_B m_{B-1} + q'\log_B m_0$.

6.2. Computation of $f_l(\alpha,\alpha',r)$.

For sake of simplicity, we shall omit the sequence r in $L(q,q')$.
Set

$$\begin{aligned}
J(q,q') &:= \alpha q + \alpha q' - \tau(q+q') \\
Q &:= \{(q,q')\in\mathbb{R}^2\ ;\ L(q,q')\le 0\}
\end{aligned}$$

Recall that $\tau(q,q',r) = \tau(q+q') - \beta\max(0,L(q,q'))$. Therefore, $f_l(\alpha,\alpha',r) = \inf_{(q,q')\in Q} J(q,q')$. The computation of f_l is quite tedious, and its value depends upon whether m_0 and m_{B-1} are equal. Since the infimum is seeked over Q, we need to derive some properties of this set depending on the values of m_0 and m_{B-1}. This is done in the following lemmas.

Lemma 6.

1. $\forall i\in\{0,\ldots,B-1\}$, $\forall q\in\mathbb{R}\quad \tau(q) < -q\log_B m_i$.
2. $m_0 = m_{B-1} \Longrightarrow Q = \mathbb{R}^2$.

Proof.

1. Clearly, for all i and q, $\sum_{k=0}^{B-1} m_k^q > m_i^q$ and thus $\tau(q) < -q\log_B m_i$.
2. $m_0 = m_{B-1} \Longrightarrow L(q,q') = \tau(q+q') + (q+q')\log_B m_0$. From the property 1, we deduce $L(q,q') < 0$ for all $q,q'\in\mathbb{R}^2$.

□

Lemma 7.

$\forall u\in\mathbb{R}\ \ \exists q\in\mathbb{R}$ *such that* $(q,u-q)\in Q$.

Proof. By definition of L and using lemma 6,

$L(q,u-q) = \tau(u) + q\log_B m_{B-1} + (u-q)\log_B m_0 < (u-q)(\log_B m_0 - \log_B m_{B-1})$

Then choose q veryfing $(u-q)(\log_B m_0 - \log_B m_{B-1}) \le 0$ (if $m_0 = m_{B-1}$, the result is straightforward). □

We will also need the following lemma

Lemma 8. *If* $m_0 \neq m_{B-1}$,

$$\forall\ u \in \mathbb{R} \ \ \exists ! q \in \mathbb{R} \text{ such that } L(q, u-q) = 0$$

Proof. Trivial. □

Let a, b be two fixed elements of $[\alpha_{\min}, \alpha_{\max}]$, $a \neq b$, and define, for all $x, y \in [\alpha_{\min}, \alpha_{\max}]$ such that $(x-y)/(b-a) \in \mathbb{R}^+ \setminus \{1\}$,

$$(E) \qquad A(x,y) := \frac{ax - by}{x - y + a - b}$$

The following results hold:

Lemma 9.
⋆ *Assume* $a > b$, *and note* (x_0, y_0) *a solution of* $A(x,y) = \alpha_{\min}$.
If $b \neq \alpha_{\min}$, $(x_0, y_0) = (\alpha_{\min}, \alpha_{\min})$.
If $b = \alpha_{\min}$, (x_0, y_0) *takes the form* $(\alpha_{\min}, y_0)$, *where* y_0 *spans* $[\alpha_{\min}, \alpha_{\max}] \setminus \{a\}$.
The same results hold when replacing $\alpha_{\min}$ *with* $\alpha_{\max}$.

⋆ *Assume* $a < b$, *and note* (x_0, y_0) *a solution of* $A(x,y) = \alpha_{\min}$.
If $a \neq \alpha_{\min}$, $(x_0, y_0) = (\alpha_{\min}, \alpha_{\min})$.
If $a = \alpha_{\min}$, (x_0, y_0) *takes the form* $(x_0, \alpha_{\min})$, *where* x_0 *spans* $[\alpha_{\min}, \alpha_{\max}] \setminus \{b\}$.
The same results hold when replacing $\alpha_{\min}$ *with* $\alpha_{\max}$.

Proof.
Let us notice some properties of the mapping A:

- A is strictly increasing in x if $y < a$.
- A is strictly decreasing in x if $y > a$.
- A is strictly increasing in y if $x > b$.
- A is strictly decreasing in y if $x < b$.
- $A(x, a) = a$ for all $x \neq b$.
- $A(b, y) = b$ for all $y \neq a$.
- $A(x, x) = x$ for all x.

Let us seek for solutions to (E) other than $(\alpha_{\min}, \alpha_{\min})$, and under the assumption $a > b$. This implies that A is defined only when $y \geq x$.

(1) <u>If $b \neq \alpha_{\min}$</u> (i.e. $b > \alpha_{\min}$).
CLAIM 1. If $x_0 \neq \alpha_{\min}$ (i.e. $x_0 > \alpha_{\min}$), then $y_0 < a$.
Indeed, if $y_0 = a$, then $b > a = A(x_0, y_0) = \alpha_{\min}$ which is contradictory. If $y_0 > a$, we have $\alpha_{\min} = A(x_0, y_0) \geq A(y_0, y_0) = y_0 > a$, which is also contradictory.
CLAIM 2. $x_0 = \alpha_{\min}$.
Assume that $x_0 > \alpha_{\min}$.

⋆ If $x_0 < b$, then $\alpha_{\min} = A(x_0, y_0) > A(x_0, a) = a$, which is contradictory.

⋆ If $x_0 > b$, and since $y_0 \geq x_0$, we have $\alpha_{\min} = A(x_0, y_0) \geq A(x_0, x_0) = x_0$, which is contradictory.

⋆ If $x_0 = b$, $\alpha_{\min} = A(x_0, y_0) = b$, which is contradictory.

Thus we proved

$$x_0 = \alpha_{\min}$$

CLAIM 3. $y_0 = \alpha_{\min}$.

Assume $y_0 > \alpha_{\min}$. Since $x_0 = \alpha_{\min}$, we have $x_0 < b$ and $y_0 > x_0$, which implies $\alpha_{\min} = A(x_0, y_0) < A(x_0, x_0) = x_0$ which is contradictory.

Hence, if $b \neq \alpha_{\min}$, then $x_0 = y_0 = \alpha_{\min}$.

(2) If $b = \alpha_{\min}$, one only needs to solve the equation $A(x, y) = \alpha_{\min}$ which yields $x_0 = \alpha_{\min}$ and y_0 is any value in $[\alpha_{\min}, \alpha_{\max}] \setminus \{a\}$.

The reasoning is similar when searching for solutions to $A(x, y) = \alpha_{\max}$: prove that if $a \neq \alpha_{\max}$, then $y_0 \neq \alpha_{\max}$ implies $x_0 > b$, and then deduce that $y_0 = \alpha_{\max}$. The result will then follow after proving that $x_0 \neq \alpha_{\max}$ cannot hold.

When $a < b$, the result is obtained from the previous one by interchanging the role of a and b, and of x and y. □

If (q, q') in Q minimizes J, then either one of the Kuhn and Tucker conditions is met :

(H1)	$\nabla J(q, q') = 0$	if $L(q, q') < 0$
(H2)	$\exists \lambda \geq 0 \;\; \nabla J(q, q') + \lambda \nabla L(q, q') = 0$	if $L(q, q') = 0$

⋆ Assume that (H1) is satisfied. This implies

$$\begin{cases} \alpha - \tau(q + q') = 0 \\ \alpha' - \tau(q + q') = 0 \end{cases}$$

If $\alpha \neq \alpha'$, there are no elements in Q minimizing J and satisfying (H1), and we are then led to treating the case (H2).

If $\alpha = \alpha'$, then

$$\begin{aligned} f_l(\alpha, \alpha', r) &= \inf_{(q,q') \in Q} \{\alpha(q + q') - \tau(q + q')\} \\ &= \inf_{q \in \mathbb{R}} \{\alpha q - \tau(q)\} \quad \text{(lemma 7)} \\ &= f_l(\alpha) \end{aligned}$$

⋆ Assume that (H2) is satisfied. The corresponding equality reads

$$(E) \;\; \exists \lambda \geq 0 \;\; \begin{cases} \alpha - \tau'(q + q') + \lambda(\tau'(q + q') + \log_B m_{B-1}) = 0 \\ \alpha' - \tau'(q + q') + \lambda(\tau'(q + q') + \log_B m_0) = 0 \\ L(q, q') = 0 \end{cases}$$

which gives

$$\alpha + \lambda \log_B m_{B-1} = \alpha' + \lambda \log_B m_0 \tag{2}$$

- If $m_0 = m_{B-1}$, (2) yields $\alpha = \alpha'$ and, by lemma 6, $Q = \mathbb{R}^2$.
  - If $\alpha \neq \alpha'$, then, as in the case (H1), the infimum is not reached, and $f_l(\alpha, \alpha', r) = -\infty$.
  - If $\alpha = \alpha'$, then

$$f_l(\alpha, \alpha, r) = \inf_{(q,q')\in\mathbb{R}^2} \{\alpha(q+q') - \tau(q+q')\} = f_l(\alpha)$$

- If $m_0 \neq m_{B-1}$, (2) reads

$$\lambda := \lambda(\alpha, \alpha') := \frac{\alpha' - \alpha}{\alpha_0 - \alpha_{B-1}}$$

  - If $\lambda(\alpha, \alpha') < 0$, then (E) does not hold, and $f_l(\alpha, \alpha', r) = -\infty$.
  - If $\lambda(\alpha, \alpha') = 1$, (E) implies $\alpha = -\log_B m_{B-1}$ and $\alpha' = -\log_B m_0$. If these two equalities are met, then for all q, q', $J(q, q') = \alpha q + \alpha q' - \tau(q+q') = -L(q, q') = 0$, and hence $f_l(\alpha, \alpha', r) = 0$.
    If $(\alpha, \alpha') \neq (-\log_B m_{B-1}, -\log_B m_0)$, then $f_l(\alpha, \alpha', r) = -\infty$.
  - If $\lambda(\alpha, \alpha') \geq 0$ and $\lambda(\alpha, \alpha') \neq 1$, then (E) implies

$$\tau'(q+q') = \alpha_c(\alpha, \alpha') \tag{3}$$

    * If $\alpha_c(\alpha, \alpha') \in (\alpha_{\min}, \alpha_{\max})$, then there exists $\bar{q} \in \mathbb{R}$ such that $\tau'(\bar{q}) = \alpha_c$, and thus, using lemma 8,

$$\exists\ (q, q') \in \partial Q \text{ such that } \tau'(q+q') = \alpha_c, \bar{q} = q + q'$$

Furthermore, we know that (proposition 1)

$$\begin{aligned}
f_l(\alpha_c) &= \alpha_c \bar{q} - \tau(\bar{q}) \\
&= \alpha_c(q+q') - \tau(q+q') \\
&= \alpha_c(q+q') + q \log_B m_{B-1} + q' \log_B m_0 \\
&= q \frac{\alpha + \log_B m_{B-1}}{1-\lambda} + q' \frac{\alpha' + \log_B m_0}{1-\lambda} \\
&= \frac{1}{1-\lambda} (\alpha q + \alpha' q' + q \log_B m_{B-1} + q' \log_B m_0) \\
&= \frac{1}{1-\lambda} (\alpha q + \alpha' q' - \tau(q+q')) \\
&= \frac{1}{1-\lambda} f_l(\alpha, \alpha', r)
\end{aligned}$$

which can be rewritten

$$f_l(\alpha, \alpha', r) = (1-\lambda) f_l(\alpha_c(\alpha, \alpha'))$$

$*$ If $\alpha_c = \alpha_{\min}$, then, by lemma 9, either $\alpha = \alpha' = \alpha_{\min}$, which gives $f_l(\alpha, \alpha, r) = f_l(\alpha)$, or $\alpha = \alpha_{\min}$ and $\alpha' > \alpha$. In this case, $f_l(\alpha, \alpha', r) = -\infty$. Indeed, let $u \in \mathbb{R}^+$ and q_u such that $L(u - q_u, q_u) = 0$ (as in lemma 8). Notice than $q_u \leq 0$. Then $J(u - q_u, q_u) = \alpha_{\min} u - \tau(u) + (\alpha' - \alpha_{\min}) q_u$, and $\lim_{u \to +\infty} J(u - q_u, q_u) = \lim_{u \to +\infty} f_l(\alpha_{\min}) + (\alpha' - \alpha_{\min}) q_u = -\infty$, and we conclude $f_l(\alpha, \alpha', r) = -\infty$.

In the same manner, if $\alpha_c = \alpha_{\max}$, then either $\alpha = \alpha' = \alpha_{\max}$, which implies $f_l(\alpha, \alpha', r) = f_l(\alpha)$, or $\alpha = \alpha_{\max}$ and $\alpha' > \alpha$, which in turn implies $f_l(\alpha, \alpha', r) = -\infty$.

6.3. Computation of $F_h(\alpha, \alpha', r)$. Let $t \in [0,1)$, $t := \sum_{i=1}^{\infty} t_i B^{-i}$ with $t_i \in \{0, \ldots, B-1\}$, and $n \in \mathbb{N}$. From now on, we shall use the following definitions:

- $n_i(t,n) := \#\{k \leq n\,; t_k = i\}$ for $i \in \{0, \ldots, B-1\}$.
- $\varphi_i(t) := \lim_n n_i(t,n)/n$ when the limit exists.
- $l(t,n) := n - 1 - \max\{i < n\,;\ t_i < B-1\}$ = number of succesive digits equal to $B-1$, starting backwards from the $(n-1)$-th digit. Notice that $l(t,n) = 0$ if $t_{n-1} < B-1$.
- $\bar{l}(t) := \limsup_{n \to +\infty} l(t, p(n))/n$.

It will be also useful to consider packets of $B-1$'s in the expansion of t. In that view, we define two sequences $(a_n)_n$ and $(b_n)_n$ as follows:

$$a_1 := \min\{i\,; t_i = B-1\} \qquad b_1 := \min\{i > a_1\,; t_i < B-1\}$$

and, for all $n > 1$,

$$a_n := \min\{i > b_{n-1}\,; t_i = B-1\} \qquad b_n := \min\{i > a_n\,; t_i < B-1\}$$

Set

$$\Theta := \bigcup_{n \geq 1} \{a_n,\ a_{n+1}, \ldots, b_n - 1\}$$

Θ is the set of indices of all packets of $B-1$'s in the B-adic expansion of t, i.e.

$$i \in \Theta \Leftrightarrow t_i = B-1$$

Lemma 10. *Let $t \in [0,1)$. Then*

$$\liminf_{n \to +\infty} \frac{l(t, p(n))}{n} = 0$$

Proof. If $\beta = 0$, the result follows from the property $l(t, p(n)) < p(n)$.
Assume $\beta > 0$, and define the sequence $(\sigma_n)_n$ as follows: $\sigma_1 := 1$, $\sigma_{n+1} := \min\{i\,; p(i) > p(\sigma_n)\}$.

CLAIM 1. $\lim \dfrac{\sigma_{n+1}}{\sigma_n} = 1$.
Clearly, $\sigma_{n+1} > \sigma_n$ since $p(\sigma_{n+1}) > p(\sigma_n)$. Thus, $\lim_n \sigma_n = +\infty$ and $\liminf_n \sigma_{n+1}/\sigma_n \geq 1$.
In addition, we have

$$p(\sigma_n) \geq p(\sigma_{n+1} - 1) \Rightarrow \limsup_n \frac{\sigma_{n+1}}{\sigma_n} = \limsup_n \frac{p(\sigma_{n+1} - 1)}{p(\sigma_n)} \leq 1$$

CLAIM 2. $\liminf\limits_{n \to +\infty} \dfrac{l(t, p(n))}{n} = 0$.
Assume $\liminf\limits_{n \to +\infty} \dfrac{l(t, p(n))}{n} > 0$. There exists a constant A such that, for large n, $l(t, p(n)) \geq A\, n > 0$. In particular, this implies

$$t_{p(\sigma_n)-1} = B - 1$$

when n is large.
Moreover, since non terminating expansions are not considered here, we have

$$\forall n \; \exists n_0 := n_0(n) \geq n \quad \text{such that} \quad l(t, p(\sigma_{n_0})) < p(\sigma_{n_0}) - p(\sigma_{n_0 - 1}) \tag{4}$$

Indeed, if (4) does not hold, then

$$\exists n_0 \; \forall n \geq n_0 \quad l(t, p(\sigma_n)) \geq p(\sigma_n) - p(\sigma_{n-1})$$

Since $t_{p(\sigma_n)-1} = B - 1$, this implies

$$l(t, p(\sigma_n)) = p(\sigma_n) - p(\sigma_{n-1})$$

and $t_n = B - 1$ for large n, i.e. the expansion is non terminating, which is contradictory.
Set $u_n := \sigma_{n_0(n)}$. Inequality (4) reads

$$\frac{l(t, p(u_n))}{u_n} < \frac{p(u_n)}{u_n} \left(1 - \frac{p(u_{n-1})}{u_{n-1}} \frac{u_n}{p(u_n)} \frac{u_{n-1}}{u_n}\right)$$

We conclude

$$\liminf_{n \to +\infty} \frac{l(t, p(n))}{n} \leq \beta \left(1 - \lim_n \frac{u_{n-1}}{u_n}\right) = 0$$

which is in contradiction with the hypothesis of claim 2. □

Lemma 11. *Let $t \in [0, 1)$. If $\varphi_{B-1}(t)$ exists, then*

$$\bar{l}(t) > 0 \Rightarrow \varphi_{B-1}(t) = 1$$

Proof.
CLAIM 1. $\bar{l}(t) > 0 \Longrightarrow \liminf_{n\to+\infty} \frac{a_n}{b_n} < 1$.
Let $(\sigma_n)_n$ be a sequence of $\mathbb{N}$ such that

$$\bar{l}(t) = \lim_n \frac{l(t, p(\sigma_n))}{\sigma_n}$$

and set $I := \{n\,; t_{p(\sigma_n)-1} = B-1\}$. Clearly, I is not empty, nor finite. Otherwise, $l(t, p(\sigma_n)) = 0$ for large n, and $\bar{l}(t) = 0$. We can assume, without loss of generality, that $t_{p(\sigma_n)-1} = B-1$ for all n, i.e. $p(\sigma_n)-1 \in \Theta$. Denote by $i(n)$ the (unique) integer such that

$$a_{i(n)} < p(\sigma_n) \leq b_{i(n)}$$

We have

$$\frac{l(t, p(\sigma_n))}{\sigma_n} = \frac{p(\sigma_n) - a_{i(n)}}{\sigma_n} \leq \frac{b_{i(n)} - a_{i(n)}}{a_{i(n)}}$$

implying

$$0 < \bar{l}(t) \leq \limsup_{n\to+\infty} \frac{b_n - a_n}{a_n}$$

CLAIM 2. $\liminf_{n\to+\infty} \frac{a_n}{b_n} < 1 \Rightarrow \varphi_{B-1}(t) = 1$.
Clearly, $\lim_n a_n = +\infty$. Otherwise, for large n, $(a_n)_n$ would be constant and $l(t, p(n)) = 0$ for large n, yielding $\bar{l}(t) = 0$.
Set $L := \liminf_{n\to+\infty} \frac{a_n}{b_n} < 1$. By definition of $(a_n)_n$ and $(b_n)_n$,

$$\forall\, n \quad n_{B-1}(t, b_n) = n_{B-1}(t, a_n) + b_n - a_n - 1$$

which gives

$$\forall\, n \quad \frac{n_{B-1}(t, b_n)}{b_n} = \frac{a_n}{b_n}\frac{n_{B-1}(t, a_n)}{a_n} + 1 - \frac{a_n}{b_n} - \frac{1}{b_n}$$

Since $\varphi_{B-1}(t)$ exists, we obtain

$$\varphi_{B-1}(t) = L\ \varphi_{B-1}(t) + 1 - L$$

which reads

$$\varphi_{B-1}(t) = 1$$

□

Lemma 12. *Let $l \in [0, \beta]$. There exists $t \in [0, 1)$ such that*

$$\varphi_{B-1}(t) = 1 \quad \textit{and} \quad \bar{l}(t) = l$$

In order to prove lemma 12, we shall first prove the following result :

Lemma 13. *Assume $\lim_n p(n) = +\infty$. Let $(u_n)_{n\in\mathbb{N}^*}$ be a sequence of $\mathbb{N}$ such that*

- $u_{n+1} \geq u_n$.
- $\lim_n u_n = +\infty$.
- $\liminf_n \frac{p(u_n)}{p(u_{n+1})} = r \in [0, 1]$

If the real number $t := \sum_{i\geq 1} t_i B^{-i}$ *is such that*

$$t_i < B-1 \Longleftrightarrow \exists\ n \in \mathbb{N}^* \quad i = p(u_n)$$

then

$$\bar{l}(t) = \beta(1-r)$$

Proof. The real t has the following B-adic expansion

$$t = 0.\underbrace{(B-1)\dots(B-1)}\bullet\underbrace{(B-1)\dots(B-1)}\bullet\underbrace{(B-1)\dots(B-1)}\bullet\underbrace{(B-1)\dots}$$

$$\uparrow \qquad\qquad \uparrow \qquad\qquad \uparrow$$

$$\text{digit } \#p(u_1) \qquad \text{digit } \#p(u_2) \qquad \text{digit } \#p(u_3)$$

where every $\bullet$ can represent any integer other that $B-1$.
Clearly,

$$\bar{l}(t) \geq \limsup_n \frac{l(t, p(u_n))}{u_n} = \limsup_n \left(\frac{p(u_n) - p(u_{n-1}) - 1}{u_n}\right) = \beta(1-r)$$

Let us prove the opposite inequality. Let $k \in \mathbb{N}^\star$ and set $n(k) := \max\{i\,;\ p(u_i) < p(k)\}$, i.e. $n(k)$ verifies $p(u_{n(k)}) < p(k) \leq p(u_{n(k)+1})$.
Notice that $\lim_k n(k) = +\infty$. Indeed, we have $i \leq n(k) \Leftrightarrow p(u_i) < p(k)$. Thus, the inequality $p(k) \leq p(k+1) \leq p(u_{n(k+1)})$ gives $n(k) < n(k+1)$. Hence, $\lim_k n(k) = +\infty$.
Furthermore,

$$l(t, p(k)) = p(k) - p(u_{n(k)}) - 1$$

and

$$\bar{l}(t) = \limsup_k \left(\frac{p(k)}{k} - \frac{p(u_{n(k)})}{k} - \frac{1}{k}\right) = \beta\left(1 - \liminf_k \frac{p(u_{n(k)})}{p(k)}\right)$$

Let us show that $\liminf_k \dfrac{p(u_{n(k)})}{p(k)} \geq r$. The definition of $n(k)$ yields

$$\frac{p(u_{n(k)})}{p(k)} \geq \frac{p(u_{n(k)})}{p(u_{n(k)+1})}$$

and since $\lim_k n(k) = +\infty$,

$$\liminf_k \frac{p(u_{n(k)})}{p(k)} \geq r$$

We obtain

$$\bar{l}(t) \leq \beta(1-r)$$

which finishes the proof. $\square$

Proof of lemma 12. We shall treat the cases $l = 0$, $l \in (0, \beta)$ and $l = \beta$ separately.

- CASE $l = 0$.
  A possible choice of t is

  $$t := 0.(B-1)0(B-1)(B-1)0(B-1)(B-1)(B-1)0(B-1)(B-1)(B-1)(B-1)0\dots$$

  It satisfies $\bar{l}(t) = 0$. To see this, denote by $(z_n)_{n\geq 1}$ the sequence of indices of the null digits, i.e. $z_n := \sum_{i=1}^n i + n = n(n+3)/2$. Let $k > 1$ and $n(k)$ be the (unique) integer such that $z_{n(k)} < k \leq z_{n(k)+1}$. It is straightforward to verify that $\lim_k n(k)/k = 0$. Then

$$l(t,k) = k - z_{n(k)} - 1 \leq z_{n(k)+1} - z_{n(k)} - 1 = \frac{n(k)-1}{2}$$

yields

$$\lim_k \frac{l(t,k)}{k} = 0$$

This in turn shows that

$$\bar{l}(t) = \limsup_{n\to+\infty} \frac{l(t,p(n))}{n} = \limsup_{n\to+\infty} \frac{l(t,p(n))}{p(n)} \frac{p(n)}{n} \leq 0 \times \beta = 0$$

The real t also satisfies $\varphi_{B-1}(t) = 1$. Indeed,

$$n_{B-1}(t,k) = k - n(k)$$

and we conclude

$$\varphi_{B-1}(t) := \lim_k \frac{n_{B-1}(t,k)}{k} = 1$$

- CASE $l \in (0,\beta)$.
  Set $r := 1 - \dfrac{l}{\beta} \in (0,1)$. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of $\mathbb{N}$ defined as

$$u_n := \min\{i\,;\; p(i) \geq \sum_{k=1}^{n} [r^{-k}] + n\} \tag{5}$$

Such a sequence is well-defined, since the hypothesis $\beta > 0$ implies $\lim_n p(n) = +\infty$.

Thus,

$$p(u_n - 1) < \sum_{k=1}^{n} [r^{-k}] + n \leq p(u_n) \tag{6}$$

The sequence $(u_n)_{n\in\mathbb{N}}$ posseses the following properties:

$$u_{n+1} \geq u_n \tag{7}$$

$$\lim_n u_n = +\infty \tag{8}$$

$$\lim_n \frac{p(u_n)}{p(u_{n+1})} = r \tag{9}$$

Inequalities (6) give

$$p(u_{n+1}) \geq \sum_{k=1}^{n+1} [r^{-i}] + n + 1 > \sum_{k=1}^{n} [r^{-k}] + n > p(u_n - 1)$$

Since the mapping $n \mapsto p(n)$ is not decreasing, we deduce $u_{n+1} > u_n - 1$ which proves (7).

Equality (8) follows from the definition (5): $u_n \geq p(u_n) \geq n$.

To prove (9), notice that

$$\lim_n \frac{p(n-1)}{p(n)} = \lim_n \frac{p(n-1)}{n-1} \frac{n}{p(n)} \frac{n-1}{n} = 1$$

Inequalities (6) yield

$$\frac{p(u_n-1)}{p(u_{n+1})} < \frac{\sum_{i=1}^{n}[r^{-i}]+n}{\sum_{i=1}^{n+1}[r^{-i}]+n+1} < \frac{p(u_n)}{p(u_{n+1}-1)}$$

which gives

$$\limsup_n \frac{p(u_n-1)}{p(u_{n+1})} \leq r \leq \liminf_n \frac{p(u_n)}{p(u_{n+1}-1)}$$

Using (8), we deduce (9).

We can now use lemma 13 to deduce

$$\bar{l}(t) = \beta(1-r) = l$$

Let us now show that $\varphi_{B-1}(t)$ exists and equals 1.

Let $k \in \mathbb{N}$ and set $n(k) := \max\{i\,;\ p(u_i) \leq k\}$, i.e. $n(k)$ is such that $p(u_{n(k)}) \leq k < p(u_{n(k)+1})$.

Let $n \geq \left[\frac{\log(k+1)}{-\log r}\right]$. We have $r^{-(n+1)} > k+1$ and

$$p(u_{n+1}) \geq \sum_{i=1}^{n+1}[r^{-i}] + n + 1 > r^{-(n+1)} > k$$

which yields

$$n(k) \leq \left[\frac{\log(k+1)}{-\log r}\right]$$

It is straightforward that $n_{B-1}(t,k) = k - n(k)$, which gives $\varphi_{B-1}(t) = 1$.

- CASE $l = \beta$.

For $n \in \mathbb{N}^*$, set

$$u_n := \min\{i\,;\ p(i) \geq \sum_{k=1}^{n} k^k + n\}$$

As for the $l \in (0,\beta)$ case, one can show that the hypotheses of lemma 13 are satisfied, with $r = 0$.

Thus,

$$\bar{l}(t) = \beta = l$$

In addition, we have $\varphi_{B-1}(t) = 1$. To see this, let $k \in \mathbb{N}^*$ and $n(k) := \max\{i\,;\ p(u_i) \leq k\}$.

Choose $n \geq [\sqrt{k}]$. We have

$$p(u_{n+1}) \geq (n+1)^{(n+1)} \geq (n+1)^2 > k$$

yielding $n(k) \leq n$. Thus, for all $k \in \mathbb{N}^*$, $n(k) \leq [\sqrt{k}]$. As in the case treated above, we deduce $\varphi_{B-1}(t) = 1$.

□

Lemma 14. *Let* $B > 2$, $I \in \{0,\ldots,B-1\}$ *and* $P := \{(\varphi_0,\ldots,\varphi_{B-1}) \in [0,1]^B\,;\ \sum_{j<B} \varphi_j = 1\}$.

1. *If* $m_I > m_k$ *for all* $k \neq I$, *or* $m_I < m_k$ *for all* $k \neq I$, *then*

$$\sum_{k<B} \varphi_k \log_B m_k = \log_B m_I \Longleftrightarrow \varphi_I = 1,\ \varphi_k = 0 \text{ for all } k \neq I$$

2. *Otherwise, there exists* $(\varphi_0, \dots, \varphi_{B-1}) \in P$ *such that* $\sum_{k<B} \varphi_k \log_B m_k = \log_B m_I$ *and* $\varphi_I < 1$.

Proof of 1. We only treat the case $m_I > m_k$ for all $k \neq I$ (the other case can be easily deduced from this one).
Suppose that $\varphi_I < 1$. Then there exists $j \neq I$ such that $\varphi_j \neq 0$. We have

$$\sum_{k<B} \varphi_k \log_B m_k = \sum_{k<B\,;\varphi_k \neq 0} \varphi_k \log_B m_k < \log_B m_I \sum_{k<B\,;\varphi_k \neq 0} \varphi_k = \log_B m_I$$

□

Proof of 2. If there exists $k \neq I$ such that $m_k = m_I$, then the proof is straightforward: choose $\varphi_k = 1$ and $\varphi_j = 0$ for all $j \neq k$.
Now consider the case where $m_k \neq m_I$ for all $k \neq I$, $m_I \neq \max_k m_k$ and $m_I \neq \min_k m_k$. Let k, j be such that $m_k < m_I < m_j$. Then choose φ_k, φ_j in $[0,1]$ such that

$$\begin{cases} \varphi_k \log \dfrac{m_k}{m_I} + \varphi_j \log \dfrac{m_j}{m_I} & = 0 \\ \varphi_k + \varphi_j \in (0,1] \end{cases}$$

This guarantees that the vector $(\varphi_0, \dots, \varphi_{B-1})$, defined by $\varphi_i = 0$ for all $i \notin \{k, j, I\}$ and $\varphi_I = 1 - \varphi_j - \varphi_k$, belongs to P, and that $\sum_{i<B} \varphi_i \log_B m_i = \log_B m_I$. □

Lemma 15. *If* $E_{-\log_B m_{B-1}} \bigcap \{t;\ \varphi_{B-1}(t) < 1\} \neq \emptyset$, *then*

$$\dim \left(E_{-\log_B m_{B-1}} \bigcap \{t;\ \varphi_{B-1}(t) < 1\}\right) = \dim E_{-\log_B m_{B-1}}$$

Proof. Clearly, $\{t;\ \varphi_{B-1}(t) = 1\} \subset E_{-\log_B m_{B-1}}$, and thus, using a theorem of Billingsley's, $\dim (E_{-\log_B m_{B-1}} \bigcap \{t;\ \varphi_{B-1}(t) = 1\}) = 0$.
Since $E_{-\log_B m_{B-1}} = (E_{-\log_B m_{B-1}} \bigcap \{t;\ \varphi_{B-1}(t) = 1\}) \bigcup (E_{-\log_B m_{B-1}} \bigcap \{t;\ \varphi_{B-1}(t) < 1\})$, we deduce

$$\dim \left(E_{-\log_B m_{B-1}} \bigcap \{t;\ \varphi_{B-1}(t) < 1\}\right) = \dim E_{-\log_B m_{B-1}}$$

□

We can now turn to the proof of propositions 2 and 3.
Let $t \in [0,1)$ be such that $\alpha(t)$ exists. Set $l := l(t, p(n))$ and $i := a_{p(n)-l-1}$.
We have (see page 17)

$$\begin{aligned} \alpha_n(t + B^{-p(n)}) &= \frac{\log(\mu_n(t + B^{-p(n)}))}{\log(B^{-n})} \\ &= -\frac{l(t,p(n)) + 1}{n} \log_B \left(\frac{m_0}{m_{B-1}}\right) - \frac{1}{n} \log_B \frac{m_{i+1}}{m_i} + \alpha_n(t) \quad (10) \end{aligned}$$

Proof of proposition 2.

Recall that $f_h(\alpha,\alpha',r) := \dim E_{\alpha,\alpha',r}$ where

$$E_{\alpha,\alpha',r} = \{t\,;\ \alpha(t)=\alpha, \lim_n \alpha_n(t+B^{-p(n)}) = \alpha' \text{ when both limits exist}\}$$

Equality (10) and lemma 10 yield $\lim_{n\to+\infty} \alpha_n(t+B^{-p(n)}) = \lim_{n\to+\infty} \alpha_n(t)$ whenever $\lim_n \dfrac{l(t,p(n))}{n}$ exists. Hence, if $\alpha \neq \alpha'$, it is clear that $f_h(\alpha,\alpha',r) = -\infty$. If $\alpha=\alpha'$, $f_h(\alpha,\alpha,r) = f_h(\alpha,\emptyset)$. Therefore, the f_h spectrum is very trivial, and a need for another function capable of a finer analysis arise. For this reason, we have defined the F_h spectrum (see page 14). □

Proof of proposition 3.

As mentionned in the remark at page 14, we can consider a $\overline{\lim}$ as well as a $\underline{\lim}$ in the definition of $\tilde{E}_{\alpha,\alpha',r}$ without loosing in generality. Switching from one definition to another is equivalent to switching the values of m_0 and m_{B-1}. Indeed, let us examine the two following cases :

(1) $m_0 \le m_{B-1}$: Equation (10) gives

$$\begin{aligned}
\limsup_{n\to+\infty} \alpha_n(t+B^{-p(n)}) &= \bar{l}(t)\ \log_B \frac{m_{B-1}}{m_0} + \alpha(t) \\
\liminf_{n\to+\infty} \alpha_n(t+B^{-p(n)}) &= \alpha(t) \qquad \text{(see lemma 10)}
\end{aligned}$$

(2) $m_0 \ge m_{B-1}$:

$$\begin{aligned}
\limsup_{n\to+\infty} \alpha_n(t+B^{-p(n)}) &= \alpha(t) \\
\liminf_{n\to+\infty} \alpha_n(t+B^{-p(n)}) &= \bar{l}(t)\ \log_B \frac{m_{B-1}}{m_0} + \alpha(t) \qquad \text{(see lemma 10)}
\end{aligned}$$

Let us consider the two following cases : $m_0 \ge m_{B-1}$ and $m_0 < m_{B-1}$.

<u>CASE $m_0 \ge m_{B-1}$</u> Thus, by lemma 10,

$$\limsup_{n\to+\infty} \alpha_n(t+B^{-p(n)}) = \alpha(t)$$

which leads to

$$\tilde{E}_{\alpha,\alpha',r} = \begin{cases} \{t\,;\alpha(t)=\alpha\} & \text{if } \alpha=\alpha' \\ \emptyset & \text{if } \alpha\neq\alpha' \end{cases}$$

We deduce the F_h spectrum :

$$F_h(\alpha,\alpha',r) := \begin{cases} f_h(\alpha) & \text{if } \alpha=\alpha' \\ -\infty & \text{if } \alpha\neq\alpha' \end{cases}$$

<u>CASE $m_0 < m_{B-1}$</u> In this case,

$$\limsup_{n\to+\infty} \alpha_n(t+B^{-n}) = \bar{l}(t)\ c_0 + \alpha(t)$$

where $c_0 := \log_B \dfrac{m_{B-1}}{m_0}$.
Set

$$L_{\alpha,\alpha',r} := \left\{ t \in [0,1), ;\ \bar{l}(t) = \frac{\alpha' - \alpha}{c_0} \right\}$$

Clearly,

$$\tilde{E}_{\alpha,\alpha',r} = E_\alpha \bigcap L_{\alpha,\alpha',r}$$

- **Case** $\dfrac{\alpha'-\alpha}{c_0} \notin [0,\beta]$: then $\tilde{E}_{\alpha,\alpha',r} = \emptyset$ and $F_h(\alpha,\alpha',r) = -\infty$.
- **Case** $\alpha' = \alpha$

  – **If** $\alpha = -\log_B m_{B-1}$: by lemma 12, there exists $t_0 \in [0,1)$ such that $\varphi_{B-1}(t_0) = 1$ and $\bar{l}(t_0) = 0$. Thus $E_\alpha \bigcap L_{\alpha,\alpha',r} \neq \emptyset$, and $\dim \tilde{E}_{\alpha,\alpha',r} \geq 0$. Either one of the following cases holds:

  * If $m_{B-1} > m_i$ for all $i < B-1$, or $m_{B-1} < m_i$ for all $i < B-1$, we have (lemma 14)

    $$E_\alpha = \{t\,;\ \varphi_{B-1}(t) = 1\}$$

    Hence,

    $$F_h(\alpha,\alpha,r) = f_h(\alpha) = 0$$

  * Otherwise, according to lemma 14, $E_\alpha \bigcap \{t\,;\ \varphi_{B-1}(t) < 1\} \neq \emptyset$. From lemma 11, we have

    $$E_\alpha \bigcap \{t\,;\ \varphi_{B-1}(t) < 1\} \subset \tilde{E}_{\alpha,\alpha,r} \subset E_\alpha$$

    and lemma 15 yields

    $$F_h(\alpha,\alpha,r) = f_h(\alpha)$$

  – **If** $\alpha \neq -\log_B m_{B-1}$. We have $t \in E_\alpha$. Then $\varphi_{B-1}(t) < 1$ and, by lemma 11, $\bar{l}(t) = 0$, and hence $t \in L_{\alpha,\alpha,r}$. This gives $E_\alpha \subset L_{\alpha,\alpha,r}$, and $\tilde{E}_{\alpha,\alpha,r} = E_\alpha$. We conclude

  $$F_h(\alpha,\alpha,r) = f_h(\alpha)$$

  In both cases, $F_h(\alpha,\alpha,r) = f_h(\alpha)$.
- **Case** $\dfrac{\alpha'-\alpha}{c_0} \in (0,\beta]$
  We have

  $$t \in E_\alpha \bigcap L_{\alpha,\alpha',r} \Rightarrow \begin{cases} \bar{l}(t) > 0 \\ \varphi_{B-1}(t) \text{ exists} \end{cases} \overset{\text{lemma 11}}{\Rightarrow} \varphi_{B-1}(t) = 1$$

  – **If** $\alpha = -\log_B m_{B-1}$
  Then $\tilde{E}_{\alpha,\alpha',r} \neq \emptyset$ by lemma 12, and since $\tilde{E}_{\alpha,\alpha',r} \subset \{t\,;\ \varphi_{B-1}(t) = 1\}$, we conclude $F_h(\alpha,\alpha',r) = 0$.

– **If** $\alpha \neq -\log_B m_{B-1}$
Let $t \in E_\alpha$. Then $\varphi_{B-1}(t) < 1$, which gives $\bar{l}(t) = 0$ (lemma 12). In other words, $E_\alpha \bigcap L_{\alpha,\alpha',r} = \emptyset$ and

$$F_h(\alpha, \alpha', r) = -\infty$$

□

6.4. Computation of $f_g(\alpha, \alpha', r)$. In the case of one point statistics, it is well known that $f_h(., \emptyset) \equiv f_l(., \emptyset)$, and hence $f_h(., \emptyset) \equiv f_g(., \emptyset) \equiv f_l(., \emptyset)$ (theorem 1, page 9. See also [21]). The f_g spectrum can be computed independently of this result, using the Gartner-Ellis theorem, and the differentiability of τ (see for instance [3]). In the case of two point statistics, we shall also use the large deviation technique, despite the fact that the function τ is neither differentiable nor steep (the vectors (q, q') where τ has no derivatives are exactly the elements of ∂Q). Nevertheless, a close look at the proof of Ellis' theorem [28] shows that these conditions on τ are not necessarely required in our case. We will discuss this point in more detail later in the computation of f_g.

Let $\mathbf{X}$ be a random variable uniformly distributed on $[0, 1)$, and define

$$Y_n(\mathbf{X}) := (-\log_B \mu(I^n(\mathbf{X})), -\log_B \mu(I^n(\mathbf{X}) + r_n))$$

and, for all $\varepsilon > 0$, set $F_\varepsilon := [\alpha - \varepsilon, \alpha + \varepsilon] \times [\alpha' - \varepsilon, \alpha' + \varepsilon]$. Then

$$\begin{aligned} \lim_{n\to+\infty} \frac{1}{n} \log \Pr\left(\frac{Y_n(\mathbf{X})}{n} \in F_\varepsilon\right) &= \lim_{n\to+\infty} \frac{1}{n} \log \frac{\text{card } K_\varepsilon^n(\alpha, \alpha', r)}{B^n} \\ &= \log B \ (f_g^\varepsilon(\alpha, \alpha', r) - 1) \end{aligned}$$

Let $(\theta_1, \theta_2) \in \mathbb{R}^2$. The moment generating function $M_n(\theta_1, \theta_2)$ of $Y_n(\mathbf{X})$ is

$$\begin{aligned} M_n(\theta_1, \theta_2) &:= \mathrm{E}(e^{<(\theta_1,\theta_2), Y_n(\mathbf{X})>}) \\ &= \mathrm{E}\left(\mu(I^n(\mathbf{X}))^{-\frac{\theta_1}{\log B}} \mu(I^n(\mathbf{X} + r_n))^{-\frac{\theta_2}{\log B}}\right) \\ &= B^{-n} \sum_{k=0}^{B^n-1} \mu(I_k^n)^{-\frac{\theta_1}{\log B}} \ \mu(I_k^n + r_n)^{-\frac{\theta_2}{\log B}} \end{aligned}$$

Set $\varphi_n(\theta_1, \theta_2) := \dfrac{1}{n} \log M_n(\theta_1, \theta_2)$. Since τ exists, we obtain

$$\varphi(\theta_1, \theta_2) := \lim_n \varphi_n(\theta_1, \theta_2) = -\log B \ \left(1 + \tau\left(-\frac{\theta_1}{\log B}, -\frac{\theta_2}{\log B}\right)\right)$$

Clearly, φ is a convex and continuous on $\mathbb{R}^2$.
Now consider the *rate function* $I(s, t) := \sup_{\theta_1,\theta_2}(s\theta_1 + t\theta_2 - \varphi(\theta_1, \theta_2))$. Notice that $I(s, t) = -\log B \ (f_l(s, t, r) - 1)$.
Ellis' theorem asserts that, if the supremum in $I(s, t)$ is reached on F_ε for ε sufficiently small, then

$$\liminf_{n\to+\infty} \frac{1}{n} \log \Pr\left(\frac{Y_n(\mathbf{X})}{n} \in F_\varepsilon\right) \geq -\inf_{(s,t)\in \overset{\circ}{F}_\varepsilon} I(s, t)$$

which yields

$$f_g(\alpha,\alpha',r) \geq \lim_{\varepsilon\to 0} \sup_{(s,t)\in \overset{\circ}{F}_\varepsilon} f_l(s,t,r)$$

which, combined with theorem 4, gives

$$\lim_{\varepsilon\to 0} \sup_{(s,t)\in \overset{\circ}{F}_\varepsilon} f_l(s,t,r) \leq f_g(\alpha,\alpha',r) \leq f_l(\alpha,\alpha',r) \tag{11}$$

In what follows, we will prove that $\lim_{\varepsilon\to 0} \sup_{(s,t)\in \overset{\circ}{F}_\varepsilon} f_l(s,t,r) = f_l(\alpha,\alpha',r)$ by examining all the possible values of α and α' as in proposition 4.

First of all, let us treat the trivial case $f_l(\alpha,\alpha',r) = -\infty$. Theorem 4 gives $f_g(\alpha,\alpha',r) = -\infty$, and hence $f_g(\alpha,\alpha',r) = f_l(\alpha,\alpha',r)$.

- CASE $\alpha = \alpha'$.
  Then $f_l(\alpha,\alpha',r) = f_l(\alpha) = f_h(\alpha) = f_h(\alpha,\alpha',r)$ and, by theorems 3 and 4, $f_g(\alpha,\alpha',r) = f_l(\alpha,\alpha',r)$.

- CASE $\alpha' \neq \alpha$, $m_0 \neq m_{B-1}$.

  - Assume that $\alpha_c \in (\alpha_{\min},\alpha_{\max}), \lambda \neq 1, \lambda \geq 0$. Let $\varepsilon > 0$ be small enough such that, for all $(s,t) \in F_\varepsilon$, we have $\alpha_c(s,t) \in (\alpha_{\min},\alpha_{\max})$. In addition, choose $\varepsilon < \frac{1}{2}|\alpha_0 - \alpha_{B-1} + \alpha - \alpha'|$. This condition ensures that for all $(s,t) \in F_\varepsilon$, we have $\lambda(s,t) \neq 1$ and $\lambda(s,t) > 0$.
    Thus, for all $(s,t) \in F_\varepsilon$, the infimum in $f_l(s,t,r) := \inf_{q,q'}\{sq + tq' - \tau(q,q',r)\}$ is reached (see the computation of $f_l(.,.,r)$ in this case, page 21).
    Since $f_l(s,t,r) = (1-\lambda)f_l(\alpha_c(s,t))$ (see page 15), $f_l(.,.,r)$ is continuous on F_ε.
    Using inequalities (11), we conclude $f_g(\alpha,\alpha',r) = f_l(\alpha,\alpha',r)$.

  - Assume that $\alpha = -\log_B m_{B-1}$ and $\alpha' = -\log_B m_0$ (then $\lambda = 1$). Take $0 < \varepsilon \leq |\alpha_0 - \alpha_{B-1}|/2$. This ensures that, for all $(s,t) \in F_\varepsilon$, $\lambda(s,t) \geq 0$, and that the infimum in $f_l(s,t,r)$ is reached (here again, see the computation of f_l for the case $\lambda(s,t) \geq 0$, page 21).
    Since $f_l(.,.,r)$ is continuous on F_ε, we conclude $f_g(\alpha,\alpha',r) = f_l(\alpha,\alpha',r)$.

This finishes the proof of proposition 5.

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