Simulations of Large Telecommunication Networks Based on Probabilistic Modeling

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Abstract: We describe a new technique for simulations of large communication networks. This technique is based on a new approach for communication networks modeling by means of point processes and stochastic geometry tools. Simulator ARC developed by the authors and described in the paper uses recent effective geometrical algorithms for computing the topology of the system model. New algorithms based on stochastic gradient technique and implemented in the simulator, allow evaluation of certain performance characteristics of the system and can be used for optimization of the system parameters. We analyze the used estimator and give its asymptotic variance.

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Simulations de grands réseaux de télécommunications basé sur une modélisation probabiliste

Résumé : Nous décrivons une nouvelle technique de simulation des grands réseaux de communication. Cette technique, fondée sur une approche originale destinée à la construction de modèles pour les réseaux de communication, s'appuie sur des processus ponctuels et des outils de géométrie stochastique. Le simulateur ARC, développé par les auteurs et décrit dans le papier, utilise des algorithmes géométriques récents et efficaces ayant pour but le calcul de la topologie du modèle. De nouveaux algorithmes, basés sur la technique du gradient stochastique et implantés dans le simulateur, permettent l'évaluation de certaines performances caractéristiques du système, et peuvent servir à optimiser les paramètres du système. Nous analysons l'estimateur utilisé et donnons sa variance asymptotique.

Mots-clé : Modélisation stochastique, modélisation macroscopique, réseaux cellulaires, pavage de Voronoï, triangulation de Delaunay, télé-trafic, processus ponctuels, processus de Poisson

1 Introduction

The necessity of modeling of large communication systems arises primarily in economic analysis and strategic planning of such systems. The global character and complexity of existing networks often prevents an analyst from a simple simulation of these systems in their integrity. First of all, such a system possesses usually a huge number of parameters that causes difficult machine resource problems. Second, complex mutual correlations between the parameters and the performance of the system often mask its essential behavior principles. All that necessitates the development of a theoretical model catching the essential characteristics of the system through a minimum number of structural parameters and which then can be put in the basis of a simulator.

A new approach to modeling of large communication systems was proposed in [2] [3]. The main idea of this approach is to represent the structural units of the network (subscribers, stations of different levels, cables etc.) as a realization of stochastic processes. The decision variables are now functionals of the realization and the network description thus reduces to studies of the distributions of the processes. By this way, the performance characteristics can sometimes be analytically expressed through their parameters. The main advantage of these probabilistic models in comparison with existing deterministic ones, is that their non-deterministic nature allows to take into account spatial structural variability of the system without changing dramatically the number of the parameters of the model.

In the cases when an analytical description is not possible or too cumbersome, this approach provides, however, possibility to study the model by means of simulations as we show in this paper.

The structure of the article is the following. In the next section we describe the probabilistic model of a large cellular communication network which lies in the basis of simulator ARC. Section 3 devoted to the statistical algorithms and their properties used in the simulator. The structure and functionalities of the simulator is the subject of Section 4. We conclude with examples of simulations and the discussion on the used algorithms in Section 5.

2 Stochastic hierarchical model of cellular systems

In real communication networks the stations play different roles depending on their level of hierarchy. We use the term *stations* in a broad sense, including, for instance, the concentration nodes where cables meet. The level of a station corresponds more or less to the minimal number of stations standing between this station and a network subscriber. The hierarchical structure of existing systems is well seen in the following chain: subscribers, distribution points, remote distribution points, remote concentrators, local exchange etc., though in reality it is rarely observed in a pure form.

In the stochastic hierarchical model the stations of level j $(j = 0, \ldots, N)$ are represented by a realization of a planar stochastic point process Π_j. Their distributions can be degenerate to include deterministic positions of stations. Another important case is when the processes Π_0, $\Pi_1, \ldots, \Pi_N$ are independent homogeneous Poisson processes with decreasing intensities: $\lambda_0 > \lambda_1 > \cdots > \lambda_N$. (For uniformity we call 0-level stations the *subscribers* themselves.)

The points of the processes (stations) are connected according to a certain rule which is one of the main parameters of the model. We distinguish two types of connections: intra-level and inter-level or hierarchical connections. Consider, as an example, the closest neighbor rule. The stations of level j $(j = 0, \ldots, N-1)$ are connected to their closest stations of level $j+1$. Conversely, with each station $x^{(j)}$ of level j there is an associated convex set $\mathcal{C}(x^{(j)})$ which consists of the points x of the plane which are

closer to $x^{(j)}$ than to any other station of the level j. Exactly those stations of the previous level $j-1$ which are situated in $\mathcal{C}(x^{(j)})$ are connected to $x^{(j)}$. The set $\mathcal{C}(x^{(j)})$ is known as the *Voronoi cell* with *nucleus* $x^{(j)}$ constructed with respect to the point set Π_j. Figure 2 shows a typical Voronoi cell. The system of all such cells constitutes a *tessellation* of the plane known as the *Voronoi tessellation*. Being a function of a random set Π_j, the Voronoi tessellation is itself a random element. The cells now model the zones served by the stations of j-th level and (as in reality) they are all different in size and shape. The described structure repeats itself on each hierarchical level providing connectness of the network.

The two level hierarchical model based on Poisson homogeneous processes was studied in [9] where some analytical results concerning distributions of such important characteristics like the number of stations or the total length of hierarchical connections within a zone were obtained. Thanks to the structural similarity, many characteristics of systems with higher number of levels can, in fact, be obtained through consideration of a two level model and then by iteration of the obtained results [2].

In addition to the described above liaisons to the closest station, auxiliary hierarchical links can also be drawn to the second, third, etc. closest station of the next hierarchical level.

Different rules can be applied to define the intra-level connections. They may, for example, be represented by the edges of the Delaunay triangulation, by the full graph, or by the minimal spanning tree (for finite point processes). The *Delaunay triangulation* is the graph dual to the Voronoi tessellation: two nuclei are connected by a Delaunay graph edge if and only if their Voronoi cells are neighboring, i.e. have a common side. If all the points are in *general position* (no 3 points are collinear and no 4 points lie on a circle) then the faces of the Delaunay graph are all triangles. For further details on the Voronoi and Delaunay tessellations see e.g., [15] and the references therein.

In simulator ARC each connection may have additional parameters like its type (logical, physical), support material (copper, optical fiber), the way it is carried over (in the ground, suspended wire) etc. The stations may also possess other characteristics: power, cost etc. as described in details in Section 4.

A typical configuration of the described model is shown in Figure 1.

3 Statistical algorithms used in ARC and their properties

In the basis of simulator ARC lies the class of hierarchical probabilistic models described in the previous section. When simulating a communication network structure we deal with the superposition $\mathbf{\Pi}$ of the processes $\Pi_0, \ldots, \Pi_N$ restricted onto a compact set W — an observation *window* (a rectangle in simulator ARC). We describe in this section the statistical algorithms used in the current version of the simulator. These algorithms allow one to estimate not only distribution characteristics of performance functionals but also their gradients with respect to the parameters of the model. Although the state space here is always a plane, we should stress that all the results of this section rest valid without any change in arbitrary finite dimensional space $\mathbb{R}^d$.

3.1 Estimation of the mean values

From now on we assume that the stochastic process $\mathbf{\Pi}$ is *stationary ergodic*. In this case a single configuration is (theoretically) sufficient to obtain the distribution of any random variable. Since the probability $\mathbf{P}\{F \le x\}$ equals the expectation $\mathbf{E}\,\mathbf{1}\{F \le x\}$ of the indicator function, the problem of

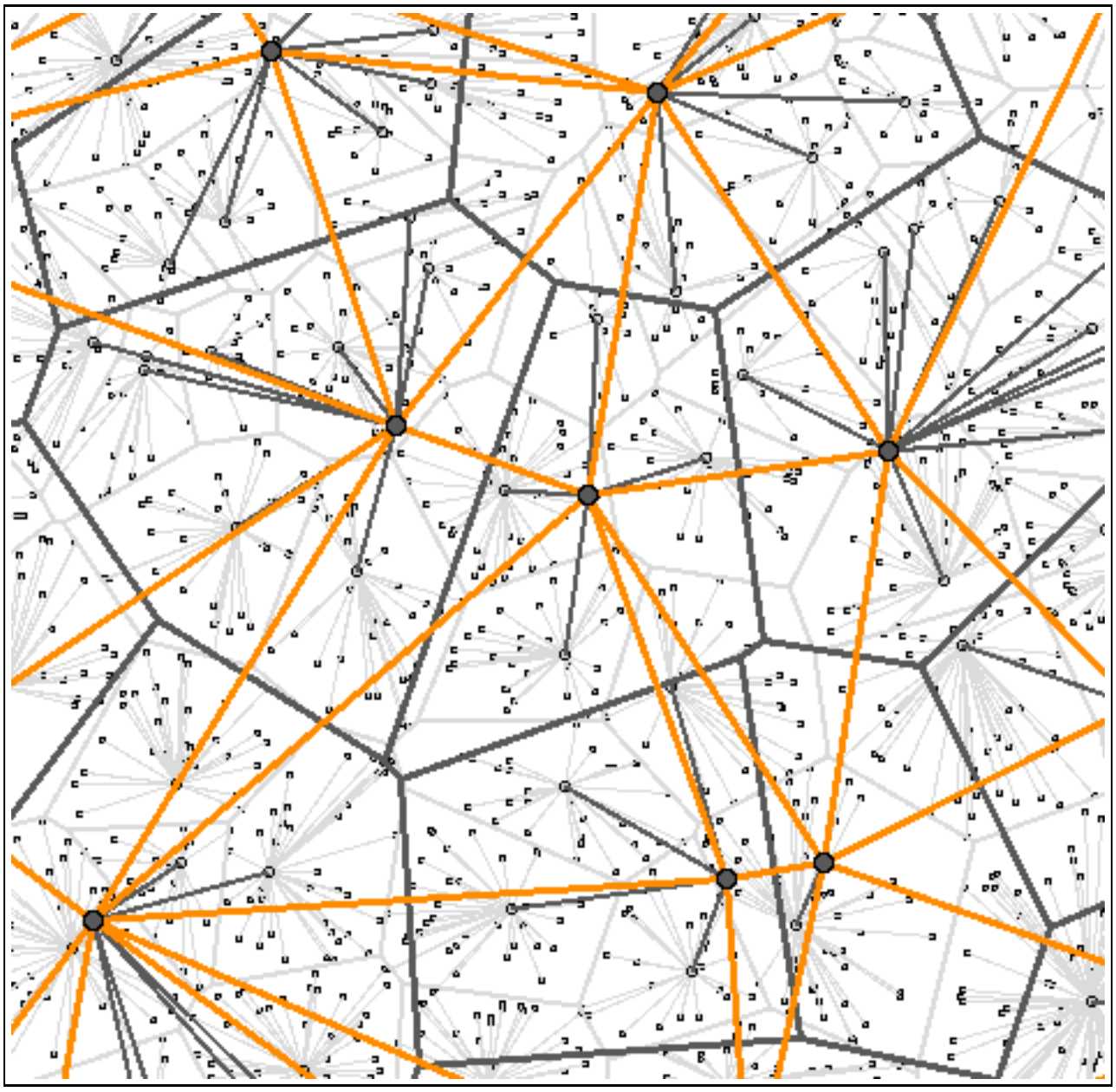

Figure 1: The model with 3 hierarchical levels of stations (the connections between the stations of the highest level follow the Delaunay triangulation)

the distribution's estimation reduces to the estimation of the expectations of random variables. By the individual ergodic theorem (cf. [7, Prop.10.2.II]) the corresponding estimator $\widehat{\mathbf{E}\,F}$ is given by

$$\widehat{\mathbf{E}\,F} = |W|^{-1} \int_W F(\theta_x \mathbf{\Pi})\, dx\,, \tag{1}$$

where $|W|$ is the area of the set W and $\theta_\bullet$ is a group of shifts parameterized by $x \in \mathbb{R}^2$. When acting on the plane $\mathbb{R}^2$, $\theta_x B = x + B = \{x + y : \ y \in B\}$ for any Borel set B. Applied to a measure μ the shifted measure $\theta_x \mu$ takes the value $(\theta_x \mu)(B) = \mu(\theta_x B)$ on a Borel B. For a unit mass measure δ_y concentrated in a point y this implies $\theta_x \delta_y = \delta_{y-x}$. We interpret a point process as a counting measure on $\mathbb{R}^2$, therefore $\theta_x \Pi = \sum_i \delta_{x_i - x}$ if $\Pi = \sum_i \delta_{x_i}$.

We distinguish two types of *statistics* (functions of the configuration) treated by means of simulations. First type are the *global* characteristics which are functions of the entire configuration $\mathbf{\Pi}$. In contrast, a statistic $F = F(\mathbf{\Pi})$ is called *local* if there exists a compact set $B = B(\mathbf{\Pi})$ depending, in general, on configuration $\mathbf{\Pi}$, such that any change of $\mathbf{\Pi}$ outside B does not affect the value of F. We call such B the *dependence region*.

The reason in inventing such classification is the following. Generally a complex problem which arises in applications of the estimator (1), is to eliminate the bias due to the restriction of the infinite configuration onto a finite region W. Local statistics provide an easy solution: the influence of the configuration $\mathbf{\Pi}$ outside the observation window W is eliminated if we restrict the integration in (1) onto the set $\widetilde{W} = \{x \in W : \ \theta_x B(\theta_x \mathbf{\Pi}) \subseteq W\}$.

A variety of local statistics provides a Voronoi cell. It can be easily seen that the geometry of a Voronoi cell $\mathcal{C}(x_0)$ constructed with respect to a set of points $\Pi = \{x_i\}$ is completely determined by a closed domain $\mathcal{F}(x_0)$ which we call the *Voronoi flower* ($\mathcal{F}(x_0)$ is also known as the *fundamental region*). It is the union of discs centered in the vertices of the cell and having the cell's nucleus x_0 on their boundary (see Figure 2). If the points of Π are in general position then these discs have exactly 3 points of Π on the boundary and no points of Π inside. This is also the fundamental property of the Delaunay triangulation: a triangle with the vertices in a three points of Π belongs to the Delaunay triangulation if and only if its circumdisc contains no points of the process in its interior. If we change the configuration Π outside $\mathcal{F}(x_0)$ this does not change the cell $\mathcal{C}(x_0)$. Therefore any geometrical characteristic of $\mathcal{C}(x_0)$ (e.g. its area, perimeter, number of edges etc.) is local. Furthermore, in the model, where the station zones are modeled by Voronoi cells, a number of statistics associated with a particular station, like the number of stations attached to a fixed station x_i, the length (or generally, the cost) of these connections and others will be also local.

Consider the case when F is a local statistic associated with stations of a level j. We are interested in estimation of the value of F in a 'typical' station. The rigorous analog provides the *Palm distribution* which can be interpreted as a conditional distribution of the process $\mathbf{\Pi}$ 'given there is a point of Π_j in the origin 0' (cf. [7, Ch.12]). $F(\Pi)$ in this case is a local statistic associated with this station 0. In what follows we fix the level j and where it does not lead to ambiguity, we omit the index j. Heuristically, to estimate the mean value of F in a typical station one should average the values of F over all the observed stations of the level j. This is exactly what implemented in simulator ARC. A theoretical basis is provided by the theorem below. In what follows $W \uparrow \mathbb{R}^2$ means that W is in fact an element of an *averaging family* $\{W_t, \ t > 0\}$, i.e. for each t set W_t is a *convex* domain containing the disc of radius $r(t)$ centered in the origin, and $r(t)$ tends to infinity as t grows.

Theorem 3.1. *Let $\mathbf{E}^0$ be the expectation corresponding to the Palm distribution of the process Π ($= \Pi_j$). Let F be a local $\mathbf{P}^0$-integrable characteristic such that its dependence region $B(\mathbf{\Pi})$ is compact $\mathbf{P}^0$-almost surely. Then as $W \uparrow \mathbb{R}^2$ an estimator of $\mathbf{E}^0 F$ is given by*

$$\widehat{\mathbf{E}^0 F} = \frac{1}{\widetilde{N}} \sum_{x_i \in \widetilde{\Pi}} F(\theta_{x_i} \mathbf{\Pi}), \tag{2}$$

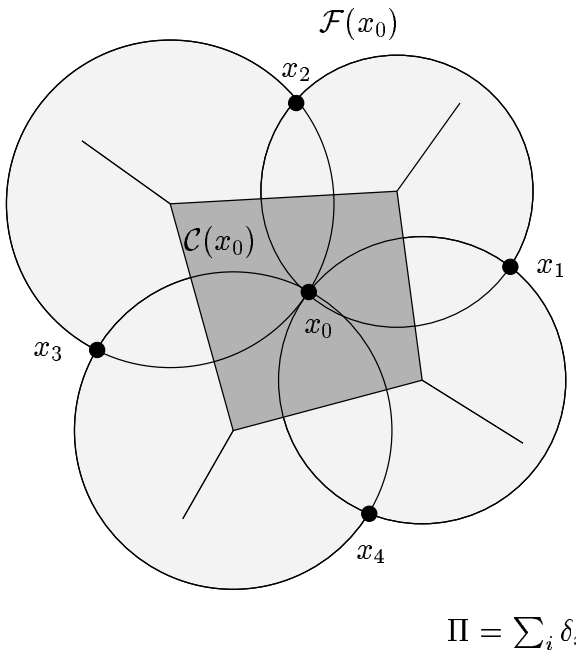

Figure 2: Geometrical structure of a Voronoi cell and its Voronoi flower.

where $\widetilde{\Pi}$ *is the subprocess* $\{x_i \in \Pi \cap W \,:\, B_i \subseteq W\}$ *with* $B_i = x_i + B(\theta_{x_i}\mathbf{\Pi})$, *and* $\widetilde{N}$ *is the number of points in* $\widetilde{\Pi}$.

Proof. We have

$$\frac{1}{\lambda|W|}\sum_{x_i\in\Pi\cap W} F(\theta_{x_i}\mathbf{\Pi})\,\mathbf{1}\{B_i\subseteq W\} = \frac{1}{\lambda|W|}\sum_{x_i\in\Pi\cap W} F(\theta_{x_i}\mathbf{\Pi}) - \frac{1}{\lambda|W|}\sum_{x_i\in\Pi\cap W} F(\theta_{x_i}\mathbf{\Pi})\,\mathbf{1}\{B_i\not\subseteq W\}.$$

By the individual ergodic theorem the first summand in the RHS of the last identity tends almost surely (a.s.) to $\mathbf{E}^0 F$ (cf. [7, eq. (12.4.3), p. 485]). Let us show now that the second term vanishes. Indeed, denote by

$$\rho(\mathbf{\Pi}) = \sup_{x\in B(\mathbf{\Pi})} |x| \text{ and } W'_R = \{x\in W \,:\, \inf_{y\in\partial W} |x-y| \geq R\}\,. \tag{3}$$

Then the absolute value of the second summand is dominated by

$$\frac{|W'_R|}{|W|}\frac{1}{\lambda|W'_R|}\sum_{x_i\in\Pi\cap W'_R} |F(\theta_{x_i}\mathbf{\Pi})|\;\mathbf{1}\{\rho(\theta_{x_i}\mathbf{\Pi})\geq R\} + \frac{|W\setminus W'_R|}{|W|}\frac{1}{\lambda|W\setminus W'_R|}\sum_{x_i\in\Pi\cap W\setminus W'_R} |F(\theta_{x_i}\mathbf{\Pi})|\,. \tag{4}$$

Since $|W'_R|/|W| \xrightarrow{a.s.} 1$ for any R, then by the ergodic theorem the first term in (4) converges a.s. to $\mathbf{E}^0|F|\,\mathbf{1}\{\rho\geq R\}$ while the second one vanishes being a product of $|W\setminus W'_R|/|W| \xrightarrow{a.s.} 0$ and the converging to $\mathbf{E}^0|F|$ factor. Since R can be taken arbitrarily large we have shown that for any F as in the conditions of the theorem

$$\frac{1}{\lambda|W|}\sum_{x_i\in\Pi\cap W} F(\theta_{x_i}\mathbf{\Pi})\,\mathbf{1}\{B_i\subseteq W\} = \frac{\widetilde{N}}{\lambda|W|}\frac{1}{\widetilde{N}}\sum_{x_i\in\widetilde{\Pi}} F(\theta_{x_i}\mathbf{\Pi}) \xrightarrow{a.s.} \mathbf{E}^0 F \text{ as } W\uparrow\mathbb{R}^2. \tag{5}$$

Taking now $F = 1$ in (5) we get

$$\widetilde{N}/(\lambda|W|) \xrightarrow{a.s.} 1 \tag{6}$$

and the statement of the theorem follows. □

3.2 Stochastic gradient algorithms

A new original feature of the simulator ARC is the gradient estimation procedure based on the stochastic gradient technique. It is implemented in ARC for the models where the stations are represented by realizations of homogeneous Poisson processes $\Pi_0, \ldots, \Pi_N$ with the intensities $\lambda_0, \ldots, \lambda_N$, respectively.

Let, as in the previous section, $F(\Pi_j)$ be a local statistic associated with a station of a fixed level j at the origin. Our goal is to estimate the derivative $\frac{d}{d\lambda}\,\mathbf{E}^0_\lambda\, F$, where $\lambda = \lambda_j$ is the intensity of the j-level stations. This information could then be used, for instance, for optimization of the density of these stations.

A certainly *bad* thing to do is to use the quantity

$$(\triangle\lambda)^{-1}\,\big[\widehat{\mathbf{E}^0_{\lambda+\triangle\lambda}\,F} - \widehat{\mathbf{E}^0_\lambda\,F}\big]$$

as an estimator of $\frac{d}{d\lambda}\,\mathbf{E}^0_\lambda\,F$. This is clear from the fact that in 'normal' cases the asymptotic variance of the estimator (2) is of order $O\big(|\widetilde{W}|^{-1}\big)$ (cf. [14]), so that the variance of the last estimator explodes when $\triangle\lambda$ vanishes.

A random variable Δ is called *stochastic gradient* for F if $\frac{d}{d\lambda}\,\mathbf{E}_\lambda\,F = \mathbf{E}_\lambda\,\Delta$. Existence of a stochastic gradient reduces estimation of the derivative above to estimation of its expectation, i.e. in our case to application of the ergodic theorem. That observation lies in the basis of a quickly developing in the last years area of *random perturbation analysis* of stochastic systems. The state-of-the-art in this domain can be found, e.g. in [10].

First expressions for the stochastic gradient in the systems driven by a Poisson process appeared in [1] for 1-dimensional case. The expression for a stationary Poisson process in a general state space was proved under certain minimal uniform integrability conditions in [4]. In our case of a finite process Π

$$\frac{d}{d\lambda}\,\mathbf{E}_\lambda\,F = \mathbf{E}_\lambda \int\limits_W \big[F(\Pi+\delta_x) - F(\Pi)\big]\,dx \tag{7}$$

that holds for any integrable function F and all $\lambda < \lambda_0$ such that the stochastic gradient $\Delta = \int\limits_W [F(\Pi + \delta_x) - F(\Pi)]\,dx$ is $\mathbf{E}_\lambda$-integrable for all $\lambda \le \lambda_0$. (Non-stationary Poisson processes were considered in [12], where the similar expressions were obtained.)

By Slivnyak's theorem $\mathbf{E}^0\,F(\Pi) = \mathbf{E}\,F(\Pi+\delta_0)$ (cf. [16, p.114–115]) therefore the relation (7) holds also for statistics associated with stations with expectation $\mathbf{E}_\lambda$ replaced by the expectation $\mathbf{E}^0_\lambda$ with respect to the Palm distribution of the process Π.

For a local statistic F one has $F(\mathbf{\Pi}+\delta_x) - F(\mathbf{\Pi}) = 0$ for all x lying outside the dependence region $B(\mathbf{\Pi})$, where $\delta^{(j)}_x$ denotes an additional station of level j added into position x. Therefore

$$\frac{d}{d\lambda}\,\mathbf{E}^0_\lambda = \int \mathbf{E}^0_\lambda\big[F(\mathbf{\Pi}+\delta^{(j)}_x) - F(\mathbf{\Pi})\big]\;\mathbf{1}_{B(\mathbf{\Pi})}(x)\,dx. \tag{8}$$

By (2) an estimator of the expectation under the integral is given by

$$\frac{1}{\widetilde{N}} \sum_{x_i \in \widetilde{\Pi}} \left[F(\theta_{x_i}\mathbf{\Pi} + \delta_x^{(j)}) - F(\theta_{x_i}\mathbf{\Pi})\right] \mathbf{1}_{B(\theta_{x_i}\mathbf{\Pi})}(x)$$
$$= \frac{1}{\widetilde{N}} \sum_{x_i \in \widetilde{\Pi}} \left[F_i(\mathbf{\Pi} + \delta_{x+x_i}^{(j)}) - F_i(\mathbf{\Pi})\right] \mathbf{1}_{B_i(\mathbf{\Pi})}(x + x_i)\,, \tag{9}$$

where $F_i(\mathbf{\Pi}) = F(\theta_{x_i}\mathbf{\Pi})$. By definition if $x_i \in \widetilde{\Pi}$ then $B_i \subseteq W$ and thus $F(\theta_{x_i}\mathbf{\Pi})$ depends only on the configuration in $B(\theta_{x_i}\mathbf{\Pi}) = B_i - x_i$, i.e. uses only information available in the window W. Now putting $y = x + x_i$ expression (9) transforms the RHS of (8) into

$$\frac{1}{\widetilde{N}} \sum_{x_i \in \widetilde{\Pi}} \int_{B_i} \left[F_i(\mathbf{\Pi} + \delta_y^{(j)}) - F_i(\mathbf{\Pi})\right] dy = \frac{1}{\widetilde{N}} \sum_{x_i \in \widetilde{\Pi}} \int_{W} \left[F_i(\mathbf{\Pi} + \delta_y^{(j)}) - F_i(\mathbf{\Pi})\right] dy.$$

The last integral can be estimated by the Monte-Carlo method finally giving the following estimator which is used in simulator ARC:

$$\widehat{\frac{d}{d\lambda}\mathbf{E}_\lambda^0 F} = \frac{|W|}{\widetilde{N}} \sum_{x_i \in \widetilde{\Pi}} \frac{1}{M} \sum_{m=1}^{M} \left[F(\theta_{x_i}(\mathbf{\Pi} + \delta_{U_m}^{(j)})) - F(\theta_{x_i}\mathbf{\Pi})\right], \tag{10}$$

where $U_m,\ m = 1, \ldots, M$ are independent uniformly distributed in W random variables.

As we show in the next section the asymptotic variance of this estimator is $O\left(|W|^{-1} + M^{-1}\right)$ which can be made arbitrarily close to the rate $O\left(|W|^{-1}\right)$ of variance of estimator (2) provided by the ergodic theorem.

In comparison with the similar estimator proposed in [4, Sec.5], estimator (10) has the following advantages: no restriction on the number M of additional points U_m 'responsible' for the estimation of the derivative, calculations are easier since one need not calculate the dependence regions for each point x_i. In addition a theoretical analysis is easier since no skew product measures arise here.

3.3 Asymptotic variance of the gradient estimator

Since $\mathbf{E}^0 F(\mathbf{\Pi}) = \mathbf{E}^0 \mathbf{E}\left[F(\mathbf{\Pi}) \,\middle|\, \Pi_j\right]$, we may assume, without loss of generality, that F is just a function of $\Pi_j = \Pi$.

In this section we obtain the asymptotic variance of the estimator

$$E = E(\Pi, \boldsymbol{U}) = \frac{1}{\lambda} \sum_{x_i \in \widetilde{\Pi}} \frac{1}{M} \sum_{m=1}^{M} D_{U_m}(\theta_{x_i}\Pi), \tag{11}$$

where $D_U(\Pi) = [F(\Pi + \delta_U) - F(\Pi)]$. It differs from (10) by a factor $\lambda|W|/\widetilde{N}$, which tends to 1 almost surely by (6). The probability measure $\mathbf{P}$ below is the direct product of the distribution $\mathbf{P}_\Pi$ of the Poisson point process and the distribution $\mathbf{P}_{\boldsymbol{U}}$ of vector $\boldsymbol{U}$ of M independent variables U_m each having uniform distribution $\mathbf{P}_U$ on W. The corresponding expectations and covariances have similar indexes.

Theorem 3.2. *Provided the integrals (12) and (13) below exist, the asymptotic variance of the estimator (11) is given by*

$$\frac{1}{\lambda|W|}\int \mathbf{cov}\left\{G(\Pi+\delta_0+\delta_y), G(\theta_y\Pi+\delta_0+\delta_{-y})\right\}(\delta_0(dy)+\lambda dy) \tag{12}$$

$$+\frac{1}{\lambda M}\int \mathbf{E}\,H(\Pi+\delta_0+\delta_y, y)\,(\delta_0(dy)+\lambda dy) = O\left(|W|^{-1}+M^{-1}\right), \tag{13}$$

where

$$G(\Pi) = |B(\Pi)|\;\mathbf{E}_U\left[\,D_U(\Pi)\;\middle|\; U\in B(\Pi)\right] \tag{14}$$

and

$$H(\Pi, y) = |B(\Pi)\cap B(\theta_y\Pi)|\;\mathbf{cov}_U\left[\{D_U(\Pi), D_U(\theta_y\Pi)\}\;\middle|\; U\in B(\Pi)\cap B(\theta_y\Pi)\right]. \tag{15}$$

The proof of this theorem is based on the following lemma which gives the asymptotic variance of the sum showing up in the estimator (2):

Lemma 1. *Let $g(\Pi)$ be a local characteristic with $\mathbf{P}^0$-a.s. compact dependence region $B(\Pi)$. Then as $W\uparrow\mathbb{R}^2$*

$$\frac{1}{\lambda|W|}\,\mathbf{var}\sum_{x_i\in\widetilde{\Pi}} g(\theta_{x_i}\Pi) \xrightarrow{a.s.} \mathbf{var}\,g(\Pi+\delta_0)+\lambda\int \mathbf{cov}\left\{g(\Pi+\delta_0+\delta_y), g(\theta_y\Pi+\delta_0+\delta_{-y})\right\}dy \tag{16}$$

$$= \int \mathbf{cov}\left\{g(\Pi+\delta_0+\delta_y), g(\theta_y\Pi+\delta_0+\delta_{-y})\right\}(\delta_0(dy)+\lambda dy)$$

(with convention $\delta_0+\delta_0=\delta_0$ for the last integral).

Proof of Lemma 1. Since the variance of a variable does not depend on an additive constant, we may assume that $\mathbf{E}^0\,g(\Pi)\,\mathbf{1}\{B(\Pi)\subseteq W\}=0$. In this case

$$\frac{1}{\lambda|W|}\,\mathbf{var}\sum_{x_i\in\widetilde{\Pi}} g(\theta_{x_i}\Pi) = \frac{1}{\lambda|W|}\,\mathbf{E}\sum_{x_i\in\Pi\cap W} g^2(\theta_{x_i}\Pi)\,\mathbf{1}\{B_i\subseteq W\}$$

$$+\frac{1}{\lambda|W|}\,\mathbf{E}\sum_{\substack{x_i,x_j\in\Pi\cap W\\ x_i\neq x_j}} g(\theta_{x_i}\Pi)\,\mathbf{1}\{B_i\subseteq W\}\,g(\theta_{x_j}\Pi)\,\mathbf{1}\{B_j\subseteq W\}, \tag{17}$$

where as before $B_i = x_i + B(\theta_{x_i}\Pi)$.

For the rest of the proof we need the following Mecke's formula for a Poisson process and its second iteration [11]:

$$\mathbf{E}\sum_{x_i\in\Pi} f(x_i,\Pi) = \lambda\int \mathbf{E}\,f(x,\Pi+\delta_x)\,dx\,; \tag{18}$$

$$\mathbf{E}\sum_{\substack{x_i,x_j\in\Pi\\ x_i\neq x_j}} f(x_i,x_j,\Pi) = \lambda^2\iint f(x,y,\Pi+\delta_x+\delta_y)\,dxdy. \tag{19}$$

With the help of (18) and using the same notations W'_R, ρ as in (3), the first term in the sum (17) can be rewritten as

$$\frac{1}{|W|}\int\limits_{W} \mathbf{E}\, g^2(\theta_x\Pi+\delta_0)\ \mathbf{1}\{B_x \subseteq W\}\, dx = \frac{1}{|W|}\int\limits_{W'_R} \mathbf{E}\, g^2(\theta_x\Pi+\delta_0)\, dx \tag{20}$$

$$- \frac{1}{|W|}\int\limits_{W'_R} \mathbf{E}\, g^2(\theta_x\Pi+\delta_0)\ \mathbf{1}\{B_x \not\subseteq W\}\, dx + \frac{1}{|W|}\int\limits_{W\setminus W'_R} \mathbf{E}\, g^2(\theta_x\Pi+\delta_0)\ \mathbf{1}\{B_x \subseteq W\}\, dx\,, \tag{21}$$

where $B_x = x + B(\theta_x\Pi+\delta_0)$. The term in the RHS of (20) equals by stationarity $|W'_R|/|W|\ \mathbf{E}\, g^2(\Pi+\delta_0)$ and converges a.s. to $\mathbf{var}\, G(\Pi+\delta_0)$.

Furthermore, the absolute value of (21) is dominated by

$$\frac{1}{|W|}\int\limits_{W'_R} \mathbf{E}\, g^2(\theta_x\Pi+\delta_0)\ \mathbf{1}\{\rho(\theta_x\Pi+\delta_0) > R\}\, dx + \frac{1}{|W|}\int\limits_{W\setminus W'_R} \mathbf{E}\, g^2(\theta_x\Pi+\delta_0)\, dx$$

or, by stationarity,

$$\frac{|W'_R|}{|W|}\,\mathbf{E}\, g^2(\Pi+\delta_0)\,\mathbf{1}\{\rho(\Pi+\delta_0) > R\} + \frac{|W\setminus W'_R|}{|W|}\,\mathbf{E}\, g^2(\Pi+\delta_0)$$

that vanishes by the same reasoning as in the proof of Theorem 3.1.

Finally, by (19) the second summand in the RHS of (17) equals

$$\frac{\lambda}{|W|}\iint\limits_{W^2} \mathbf{E}\, g(\theta_x\Pi+\delta_0+\delta_{y-x})\,\mathbf{1}\{x + B(\theta_x\Pi+\delta_0+\delta_{y-x}) \subseteq W\}$$

$$\times\, g(\theta_y\Pi+\delta_0+\delta_{x-y})\,\mathbf{1}\{y + B(\theta_y\Pi+\delta_0+\delta_{x-y}) \subseteq W\}\, dxdy$$

Proceeding the same way as above we find that the main contribution gives the term

$$\frac{\lambda}{|W|}\iint\limits_{(W'_R)^2} \mathbf{E}\, g(\theta_x\Pi+\delta_0+\delta_{y-x})\, g(\theta_y\Pi+\delta_0+\delta_{x-y})\, dxdy\,.$$

By stationarity this expression can be rewritten as

$$\frac{\lambda}{|W|}\int\limits_{W'_R} dx \int \mathbf{E}\, g(\Pi+\delta_0+\delta_y)\, g(\theta_y\Pi+\delta_0+\delta_{-y})\ \mathbf{1}_{\{W'_R-x\}}(y)\, dy$$

that is seen to converge to the second summand in (16) as $W \uparrow \mathbb{R}^2$. □

Proof of Theorem 3.2. For any random variable ξ and σ-algebra σ one has:

$$\mathbf{var}\,\xi = \mathbf{E}\Big(\mathbf{E}\big[\xi \;\big|\; \sigma\big]\Big)^2 - \Big(\mathbf{E}\,\mathbf{E}\big[\xi \;\big|\; \sigma\big]\Big)^2 + \mathbf{E}\,\mathbf{E}\big[\xi^2 \;\big|\; \sigma\big] - \mathbf{E}\Big(\mathbf{E}\big[\xi \;\big|\; \sigma\big]\Big)^2 = \mathbf{var}\,\mathbf{E}\big[\xi \;\big|\; \sigma\big] + \mathbf{E}\,\mathbf{var}\big[\xi \;\big|\; \sigma\big].$$

Applying to our situation

$$\mathbf{var}\, E = \mathbf{var}_\Pi\, \mathbf{E}_U\big[E \;\big|\; \sigma(\Pi)\big] + \mathbf{E}_\Pi\, \mathbf{var}_U\big[E \;\big|\; \sigma(\Pi)\big]\,,$$

where $\sigma(\Pi)$ is the σ-algebra generated by the process Π. The first term reads

$$\mathbf{E}_U\left[E \;\middle|\; \sigma(\Pi)\right] = \frac{1}{\lambda}\sum_{x_i\in\widetilde{\Pi}} \mathbf{E}_U\, D_U(\theta_{x_i}\Pi) = \frac{1}{\lambda|W|}\sum_{x_i\in\widetilde{\Pi}} |B(\theta_{x_i}\Pi)|\; \mathbf{E}_U\left[D_U(\theta_{x_i}\Pi) \;\middle|\; U\in B(\theta_{x_i}\Pi)\right]$$

and applying Lemma 1 with $g(\Pi)$ replaced with (14) we obtain for $\mathbf{var}_\Pi\, \mathbf{E}_U\left[E \;\middle|\; \sigma(\Pi)\right]$ expression (12).

For the second term we have

$$\mathbf{var}_U\left[E \;\middle|\; \sigma(\Pi)\right] = \frac{1}{\lambda^2 M}\,\mathbf{var}_U \sum_{x_i\in\widetilde{\Pi}} D_U(\theta_{x_i}\Pi) = \frac{1}{\lambda^2 M}\sum_{x_i,x_j\in\widetilde{\Pi}} \mathbf{cov}_U\{D_U(\theta_{x_i}\Pi), D_U(\theta_{x_j}\Pi)\}\,.$$

Note that the covariance above is zero if U does not lie in both B_i and B_j. Therefore conditioning on the event $U\in B_i\cap B_j$ we may rewrite the expectation of last expression as

$$\frac{1}{\lambda^2 M|W|}\,\mathbf{E}\sum_{x_i,x_j\in\widetilde{\Pi}} |B_i\cap B_j|\,\mathbf{cov}_U\left[\{D_U(\theta_{x_i}\Pi), D_U(\theta_{x_j}\Pi)\} \;\middle|\; U\in B_i\cap B_j\right]$$

in which we recognize the same type of sum as in the RHS of (17). Therefore proceeding the same way as in the proof of Lemma 1 we obtain

$$\mathbf{E}_\Pi\,\mathbf{var}_U\left[E \;\middle|\; \sigma(\Pi)\right] \xrightarrow{a.s.} \frac{1}{\lambda M}\int \mathbf{E}_\Pi\, H(\Pi+\delta_0+\delta_y, y)\,(\delta_0(dy)+\lambda dy)\,,$$

where H is given by (15). Combining all the above we obtain the statement of the theorem. □

4 The simulator ARC

4.1 ARC functionalities

A prototype ARC is designed as a testbed for a simulator of large communication networks and uses the new statistical algorithms described above. Its current version allows the user to generate a new model or to load a previously stored one from a file, to save a model to a file, to visualize the model graphically, and to perform computations based on the model parameters like the evaluation of some performance characteristics and/or optimization of a given cost function. We describe these capabilities in more details in Section 5 below.

All interactions with the user are organized via a successive graphical panels made up by a set of buttons and data entry fields. The whole model or only a part of it can be visualized in a graphical window, the color of each displayed object can be chosen by the user. We use the LEDA data structures [13] and graphical library for all the graphical aspects of ARC.

4.2 Basic objects

ARC is written in C++ language. The basic objects handled in it, each being a C++ class, are *subscribers, stations, physical links, logical links, communication flows* and *cells.*

A station, as well as a subscriber, has the following attributes: its coordinates in the plane, the hierarchical level to which it belongs (level 0 for a subscriber), its associated cell, i.e. the zone it serves

(not present for a subscriber), a list of links attached to it (physical and logical links to the upper, the lower and the same hierarchy levels), and a list of communication flows relative to it. A subscriber has a specific attribute indicating whether it is fixed or mobile. In what follows, a *node* or *site* indicates a station or a subscriber.

The physical links represent transmission links and the logical links represent commutation links of the modeled communication network. A logical link may comprise several physical links which are listed in a specific attribute. Conversely, a physical link has a specific attribute listing all the logical links that use it. Their common attributes are : their length, the objects at their two extremities (a subscriber and a station or two stations), and their support material (copper, optical fiber etc., suspended or in a trench). The length of a physical link is the Euclidean distance between its two extremities while the length of a logical link is the sum of the physical links' lengths composing it.

A communication flow is established between two nodes of the model, and uses a path composed of logical links between them. A flow's attributes are : the two concerned nodes, three kinds of distances between this nodes (the Euclidean distance, the sum of the lengths of the logical links in use and the difference in level between the two nodes following the path), and a list of the links used by it.

A cell of a site is a polygon that models the zone served by the corresponding station. The attributes of a cell are : its corresponding site and a list of the polygon's vertices. The current version uses only Voronoi cells as the station zones' model, although any other type of polygons can be defined as the cell objects in ARC.

4.3 Specifications

The needed data structures as well as the operations to be performed over the objects, for the construction of the model and for the implementation of the algorithms described above, can be enumerated as follows.

- The simulator needs access to every node of the model, for instance when computing the mean values of parameters or when displaying the whole model. A simple list of each object type for each hierarchy level is sufficient for this operation.
- The simulator needs access to every node contained in a rectangular sub-window of the whole space, for instance when displaying only a small part of the space. Since the point processes used in this first version of ARC for generation of nodes are homogeneous Poisson processes, the data structure to be used for this access operation is a quaternary tree that does not need to be balanced thanks to the uniform distribution of the nodes in space. In the future implementations, the quaternary tree may no longer be appropriate if the spatial distribution of the nodes is not uniform.
- The simulator needs to find the nearest node of a given node, and needs to construct a space tessellation with respect to a large number of nodes. Therefore the chosen algorithm should be robust and fast. The *Delaunay tree* detailed below is a suitable for such operations data structure.

4.4 The Delaunay Tree

The Delaunay tree is a hierarchical data structure which allows an "on-line" construction of the Delaunay triangulation of a finite set of sites without knowing them in advance. It first appeared back in 1986 in a paper by Boissonnat and Teillaud [5]; full details concerning this structure are available in another paper

by the same authors [6]. The Delaunay Tree is an incremental algorithm : each site is introduced one after another and the triangulation is updated after each insertion. Let x_i be a new site to be introduced in the triangulation. By the characteristic property of the Delaunay tessellation all the triangles whose circumscribing circles contain x_i in their interior can no longer be triangles of the triangulation. These triangles which are *in conflict* with x_i are not eliminated from the Delaunay Tree, but labeled *dead*. So, at each step, relationships between the old and the new triangles is defined and stored as a *history* of the construction. This sequential character of the Delaunay Tree is especially convenient for implementation of the stochastic gradient algorithms described in Section 3.2 and was decisive for the choice of this particular structure.

As long as the Delaunay triangulation is constructed, the Voronoi cell associated with site x_i is the polygon whose vertices are the centers of the circumdiscs around the Delaunay triangles having x_i as a vertex. So each time we need to construct the Voronoi cell of x_i we just have to locate one triangle that contains this node and then to turn around x_i to obtain all the others. The Delaunay Tree enables to do such operations in a very easy and fast way.

The Delaunay Tree had been tested a lot, as confirmed by the technical report written by Devillers [8]. Even if we won't give any details here, we can say that it is possible to compute the Delaunay triangulation of 100.000 sites on an ordinary workstation in a reasonable time (a few minutes on a Sun Sparc Station 5 with 32 Mo RAM). Moreover, the Delaunay Tree is a structure easy to implement, which can be adapted to more general contexts, like k-Delaunay tessellations. This kind of tessellations enables to compute k-order Voronoi diagrams, which will be used in future versions of ARC to solve queries such as the k nearest neighbors of a given site. They arise in the models where a station may be connected not only to nearest station in the upper level, but to the k nearest ones.

A randomized analysis of the algorithm proves that the Delaunay Tree (and thus the Delaunay triangulation) of n sites can be incrementally constructed in expected time $O(n \log n)$ (for two dimensional space) with an expected storage $O(n)$. It should be emphasized that this order of the expected time has been proved to be optimal.

4.5 Model construction

The sites of each hierarchy level are randomly and successively generated according to the distribution of the corresponding point process, and successively put in a storage list, in the quaternary tree and in the Delaunay Tree. These data structures are used afterwards for the generation of the other objects (cells, links, and communication flows).

The figure 3 shows the different parameters the user has to give (inside a parallelogram) to generate and construct each part of the model (inside a corner-rounded box). A box at the end of an arrow needs the box at the beginning of the arrow to be constructed. In more details,

- the set of sites needs the specification of the point processes to be generated;
- the intra-level connection needs the set of nodes and the specification of the connection law (Delaunay triangulation, full connection etc.);
- the inter-level connections also need the set of nodes, the number of levels and the space tessellation rule (Voronoi, the power diagram etc.);
- the construction of communication flows needs all intra- and inter-level connections, and the rules that characterize the communication requests in the model.

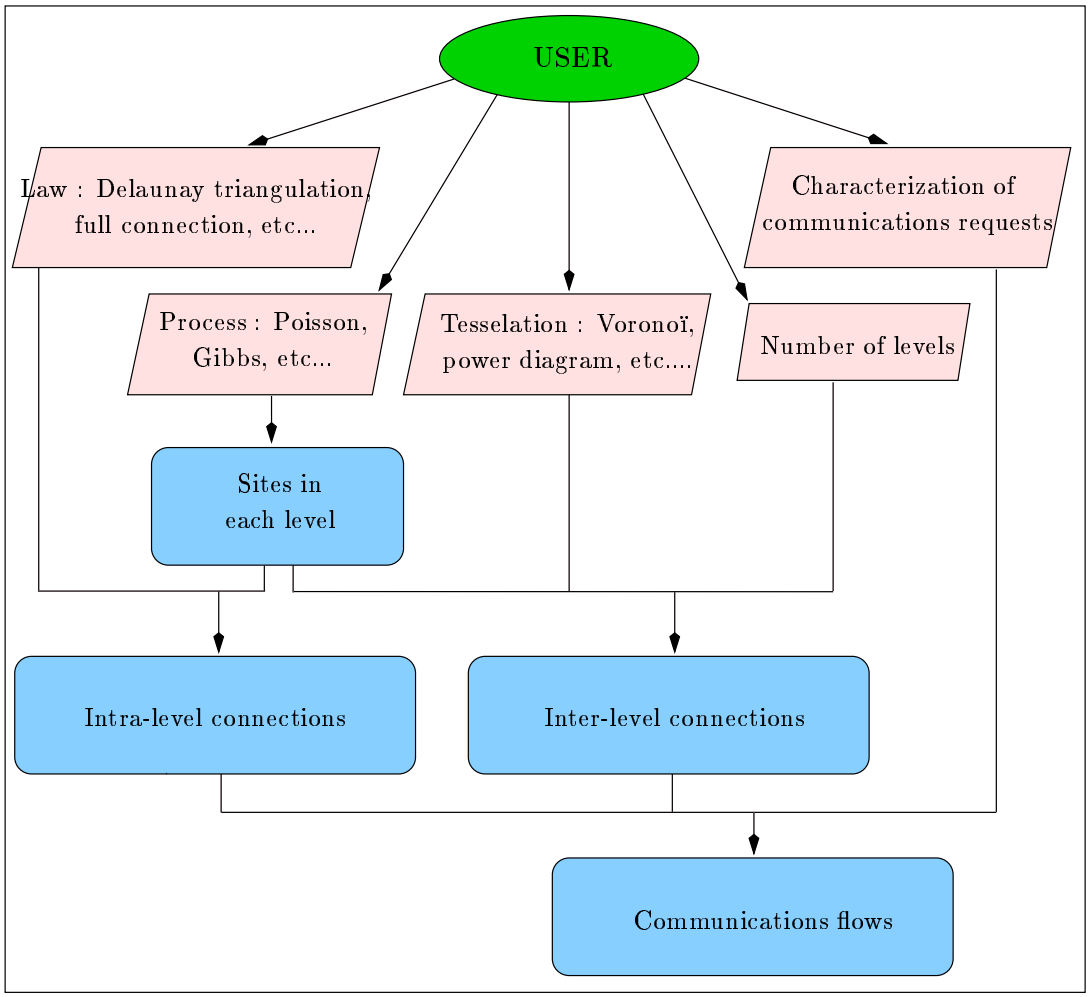

Figure 3: Model construction

5 Examples of computations and discussion

To be able to evaluate the quality of the estimation algorithms used in ARC, we have chosen the model and the characteristics for which the theoretical values were known. Namely, we considered a 3-level hierarchical model described in Section 2, where the stations were represented by realizations of independent homogeneous Poisson processes.

The evaluated characteristics, expressed as functionals of model structuring objects, were the following:

- the geometrical characteristics of the cell or the zone served by a typical station on a given hierarchy level: the surface, the length of the boundary and the number of neighboring stations;
- The cost of all the hierarchical connection of a typical second level station via intermediate 1-level stations down to the subscribers' 0-level.

The expectations and the gradients of these functionals were estimated on a *single* realization of the process by means of estimators (2) and (10), respectively. For the cells characteristics on a given hierarchy level, the gradients are estimated with respect to the intensity of the point process of this level. For the cost, the gradient is estimated with respect to the intensity of the intermediate level (level 1) process intensity. All results of computations in ARC are directly compared to the theoretical results.

The first statistics are the geometrical characteristics of a typical Voronoi cell constructed with respect to a Poisson process. The theoretical values of their expectations can be found e.g., in [15, p.287]. In the second case the cost function includes the cost of the stations and the cost of wires from the subscribers to the 2-level station (via 1-level stations) and the cost of engineering environment (trenches) between the 1-level stations and the the 2-level station. The problem of optimization of the intensity of the intermediate 1-level sites was considered in [3] where an explicit formula for the cost function was given.

The model parameters used in simulations are:

- the window size : (0, 50) x (0, 50)
- the subscribers' level process intensity : 10.91
- level 1 process intensity : 0.27
- level 2 process intensity : 0.018
- the cost of the nodes are : level $1-18$, level $0-4$. The cost of 2-level nodes is not pertinent to the gradient evaluation;
- the cost of a length unity of trenches : 9
- the cost of a length unity of wires : 4

The following are the results of the estimation and the gradients computation of the characteristics of a typical cell. The number of nodes in the window used in the estimation : 494. The number M of additional points U_m used for the gradient estimation : 500.

	Mean	Theor. mean	Mean/Theor.mean
Surface	3.571713894003	3.666670333337	0.974102815170
Frontier length	7.530710107113	7.659420691762	0.983195780748
Neighbors	5.967611336032	6.000000000000	0.994601889339

	Gradient	Theor.grad.	Grad/Theor.grad
Surface	-12.417761137227	-13.444471333374	0.923633278640
Frontier length	-12.322017528162	-14.042285310516	0.877493745191
Neighbors	0.311845077197	0.000000000000	

In the first table ne can see the order of accuracy of the ergodic theorem estimator (2) which is the best one may expect from a single configuration data. In the second table we observe a slight degradation of the accuracy which is natural for that number of additional uniform points. Below are successive computation of the cost function and its gradient. One can observe how the optimization of the intermediate level intensity goes on.

```
-------------------------------------------------------------------------------
Level 1 process intensity: 0.27272700
Number of considered sites:       15  (on the highest hierarchical level)

      MEAN COST:      11961.50352868

      COST GRADIENT : -7361.35478162
===============================================================================
Level 1 process intensity: 0.50000000
Number of considered sites:       16  (on the highest hierarchical level)

      MEAN COST:      10658.34028128

      COST GRADIENT : -1401.71686228
===============================================================================
Level 1 process intensity: 0.60000000
Number of considered sites:       16  (on the highest hierarchical level)

      MEAN COST:      10851.15125511

      COST GRADIENT : 523.04066513
===============================================================================
Level 1 process intensity: 0.55000000
Number of considered sites:       16  (on the highest hierarchical level)

      MEAN COST:      10510.39609885

      COST GRADIENT : -168.12384335
-------------------------------------------------------------------------------
```

As a concluding remark we would like to emphasize that the considered algorithms allow estimation of local characteristics on a *single* configuration. This opens possibility to apply them to the existing configurations of stations that could give an idea of directions to the system cost reduction. Of course, recommendations based on that kind of observations should be met with some criticism: why, after all, the stations should follow a Poisson process? Whether this hypothesis is statistically plausible or not should be decided on a case-by-case basis. The authors studied available statistics on the station locations in a large city and, when restricted to the residential area, we found no contradiction to the hypothesis for the homogeneous Poisson process distribution.

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