Application of Backstepping Techniques to the Time-varying Exponential Stabilization of Chained Form Systems

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Abstract: It is known that the kinematic model of several nonholonomic systems can be converted into a *chained form* control system. Asymptotical stabilization of any equilibrium point of this system cannot be achieved by means of a continuous pure state feedback, but can be obtained by using a *time-varying* continuous feedback [20]. In the present paper, a backstepping technique is used to derive explicit time-varying feedbacks that ensure *exponential* stability of the closed-loop system. Two classes of control laws are proposed, with one of them involving a dynamic extension of the original chained system. Like in other recent studies on the same topic, exponential convergence is obtained by using the properties associated with homogeneous systems. The control laws so obtained are continuous in both the state and time variables. A complementary and novel feature of the proposed control design technique lies in the estimation of a lowerbound of the asymptotical rate of convergence as a function of a reduced set of control parameters which is independent of the system's dimension. Moreover, the fact that this lowerbound may take any positive value indicates that any prespecified exponential rate of convergence can be achieved via a suitable choice of the control parameters.

Key-words: Nonholonomic systems, chained systems, time-varying continuous feedback, asymptotical stabilization, dynamic extension.

(Résumé : tsvp)

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Application de Techniques de Backstepping à la Stabilisation Exponentielle Instationnaire de Systèmes en Forme Chaînée

Résumé : Les modèles cinématiques de plusieurs systèmes non-holonomes peuvent être exprimés sous forme canonique dite “chaînée”. La stabilisation des points d'équilibre de ces systèmes ne peut être obtenue par des retours d'état continus fonctions seulement de l'état du système. Elle est cependant possible par des retours d'état continus dépendant également du temps [20]. Nous utilisons içi une technique de “backstepping” pour synthétiser des retours d'état assurant une stabilisation *exponentielle* du système en boucle fermée. Cette propriété de stabilisation exponentielle découle des propriétés associées aux systèmes homogènes. Différentes lois de commande sont proposées, l'une d'entre elles faisant intervenir une extension dynamique exogène du système chaîné de départ. Ces lois de commande sont continues. La méthode de synthèse présente également l'avantage de fournir une borne inférieure du taux de convergence asymptotique. Cette borne est fonction d'un petit nombre de paramètres de commande, est indépendante de la dimension du système, et peut être augmentée arbitrairement.

Mots-clé : Systèmes non-holonomes, systèmes chaînés, retour d'état continu instationnaire, stabilisation asymptotique, extension dynamique.

1 Introduction

The study of nonholonomic mechanical systems has recently received much attention. The modelling kinematic equations of several of these systems (unicycle-type carts, car-like vehicles with trailers,...) can be converted into canonical *chained form* control equations (see e.g. [22, 20, 15]). This has motivated a thorough investigation of this class of systems. For instance, strategies have been proposed to solve trajectory planning problems (see e.g. [25, 13]). The problem of asymptotical stabilization of a given equilibrium point (i.e. a given fixed configuration of the mechanical system) has also given rise to a growing litterature. The present study is a contribution to the solution of this problem.

It is known that, due to the non-satisfaction of Brockett's necessary condition [3], nonholonomic systems cannot be asymptotically stabilized by using continuous pure state feedbacks. In order to circumvent this obstruction, two main research directions have been followed. The first one, pioneered by Bloch and McClamroch in [2], consists of using discontinuous feedbacks. The second solution consists of using *time-varying* continuous feedbacks. The latter possibility has first been proposed and investigated by Samson in [19], for the stabilization of a unicycle-type vehicle whose equations can be converted into a three-dimensional chained system. This initial result has triggered a quickly expanding research on time-varying feedback stabilization. For example, general existence theorems have since then been established by Coron [5]-[6]. They basically state that any driftless controllable system (such as chained systems) can be asymptotically stabilized, eventually in finite time, by means of a time-varying continuous feedback. In spite of this result, the first constructive approaches (see e.g. [19, 16, 20, 21, 24]), which involved "smooth" feedback control laws, displayed slow (polynomial) asymptotical convergence. In order to obtain faster convergence (say exponential) different solutions have been devised. One approach, considered for example in [4, 23] or in [1], consists of using feedback controls which are discontinuous in the state variable. Another solution, initially proposed by M'Closkey and Murray in [10], is based on the design of continuous time-varying feedbacks which make the closed loop system homogeneous of degree zero. The solution here proposed fits within the latter category. Two classes of control laws are designed and analyzed. One of them involves a dynamic extension of the original chained form system.

By comparison with time-varying control design studies previously reported in the litterature on the subject, the proposed method presents the advantage of providing a lowerbound of the asymptotical rate of convergence. This lowerbound is calculated from a reduced set of control design gain parameters, when other parameters are adequately chosen large enough. This is very similar to what is achieved in the case of linear control systems with classical backstepping techniques based on the use of large gains. As a matter of fact, the proposed solution may be interpreted as an adaptation of these techniques when applied to the n-th order linear integrator which is obtained by setting one of the chained system control inputs equal to one. We believe that the possibility of achieving any prespecified exponential rate of convergence is an important feature which, added to the basic property of stabilization, enhances the applicability of any nonlinear feedback control scheme. For example, the solution presented in [17] was not satisfatory in this respect: a correct adjustment of the control design parameters and the monitoring of the rate of convergence proved to be quite difficult.

The paper is organized as follows. In Section 2, general properties and stabilization results associated with homogeneous systems, which are useful to establishing our main result, are recalled. The main contribution of the paper is summarized in Section 3, where new continuous feedback laws for the exponential stabilization of the chained system are proposed. Due to its lenght, the corresponding stability proof is reported in the paper's Appendix. Finally, simulations of a car-like mobile robot, corresponding to the four-dimensional chained system, are presented in Section 5. Comparision with the control simulated in [17] shows that the vehicle's motion is more "natural" and alike the one performed by a human driver during a parking manoeuvre. This qualitative improvement of the system's transcient behaviour was also one of the objectives pursued by the authors.

Throughout the paper, the following notations are used:

- $|\,.\,|$ denotes the Euclidean norm.
- $\mathbb{R}^+$ denotes the set $\{x \in \mathbb{R},\ x > 0\}$.
- A function $f : \mathbb{R}^n \longmapsto \mathbb{R}^p$ is of class C^p (resp. C^∞) if it has continuous partial derivatives up to the order p (resp. at any order).

When refering to the concept of asymptotic stabilization of an equilibrium point of a given differential system, it will always be in the usual sense of Lyapunov (see the definition in [7] for example).

2 Homogeneity and exponential stabilization

Let us first recall some definitions about homogeneous systems. For a more complete exposition, the reader is referred to [9] or [8].
For any $\lambda > 0$ and any set of real parameters $r_i > 0$ $(i = 1, \ldots, n)$, one defines the following "dilation" operator $\delta^r_\lambda : \mathbb{R}^n \longmapsto \mathbb{R}^n$ by

$$\delta^r_\lambda(x_1, \ldots, x_n) = (\lambda^{r_1}\, x_1, \ldots, \lambda^{r_n}\, x_n).$$

An *homogeneous* norm associated with this dilation operator is:

$$\rho^r_p(x) = (\sum_{j=1}^{n} |x_j|^{\frac{p}{r_j}})^{\frac{1}{p}} \qquad with\ p > 0.$$

A continuous function $f : \mathbb{R}^n \longmapsto \mathbb{R}$ is homogeneous of degree $\tau \geq 0$ with respect to the dilation δ^r_λ if :

$$\forall \lambda > 0, \quad f(\delta^r_\lambda(x)) = \lambda^\tau\, f(x).$$

A differential system $\dot{x} = f(x)$ (or a vector field f), with $f : \mathbb{R}^n \longmapsto \mathbb{R}^n$ continuous, is homogeneous of degree $\tau \geq 0$ with respect to the dilation δ^r_λ if :

$$\forall \lambda > 0, \quad f_i(\delta^r_\lambda(x)) = \lambda^{\tau + r_i}\, f_i(x) \qquad (i = 1, \ldots, n).$$

The above definitions can be extended to time-dependant functions and systems. Such an extension has already been considered in [17] and simply follows by considering the extended dilation operator:

$$\delta^r_\lambda(x_1, \ldots, x_n, t) = (\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n, t).$$

The definitions remain unchanged.
Note that for any positive T-periodic continuous function $V(x,t)$ which is homogeneous of degree $\alpha > 0$ with respect to $\delta^r_\lambda(x,t)$ and vanishes only at $x = 0$, there are two strictly positive numbers $\gamma_{1,p}$ and $\gamma_{2,p}$ such that:

$$\gamma_{1,p}\ (\rho^r_p(x))^\alpha \ \leq\ V(x,t)\ \leq\ \gamma_{2,p}\ (\rho^r_p(x))^\alpha\ , \qquad \forall (x,t)$$

The following result, which is a particular case of a proposition by Pomet and Samson, establishes the existence of homogeneous Lyapunov functions for time-varying asymptotically stable systems which are homogeneous of degree zero with respect to some dilation. This proposition extends a theorem that Rosier [18] had established for autonomous (time-invariant) systems.

Proposition 1 (Pomet, Samson [17]) *Let $f(x,t): \mathbb{R}^n \times \mathbb{R} \longmapsto \mathbb{R}^n$ a T-periodic continuous function ($f(x,t+T) = f(x,t)$). Assume that the system:*

$$\dot{x} = f(x,t) \tag{1}$$

is homogeneous of degree zero with respect to a dilation $\delta_\lambda^r(x,t)$ and that $x = 0$ is an asymptotically stable equilibrium of (1).
Then, for any $\alpha > 0$ and $p < \frac{\alpha}{max\{r_j\}}$, there exists a function $V(x,t): \mathbb{R}^n \times \mathbb{R} \longmapsto \mathbb{R}$ such that:

- *V is of class C^p on $\mathbb{R}^n \times \mathbb{R}$ and of class C^∞ on $(\mathbb{R}^n - \{0\}) \times \mathbb{R}$.*
- *V is T-periodic ($V(x,t+T) = V(x,t)$).*
- *V est homogeneous of degree α with respect to the dilation δ_λ^r:*
$$V(\delta_\lambda^r(x,t)) = \lambda^\alpha V(x,t)$$
- *$V(x,t) > 0$ if $x \neq 0$, $V(0,t) = 0$*
- *$V(x,t)$ is "proper" with respect to x:*
  $\forall t \;:\; V(x,t) \longmapsto +\infty$ when $|x| \longmapsto +\infty$.
- *$\exists \tau > 0 \;:\; \frac{\partial V}{\partial t}(x,t) + \frac{\partial V}{\partial x}(x,t)\; f(x,t) \leq -\tau V(x,t)$*
  so that, along any system's trajectory: $V(x(t),t) \leq e^{-\tau t}\; V(x(0),0)$.

Note that this proposition implies that, in the case of systems which are homogeneous of degree 0, the asymptotical stability of an equilibrium point is always *global*. The last inequality of the proposition also implies that each state variable $x_i(t)$ ($i = 1, \ldots, n$) converges to zero with an exponential rate better than $\tau\frac{r_i}{\alpha}$.
However, since $\rho_p^r(x)$ is not equivalent to any usual p-norm (except when all "weights" r_i are equal), the proposition does not imply that $x = 0$ is asymptotically exponentially stable in the classical sense.

The next proposition is the autonomous version of a "backstepping" result proved by the authors in [14].

Proposition 2 *Consider the following system :*

$$\dot{x} = f(x, v(x^1)) \tag{2}$$

with $f : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$ *a continuous function,* $x^1 = (x_1, \ldots, x_m)$, $m \leq n$, *and* $v : \mathbb{R}^m \mapsto \mathbb{R}$ *a continuous function, of class* C^1 *on* $(\mathbb{R}^m - \{0\})$, *homogeneous of degree* q *with respect to a dilation* $\delta^r(x)$.
Assume that:
i) the system (2) is homogeneous of degree 0 with respect to the dilation $\delta^r(x)$, *ii) the origin* $x = 0$ *of this system is asymptotically stable, so that there exists, for any odd integer* $p > max\{\frac{r_i}{q},\ 1 \leq i \leq m\}$, *a Lyapunov function* $V(x)$ *homogeneous of degree* $(p+1)q$ *and a positive real number* τ *such that:*

$$\frac{\partial V}{\partial x} f(x, v(x^1)) \ \leq\ -\tau\, V(x) \tag{3}$$

Then, there exists a positive continuous function γ *defined on* $(0, \tau)$, *which depends continuously* [1] *on the vector field* f *and such that for any* $\epsilon \in (0, \tau)$ *and any* $k \geq \gamma(\epsilon)$:
i) the origin $(x = 0, y = 0)$ *of the system*

$$\left\{ \begin{array}{rcl} \dot{x} & = & f(x, y) \\ \dot{y} & = & -k(y - v(x^1)) \end{array} \right. \tag{4}$$

is asymptotically stable,
ii) a Lyapunov function for the latter system is:

$$W(x, y) \ =\ V(x) + \frac{1}{\sqrt{k}} \int_{v(x^1)}^{y} (s^p - v^p(x^1))\, ds \tag{5}$$

iii) this function verifies the following inequality:

$$\frac{\partial W}{\partial x} f(x, y) - k \frac{\partial W}{\partial y}(y - v(x^1)) \ \leq\ -(\tau - \epsilon)\, W(x, y) \tag{6}$$

which implies that along any solution to the system (4):

$$\dot{W} \ \leq\ -(\tau - \epsilon) W$$

[1] The continuous dependence of γ with respect to f is here defined on the set of C^o vector fields, homogeneous of degree 0 with respect to the dilation $\delta(x, y) = (\delta^r(x), \lambda^q y)$, and with the topology of uniform convergence on $S^{n-1} = \{(x, y) :\ \rho(x, y) = 1\}$ (where ρ is any homogeneous norm with respect to the dilation δ).

Remark 1: In the case where $v(x^1)$ is of class C^1 everywhere, then the Lyapunov function (5) can be replaced by:

$$W(x,y) \;=\; V(x) + \frac{1}{\sqrt{k}}|y - v(x^1)|^{p+1} \tag{7}$$

Remark 2: The above proposition gives a simple solution to the problem of determining a stabilizing feedback $u(x,y)$ for a system in the form:

$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= u \end{aligned}$$

from the knowledge of a feedback $v(x)$ which stabilizes the reduced system:

$$\dot{x} \;=\; f(x,v)$$

provided that the right homogeneity properties are met.
This solution is: $u(x,y) = -k(y - v(x))$, with $k > 0$ chosen large enough.
The design of the stabilizing feedback $u(x,y)$ thus involves two steps: i) the design of a stabilizing feedback $v(x)$ (of class C^1 on $\mathbb{R}^n - \{0\}$) for the reduced-order system obtained by taking y, instead of $\dot{y}$, as the control variable , and ii) the product of $(v(x) - y)$ by a large enough positive coefficient. This method may be interpreted as a *backstepping* technique, based on the use of a large control gain, applied to the classical problem of "adding an integrator at the input level".

We will conclude this section with a corollary of the previous proposition. This corollary is the basic technical result used for the forthcoming control design and analysis.

Corollary 1 *Consider the following system:*

$$\dot{x} = \rho(x)g(x,u(x)) \tag{8}$$

with $\rho(x)$ an homogeneous norm associated with some dilation $\delta^r(x)$, $g : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$ a continuous function, and $u : \mathbb{R}^n \mapsto \mathbb{R}$ a continuous function, of class C^1 on $(\mathbb{R}^n - \{0\})$, homogeneous of degree $q > 1$ with respect to δ^r.
Assume further that the system (8) is homogeneous of degree 0 with respect to the dilation δ^r and that the origin $x = 0$ of the system (8) is asymptotically stable, so that there exists, for any odd integer $p > max\{\frac{r_i}{q},\ 1 \leq i \leq m\}$, a Lyapunov function

$V(x)$ homogeneous of degree $(p+1)q$ with respect to the dilation δ^r and a positive real number τ such that:

$$\frac{\partial V}{\partial x}\rho(x)\, g(x,u(x)) \;\leq\; -\tau\, V(x). \tag{9}$$

Consider also an homogeneous norm $\rho_e(x,z,y)$ associated with the extended dilation $\delta_e^r(x,z,y) = (\delta^r(x), \lambda^q z, \lambda^q y)$ and which verifies the following inequality:

$$\forall (x,z,y), \;\; \rho_e(x,z,y) \geq \rho(x). \tag{10}$$

Then, there exists two positive continuous functions β' and β with support included in $(0,\tau)$ and $(0,\tau)\times \mathbb{R}^+$ respectively and such that the following two points hold for $\epsilon \in (0,\tau)$, $k' \geq \beta'(\epsilon)$ and $k \geq \beta(\epsilon, k')$:

i) the origin $(x=0, z=0, y=0)$ of the system:

$$\left\{ \begin{array}{l} \dot{x} = \rho_e(x,z,y) g(x,y) \\ \dot{z} = \rho_e(x,z,y)(-\dfrac{k'}{\rho_e(x,z,0)}(z-u(x))) \\ \dot{y} = -k(y-z) \end{array} \right. \tag{11}$$

is asymptotically stable.

ii) the function:

$$W(x,z,y) \;=\; V(x) + \frac{1}{\sqrt{k'}}\int_{u(x)}^{z} (s^p - u^p(x))\, ds + \frac{1}{\sqrt{k}}|y-z|^{p+1} \tag{12}$$

is a Lyapunov function associated with the system (11), homogeneous of degree $(p+1)q$ with respect to the dilation δ_e^r, and such that:

$$\begin{array}{c} \dfrac{\partial W}{\partial x}\,\rho_e(x,z,y) g(x,y) \;+\; \dfrac{\partial W}{\partial z}\,\rho_e(x,z,y)(-\dfrac{k'}{\rho_e(x,z,0)}(z-u(x))) \\ +\dfrac{\partial W}{\partial y}(-k(y-z)) \;\leq\; -(\tau-\epsilon)\, W(x,z,y) \end{array} \tag{13}$$

Therefore, along any solution of the system (11):

$$\dot{W} \;\leq\; -(\tau - \epsilon)\, W$$

Proof
The point i) is an obvious consequence of the point ii). Let us thus concentrate on the proof of ii).
It follows from (10) and (9)that:

$$\frac{\partial V}{\partial x}\rho_e(x,u(x),0)g(x,u(x)) \ \leq \ -\tau\, V(x). \tag{14}$$

Therefore, $V(x)$ is also a Lyapunov function for the system:

$$\dot{x} = \rho_e(x,u(x),0)g(x,u(x)) \tag{15}$$

and the origin $x=0$ of this system is asymptotically stable.
By application of Proposition 2 to the system (15) it follows that there exists a positive continuous function γ_1 defined on $(0,\tau)$, such that for any $\epsilon_1 \in (0,\tau)$ and any $k' \geq \gamma_1(\epsilon_1)$, the origin of the system:

$$\left\{ \begin{array}{l} \dot{x} = \rho_e(x,z,0)g(x,z) \\ \dot{z} = -k'(z-u(x)) \end{array} \right. \tag{16}$$

is asymptotically stable. Moreover, the function:

$$V'(x,z) = V(x) + \frac{1}{\sqrt{k'}}\int_{u(x)}^{z} s^p - u^p(x)\, ds \tag{17}$$

is a Lyapunov function associated with the system (16), such that:

$$\frac{\partial V'}{\partial x}\,\rho_e(x,z,0)g(x,z) \ \ + \ \ \frac{\partial V'}{\partial z}\,(-k'(z-u(x))) \leq \ -(\tau-\epsilon_1)\,V'(x,z). \tag{18}$$

Multiplying both sides of (18) by $\frac{\rho_e(x,z,z)}{\rho_e(x,z,0)}$ and using the fact that $\rho_e(x,z,z) \geq \rho_e(x,z,0)$ one obtains:

$$\frac{\partial V'}{\partial x}\,\rho_e(x,z,z)g(x,z) + \frac{\partial\, V'}{\partial z}\,(\rho_e(x,z,z)(-\frac{k'}{\rho_e(x,z,0)}(z-u(x)))) \leq \ \ -(\tau-\epsilon_1)\,V'(x,z) \tag{19}$$

which proves that $V'(x,z)$ is a Lyapunov function for the system:

$$\left\{ \begin{array}{l} \dot{x} = \rho_e(x,z,z)g(x,z) \\ \dot{z} = \rho_e(x,z,z)(-\frac{k'}{\rho_e(x,z,0)}(z-u(x)) \end{array} \right. \tag{20}$$

and that the origin $(x = 0, z = 0)$ of this system is asymptotically stable.
Let us remark that the quotient $\dfrac{z - u(x)}{\rho_e(x,z,0)}$ appearing in the system (20) is homogeneous of positive degree $(q-1)$ with respect to the dilation $\delta_e^r(x,z,0)$, and is thus well defined and continuous everywhere.

For any $\epsilon_1 \in (0,\tau)$ and any $k' \geq \gamma_1(\epsilon_1)$ there exists, by application of Proposition 2 to the system (20), another positive continuous fonction γ_2 defined on $(0, \tau - \epsilon_1)$ which also depends continuously on k', and such that for any $\epsilon_2 \in (0, \tau - \epsilon_1)$ and $k \geq \gamma_2(\epsilon_2, k')$, the function (12) satisfies the inequality (13) with ϵ replaced by $\epsilon_1 + \epsilon_2$. Let us define β' and β by $\beta'(\epsilon) = \gamma_1(\frac{\epsilon}{2})$ and $\beta(\epsilon, k') = \gamma_2(\frac{\epsilon}{2}, k')$. One easily verifies that β' and β satisfy the conditions specified in the corollary.
(end of Proof).

3 Exponential stabilization of the chained form system

3.1 With dynamic extension

Let us consider the two-inputs chained form control system:

$$\left\{ \begin{array}{lcl} \dot{x}_1 & = & u_1 \\ \dot{x}_2 & = & u_1\, x_3 \\ & \vdots & \\ \dot{x}_j & = & u_1\, x_{j+1} \\ & \vdots & \\ \dot{x}_{n-1} & = & u_1\, x_n \\ \dot{x}_n & = & u_2 \end{array} \right. \tag{21}$$

For reasons that will be discussed later, we associate the following exogenous extension with the system (21):

$$\left\{ \begin{array}{lcl} \dot{z}_3 & = & v_3 \\ & \vdots & \\ \dot{z}_n & = & v_n \end{array} \right. \tag{22}$$

In the remaining of the paper we will denote the vectors $(x_1, \ldots, x_n)$, $(z_3, \ldots, z_n)$, (u_1, u_2) and $(v_3, \ldots, v_n)$ as x, z, u and v, respectively.
The dilation:

$$\delta(x,z,t) \;=\; (\lambda^{r_1}\, x_1, \ldots, \lambda^{r_n}\, x_n, \lambda^{r_3}\;\; z_3, \ldots, \lambda^{r_n}\, z_n, t) \tag{23}$$

will also be used with the following specific choice of the "weights" r_i:

$$\left\{ \begin{array}{lcl} r_1 & = & 1 \\ r_j & = & n - j + 2, \quad j = 2, \ldots, n \end{array} \right. \tag{24}$$

The following homogeneous partial norms associated with the dilation (23):

$$\begin{array}{lcl} \rho_2(x_2) & = & |x_2|^{\frac{1}{n}} \\ \rho'_k(x_2, \ldots, x_{k-1}, z_3, \ldots, z_k) & = & (\sum_{i=2}^{k-1} |x_i|^{\frac{p}{r_i}} + \sum_{j=3}^{k} |z_j|^{\frac{p}{r_j}})^{\frac{1}{p}} \quad k = 3, \ldots, n \\ \rho_k(x_2, \ldots, x_k, z_3, \ldots, z_k) & = & (\sum_{i=2}^{k} |x_i|^{\frac{p}{r_i}} + \sum_{j=3}^{k} |z_j|^{\frac{p}{r_j}})^{\frac{1}{p}} \quad k = 3, \ldots, n \end{array} \tag{25}$$

will appear in the control expression, with the parameter p being arbitrary except that it should be *larger than* n in order to ensure that the feedback control law is defined everywhere.

The present study focuses on the determination of continuous feedback laws $u(x, z, t)$, $v(x, z, t)$ which make the origin of the system (21)-(22) exponentially asymptotically stable (in the sense of Proposition 1).
The main result is summarized in the following proposition.

Proposition 3 *Consider the time-varying feedback law:*

$$\left\{ \begin{array}{lcl} u_1(x, z, t) & = & k_2 g(t)[k_1(-x_1 g(t) + |x_1 g(t)|) + \rho_n(x, z)] \\ u_2(x, z, t) & = & -\frac{k_n}{\rho_n}(|u_1| x_n - u_1 z_n) \\ v_3(x, z, t) & = & -\frac{k'_3}{\rho'_3}(|u_1| z_3 + \frac{1}{\rho_2}|u_1| x_2) \\ v_j(x, z, t) & = & -\frac{k'_j}{\rho'_j}(|u_1| z_j + \frac{k_{j-1}}{\rho_{j-1}}(|u_1| x_{j-1} - u_1 z_{j-1})) \quad j = 4, \ldots, n. \end{array} \right. \tag{26}$$

with:

- $k = (k_1, k_2, k'_3, k_3, \ldots, k'_n, k_n)$ *a vector of design parameters,*
- g *a* T*-periodic piecewise continuous function, such that:*

$$\forall t \in [0, \frac{T}{2}], \quad 0 \leq g(t) \leq G, \quad g(t + T/2) = -g(t) \;\; \textit{and} \;\; \frac{2}{T} \int_0^{T/2} g(s)\, ds > 0 \tag{27}$$

Define the two positive numbers σ_1 *and* σ_2 *as follows:*

$$\sigma_i \; = \; \frac{2}{T}\int_0^{T/2} g^i(s)\,ds, \quad i = 1,2. \tag{28}$$

Then, there exist positive continuous functions $\gamma_3', \gamma_3, \ldots, \gamma_n', \gamma_n$ *such that for any* $\epsilon_2 \in (0, \sigma_1)$*, any* $\bar{k}_2 > 0$*, and any* k *such that:*

$$\begin{array}{l} k_1 \; > \; 0, \quad k_2 \geq \bar{k}_2 \\ k_3' \geq \gamma_3'(\epsilon_2), \quad k_3 \geq \gamma_3(\epsilon_2, \bar{k}_2, k_3') \\ k_4' \geq \gamma_4'(\epsilon_2, \bar{k}_2, k_3', k_3), \quad k_4 \geq \gamma_4(\epsilon_2, \bar{k}_2, k_3', k_3, k_4') \\ \vdots \\ k_n' \geq \gamma_n'(\epsilon_2, \bar{k}_2, k_3', k_3, \ldots, k_{n-1}', k_{n-1}), \quad k_n \geq \gamma_n(\epsilon_2, \bar{k}_2, k_3', k_3, \ldots, k_{n-1}', k_{n-1}, k_n') \end{array} \tag{29}$$

[1^o)] Asymptotical stability*:*
The feedback controls (26) globally exponentially stabilize (in the sense of Proposition 1) the origin $(x = 0, z = 0)$ *of the system (21)-(22).*

[2^o)] Exponential convergence rates*:*
Along any trajectory of the closed loop system (21)-(22)-(26)

i) the variable x_1 *exponentially converges to zero with a rate better than* $k_2\, Min(k_1\,\sigma_2 - \epsilon_1, \dfrac{\sigma_1 - \epsilon_2}{n})$*, where* ϵ_1 *is any arbitrary small positive number.*

ii) one has

$$V_+(Y(jT)) \; \leq \; e^{-\frac{p}{n}k_2(\sigma_1 - \epsilon_2)jT}\, V_+(Y(0)), \quad \forall j \in \mathbb{N}, \tag{30}$$

with the function $V_+(Y), (Y = (x_2, z_3, x_3, \ldots, z_n, x_n))$ *defined by:*

$$\begin{aligned} V_+(Y) \; &= \; |x_2|^{\frac{p}{r_2}} + \frac{1}{\sqrt{k_3'}}\int_{-\frac{x_2}{\rho_2}}^{z_3} s^{\frac{p}{r_3}-1} + (\frac{x_2}{\rho_2})^{\frac{p}{r_3}-1}\,ds \\ &+ \sum_{i=4}^{n} \frac{1}{\sqrt{k_i'}}\int_{-\frac{k_{i-1}}{\rho_{i-1}}(x_{i-1}-z_{i-1})}^{z_i} s^{\frac{p}{r_i}-1} + (\frac{k_{i-1}}{\rho_{i-1}}(x_{i-1} - z_{i-1}))^{\frac{p}{r_i}-1}\,ds \\ &+ \sum_{i=3}^{n} \frac{1}{\sqrt{k_i}}|x_i - z_i|^{\frac{p}{r_i}} \end{aligned} \tag{31}$$

(with p any integer such that $p > n$ and $\frac{p}{r_i}$ $(i = 3, \ldots, n)$ is even).
Relation (30) implies that each variable x_i, $(i = 2, \ldots, n)$ exponentially converges to zero with a rate better than $\frac{k_2 r_i}{n}(\sigma_1 - \epsilon_2)$.

The proof is given in the Appendix.
This proposition calls for some comments and remarks.

- In order to better put into perspective the fact that the stabilization of the exogeneous extension (22) is only a technical step used for the stabilization of the chained system (21), one may rewrite the dynamical feedbacks $u_1(x, z, t)$ and $u_2(x, z, t)$ of relation (26) without introducing the auxiliary controls $v_j(x, z, t)$ $(j = 3, \ldots, n)$ explicitly. This yields the following closed-form of the control expressions:

$$\text{(I)} \quad \left\{ \begin{array}{lcl} u_1(x,z,t) & = & k_2 g(t)[k_1(-x_1 g(t) + |x_1 g(t)|) + \rho_n(x,z)] \\ u_2(x,z,t) & = & -\frac{k_n}{\rho_n}|u_1|(x_n - sign(u_1) z_n) \end{array} \right. \tag{32}$$

with:

$$\dot{z}_i = -\frac{k'_i}{\rho'_i}|u_1|(z_i - w_i), \quad (i = 3, \ldots, n) \tag{33}$$

and:

$$\left\{ \begin{array}{lcl} w_3 & = & -\frac{x_2}{\rho_2} \\ w_{i+1} & = & -\frac{k_i}{\rho_i}(x_i - sign(u_1) z_i), \quad i = 3, \ldots, n-1. \end{array} \right. \tag{34}$$

- The controls functions (26) are i) well defined everywhere, ii) continuous in the variables x and z, due to the fact that the weight r_i of each variable x_i and z_i, $(i = 3, \ldots, n)$ is strictly greater than 1, and iii) uniformly continuous (resp. differentiable) in the variable t when the function $g(t)$ is uniformly continuous (resp. differentiable). They are not differentiable on the sets defined by $\rho_i(x, z) = 0$ $(i = 2, \ldots, n)$ and $\rho'_i(x, z) = 0$ $(i = 3, \ldots, n)$. In particular, they are not differentiable at the origin $(x = 0, z = 0)$.

- The sign of $u_1(x, z, t)$ is given by the sign of $g(t)$. It thus changes periodically along any solution to the controlled system. The stability analysis much relies upon this property.

- A weakness of the proposed result is that the functions γ_i' and γ_i $(i = 4, \ldots, n)$ are not explicited. In fact, by working some more on the proof, it would have been possible to determine such functions explicitly. However, the expressions of these candidate functions would have been quite complicated, and also their use for the determination of the control coefficients k_i and k_i' $(i = 3, \ldots, n)$ would have lead to very large and conservative values. The simulation experiments that we have conducted tend to show that, for small orders of the system $(n \leq 5)$, it is not very difficult to find values that ensure closed-loop stability. Nevertheless, it has also been verified in simulation that the positivity of all coefficients is not by itself sufficient.

  In order to strenghten the practical usefulness of the proposition, we propose below a conjecture which, if it were proved to be true, would suggest an iterative method that would facilitate the determination of the control coefficients.

  Conjecture 1 *Consider two positive coefficients k_1 and k_2, and coefficients k_3', k_3, ..., k_m', k_m $(m \geq 3)$ for which the controls (26) stabilize the origin of the system (21)-(22) when $n = m$.*
  Assume that the same coefficients are used in the controls when considering the case where $n = m + 1$.
  It is conjectured that there exist positive coefficients k_{m+1}' and k_{m+1} for which the controls (26) stabilize the origin of the system (21)-(22) when $n = m + 1$.

  Following this conjecture, the coefficients k_3', k_3 could be determined from a simulation study of the three-dimensional chained system $(n = 3)$. Then k_4' and k_4 would be determined in the same way for the system of order $n = 4$ (while keeping the same values for k_3', k_3), and so on until the desired order is reached.

- The proposition provides us with lowerbounds of the asymptotical exponential convergence rates associated with the system's state variables. These lowerbounds depend on the sole parameters k_1 and k_2, once the function $g(t)$ has been chosen, and can be increased as much as desired via the choice of these parameters. Note also that the lowerbounds obtained for the convergence rates associated with the variables x_i $(i = 2, \ldots, n)$ only depend on k_2, and that increasing the value of k_1 beyond $\frac{\sigma_1}{n\sigma_2}$ does not make the lowerbound of the convergence rate of x_1 larger.

- The dependence of the functions γ_i and γ_i' upon the parameter $\bar{k}_2$ gives two indications: i) when k_2 is changed but always kept larger than some (arbitrary) threshold, there exist fixed control coefficients k'_3, k_3, ..., k'_n, k_n for which the asymptotical stability of the origin is uniformly ensured, and ii) larger coefficients have to be used when the threshold becomes smaller than some value $\bar{k}_{2,min}$. In other words, it is somewhat more difficult to stabilize the system when k_2 is small.

- The conditions upon the function $g(t)$ are easily met. Take, for example, $g(t) = sin(\omega t)$, or $g(t) = sign(sin(\omega t))$. Note that the asymptotical rates of convergence which are estimated in the proposition are little influenced by the period T of this function. In this respect, there is no obvious advantage of choosing a short period. Nonetheless, the choice of ω has a significant influence on the "shape" of the systems solutions. For instance, when monitoring the trajectory of a mobile robot in the (x_1, x_2) plan, it can be observed from simulation results that before $(x_1(t), x_2(t))$ enters a small neighborhood of zero, the number of "cusp like" points on the trajectory increases with ω, while the amplitude of $x_1(t)$ (which represents the x coordinate of the mobile robot) between successive cusps is smaller.

3.2 Without dynamic extension

A complication introduced in the previous result concerns the dynamic extension involved in the design of the proposed feedback controls, and one may question the necessity of it. This complication here results from the technical difficulty that we have met when trying to prove the closed-loop system's stability when considering the following simpler control law (which does not involve any dynamic extension):

$$(\mathrm{II}) \quad \left\{ \begin{array}{lcl} u_1(x,t) & = & k_2 g(t)[k_1(-x_1 g(t) + |x_1 g(t)|) + \tilde{\rho}_n(x)] \\ u_2(x,t) & = & -\dfrac{k_n}{\tilde{\rho}_n}|u_1|(x_n - sign(u_1) w_n(x,t)) \end{array} \right. \tag{35}$$

where w_n is defined from the following recursive expression:

$$\left\{ \begin{array}{lcl} w_3 & = & -\dfrac{x_2}{\tilde{\rho}_2} \\ w_{i+1} & = & -\dfrac{k_i}{\tilde{\rho}_i}(x_i - sign(u_1) w_i), \quad (i = 3, \ldots, n-1) \end{array} \right. \tag{36}$$

and $\tilde{\rho}_k(x_2, \ldots, x_k) = (\sum_{i=2}^{k} |x_i|^{\frac{p}{r_i}})^{\frac{1}{p}}$, $(k = 2, \ldots, n)$.

Note that if each variable z_i, involved in the control (I) considered in the previous section, were identically equal to w_i (which, in an abstract way, would correspond to choosing $k'_i = +\infty$ in (33)), then the control (I) would basically simplify into the control (II).

If the result of Proposition 2 had held for functions $v(.)$ that are only continuous (instead of being of class C^1 everywhere except at zero, as required in the proof of the proposition), then the iterative technique used in the proof of Proposition 3 would have also applied to the control (II), yielding similar stability results and estimations of the convergence rates.
Simulations conducted in the case where $n = 4$ (see the simulation results in section 4) also tend to indicate that the difficulty may be purely technical, and that the control (II) is likely to perform as well as the control (I).
Nevertheless, the stabilization properties of the control (II) will remain conjectural until a proper proof is found.

Instead of the control law (II), it is also possible to consider slightly more complicated feedback laws which do not involve a dynamic extension either and whose stabilization properties can be proved in the same way as in the case of the dynamic extension. This possibility is summarized in the next proposition.

Proposition 4 *Consider the time-varying feedback law:*

$$(\mathrm{III}) \quad \left\{ \begin{array}{rcl} u_1(x,t) & = & k_2 g(t)[k_1(-x_1 g(t) + |x_1 g(t)|) + \tilde{\rho}_n(x)] \\ u_2(x,t) & = & -\dfrac{k_n}{\tilde{\rho}_n}|u_1|(x_n - sign(u_1) w_n(x,t)) \end{array} \right. \tag{37}$$

where w_n is defined according to the following recursive expression:

$$\left\{ \begin{array}{rcl} w_3 & = & -\dfrac{x_2}{\tilde{\rho}_2} \\ w_{i+1} & = & -\dfrac{k_i}{\tilde{\rho}_i} \dfrac{(x_i^{\alpha_i} - sign(u_1) w_i^{\alpha_i})}{\tilde{\rho}_i^{r_i(\alpha_i - 1)}}, \quad (i = 3, \ldots, n-1) \end{array} \right. \tag{38}$$

with:

- $\{\alpha_i\}$, $(\ i = 3, \ldots, n-1)$ *a set of odd integers such that:*

$$\forall\, i \quad \alpha_i r_i > n \tag{39}$$

- $k = (k_1, k_2, \ldots, k_n)$ *a vector of design parameters,*

- g, σ_1, *and* σ_2 *defined as in Proposition 3.*

There exist positive continuous functions $\gamma_3, \ldots, \gamma_n$ such that for any $\epsilon_2 \in (0, \sigma_1)$, any $\bar{k}_2 > 0$, and any k such that:

$$\begin{array}{l} k_1 > 0, \quad k_2 \geq \bar{k}_2 \\ k_3 \geq \gamma_3(\epsilon_2, \bar{k}_2) \\ k_4 \geq \gamma_4(\epsilon_2, \bar{k}_2, k_3) \\ \vdots \\ k_n \geq \gamma_n(\epsilon_2, \bar{k}_2, k_3, \ldots, k_{n-1}) \end{array} \tag{40}$$

[1^o)] Asymptotical stability*:*
The feedback control (37)-(38) globally exponentially stabilize (in the sense of Proposition 1) the origin $x = 0$ of the system (21).

[2^o)] Exponential convergence rates*:*
Along any trajectory of the closed loop system (21)-(37)-(38)

i) the variable x_1 exponentially converges to zero with a rate better than $k_2\, Min(k_1\, \sigma_2 - \epsilon_1, \dfrac{\sigma_1 - \epsilon_2}{n})$, where ϵ_1 is any arbitrary small positive number.

ii) one has

$$V_+(Y(jT)) \leq e^{-\frac{p}{n} k_2(\sigma_1 - \epsilon_2) jT} V_+(Y(0)), \quad \forall j \in \mathbb{N}, \tag{41}$$

with the function $V_+(Y), (Y = (x_2, x_3, \ldots, x_n))$ defined by:

$$V_+(Y) = |x_2|^{\frac{p}{r_2}} + \sum_{i=3}^{n} \frac{1}{\sqrt{k_i}} \int_{w_i}^{x_i} s^{\frac{p}{r_i}-1} - w_i^{\frac{p}{r_i}-1}\, ds \tag{42}$$

(with p any integer such that $p > n$ and $\dfrac{p}{r_i}$ $(i = 3, \ldots, n)$ is even).
Relation (41) implies that each variable x_i $(i = 2, \ldots, n)$ exponentially converges to zero with a rate better than $\dfrac{k_2 r_i}{n}(\sigma_1 - \epsilon_2)$.

The proof of this result closely follows the proof of Proposition 3 and will not be reproduced for this reason. Let us just indicate that it only requires a slight modification of Proposition 2 so that the conclusion of this proposition still holds when the system (4) is replaced by the system:

$$\left\{ \begin{array}{rcl} \dot{x} & = & f(x, y) \\ \dot{y} & = & -k \dfrac{(y^p - v^p(x^1))}{\rho^{(p-1)q}(x, y)} \end{array} \right. \tag{43}$$

It may be noted that the simpler control law (II) can be interpreted as a particular case of the control law (III) with $\alpha_i = 1$ $(i = 3, \ldots, n-1)$. The reason why Proposition 4 does not apply to this case is that the condition (39) is not satisfied with this choice of α_i.

4 Simulation results

Simulations have been carried out for the three different control laws (I) – (III) proposed in section 3. We present hereafter some results obtained for the dimension n=4. Let us recall that the four-dimensional chained form can be used to represent the kynematic model of a car-like vehicle. In this case, the variables x_1 and x_2 represent the coordinates x and y of the point located at the middle of the rear-wheels axis, and $x_3 = tan(\theta)$ where θ is the angle representing the orientation of the vehicle.

The simulations have been made with the following coefficients:

$$k_1 = 0.5, \quad k_2 = 0.2 \tag{44}$$

which, according to the stability analysis, basically determine the lowerbound of the exponential rate of convergence once the T-periodic function $g(t)$ has been chosen. The period of this function itself determines the frequency according to which the sign of the vehicle's translational velocity changes. The following function is used:

$$g(t) \; = \; sin(\omega t), \; \omega = 0.3$$

so that:

$$\sigma_1 \; = \; \frac{\omega}{\pi} \int_0^{\frac{\pi}{\omega}} sin(\omega s) \; ds \; = \; \frac{1}{\pi} \int_0^{\pi} sin(u) \, du \; = \; \frac{2}{\pi} \tag{45}$$

and:

$$\sigma_2 \; = \; \frac{\omega}{\pi} \int_0^{\frac{\pi}{\omega}} sin^2(\omega s) \; ds \; = \; \frac{1}{\pi} \int_0^{\pi} sin^2(u) \, du \; = \; \frac{1}{2} \tag{46}$$

The initial conditions are:

$$x(0) \; = \; (0, 1, 0, 0.4) \tag{47}$$

In addition, for the control law (I), the initial conditions relative to the vector z are:

$$z(0) \; = \; (-\frac{x_2}{\rho_2}(0), -\frac{k_3}{\rho_3(0)}(x_3(0) - g(0)z_3(0))) \tag{48}$$

In order to illustrate the fact that the gain parameters (k_3', k_3, k_4', k_4) for the control law (I) and (k_3, k_4) for the control laws (II) $-$ (III) do not necessarily have to be chosen "very large" (in order to ensure the asymptotic stability of the origin of the closed loop system), simulations have been carried out with the following choice of these parameters:

Control law (I):

$$(k_3', k_3, k_4', k_4) \;=\; (2, 4, 8, 12) \tag{49}$$

Control laws (II) $-$ (III):

$$(k_3, k_4) \;=\; (3, 8) \tag{50}$$

The figures **1.a)-3.a)** show the trajectories of the vehicle in the (x, y) plane and the figures **4.a)-6.a)** show the evolution of the logarithm of the homogeneous norm, this norm beeing defined by:

$$\rho(x, z) \;=\; (\, x_1^{12} + |x_2|^3 + x_3^4 + x_4^6 + z_3^4 + z_4^6)^{\frac{1}{12}} \tag{51}$$

for the simulation of the system (21) controlled by (I) (with dynamic extension), and by:

$$\tilde{\rho}(x) \;=\; (\, x_1^{12} + |x_2|^3 + x_3^4 + x_4^6)^{\frac{1}{12}} \tag{52}$$

for the simulation of the system controlled by (II) and (III) respectively (with no dynamic extension).

Another group of simulations (**Simulations b-d**) has been carried out with larger gain parameters:
Control law (I):

$$(k_3', k_3, k_4', k_4) \;=\; (6, 20, 60, 150) \tag{53}$$

Control laws (II) $-$ (III):

$$(k_3, k_4) \;=\; (6, 60) \tag{54}$$

The figures **1** to **3** show the evolution of the variables x, y and θ.
The figures **4** show the control inputs u_1 and u_2.
The figures **5** show the trajectory of the vehicle in the (x, y) plane.
Finally, the figures **6** show the evolution of the logarithm of the homogeneous norm.
Let us compare the convergence rate displayed by the simulations to the estimated convergence rates provided by the stability analysis.

Control law (I):
The figure **6.b)** shows that the variable $\rho(x,z)$ converges to zero, along the trajectory of the system (21) controlled by (I), with a rate of about $\frac{6}{100}$.
From Proposition 3, the estimated convergence rate associated with the variables x_1 and $\rho_4(x_2,x_3,x_4,z_3,z_4) = (|x_2|^3 + x_3^4 + x_4^6 + z_3^4 + z_4^6)^{\frac{1}{12}}$ is equal to $k_2\,Min\{k_1\sigma_2 - \epsilon_1, \frac{\sigma_1-\epsilon_2}{4}\}$ and $\frac{k_2}{4}(\sigma_1 - \epsilon_2)$ respectively. Since $\rho(x,z)$ is homogeneous of the same degree as x_1 and ρ_4, the estimated convergence rate associated with the variable $\rho(x,z)$ is equal to the smallest of the above rates. Therefore, when ϵ_1 and ϵ_2 are taken very small, which implies that the parameters k_3', k_3, k_4' and k_4 must be chosen large enough, this rate is given by $k_2\,Min\{k_1\sigma_2, \frac{\sigma_1}{4}\}$. Replacing k_1, k_2, σ_1, and σ_2 by their numerical values yields the following estimation of the convergence rate of ρ:

$$0.2\,Min\{0.25, \frac{1}{2\pi}\} \;=\; \frac{0.2}{2\pi} \;\simeq\; \frac{3.2}{100}$$

Control law (III):
From Proposition 4, one easily verifies that the same estimation of the convergence rate also holds for the variable $\tilde{\rho}$ along the trajectories of the system (21) controlled by (III).
The figure **6.d)** shows that the variable $\tilde{\rho}(x)$ converges to zero, along the trajectory of this system, with a rate of about $\frac{4}{100}$.

As a conclusion, for both control laws (I) and (III), the rates estimated from the stability analysis have the same order of magnitude but are smaller than the rates observed from the simulations. This is in accordance with the stability analysis since Propositions 3 and 4 only aim at providing lowerbounds of the convergence rates.

5 Appendix: proof of Proposition 3

The proof is here given for $n = 4$. Since the mechanism of the proof is iterative we leave to the reader the task of verifying that the general case is a direct extension of this case.
Before, starting with the core of the proof, we give below a technical lemma which will be useful at some point in the proof.

5.1 Technical lemma

Lemma 1 *Consider the following function:*

$$\begin{aligned} V_+(Y) \;&=\; |x_2|^3 \;+\; \frac{1}{\sqrt{k_3'}}\int_{-\frac{x_2}{\rho_2}}^{z_3} (s^3+(\frac{x_2}{\rho_2})^3)\,ds \;+\; \frac{1}{\sqrt{k_3}}|x_3-z_3|^4 \\ &+\frac{1}{\sqrt{k_4'}}\int_{-\frac{k_3}{\rho_3}(x_3-z_3)}^{z_4} (s^5+(\frac{k_3}{\rho_3}(x_3-z_3))^5)\,ds \;+\; \frac{1}{\sqrt{k_4}}|x_4-z_4|^6 \end{aligned} \tag{55}$$

where k_3', k_3, k_4', and k_4 are positive real numbers, and $Y=(x_2,z_3,x_3,z_4,x_4)$. Consider also the function $V_-(Y)$ defined by:

$$V_-(Y) \;=\; V_+(x_2,z_3,-x_3,-z_4,x_4) \tag{56}$$

Then, there exist positive continuous functions $\nu_3(\epsilon_2,\bar{k}_2,k_3')$, $\nu_4'(\epsilon_2,\bar{k}_2,k_3',k_3)$, and $\nu_4(\epsilon_2,\bar{k}_2,k_3',k_3,k_4')$, defined for any $\epsilon_2>0$ and $\bar{k}_2>0$, nonincreasing in $\bar{k}_2$, and such that:

$$\left\{\begin{array}{lcl} k_3 &\geq& \nu_3(\epsilon_2,\bar{k}_2,k_3') \\ k_4' &\geq& \nu_4'(\epsilon_2,\bar{k}_2,k_3',k_3) \\ k_4 &\geq& \nu_4(\epsilon_2,\bar{k}_2,k_3',k_3,k_4') \end{array}\right. \quad\Longrightarrow\quad \frac{|V_+(Y)-V_-(Y)|}{V_-(Y)} \leq \delta(\bar{k}_2,\epsilon_2) \quad \forall\, Y\neq 0. \tag{57}$$

where:

$$\delta(\bar{k}_2,\epsilon_2) \;=\; \frac{e^{3\bar{k}_2\,\epsilon_2\,\frac{T}{4}}-1}{e^{3\bar{k}_2\,\epsilon_2\,\frac{T}{4}}+1} \tag{58}$$

Proof:
Let us first show that there exist two strictly positive constants α_1 and α_2, independent of the parameters k_i and k_i', such that :

$$\begin{aligned} V_-(Y) \;&\geq\; |x_2|^3 \;+\; \frac{\alpha_1}{\sqrt{k_3'}}|z_3+\frac{x_2}{\rho_2}|^4 \;+\; \frac{1}{\sqrt{k_3}}|x_3+z_3|^4 \;+\; \frac{\alpha_2}{\sqrt{k_4'}}|z_4+\frac{k_3}{\rho_3}(x_3+z_3)|^6 \\ &+\frac{1}{\sqrt{k_4}}|x_4+z_4|^6 \end{aligned} \tag{59}$$

To this purpose, it is clearly sufficient to show, in view of the definitions of V_- and V_+, that for any odd integer p, there exists a positive number α such that:

$$\int_v^z s^p - v^p\,ds \;\;\geq\;\; \alpha\,(z-v)^{p+1} \tag{60}$$

By considering the following function:

$$h(x) = \quad 2^{p-1}[(1+x)^p - x^p] - 1 \tag{61}$$

the positivity of which is easily established, and setting $x = \dfrac{v}{s-v}$, one obtains:

$$2^{p-1}\frac{(s^p - v^p)}{(s-v)^p} - 1 \geq 0. \tag{62}$$

Therefore:

$$\int_v^z s^p - v^p\, ds \;=\; \int_v^z \frac{s^p - v^p}{(s-v)^p}(s-v)^p\, ds \;\geq\; \int_v^z \frac{1}{2^{p-1}}(s-v)^p\, ds \;=\; \frac{(z-v)^{p+1}}{2^{p-1}(p+1)} \tag{63}$$

which proves (60).

Let us now estimate an upperbound for $|V_+(Y) - V_-(Y)|$.
From the definitions of V_- and V_+, and using the fact that:

$$\int_v^z s^p - v^p\, ds \;=\; \frac{z^{p+1}}{p+1} + p\frac{v^{p+1}}{p+1} - v^p z, \tag{64}$$

one obtains:

$$V_+ - V_- \;=\; \frac{1}{\sqrt{k_3}}P_3 \;+\; \frac{1}{\sqrt{k_4'}}P_4' \;+\; \frac{1}{\sqrt{k_4}}P_4 \tag{65}$$

with:

$$\begin{aligned} P_3 &= (x_3 - z_3)^4 - (x_3 + z_3)^4 \\ P_4' &= \frac{5k_3^6}{6\rho_3^6}\left((x_3 - z_3)^6 - (x_3 + z_3)^6\right) + \frac{k_3^5}{\rho_3^5}\, z_4\left((x_3 - z_3)^5 - (x_3 + z_3)^5\right) \\ P_4 &= (x_4 - z_4)^6 - (x_4 + z_4)^6 \end{aligned} \tag{66}$$

Let us prove that for an adequate choice of the parameters k_3, k_4' and k_4, the three following inequalities hold:

$$|P_3| \;\leq\; \frac{\delta(\bar{k}_2, \epsilon_2)}{3}\sqrt{k_3}\, V_- \tag{67}$$

$$|P_4'| \;\leq\; \frac{\delta(\bar{k}_2, \epsilon_2)}{3}\sqrt{k_4'}\, V_- \tag{68}$$

$$|P_4| \;\leq\; \frac{\delta(\bar{k}_2, \epsilon_2)}{3}\sqrt{k_4}\, V_- \tag{69}$$

We first show bellow the existence of positive nonincreasing functions K_i ($i \in \mathbb{N}, i \geq 1$) such that:

$$|(x-z)^i - (x+z)^i| \;\leq\; \epsilon|x+z|^i + K_i(\epsilon)|z|^i \qquad ,\; \forall x \in \mathbb{R},\; \forall z \in \mathbb{R},\; \forall \epsilon > 0. \tag{70}$$

Indeed, using the fact that $a^i - b^i \;=\; (a-b)\,\sum_{k=0}^{i-1} a^{i-1-k}\, b^k$, one obtains that:

$$|(x-z)^i-(x+z)^i| \;\leq\; 2|z|\,\sum_{k=0}^{i-1}|x-z|^{i-1-k}|x+z|^k \;\leq\; 2|z|\,\sum_{k=0}^{i-1}(|x+z|+2|z|)^{i-1-k}|x+z|^k \tag{71}$$

By the convexity of the function x^p ($p \in \mathbb{N},\; x \geq 0$) which implies that:

$$(\frac{x+y}{2})^p \;\leq\; \frac{1}{2}(x^p+y^p), \quad \forall x \geq 0,\; \forall y \geq 0 \tag{72}$$

it follows from (71) that:

$$|(x-z)^i - (x+z)^i| \;\leq\; 2|z|\,\sum_{k=0}^{i-1} 2^{i-2-k}(|x+z|^{i-1-k} + 2|z|^{i-1-k})|x+z|^k \tag{73}$$

Using Young's inequality, it directly follows that, for any α with $0 < \alpha \leq i$ and any $\epsilon > 0$,

$$|z|^\alpha |x+z|^{i-\alpha} \leq (\frac{\epsilon i}{i-\alpha})^{\frac{i-\alpha}{i}}|x+z|^{i-\alpha}\,(\frac{i-\alpha}{\epsilon i})^{\frac{i-\alpha}{i}}|z|^\alpha \;\leq \epsilon|x+z|^i \;+\; K(\epsilon)|z|^i \tag{74}$$

with $K(\epsilon) = \frac{\alpha}{i}(\frac{i-\alpha}{\epsilon i})^{\frac{i-\alpha}{\alpha}}$ non increasing. Inequality (70) directly follows from (73) and (74).

Then, for any homogeneous function $g(x_2, z_3, x_3)$ of degree q, there exists a constant $\bar{K}$ such that:

$$|g(x_2, z_3, x_3)| \leq \bar{K}(|x_2|^{\frac{q}{4}} + |z_3 + \frac{x_2}{\rho_2}|^{\frac{q}{3}} + |x_3 + z_3|^{\frac{q}{3}}) \tag{75}$$

This is due to the fact that the function:

$$\frac{|g(x_2, z_3, x_3)|}{|x_2|^{\frac{q}{4}} + |z_3 + \frac{x_2}{\rho_2}|^{\frac{q}{3}} + |x_3 + z_3|^{\frac{q}{3}}} \tag{76}$$

is continuous, homogeneous of degree 0, well defined for $(x_2, z_3, x_3) \neq (0,0,0)$ and, as a consequence, reaches its maximum $\bar{K}$ on the compact set
$\{(x_2, z_3, x_3),\ \rho_3(x_2, z_3, x_3) = 1\}$.
Using (70), (72), (75) and Young's inequality, it follows from (66) that:

$$\begin{aligned} |P_3| &= |(x_3 - z_3)^4 - (x_3 + z_3)^4| \leq \epsilon |x_3 + z_3|^4 + K_4(\epsilon)|z_3|^4 \\ &\leq \epsilon |x_3 + z_3|^4 + K_4(\epsilon)(|z_3 + \frac{x_2}{\rho_2}| + |\frac{x_2}{\rho_2}|)^4 \\ &\leq \epsilon |x_3 + z_3|^4 + 2^3\, K_4(\epsilon)\,(|z_3 + \frac{x_2}{\rho_2}|^4 + |x_2|^3) \end{aligned} \tag{77}$$

$$\begin{aligned} |P_4'| &\leq \frac{5k_3^6}{6} \frac{|(x_3 - z_3)^6 - (x_3 + z_3)^6|}{\rho_3^6} + |z_4|\, k_3^5 \frac{|(x_3 - z_3)^5 - (x_3 + z_3)^5|}{\rho_3^5}| \\ &\leq \frac{5k_3^6}{6} \bar{K}_1\,(|x_2|^3 + |z_3 + \frac{x_2}{\rho_2}|^4 + |x_3 + z_3|^4) \\ &+ |z_4 + \frac{k_3}{\rho_3}(x_3 + z_3)|\, k_3^5 \frac{|(x_3 - z_3)^5 - (x_3 + z_3)^5|}{\rho_3^5} \\ &+ k_3^6 \frac{|x_3 + z_3|}{\rho_3}\, \frac{|(x_3 - z_3)^5 - (x_3 + z_3)^5|}{\rho_3^5} \\ &\leq \frac{5k_3^6}{6} \bar{K}_1\,(|x_2|^3 + |z_3 + \frac{x_2}{\rho_2}|^4 + |x_3 + z_3|^4) \\ &+ \epsilon |z_4 + \frac{k_3}{\rho_3}(x_3 + z_3)|^6 + \frac{5\epsilon k_3^6}{(6\epsilon)^{\frac{6}{5}}} \frac{|(x_3 - z_3)^5 - (x_3 + z_3)^5|^{\frac{6}{5}}}{\rho_3^6} \\ &+ k_3^6\, \bar{K}_2(|x_2|^3 + |z_3 + \frac{x_2}{\rho_2}|^4 + |x_3 + z_3|^4) \\ &\leq \frac{5k_3^6}{6} \bar{K}_1\,(|x_2|^3 + |z_3 + \frac{x_2}{\rho_2}|^4 + |x_3 + z_3|^4) \\ &+ \epsilon |z_4 + \frac{k_3}{\rho_3}(x_3 + z_3)|^6 + \frac{5\epsilon k_3^6}{(6\epsilon)^{\frac{6}{5}}} \bar{K}_3\,(|x_2|^3 + |z_3 + \frac{x_2}{\rho_2}|^4 + |x_3 + z_3|^4) \\ &+ k_3^6\, \bar{K}_2(|x_2|^3 + |z_3 + \frac{x_2}{\rho_2}|^4 + |x_3 + z_3|^4) \\ &\leq \epsilon |z_4 + \frac{k_3}{\rho_3}(x_3 + z_3)|^6 \ + k_3^6 K_0(\epsilon)\,(|x_2|^3 + |z_3 + \frac{x_2}{\rho_2}|^4 + |x_3 + z_3|^4) \end{aligned} \tag{78}$$

with $K_0(\epsilon) = \frac{5}{6}\bar{K}_1 + \frac{5\epsilon}{(6\epsilon)^{\frac{6}{5}}}\bar{K}_3 + \bar{K}_2$, and

$$\begin{aligned}
|P_4| &= |(x_4 - z_4)^6 - (x_4 + z_4)^6| \le \epsilon |x_4 + z_4|^6 + K_6(\epsilon)|z_4|^6 \\
&\le \epsilon |x_4 + z_4|^6 + K_6(\epsilon)\,(|z_4 + \frac{k_3}{\rho_3}(x_3 + z_3)| + |\frac{k_3}{\rho_3}(x_3 + z_3)|)^6 \\
&\le \epsilon |x_4 + z_4|^6 + 2^5\,K_6(\epsilon)\,(|z_4 + \frac{k_3}{\rho_3}(x_3 + z_3)|^6 + |\frac{k_3}{\rho_3}(x_3 + z_3)|^6) \\
&\le \epsilon |x_4 + z_4|^6 + 2^5\,K_6(\epsilon)\,|z_4 + \frac{k_3}{\rho_3}(x_3 + z_3)|^6 \\
&+ 2^5\,K_6(\epsilon)\,k_3^6\,\bar{K}_5\,(|x_2|^3 + |z_3 + \frac{x_2}{\rho_2}|^4 + |x_3 + z_3|^4)
\end{aligned} \tag{79}$$

where the $\bar{K}_i$, $(i = 1, \ldots, 5)$ are constant values, and the K_i, $(i = 0, 4, 6)$ are nonincreasing functions.
Let:

$$\epsilon = Min\{\frac{\delta(\bar{k}_2, \epsilon_2)}{3}, \frac{\delta(\bar{k}_2, \epsilon_2)\alpha_2}{3}\} \tag{80}$$

The functions ν_3, ν_4' and ν_4 are recursively defined as follows. First, let:

$$\nu_3(\epsilon_2, \bar{k}_2, k_3') = Max\{(\frac{24K_4(\epsilon)}{\delta(\bar{k}_2, \epsilon_2)})^2, (\frac{24K_4(\epsilon)\sqrt{k_3'}}{\delta(\bar{k}_2, \epsilon_2)\alpha_1})^2\} \tag{81}$$

Then, for $k_3 \ge \nu_3$, the inequality (67) follows from (59), (80) and (77). Next, let:

$$\nu_4'(\epsilon_2, \bar{k}_2, k_3', k_3) = Max\{(\frac{3k_3^6 K_0(\epsilon)}{\delta(\bar{k}_2, \epsilon_2)})^2, (\frac{3k_3^6 K_0(\epsilon)\sqrt{k_3'}}{\delta(\bar{k}_2, \epsilon_2)\alpha_1})^2, (\frac{3k_3^6 K_0(\epsilon)\sqrt{k_3}}{\delta(\bar{k}_2, \epsilon_2)})^2\} \tag{82}$$

then, for $k_4' \ge \nu_4'$, the inequality (68) follows from (59), (80) and (78). Finally, let:

$$\begin{aligned}
\nu_4(\epsilon_2, \bar{k}_2, k_3', k_3, k_4') = Max\{ & (\frac{96K_6(\epsilon)\sqrt{k_4'}}{\delta(\bar{k}_2, \epsilon_2)\alpha_2})^2, (\frac{96k_3^6 K_6(\epsilon)\bar{K}_5}{\delta(\bar{k}_2, \epsilon_2)})^2, \\
& (\frac{96k_3^6 K_6(\epsilon)\bar{K}_5\sqrt{k_3'}}{\delta(\bar{k}_2, \epsilon_2)\alpha_1})^2, (\frac{96K_6(\epsilon)\sqrt{k_3}}{\delta(\bar{k}_2, \epsilon_2)})^2\}
\end{aligned} \tag{83}$$

then, for $k_4 \ge \nu_4$, the inequality (69) follows from (59), (80) and (79).
The relation (57) is a direct consequence of (65), (67), (68) and (69).
Finally since, in view of (58), δ is an increasing function of $\bar{k}_2$, it follows from (80) that ϵ is also an increasing function of $\bar{k}_2$, which implies that the $K_i(\epsilon)$ are non increasing in $\bar{k}_2$. In view of (81), (82) and (83), this implies that ν_3, ν_4' and ν_4 are

also non increasing in $\bar{k}_2$.
(end of Proof of Lemma 1).

We may now proceed with the proof of **Proposition 3**.

5.2 Exponential stabilization of $(x_2, z_3, x_3, z_4, x_4) = 0$

Let us first recall the system's equations in the case where $n = 4$:

$$\left\{\begin{array}{lcl} \dot{x}_1 & = & u_1 \\ \dot{x}_2 & = & u_1\, x_3 \\ \dot{z}_3 & = & -\dfrac{k_3'}{\rho_3'}(|u_1|z_3 + \dfrac{1}{\rho_2}\,|u_1|x_2) \\ \dot{x}_3 & = & u_1\, x_4 \\ \dot{z}_4 & = & -\dfrac{k_4'}{\rho_4'}(|u_1|z_4 + \dfrac{k_3}{\rho_3}\,(|u_1|x_3 - u_1 z_3)) \\ \dot{x}_4 & = & -\dfrac{k_4}{\rho_4}(|u_1|x_4 - u_1 z_4) \end{array}\right. \tag{84}$$

As before, the vector $(x_2, z_3, x_3, z_4, x_4)$ will be denoted as Y.
We will proceed as follows. Firstly, regardless of the variable x_1, we will show the convergence of the variables $Y(t)$ to zero. More precisely, we will construct a Lyapunov function $V_+(Y)$ which is exponentially decreasing along any trajectory of the closed-loop system, provided that the control coefficients satisfy the conditions specified in the proposition. This is the subject of the present subsection. Then, we will show in the subsection 5.3 that the variable x_1 also exponentially tends to zero. A similar two-steps analysis can be found, for example, in [20] and in [17].

Lemma 2 *Consider the function $V_+(Y)$ defined by (55). There exist functions γ_3', γ_3, γ_4' and γ_4 as specified in Proposition 3, such that, for k satisfying (29) and along any trajectory of the closed-loop system:*

$$V_+(Y(jT+t)) \;\leq\; e^{-3k_2(\sigma_1-\epsilon_2)jT}\, V_+(Y(0)) \quad \forall j \in \mathbb{N}, \forall t \in [0,T]. \tag{85}$$

Proof
Let us distinguish two cases according to whether the sign of $g(t)$ is positive or negative.

Case 1: $t \in [jT, jT+T/2]$. In this case, $g(t)$ is positive, which implies, in view of (26), that $u_1(t)$ is also positive. The five last equations of the system (84) can then be rewritten as follows:

$$\left\{\begin{array}{lcl} \dot{x}_2 & = & |u_1|\, x_3 \\ \dot{z}_3 & = & |u_1|\, (-\dfrac{k_3'}{\rho_3'}(z_3 + \dfrac{x_2}{\rho_2})) \\ \dot{x}_3 & = & |u_1|\, x_4 \\ \dot{z}_4 & = & |u_1|\, (-\dfrac{k_4'}{\rho_4'}(z_4 + \dfrac{k_3}{\rho_3}\,(x_3 - z_3))) \\ \dot{x}_4 & = & |u_1|\, (-\dfrac{k_4}{\rho_4}(x_4 - z_4)) \end{array}\right. \tag{86}$$

By application of Corollary 1 to the system:

$$\dot{x}_2 = \rho_2(x_2)(-\frac{x_2}{\rho_2(x_2)}) \tag{87}$$

for which $V(x_2) = x_2^3$ is a Lyapunov function ($\dot{V} = -3V$ along any trajectory), one deduces the existence of positive continuous function β_3' and β_3 such that, for any (k_3', k_3) verifying:

$$k_3' \geq \beta_3'(\epsilon), \quad k_3 \geq \beta_3(\epsilon, k_3'), \quad \epsilon \in (0,3) \tag{88}$$

the following two points hold:

1^o) the origin of the system:

$$\left\{\begin{array}{lcl} \dot{x}_2 & = & \rho_3\, x_3 \\ \dot{z}_3 & = & \rho_3\, (-\dfrac{k_3'}{\rho_3'}(z_3 + \dfrac{x_2}{\rho_2})) \\ \dot{x}_3 & = & \rho_3\, (-\dfrac{k_3}{\rho_3}(x_3 - z_3)) \end{array}\right. \tag{89}$$

is asymptotically stable.

2^o) the function:

$$W(x_2, z_3, x_3) \;=\; |x_2|^3 + \frac{1}{\sqrt{k_3'}} \int_{-\frac{x_2}{\rho_2}}^{z_3} (s^3 + (\frac{x_2}{\rho_2})^3)\; ds + \frac{1}{\sqrt{k_3}} |x_3 - z_3|^4 \tag{90}$$

which is homogeneous of degree 12 with respect to the dilation $\delta_3(x_2, z_3, x_3) = (\lambda^{r_2} x_2, \lambda^{r_3} z_3, \lambda^{r_3} x_3)$, is such that:

$$\frac{\partial W}{\partial x_2}\,\rho_3 x_3 \;+\; \frac{\partial W}{\partial z_3}\,\rho_3(-\frac{k_3'}{\rho_3'}(z_3+\frac{x_2}{\rho_2})) \;+\; \frac{\partial W}{\partial x_3}(-k_3(x_3-z_3)) \;\le\; -(3-\epsilon)\,W \tag{91}$$

Applying once again the Corollary 1 to the system (89), one deduces the existence of positive continuous functions β_4' and β_4 which depend continuously on k_3' and k_3 ,and such that, for any (k_3', k_3) satisfying (88) and any (k_4', k_4) satisfying:

$$k_4' \ge \beta_4'(\epsilon', k_3', k_3), \quad k_4 \ge \beta_4(\epsilon', k_3', k_3, k_4'), \quad \epsilon' \in (0, 3-\epsilon) \tag{92}$$

the following two points hold:

1^o) the origin of the system:

$$\left\{\begin{array}{rcl} \dot{x}_2 &=& \rho_4\, x_3 \\ \dot{z}_3 &=& \rho_4\,(-\frac{k_3'}{\rho_3'}(z_3+\frac{x_2}{\rho_2})) \\ \dot{x}_3 &=& \rho_4\, x_4 \\ \dot{z}_4 &=& \rho_4\,(-\frac{k_4'}{\rho_4'}(z_4+\frac{k_3}{\rho_3}(x_3-z_3))) \\ \dot{x}_4 &=& \rho_4\,(-\frac{k_4}{\rho_4}(x_4-z_4)) \end{array}\right. \tag{93}$$

is asymptotically stable,

2^o) the function V_+ given by (55), which is homogeneous of degree 12 with respect to the dilation

$$\delta_4(Y) = (\lambda^{r_2} x_2, \lambda^{r_3} z_3, \lambda^{r_3} x_3, \lambda^{r_4} z_4, \lambda^{r_4} x_4) \tag{94}$$

is such that:

$$\begin{gathered}\frac{\partial V_+}{\partial x_2}\,\rho_4 x_3 \;+\; \frac{\partial V_+}{\partial z_3}\,\rho_4(-\frac{k_3'}{\rho_3'}(z_3+\frac{x_2}{\rho_2})) \;+\; \frac{\partial V_+}{\partial x_3}\,\rho_4\, x_4 \;+ \\ \frac{\partial V_+}{\partial z_4}\rho_4\,(-\frac{k_4'}{\rho_4'}(z_4+\frac{k_3}{\rho_3}(x_3-z_3))) + \frac{\partial V_+}{\partial x_4}\,\rho_4\,(-\frac{k_4}{\rho_4}(x_4-z_4)) \le -(3-\epsilon-\epsilon')V_+ .\end{gathered} \tag{95}$$

Let us define some new functions $\eta'_3, \eta_3, \eta'_4, \eta_4$ as follows:

$$
\begin{array}{l}
\eta'_3(\epsilon_2) \;=\; \beta'_3(\frac{3\epsilon_2}{4\sigma_1}),\; \eta_3(\epsilon_2, k'_3) \;=\; \beta_3(\frac{3\epsilon_2}{4\sigma_1}, k'_3), \\
\eta'_4(\epsilon_2, k'_3, k_3) \;=\; \beta'_3(\frac{3\epsilon_2}{4\sigma_1}, k'_3, k_3),\; \eta_4(\epsilon_2, k'_3, k_3, k'_4) \;=\; \beta_4(\frac{3\epsilon_2}{4\sigma_1}, k'_3, k_3, k'_4)
\end{array}
\tag{96}
$$

then, in view of the definition of the β'_i and β_i, $(i = 3, 4)$, these functions are well defined for $\epsilon_2 \in (0, \sigma_1)$ and are such that for k'_3, k_3, k'_4 and k_4 satisfying:

$$
k'_3 \geq \eta'_3(\epsilon_2),\; k_3 \geq \eta_3(\epsilon_2, k'_3),\; k'_4 \geq \eta'_4(\epsilon_2, k'_3, k_3),\; k_4 \geq \eta_4(\epsilon_2, k'_3, k_3, k'_4), \quad \epsilon_2 \in (0, \tau)
\tag{97}
$$

the time derivative $\dot{V}_+(Y(t))$ of the function V_+, along any trajectory $Y(t)$ of the system (86), satisfies:

$$
\dot{V}_+(Y(t)) \;\leq\; -(3 - \frac{3\epsilon_2}{2\sigma_1})V_+(Y(t))
$$

From the above inequality, and using the fact that:

$$
|u_1| \;\geq\; k_2|g(t)|\rho_4,
\tag{98}
$$

it follows that:

$$
\dot{V}_+(Y(jT + t)) \;\leq\; -(3 - \frac{3\epsilon_2}{2\sigma_1})k_2|g(t)|\, V_+(Y(jT + t)) \qquad \forall t \in [0, \frac{T}{2}].
\tag{99}
$$

This implies, using also the definition (27) of σ_1, that along any trajectory of the system (86):

$$
\left\{
\begin{array}{l}
V_+(Y(jT + \frac{T}{2})) \;\leq\; e^{-3k_2(\sigma_1 - \frac{\epsilon_2}{2})\frac{T}{2}}\, V_+(Y(jT)) \\
V_+(Y(jT + t)) \;\leq V_+(Y(jT)) \qquad \forall t \in [0, \frac{T}{2}].
\end{array}
\right.
\tag{100}
$$

Case 2: $t \in [jT + T/2, (j+1)T]$.
In this case $g(t)$ is negative, and so is $u_1(t)$. The five last equations of the system

(84) can then be rewritten as follows:

$$\left\{\begin{array}{rcl} \dot{x}_2 & = & |u_1|\,(-x_3) \\ \dot{z}_3 & = & |u_1|(-\dfrac{k'_3}{\rho'_3}(z_3+\dfrac{x_2}{\rho_2})) \\ \dot{x}_3 & = & |u_1|\,(-x_4) \\ \dot{z}_4 & = & |u_1|(-\dfrac{k'_4}{\rho'_4}(z_4+\dfrac{k_3}{\rho_3}\,(x_3+z_3))) \\ \dot{x}_4 & = & |u_1|(-\dfrac{k_4}{\rho_4}(x_4+z_4)) \end{array}\right. \tag{101}$$

By remarking that the following change of coordinates:

$$(x_2, z_3, x_3, z_4, x_4) \longmapsto (\tilde{x}_2, \tilde{z}_3, \tilde{x}_3, \tilde{z}_4, \tilde{x}_4) = (x_2, z_3, -x_3, -z_4, x_4) \tag{102}$$

transforms the system (101) into the system (86) which was considered in the Case 1, and by setting:

$$V_-(x_2, z_3, x_3, z_4, x_4) \;=\; V_+(x_2, z_3, -x_3, -z_4, x_4), \tag{103}$$

one deduces that under the same conditions as before upon the coefficients k'_3, k_3, k'_4 and k_4 (i.e. (97)), and along any trajectory of the system (84):

$$\dot{V}_-(Y(jT+\frac{T}{2}+t)) \;\le\; -(3-\frac{3\epsilon_2}{2\sigma_1})k_2\,|g(t)|\; V_-(Y(jT+\frac{T}{2}+t)) \qquad \forall t \in [0, \frac{T}{2}]. \tag{104}$$

Therefore:

$$\left\{\begin{array}{l} V_-(Y((j+1)T)) \;\le\; e^{-3k_2(\sigma_1-\frac{\epsilon_2}{2})\frac{T}{2}}\, V_-(Y(jT+\dfrac{T}{2})) \\ V_-(Y(jT+\dfrac{T}{2}+t)) \;\le V_-(Y(jT+\dfrac{T}{2})) \qquad \forall t \in [0, \dfrac{T}{2}]. \end{array}\right. \tag{105}$$

We may note at this point that if the functions V_+ and V_- had been equal then, in view of (100) and (105), the proof of the Lemma 2 would have been finished. Although these functions are not equal, we have already established in the Lemma 1 that their relative difference can be kept as small as desired by an adequate choice of the control coefficients k_3, k'_4, and k_4.

Let us then define the functions:

$$
\begin{array}{rcl}
\gamma_3'(\epsilon_2) & = & \eta_3'(\epsilon_2) \\
\gamma_3(\epsilon_2,\bar{k}_2,k_3') & = & Max\{\eta_3(\epsilon_2,k_3'),\ \nu_3(\epsilon_2,\bar{k}_2,k_3'),\ (1+\epsilon_3)\inf_{\bar{k}_2}\ \nu_3(\epsilon_2,\bar{k}_2,k_3')\} \\
\gamma_4'(\epsilon_2,\bar{k}_2,k_3',k_3) & = & Max\{\eta_4'(\epsilon_2,k_3',k_3),\ \nu_4'(\epsilon_2,\bar{k}_2,k_3',k_3), \\
 & & \quad (1+\epsilon_3)\inf_{\bar{k}_2}\ \nu_4'(\epsilon_2,\bar{k}_2,k_3',k_3)\} \\
\gamma_4(\epsilon_2,\bar{k}_2,k_3',k_3,k_4') & = & Max\{\eta_4(\epsilon_2,k_3',k_3,k_4'),\ \nu_4(\epsilon_2,\bar{k}_2,k_3',k_3,k_4'), \\
 & & \quad (1+\epsilon_3)\inf_{\bar{k}_2}\ \{\nu_4(\epsilon_2,\bar{k}_2k_3',k_3,k_4')\}
\end{array}
\tag{106}
$$

with ϵ_3 any strictly positive value.

One verifies easily that these functions have the properties announced in the Proposition 3. In particular, they are constant with respect to $\bar{k}_2$ when $\bar{k}_2$ is larger than some value $\bar{k}_{2,min}$ (which depends on ϵ_3).

Let us now consider any $\epsilon_2 \in (0,\sigma_1)$, any $\bar{k}_2 > 0$ and any (k_2,k_3',k_3,k_4',k_4) satisfying (29). Then, in view of (106), and Lemma 2,

$$
\frac{|V_+(Y)-V_-(Y)|}{V_-(Y)} \leq \delta(\bar{k}_2,\epsilon_2) \quad \forall\, Y \neq 0.
$$

which implies that:

$$
(1-\delta(\bar{k}_2,\epsilon_2))\,V_- \leq V_+ \leq (1+\delta(\bar{k}_2,\epsilon_2))\,V_- \tag{107}
$$

Moreover, the inequalities (100) and (105) also hold. As a consequence, for any $j \in \mathbb{N}$ and any $t \in [0,\frac{T}{2}]$,

$$
\begin{array}{rcl}
V_+(Y((jT+\dfrac{T}{2}+t))) & \leq & V_-(Y((jT+\dfrac{T}{2}+t)))\,(1+\delta(\bar{k}_2,\epsilon_2)) \\
 & \leq & V_-(Y(jT+\dfrac{T}{2}))(1+\delta(\bar{k}_2,\epsilon_2)) \\
 & \leq & V_+(Y(jT+\dfrac{T}{2}))\dfrac{1+\delta(\bar{k}_2,\epsilon_2)}{1-\delta(\bar{k}_2,\epsilon_2)} \\
 & \leq & V_+(Y(jT))\,e^{-3k_2\,(\sigma_1-\frac{\epsilon_2}{2})\,\frac{T}{2}}\,\dfrac{1+\delta(\bar{k}_2,\epsilon_2)}{1-\delta(\bar{k}_2,\epsilon_2)} \\
 & \leq & V_+(Y(jT))e^{-3k_2\,(\sigma_1-\frac{\epsilon_2}{2})\,\frac{T}{2}}\,e^{3\bar{k}_2\,\frac{\epsilon_2}{2}\,\frac{T}{2}} \;\leq\; V_+(Y(jT))
\end{array}
\tag{108}
$$

and,

$$
\begin{aligned}
V_+(Y((j+1)T)) &\leq V_-(Y((j+1)T))\,(1+\delta(\bar{k}_2,\epsilon_2)) \\
&\leq V_-(Y(jT+\frac{T}{2}))e^{-3k_2\,(\sigma_1-\frac{\epsilon_2}{2})\,\frac{T}{2}}(1+\delta(\bar{k}_2,\epsilon_2)) \\
&\leq V_+(Y(jT+\frac{T}{2}))e^{-3k_2\,(\sigma_1-\frac{\epsilon_2}{2})\,\frac{T}{2}}\frac{1+\delta(\bar{k}_2,\epsilon_2)}{1-\delta(\bar{k}_2,\epsilon_2)} \\
&\leq V_+(Y(jT))e^{-3k_2\,(\sigma_1-\frac{\epsilon_2}{2})\,T}\frac{1+\delta(\bar{k}_2,\epsilon_2)}{1-\delta(\bar{k}_2,\epsilon_2)} \\
&\leq V_+(Y(jT))e^{-3k_2\,(\sigma_1-\frac{\epsilon_2}{2})\,T}e^{3\bar{k}_2\,\frac{\epsilon_2}{2}\,\frac{T}{2}} \\
&\leq V_+(Y(jT))e^{-3k_2\,(\sigma_1-\epsilon_2)\,T}
\end{aligned}
\tag{109}
$$

It directly follows, from (108) and (109), that:

$$
V_+(Y(jT+t)) \;\leq\; e^{-3k_2(\sigma_1-\epsilon_2)jT}\,V_+(Y(0\,)) \quad \forall j\in\mathbb{N}, \forall t\in[0,T]
$$

which is the result announced in Lemma 2.

5.3 Exponential stabilization of $x_1 = 0$

The exponential stabilization of $x_1 = 0$ follows from the expression of the control u_1 and the already established exponential stabilization of $Y = 0$. More precisely, we have the following result:

Lemma 3 *Consider the positive continuous function:*

$$
V_1(x_1,t) = x_1^2\,(\,f_+^2(t)\,\chi(x_1)\;+\;f_-^2(t)\,(1-\chi)(x_1)\,)
\tag{110}
$$

with:

$$
\begin{aligned}
f_+(t) &= e^{-k_2k_1\int_0^t \sigma_2-g(s)\,(g(s)-|g(s)|)\,ds} \\
f_-(t) &= e^{-k_2k_1\int_0^t \sigma_2-g(s)\,(g(s)+|g(s)|)\,ds}
\end{aligned}
\tag{111}
$$

and χ the function defined by:

$$
\chi(x) = \begin{cases} 1\;if\;x\geq 0 \\ 0\;if\;x<0 \end{cases}
\tag{112}
$$

Then,

1^o) *there exist two constants c and d such that*

$$c\,x_1^2 \;\leq V(x_1,t) \leq d\,x_1^2\,. \tag{113}$$

2^o) *there exists a function $K : \mathbb{R}^4 \times (\mathbb{R} - \{0\}) \longmapsto \mathbb{R}^+$ such that:*

i) $\forall \epsilon_1 > 0$, $K(.,\epsilon_1)$ *is continuous and* $K(0,\epsilon_1) = 0$,

ii) for any trajectory $(x_1(t), Y(t))$ of the closed-loop system and any arbitrary small positive number ϵ_1, the derivative $\dot{\psi}(t)$ of the function $\psi(t) = V_1(x_1(t),t)$ is defined for allmost all t, and is such that:

$$\dot{\psi}(t) \;\leq\; -2(k_2k_1\sigma_2 - \epsilon_1)\,\psi(t) \;+\; K(Y(0),\epsilon_1)\,e^{-\frac{k_2}{2}\,(\sigma_1-\epsilon_2)t} \tag{114}$$

3^o) *the variable x_1 exponentially tends to zero for any $\epsilon_1 < k_2k_1\sigma_2$ with a rate better than $k_2\,Min\,\{k_1\sigma_2 - \epsilon_1, \dfrac{\sigma_1-\epsilon_2}{4}\}$.*

Proof

Let us first prove 1^o).

From the conditions imposed on $g(t)$ (see (27)) and the definition of σ (see (28)):

$$\begin{aligned} \int_0^T \sigma_2 - g(s)\,(g(s) - |g(s)|)\,ds \;&=\; \sigma_2\,T - \int_0^T g^2(s)\,(1 - sign(g))\,ds \\ &=\; \sigma_2\,T - 2\int_{\frac{T}{2}}^T g^2(s)\,ds \\ &=\; \sigma_2\,T - 2\int_0^{\frac{T}{2}} g^2(s)\,ds \;=\; 0 \end{aligned} \tag{115}$$

As a consequence, the function $f_+(t)$ is T-periodic and satisfies

$$0 < c_+ \leq f_+ \leq d_+ < +\infty \tag{116}$$

Similarly, one can show that

$$0 < c_- \leq f_- \leq d_- < +\infty \tag{117}$$

The inequality (113) follows with $c = \;Min(c_+,c_-)$ and $d = \;Max(d_+,d_-)$.

We now proceed with the proof of 2^o).

Let us first remark that the function ψ is everywhere continuous, and differentiable on each interval (t_i, t_{i+1}) on which g is continuous. Let t belong to such an interval. We first assume that $x_1(t) > 0$. Then, in view of (110), $V_1 = x_1^2 f_+^2$, which implies using (26) that:

$$
\begin{array}{rcl}
\dot{\psi}(t) & = & 2x_1 f_+^2 \left(k_2 g(t)[k_1(-x_1 g(t) + |x_1 g(t)|) + \rho_4(Y)]\right) \\
 & & -2x_1^2 f_+^2 \left(k_2 k_1(\sigma_2 - g(t)(g(t) - |g(t)|))\right) \\
 & = & -2k_2 k_1 \sigma_2 x_1^2 f_+^2(t) + 2x_1 f_+^2(k_2 g(t)\rho_4(Y)) \\
 & = & -2k_2 k_1 \sigma_2 \psi(t) + 2x_1 f_+^2(k_2 g(t)\rho_4(Y)) \\
 & = & -2k_2 k_1 \sigma_2 \psi(t) + 2\sqrt{\epsilon_1 c}\, x_1 \dfrac{1}{\sqrt{\epsilon_1 c}} f_+^2(k_2 g(t)\rho_4(Y))
\end{array}
\tag{118}
$$

where c is defined by (113) and ϵ_1 is any strictly positive value which may be chosen as small as desired.
Since

$$
|2\sqrt{\epsilon_1 c}\, x_1 \frac{1}{\sqrt{\epsilon_1 c}} f_+^2(k_2 g(t)\rho_4(Y))| \;\leq\; 2\epsilon_1 c x_1^2 + \frac{1}{2\epsilon_1 c} f_+^4(k_2^2 g^2(t)\rho_4^2(Y)) \tag{119}
$$

it follows from (118) and (113) that:

$$
\dot{\psi}(t) \;\leq\; -2(k_2 k_1 \sigma_2 - \epsilon_1)\psi(t) + \frac{1}{2\epsilon_1 c} f_+^4(k_2^2 g^2(t)\rho_4^2(Y)) \tag{120}
$$

Using (116) and (27),

$$
\dot{\psi}(t) \;\leq\; -2(k_2 k_1 \sigma_2 - \epsilon_1)\psi(t) + \frac{1}{2\epsilon_1 c} d_+^4 k_2^2 G^2 \rho_4^2(Y) \tag{121}
$$

Since ρ_4^2 is homogeneous of degree two with respect to the dilation (94) and V_+ is homogeneous of degree twelve with respect to this dilation and vanishes only at $Y = 0$, there exists a value M such that:

$$
\rho_4^2(Y) \;\leq\; M\, V_+^{\frac{1}{6}}(Y), \quad \forall\, Y \tag{122}
$$

and it follows from (85) that:

$$
V_+(Y(t)) \;\leq\; e^{3k_2(\sigma_1 - \epsilon_2)T} V_+(Y(0)) e^{-3k_2(\sigma_1 - \epsilon_2)t} \tag{123}
$$

This implies, from (122) that:

$$
\rho_4^2(Y(t)) \;\leq\; M\,(\, e^{3k_2(\sigma_1 - \epsilon_2)T} V_+(Y(0))\,)^{\frac{1}{6}} e^{-\frac{k_2}{2}(\sigma_1 - \epsilon_2)t} \tag{124}
$$

Define now the function K_+ in the following way:

$$K_+(Y, \epsilon_1) = \frac{1}{2\epsilon_1 c} d_+^4 k_2^2 G^2 \; M \, (\, e^{3k_2(\sigma_1 - \epsilon_2)T} V_+(Y) \,)^{\frac{1}{6}} \tag{125}$$

From (121), (124) and (125), one obtains that:

$$\dot{\psi}(t) \;\leq\; -2(k_2 k_1 \sigma_2 - \epsilon_1)\psi(t) + K_+(Y(0), \epsilon_1) e^{-\frac{k_2}{2}(\sigma_1 - \epsilon_2)t} \tag{126}$$

If $x_1(t) < 0$, one can prove, similarly, that:

$$\dot{\psi}(t) \;\leq\; -2(k_2 k_1 \sigma_2 - \epsilon_1)\psi(t) + K_-(Y(0), \epsilon_1) e^{-\frac{k_2}{2}(\sigma_1 - \epsilon_2)t} \tag{127}$$

with

$$K_-(Y, \epsilon_1) = \frac{1}{2\epsilon_1 c} d_-^4 k_2^2 G^2 \; M \, (\, e^{3k_2(\sigma_1 - \epsilon_2)T} V_+(Y) \,)^{\frac{1}{6}} \tag{128}$$

Since $\dot{\psi}(t) = 0$ when $x_1(t) = 0$, it follows from (126) and (127) that the lemma's inequality (114) is satisfied with $K = Max\{K_+, K_-\}$. This concludes the proof of part 2^o).

Finally, the estimated convergence rate announced in the part 3^o) of the lemma follows from the comparison of $\psi(t)$ with the solution $x(t)$ of the differential equation

$$\left\{ \begin{array}{rcl} \dot{x} & = & -2(k_2 k_1 \sigma_2 - \epsilon_1)x + K(Y(0), \epsilon_1) e^{-\frac{k_2}{2}(\sigma_1 - \epsilon_2)t} \\ x(0) & = & \psi(0) \end{array} \right. \tag{129}$$

since, in view of (114) and the continuity of ψ, $0 \leq \psi(t) \leq x(t)$, $\forall t$.
(end of Proof).

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Simulation a: Gain parameters (49)-(50).

Fig. 1.a): Control (I): motion in the (x,y) plane

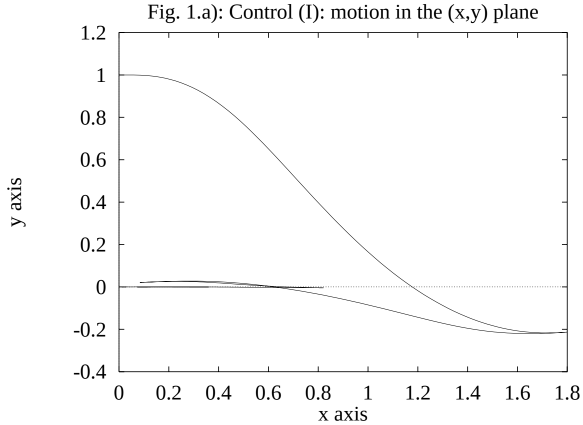


Fig. 4.a): Control (I): evolution of Log (ρ(x,z))

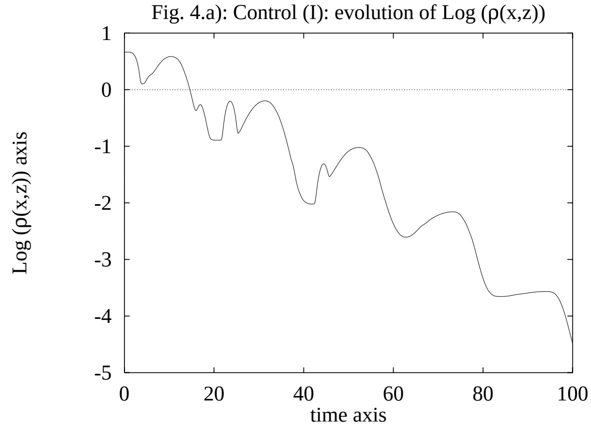


Fig. 2.a): Control (II): motion in the (x,y) plane

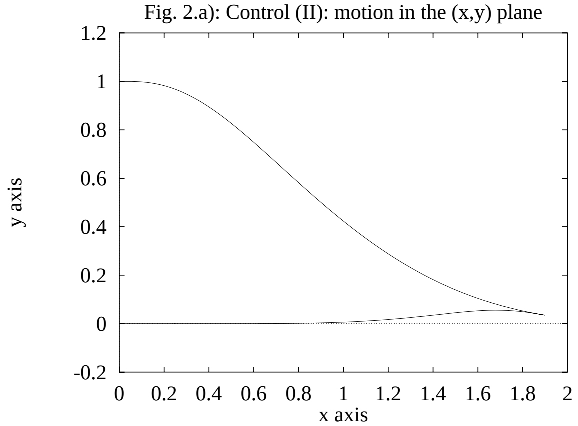


Fig. 5.a): Control (II): evolution of Log (ρ(x))

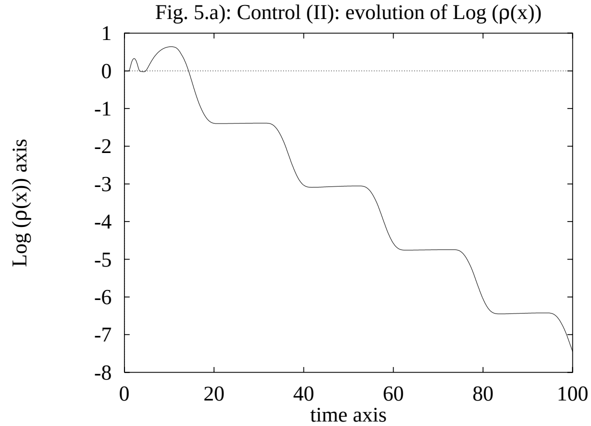


Fig. 3.a): Control (III): motion in the (x,y) plane

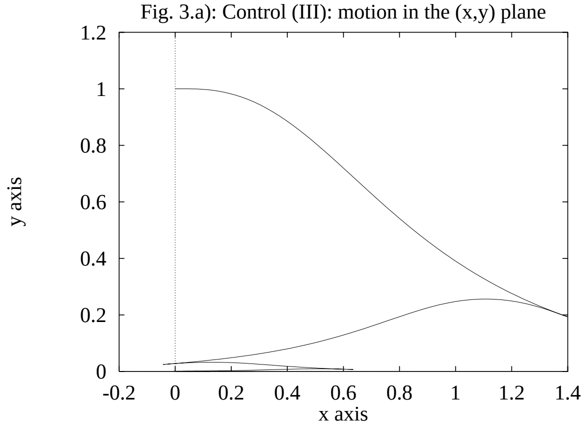
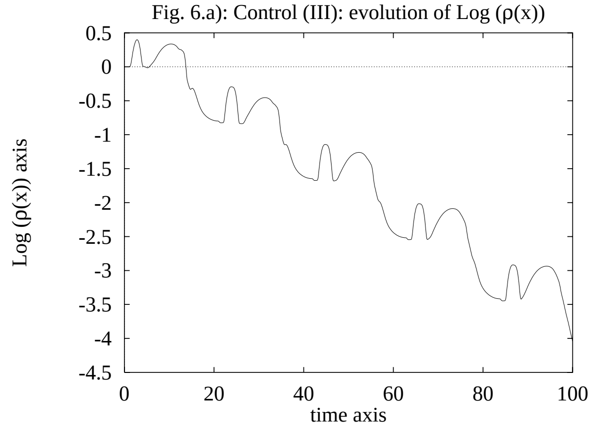


Fig. 6.a): Control (III): evolution of Log (ρ(x))



Simulation b: Gain parameters (53), control law (I)**.**

Fig. 1.b): evolution of x

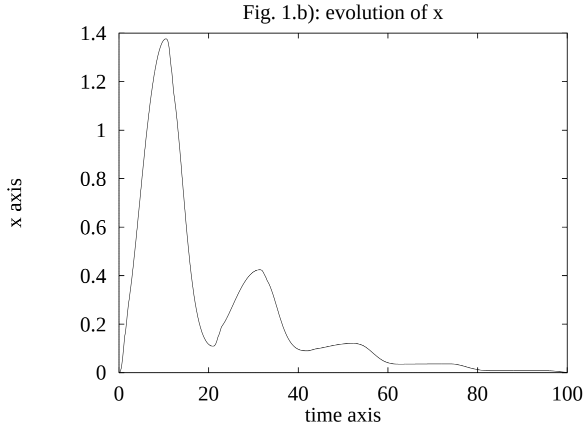


Fig. 2.b): evolution of y

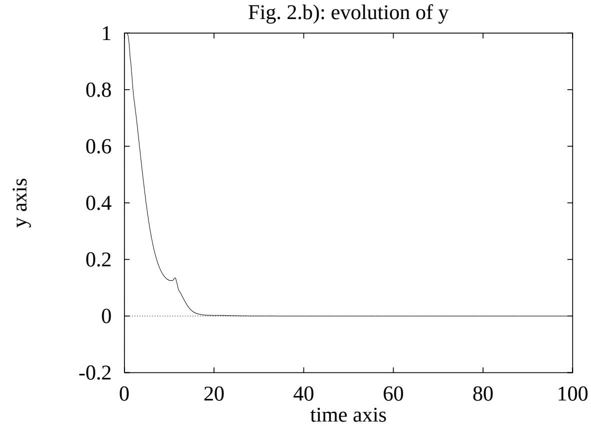


Fig. 3.b): evolution of θ

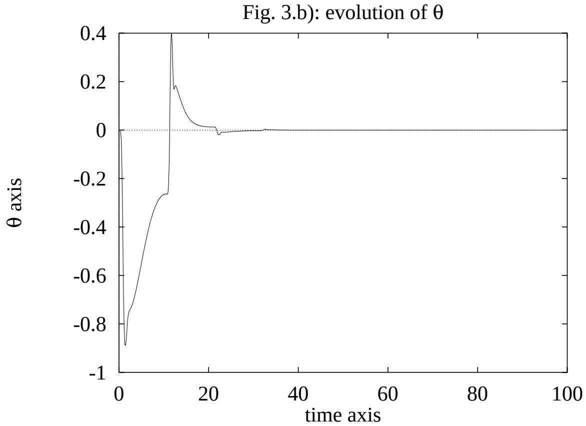


Fig. 4.b): evolution of u1 and u2

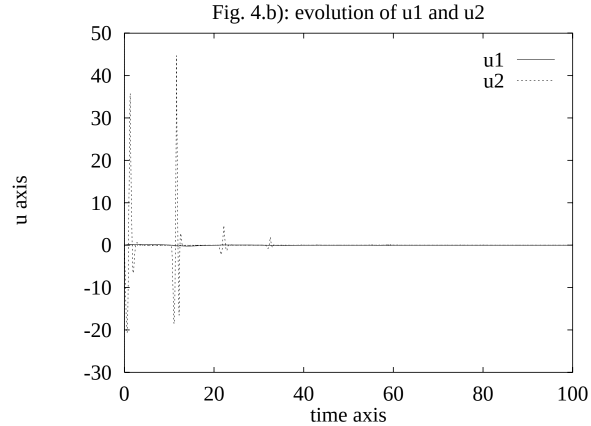


Fig. 5.b): Motion of the vehicle in the (x,y) plane

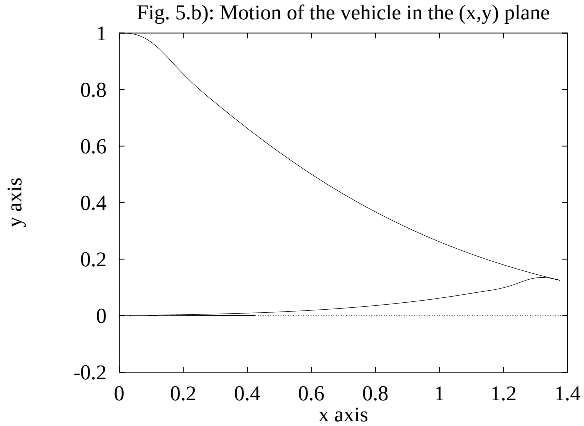
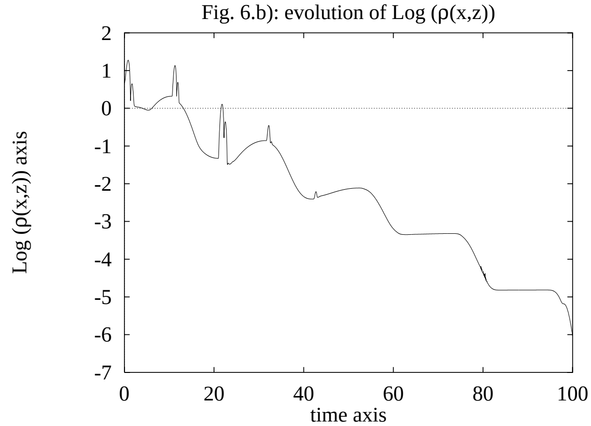


Fig. 6.b): evolution of Log (ρ(x,z))



Simulation c: Gain parameters (54), control law (II).

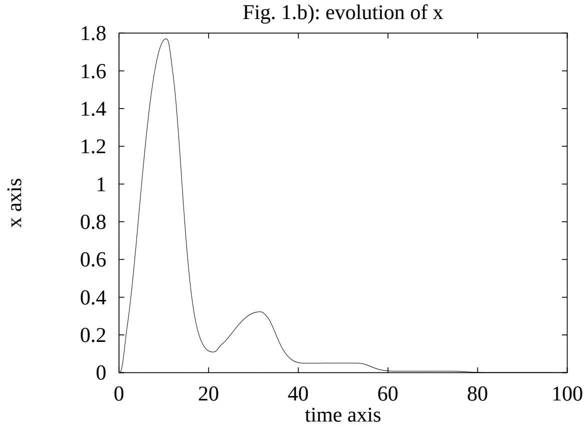


Fig. 1.b): evolution of x

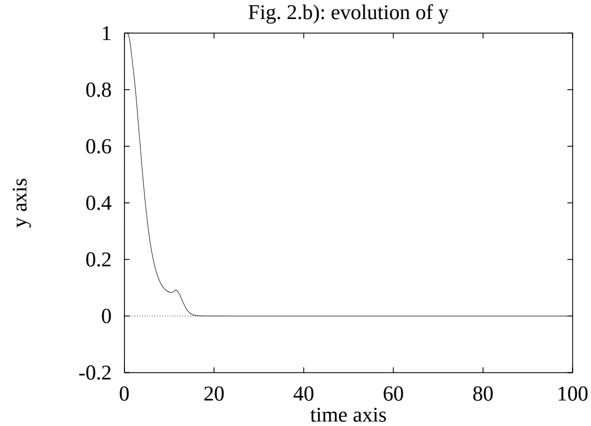


Fig. 2.b): evolution of y

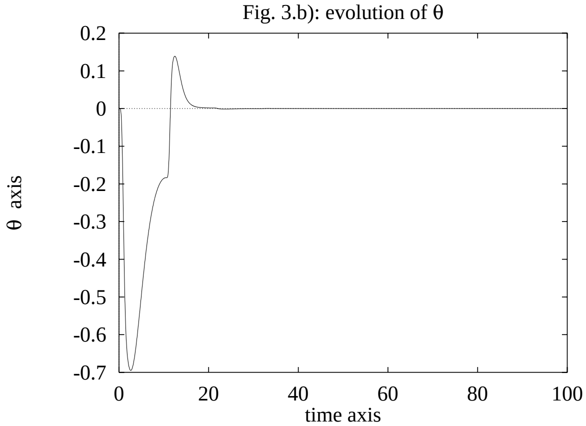


Fig. 3.b): evolution of θ

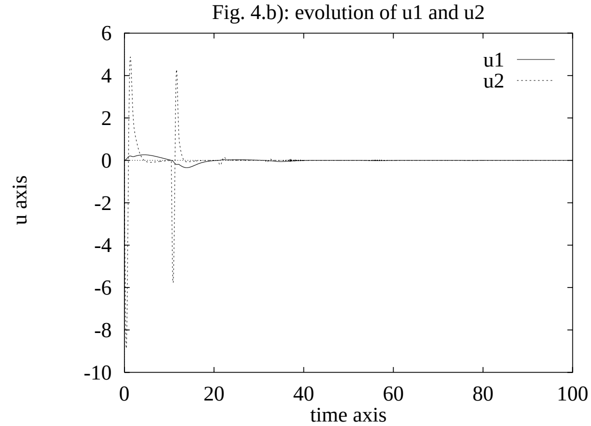


Fig. 4.b): evolution of u1 and u2

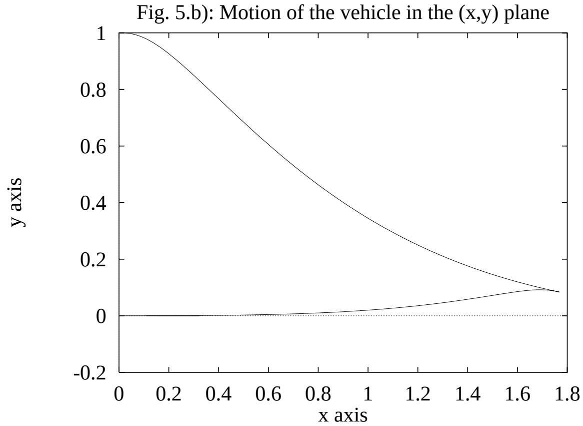


Fig. 5.b): Motion of the vehicle in the (x,y) plane

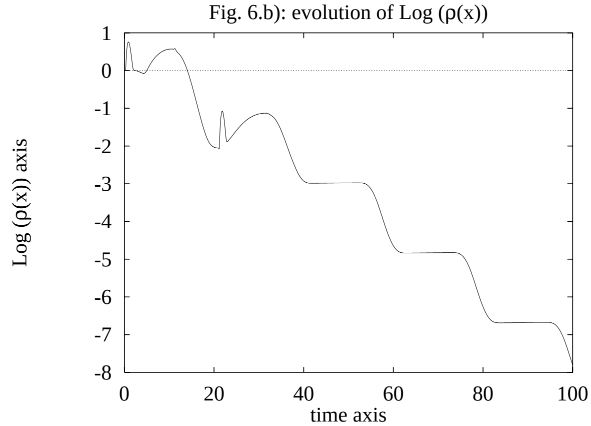


Fig. 6.b): evolution of Log (ρ(x))

Simulation d: Gain parameters (54), control law (III).

Fig. 1.c): evolution of x

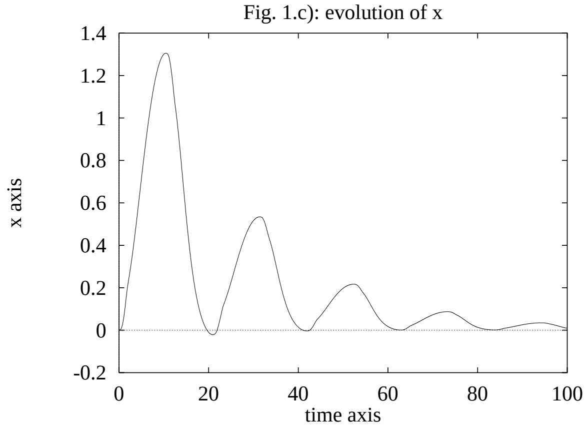


Fig. 2.c): evolution of y

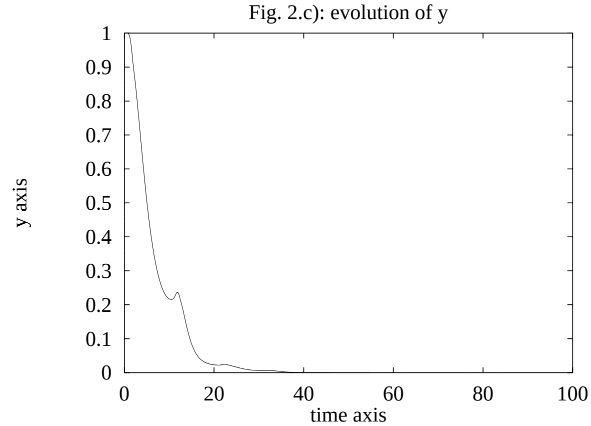


Fig. 3.c): evolution of θ

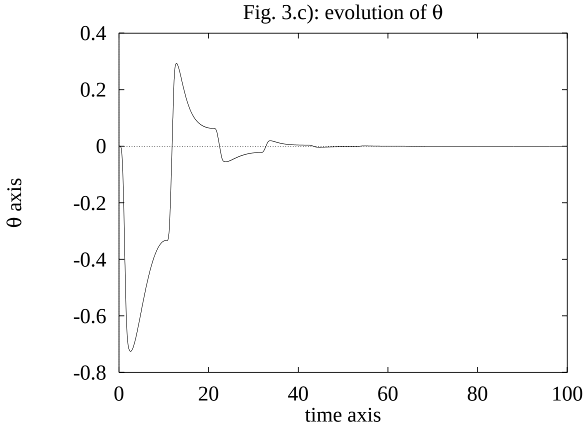


Fig. 4.c): evolution of u1 and u2

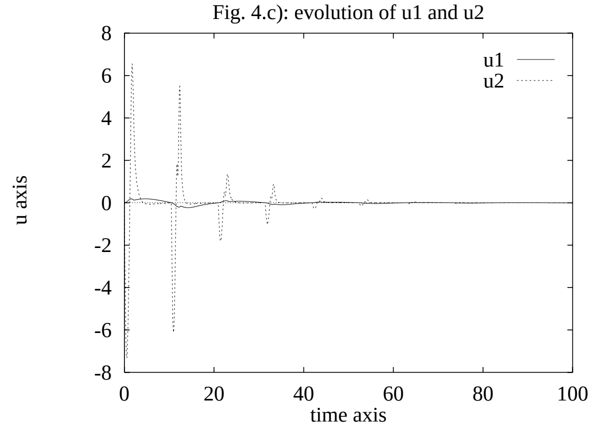


Fig. 5.c): Motion of the vehicle in the (x,y) plane

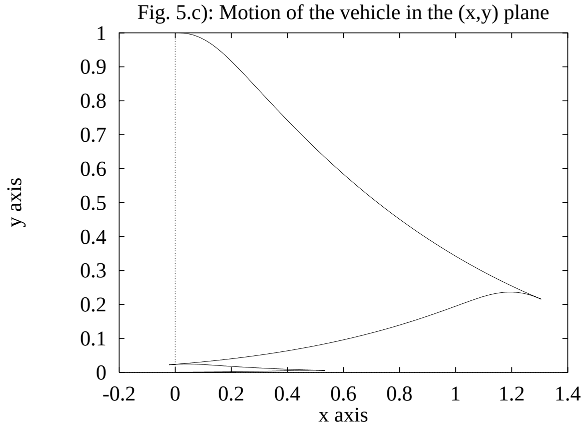
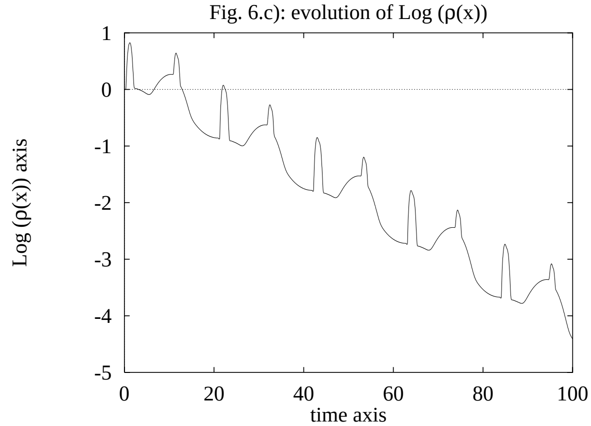


Fig. 6.c): evolution of Log (ρ(x))



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