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Robust Interpolation and Approximation for $A()$ Functions on

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Abstract: For the robust H^∞ identification of linear shift-invariant systems from frequency responses, the so-called two-stage algorithms are widely used. If the second step of such algorithms is generally reduced to the resolution of a nonlinear Nehari extension problem, the first stage may lead to specific interpolation or approximation techniques depending on the nature of the available data (density, corruptness, distribution, ...). In this paper we present two approximation techniques. In section 1, we study on a subset of the unit circle, the robust polynomial approximation based on a least-deviation problem. In section 2, we generalize Partington's results about robust Jackson and de la Vallée Poussin polynomial approximation, to the case of non-equally spaced points densely distributed on a subset of the unite circle.

Key-words: Polynomial approximation and interpolation, Jackson and de la Vallée Poussin Polynomials, robust H^∞ and H^2 identification from frequency responses.

(Résumé : tsvp)

Interpolation et approximation robustes des fonctions de $A(\mathbb{D})$ sur un arc du cercle

Résumé : Pour l'identification robuste H^∞ des systèmes linéaires invariants, les algorithmes dits en deux étapes sont largement utilisés. Si la seconde étape de tels algorithmes peut être réduite à la résolution d'un problème de Nehari, la première étape nécessite une technique d'approximation ou d'interpolation spécifique tenant compte de la nature des données disponibles (densité, perturbations, distribution, ...). Dans ce papier, nous présentons deux techniques d'approximation. En section 1, nous étudions sur un arc du cercle unité, l'approximation polynômiale robuste basée sur le problème des moindres déviations. En section 2, nous généralisons les résultats de Partington concernant l'approximation robuste par les polynômes de Jackson et de la Vallée Poussin, au cas des points non équi-répartis distribués d'une façon dense sur un sous-ensemble du cercle unité.

Mots-clé : Approximation et interpolation polynômiales, polynômes de Jackson et de de la Vallée Poussin, identification robuste dans H^∞ et identification dans H^2 à partir des réponses fréquentielles.

Introduction

Our motivation, as well as our framework and assumptions, in the study of robust interpolation and approximation for $A(\mathbb{D})$-functions on subsets of the circle [4] come from the robust H^∞ identification problem of stable linear shift invariant dynamical control systems . Given corrupted frequency response measurements available at some set of frequencies corresponding to the bandwidth of the unknown system, the worst-case deterministic (or simply the robust H^∞) identification problem is solved, constructing a robustly convergent identified model.

The robust convergence as stated in [25] quantifies the requirement for the identification algorithm to recover exactly the system if the available information about the system becomes more complete (as the number of measurements goes to infinity) and less corrupted (as the noise level goes to zero). As described in [9], the worst-case formulation of the identification problem is not only motivated by the identification framework but also by the modern robust control design techniques.

The so-called two-stage algorithm is widely studied in the worst-case H^∞ identification problem [15, 8, 25, 26]. While the second stage can be reduced to a Nehari problem to obtain the best analytic extension of the approximation computed during the first stage, the latter consists of determining an interpolation (or simply approximation) of the given data. Different techniques are available such as truncated spline Fourier series coupled with attenuation factors (see [8, 10, 11], and also [9, sect. 3.2] and [15] for more details about interpolating method using FFT, DFT, ...), rational wavelets with orthonormal basis [34], Fejér and de la Vallée Poussin operators [25, 18, 15] and inverse FFT and window functions [16]. To ensure convergence of the whole scheme in worst-case identification as emphasized in [16, 22], we require the interpolation step to be robustly convergent . An interpolating method is robustly convergent if and only if any function can be identified exactly as the noise level tends to zero and as the interpolating values are densely distributed over the domain of this function.

When the points of measurements are equally spaced on the unit circle, the cited methods are simple. The use of the Jackson and de la Vallée Poussin polynomials combined with the Nehari step yields an algorithm for identification which is robustly convergent [25]. The Jackson polynomials as defined in [33] were also considered in [15] using the idea of Cesaro sums and in [16] using window functions.

However, the non-equally spaced points case is computationally harder. In [26], Partington proposed for unequally spaced measurements a two-stage algorithm of identification (see also [27] for a more general result about interpolation in normed spaces from the values of linear functionals).

In this paper we will present two simple and robustly convergent polynomial approximation schemes for non-equally spaced points on a subset of the unit circle. The robust polynomial approximation which will be studied in section 1, solving a problem of smallest deviation, is an adaptation, to a subset of the unit circle, of the classical robust polynomial approximation on the whole circle. We will be more interested in section 2 by robust Jackson and de la Vallée Poussin polynomials. We will extend the use of these polynomials to a set of non-equally spaced points over a subset of the unit circle. In section 3 we will investigate how the Jackson and de la Vallée Poussin polynomials can be used in robust H^∞ approximation and in analytic H^2 completion ; some results from [4] and [3] will be presented. These sections 1 and 2 will be self-contained and some notations will be redundant.

1 Robust least deviation polynomial approximation

For every integer m, we consider a set $\{x_1^{(m)}, \dots, x_m^{(m)}\}$ of m points in increasing order over $[a,b] \subset [0, 2\pi]$ where a and b are respectively the lower limit of the decreasing sequence $(x_1^{(m)})_{m\geq 1}$ and the upper limit the increasing sequence $(x_m^{(m)})_{m\geq 1}$.

Let $\delta_m = \sup_{1\leq k\leq m-1}\left(x_{k+1}^{(m)} - x_k^{(m)}\right)$ be the maximum gap between every pair of consecutive points in the set $\{x_1^{(m)}, \dots, x_m^{(m)}\}$. Let I be the arc of the unit circle $\mathbb{T}$ defined by : $I = e^{i\,[a,b]}$. Define, for all $k = 1, \dots, m$, the element $z_k^{(m)}$ of I by $z_k^{(m)} = e^{i\,x_k^{(m)}}$.

For every integer m, let d be any increasing integer-valued function such that $d(m) < m$ and define $\mathcal{P}_m$ to be the set of all trigonometric polynomials of degree at most m.

We suppose that, for some unknown continuous function $f \in C(I)$, we are given m noisy data $y_k^{(m)} = f(z_k^{(m)}) + \eta_k^{(m)}$, for $k = 1, \dots, m$ where $z_k^{(m)}$ belongs to the arc I and the noise $(\eta_k^{(m)})_{k>0}$ is assumed to be any bounded sequence in l^∞. Let $\|\eta^{(m)}\|_\infty = \sup_{1\leq k\leq m} |\eta_k^{(m)}|$.

Consider the following minimization problem : seek the approximation polynomial Q_m, of degree at most $d(m)$, which realizes the smallest deviation from the noisy measurements $(y_k^{(m)})$ of f at the evaluation points $z_1^{(m)}, \dots, z_m^{(m)}$ in the l^∞ sense :

$$\inf_{P\in\mathcal{P}_{d(m)}} \left\{ \sup_{1\leq k\leq m} |P(z_k^{(m)}) - y_k^{(m)}| \right\} \quad = \quad \sup_{1\leq k\leq m} |Q_m(z_k^{(m)}) - y_k^{(m)}|\,. \tag{1}$$

Moreover, if there exists such an approximating polynomial Q_m, we will require this approximation procedure to be robustly convergent on I, that is :

$$\lim_{\substack{m\to\infty\\ \nu\to 0}} \left\{ \sup_{\|\eta^{(m)}\|_\infty \leq \nu} \|Q_m - f\|_{L^\infty(I)} \right\} \quad = \quad 0\,.$$

The following theorem states the existence and the robust convergence of such a polynomial approximation.

Theorem 1 *Let f be a continuous function on I (and notations as above).*

1. *For every integer m, there exists a polynomial Q_m belonging to the set $\mathcal{P}_{d(m)}$ of trigonometric polynomials of degree at most $d(m)$ such that :*

$$\inf_{P\in\mathcal{P}_{d(m)}} \left\{ \sup_{1\leq k\leq m} |P(z_k^{(m)}) - y_k^{(m)}| \right\} \quad = \quad \sup_{1\leq k\leq m} |Q_m(z_k^{(m)}) - y_k^{(m)}| \,. \tag{2}$$

2. *Moreover, if $d(m)$ tends to infinity and $d^2(m)\,\delta_m$ tends to zero when m goes to infinity then, Q_m converges to f robustly on I :*

$$\lim_{\substack{m\to\infty\\ \nu\to 0}} \left\{ \sup_{\|\eta^{(m)}\|_\infty\leq\nu} \|Q_m - f\|_{L^\infty(I)} \right\} \qquad = \qquad 0\,. \tag{3}$$

The remaining of section 1 is dedicated to the proof of the previous theorem. This will require to establish the inequality of Bernstein for trigonometric polynomials on proper closed subsets of $[0, 2\pi]$.

1.1 Inequalities of Bernstein and Markov

In this approximation scheme, we will make use of the fact that a trigonometric polynomial of fixed degree cannot change "too rapidly". For a trigonometric polynomial defined on the interval $[0, 2\pi]$, Bernstein's inequality provides an upper bound of the derivative with respect to the L^∞ norm of the polynomial itself and its degree [17, 19, 5, 21].

Theorem 2 *(Bernstein's inequality)*
For every complex-valued trigonometric polynomial T_n of degree n , the following inequality holds :

$$|T_n'(t)| \quad\leq\quad n\ \|T_n\|_{L^\infty([0,2\pi])}\,, \qquad 0\leq t\leq 2\pi\,. \tag{4}$$

This inequality can be extended for trigonometric polynomials defined on subsets of $[0, 2\pi]$. To do this, we will first adapt to subsets of $[-1, 1]$ the well-known Markov's and Bernstein's inequalities for algebraic polynomials, stated bellow.

Theorem 3 *(Bernstein's inequality)*
For every complex-valued algebraic polynomial P_n of degree n, the following inequality holds :

$$|P_n'(x)| \quad\leq\quad \frac{n}{\sqrt{1-x^2}}\ \|P_n\|_{L^\infty([-1,1])}\,, \qquad -1 < x < 1\,. \tag{5}$$

Theorem 4 *(Markov's inequality)*
For every complex-valued algebraic polynomial P_n of degree n, the following inequality holds :

$$|P_n'(x)| \quad\leq\quad n^2\ \|P_n\|_{L^\infty([-1,1])}\,, \qquad -1\leq x\leq 1\,. \tag{6}$$

The following two propositions are simple generalizations to subsets of $[-1,1]$ and $[0,2\pi]$ respectively of the above results.

Proposition 1 *Let $K=[a,b]$, $a<b$, be a proper closed subset of $[-1,1]$. For every algebraic polynomial P_n of degree n, the following inequalities hold :*

$$|P_n'(x)| \quad \leq \quad n^2\,\frac{2}{b-a}\,\|P_n\|_{L^\infty(K)}, \qquad a\leq x\leq b \tag{7}$$

and

$$|P_n'(x)| \quad \leq \quad n\,\frac{1}{\sqrt{(x-a)(b-x)}}\,\|P_n\|_{L^\infty(K)}, \quad a<x<b\,. \tag{8}$$

Proof of proposition 1: Let P_n be an algebraic polynomial of degree n. Let ψ be the linear function mapping $[-1,1]$ onto $[a,b]$ and given by :

$$\psi(y) \quad = \quad \frac{b-a}{2}y+\frac{a+b}{2}, \qquad \forall\, y\in[-1,1],$$

and define the (algebraic) polynomial Q_n of degree n on $[-1,1]$ by :

$$Q_n(y) \quad = \quad P_n(\psi(y)), \qquad \forall\, y\in[-1,1].$$

Then we have :

$$P_n'(x) \quad = \quad \frac{2}{b-a}Q_n'(y), \qquad x \quad = \quad \psi(y), \quad x\in[a,b], \quad \forall\, y\in[-1,1]$$

and

$$\|Q_n\|_{L^\infty([-1,1])} \qquad = \qquad \|P_n\|_{L^\infty([a,b])}.$$

Proposition 1 can now be deduced from inequalities (5) and (6). ∎

Proposition 2 *Let $K=[a,b]$, $a<b$, be a proper closed subset of $[0,2\pi]$. Let θ_1, θ_2, θ_3 and θ_4 be in $[0,\pi]$ such that $\cos[a,b]=[\cos\theta_1,\cos\theta_2]$ and $\sin[a,b]=[\sin\theta_3,\sin\theta_4]$. For every real-or complex-valued trigonometric polynomial T_n of degree n, the following inequalities hold for $a<t<b$:*

$$|T_n'(t)| \quad \leq \quad n^2\left(\frac{1}{\sin\frac{\theta_1+\theta_2}{2}\sin\frac{\theta_1-\theta_2}{2}}+\frac{1}{\cos\frac{\theta_3+\theta_4}{2}\cos\frac{\theta_4-\theta_3}{2}}\right)\,\|T_n\|_{L^\infty(K)}, \tag{9}$$

and

$$|T_n'(t)|\leq\frac{n}{2}\,\left(\frac{|\sin t|\,\|T_n\|_{L^\infty(K)}}{\sqrt{\sin\frac{t+\theta_1}{2}\,\sin\frac{t+\theta_2}{2}\sin\frac{t-\theta_1}{2}\,\sin\frac{\theta_2-t}{2}}}+\frac{|\cos t|\,\,\|T_n\|_{L^\infty(K)}}{\sqrt{\sin\frac{t-\theta_4}{2}\,\sin\frac{\theta_3-t}{2}\cos\frac{t+\theta_4}{2}\,\cos\frac{\theta_3+t}{2}}}\right). \tag{10}$$

Proof of proposition 2 : We derive proposition 2 from proposition 1 using the usual substitutions $x = \cos t$ and $y = \sin t$.
Let P_n be the even algebraic polynomial of degree n defined on $[\cos\theta_1, \cos\theta_2]$ by

$$P_n(x) \quad = \quad \frac{1}{2}\Big(T_n(t) + T_n(-t)\Big), \qquad \text{with} \qquad x \quad = \quad \cos t\,, \quad \forall\; t \;\in\; [a,b]\,,$$

and let Q_n be the odd algebraic polynomial of degree n defined on $[\sin\theta_3, \sin\theta_4]$ by

$$Q_n(y) \quad = \quad \frac{1}{2}\Big(T_n(t) - T_n(-t)\Big), \qquad \text{with} \qquad y \quad = \quad \sin t\,, \quad \forall\; t \;\in\; [a,b]\,.$$

It's then obvious that :

$$\|P_n\|_{L^\infty(\cos(K))} \;\leq\; \|T_n\|_{L^\infty(K)}\,, \qquad \|Q_n\|_{L^\infty(\sin(K))} \;\leq\; \|T_n\|_{L^\infty(K)}\,,$$

and that the derivative T'_n of T_n is given by :

$$T'_n(t) \;=\; -\sin t\;\, P'_n(x) \;+\; \cos t\;\, Q'_n(y)\,, \qquad x \;=\; \cos t \quad \text{and} \quad y \;=\; \sin t \quad \forall t \in [a,b]\,.$$

This leads to the following inequalities :

$$|T'_n(t)| \;\leq\; \sup_{x\in[\cos\theta_1\,,\,\cos\theta_2]} |P'_n(x)| \;+\; \sup_{y\in[\sin\theta_3\,,\,\sin\theta_4]} |Q'_n(y)|\,, \quad \forall t \in [a,b]\,.$$

and

$$|T'_n(t)| \;\leq\; |\sin t|\,|P'_n(x)| \;+\; |\cos t|\,|Q'_n(y)|\,, \quad \forall t\in[a,b]\,.$$

Owing to proposition 1, it results from previous inequalities that for $a \leq t \leq b$:

$$|T'_n(t)| \;\leq\; \frac{2\,n^2}{\cos\theta_2 - \cos\theta_1}\,\|P_n\|_{L^\infty(\cos(K))} \;+\; \frac{2\,n^2}{\sin\theta_4 - \sin\theta_3}\,\|Q_n\|_{L^\infty(\sin(K))}$$

and

$$|T'_n(t)| \;\leq\; \frac{n\;|\sin t|\;\|P_n\|_{L^\infty(\cos(K))}}{\sqrt{(\cos t - \cos\theta_1)\,(\cos\theta_2 - \cos t)}} \;+\; \frac{n\;|\cos t|\;\|Q_n\|_{L^\infty(\sin(K))}}{\sqrt{(\sin t - \sin\theta_4)\,(\sin\theta_3 - \sin t)}}\,,$$

which, after simplification, achieve the proof of proposition 2. ∎

1.2 Proof of theorem 1

Introduce the following two maps on $\mathcal{P}_{d(m)}$:

$$\begin{array}{rccl} \Gamma: & \mathcal{P}_{d(m)} & \longrightarrow & \mathbb{R}^+ \\ & P & \longmapsto & \sup_{1\leq k\leq m}\Big(|P(z_k^{(m)})|\Big) \end{array}$$

and

$$\begin{array}{rccl} \Delta: & \mathcal{P}_{d(m)} & \longrightarrow & \mathbb{R}^+ \\ & P & \longmapsto & \sup_{1\leq k\leq m}\Big(|P(z_k^{(m)}) - y_k^{(m)}|\Big) \end{array}$$

Then, the following lemma will be used to prove the first part of theorem 1 :

Lemma 1

$$\lim_{\|P\|_{L^\infty(I)}\to\infty} \Gamma(P) \;=\; \infty, \qquad \lim_{\|P\|_{L^\infty(I)}\to\infty} \Delta(P) \;=\; \infty.$$

Proof of lemma 1 : Let m be any fixed integer. Since polynomials in $\mathcal{P}_{d(m)}$ are of degree at most $d(m) < m$, the function Γ vanishes only at 0 on $\mathcal{P}_{d(m)}$. Moreover, since for every polynomial $P \in \mathcal{P}_{d(m)}$, $\Gamma(P) \leq \sup_{z\in I} |P(z)| = \|P\|_{L^\infty(I)}$, the function Γ is continuous on $\mathcal{P}_{d(m)}$. Thereby on the compact set $\mathcal{E}_1$:

$$\mathcal{E}_1 \;=\; \{\, P \;\in\; \mathcal{P}_{d(m)}\,,\; \|P\|_{L^\infty(I)} \;=\; 1\}$$

where Γ does not vanish, there exists a positive constant γ_{min} such that :

$$0 \;<\; \gamma_{min} \;=\; \inf_{P\in\mathcal{E}_1} \Gamma(P)\,.$$

Thus for all polynomial $P \in \mathcal{P}_{d(m)}$ except zero, $\Gamma(P/\|P\|_{L^\infty(I)}) \geq \gamma_{min}$, and then

$$\forall P \in \mathcal{P}_{d(m)}, \qquad \Gamma(P) \;\geq\; \gamma_{min}\,\|P\|_{L^\infty(I)}.$$

Consequently,

$$\lim_{\|P\|_{L^\infty(I)}\to\infty} \Gamma(P) \;=\; \infty.$$

In addition, since

$$\Delta(P) \;=\; \sup_{1\leq k\leq m} (|P(z_k^m) - y_k^m|) \;\geq\; \Gamma(P) - \sup_{1\leq k\leq m} (|y_k^m|),$$

then

$$\lim_{\|P\|_{L^\infty(I)}\to\infty} \Delta(P) \;=\; \infty.$$

and lemma 1 is proved.∎

Proof of theorem 1 : Since the function f is continuous on I and the noise sequences $(\eta_k^{(m)})_1^m$ is uniformly are bounded, the sequences $(y_k^{(m)})_1^m$ are bounded independently of m. By lemma 1 there exists a positive constant C such that for all polynomial $P \in \mathcal{P}_{d(m)}$, if $\|P\|_{L^\infty(I)} > C$, then $\Delta(P) > \sup_{1\leq k\leq m} \left(|y_k^{(m)}|\right)$. Hence,

$$\inf_{\|P\|_{L^\infty(I)}\leq C} \Delta(P) \;\leq\; \Delta(0) \;=\; \sup_{1\leq k\leq m} \left(|y_k^{(m)}|\right) \;\leq\; \inf_{\|P\|_{L^\infty(I)}> C} \Delta(P)$$

so that :

$$\inf_{P\in\mathcal{P}_{d(m)}} \Delta(P) = \inf_{P\in\mathcal{P}_{d(m)},\,\|P\|_{L^\infty(I)}\leq C} \Delta(P).$$

Moreover the function Δ, defined as the supremum of a finite set of continuous functions $P \mapsto |P(z_k^{(m)}) - y_k^{(m)}|$, is continuous. Therefore, on the compact $\{P \in \mathcal{P}_{d(m)}, \|P\|_{L^\infty(I)} \leq C\}$, the function Δ attains its infimum, that is there exists a polynomial Q_m belonging to $\mathcal{P}_{d(m)}$ such that :

$$\inf_{P\in\mathcal{P}_{d(m)}}\left\{\sup_{1\le k\le m}|P(z_k^{(m)})-y_k^{(m)}|\right\} \quad = \quad \sup_{1\le k\le m}|Q_m(z_k^{(m)})-y_k^{(m)}|.$$

This establishes equality (2) of the theorem 1.
In order to prove the second equality (3), we first establish the following lemma :

Lemma 2 *Let $\{Q_m\}_{m\ge1}$ be a sequence of minimizing trigonometric polynomials, that is, for every integer m :*

$$\min_{P\in\mathcal{P}_{d(m)}}\left\{\sup_{1\le k\le m}|P(z_k^{(m)})-y_k^{(m)}|\right\} \quad = \quad \sup_{1\le k\le m}|Q_m(z_k^{(m)})-y_k^{(m)}|\,.$$

Then $\{Q_m\}_{m\ge1}$ is bounded in $L^\infty(I)$ if $\|\eta^{(m)}\|_\infty$ is.

Proof of lemma 2 :
Suppose the sequence $\{Q_m\}_{m\ge1}$ unbounded in $L^\infty(I)$. Let $z_m = e^{i\,x_m}$ be an element of I such that $|Q_m(z)|$ is maximum. Then there exists an integer $k<m$ for which $x_k^{(m)} \le x_m \le x_{k+1}^{(m)}$.

Since the trigonometric polynomial Q_m is smooth on I, we have :

$$Q_m(z_k^{(m)}) \quad = \quad Q_m(z_m) - \int_{z_k^{(m)}}^{z_m} Q'_m(\sigma)\,d\sigma$$

and then by proposition 2, we get :

$$\forall k\in\{1,\ldots,m\},\quad |Q_m(z_k^{(m)})| \quad \ge \quad \left(1-d^2(m)\,\delta_m\,C\right)\|Q_m\|_{L^\infty(I)}\,,$$

where C is a positive constant independent of m.

Since $d^2(m)\,\delta_m$ converges to zero when m tends to infinity, then $|Q_m(z_k^{(m)})|$ tends to infinity with m, and this is also the case for

$$\sup_{1\le k\le m}|Q_m(z_k^{(m)})-y_k^{(m)}| \quad \ge \quad \sup_{1\le k\le m}|Q_m(z_k^{(m)})| - \sup_{1\le k\le m}|y_k^{(m)}|\,.$$

Hence, for m large enough

$$\sup_{1\le k\le m}|Q_m(z_k^{(m)})-y_k^{(m)}| \quad \ge \quad \sup_{1\le k\le m}|y_k^{(m)}|\,.$$

This shows that the zero polynomial does better than Q_m in the minimization problem (2) which contradicts the optimality of Q_m. ∎

Proof of theorem 1 (continued) : Let $z = e^{ix}$ be any element in I and let $\epsilon > 0$. Using $z_k^{(m)} = e^{i x_k^{(m)}}$ such that $x_k^{(m)} \leq x \leq x_{k+1}^{(m)}$, write $f(z) - Q_m(z)$ in three steps :

$$f(z) - Q_m(z) \quad = \quad \Big(f(z) - f(z_k^{(m)})\Big) + \Big(f(z_k^{(m)}) - Q_m(z_k^{(m)})\Big) + \Big(Q_m(z_k^{(m)}) - Q_m(z)\Big).$$

First, due to the continuity of the function f on I (and thus to its uniform continuity), there exists a positive real α such that for all x and y in I and whenever $|x - y| < \alpha$, we have $|f(x) - f(y)| < \epsilon/3$. Also, since $\delta_m \leq \delta_m\, d^2(m)$, then δ_m tends to zero as m tends to infinity. Hence, there is an integer m_0 such that for all $m \geq m_0$, we have : $|f(z) - f(z_k^{(m)})| < \epsilon/3$ for all $z \in I$ such that : $|z - z_k^{(m)}| < \alpha$.
In addition, using proposition 2 and lemma 2, we derive from the inequality :

$$|Q_m(z_k^{(m)}) - Q_m(z)| \quad \leq \quad \int_z^{z_k^{(m)}} |Q_m^{'}(\sigma)\, d\sigma|,$$

the following one :

$$|Q_m(z_k^{(m)}) - Q_m(z)| \quad \leq \quad d_m^2\, \delta_m\, C\, M\,.$$

Hence, due to the assumption that $d^2(m)\, \delta_m \to 0$ when $m \to \infty$, there exists an integer $m_1 \geq m_0$ such that for all $m > m_1$ we have : $|Q_m(z_k^{(m)}) - Q_m(z)| < \epsilon/3$.
On the other hand,

$$\begin{aligned} |Q_m(z_k^{(m)}) - f(z_k^{(m)})| \quad &\leq \quad |Q_m(z_k^{(m)}) - y_k^{(m)}| + |\eta_k^{(m)}| \\ &\leq \quad \min_{P \in \mathcal{P}_{d(m)}} \Big\{ \sup_{1 \leq k \leq m} |P(z_k^{(m)}) - y_k^{(m)}| \Big\} + \sup_{1 \leq k \leq m} |\eta_k^{(m)}| \end{aligned}$$

and then

$$|Q_m(z_k^{(m)}) - f(z_k^{(m)})| \quad \leq \quad \min_{P \in \mathcal{P}_{d(m)}} \sup_{1 \leq k \leq m} |P(z_k^{(m)}) - f(z_k^{(m)})| + 2 \sup_{1 \leq k \leq m} |\eta_k^{(m)}|.$$

Since the function f is continuous on I and $d(m)$ goes to infinity when m tends to infinity, then by Weierstrass approximation theorem [28, thm1.1], there is an integer $m_2 > m_1$ such that for all $m > m_2$, there exists a polynomial R in $\mathcal{P}_{d(m)}$ for which : $\|R - f\|_{L^\infty(I)} < \epsilon/3$, and then :

$$|Q_m(z_k^{(m)}) - f(z_k^{(m)})| \quad \leq \quad \epsilon/3 + 2\, \|\eta^{(m)}\|_\infty\,.$$

Finally, if $\|\eta^{(m)}\|_\infty < \nu$, then for all $m > m_2$ and all $z \in I$: $|f(z) - Q_m(z)| < \epsilon + 2\,\eta$.
Thus :

$$\forall\, \epsilon > 0, \quad \exists\, m_2 \geq 1, \quad \forall\, m > m_2 \;:\; \|f - Q_m\|_{L^\infty(I)} \;<\; \epsilon + 2\,\eta.$$

This achieves the proof of equality (3) of theorem 1.∎

2 Robust Jackson and de la Vallée Poussin polynomials

For every integer m, we consider a set $\{x_1^{(m)}, \ldots, x_m^{(m)}\}$ of m points in increasing order over $[0, 2\pi]$ such that $(x_1^{(m)})$ is a decreasing sequence while $(x_m^{(m)})$ is an increasing one. Let

$$I \quad = \quad \left[\inf_{m\geq 1} x_1^{(m)}, \sup_{m\geq 1} x_m^{(m)}\right] \qquad \text{and} \qquad \delta_m \quad = \sup_{1\leq k\leq m-1} \frac{x_{k+1}^{(m)} - x_k^{(m)}}{2\pi}\,.$$

The compact I is the smallest one containing all points $\{x_k^{(m)}\}_{k=1}^m$ while $2\pi\delta_m$ represents the maximum gap between consecutive points in the set $\{x_1^{(m)}, \ldots, x_m^{(m)}\}$, for every integer m. Let φ_m be the continuous function, linear on $]x_k^{(m)}, x_{k+1}^{(m)}[$, for $k = 1, \ldots, m-1$, and satisfying

$$\varphi_m(0) = 0, \quad \varphi_m(2\pi) = 2\pi \;\text{ and }\; \varphi_m(x_k^{(m)}) = t_k^{(m)} = t_1^{(m)} + \frac{2\pi(k-1)}{m},\; \forall\, k = 1, \ldots, m\,, \tag{11}$$

where $\{t_1^{(m)}, \ldots, t_m^{(m)}\}$ are m equally spaced points over $[0, 2\pi]$.

We suppose that, for some unknown bounded function f on I, we are given m noisy values

$$y_k^{(m)} \quad = \quad f(x_k^{(m)}) + \eta_k^{(m)}, \qquad \forall\, k = 1, \ldots, m$$

and let $\|\eta^{(m)}\|_\infty = \sup_{1\leq k\leq m} \eta_k^{(m)}$.

We define the Jackson *pseudo-polynomial* $\tilde{J}_{n,m}$ of order n associated to the corrupted function $f + \eta$ at these points $\{x_1^{(m)}, \ldots, x_m^{(m)}\}$ by :

$$\tilde{J}_{n,m}(x) \quad = \quad \tilde{J}_{n,m}(x, f+\eta) \quad = \quad \frac{1}{m}\sum_{k=1}^{m} y_k^{(m)}\, K_n(\varphi_m(x) - t_k^{(m)}) \tag{12}$$

where n is any integer less then m and K_n is the Fejér kernel of order n that we recall later. Also, we adopt for the de la Vallée Poussin *pseudo-polynomials* $\tilde{V}_{n,m}$, the expression used in [25, §3] as a discretization of the de la Vallée Poussin kernels [33, chap.IV] :

$$\tilde{V}_{n,m} \quad = \quad \frac{2\,n+1}{n}\,\tilde{J}_{2n,m} - \frac{n+1}{n}\,\tilde{J}_{n,m}\,.$$

We will misuse the term polynomial to design also such pseudo-polynomial. Then,

Theorem 5 *Let f be a bounded function on I (and notations as above).*
If $\delta_m = O(m^{-1})$ then $\tilde{J}_{n,m}(x, f+\eta)$ (resp. $\tilde{V}_{n,m}(x, f+\eta)$) converges to $f(x)$ at every point $x \in I$ of continuity of f as $n \to \infty$, m remaining always greater than n (resp. $2n$), and as the noise level ϵ goes to zero :

$$\lim_{\substack{n\to\infty \\ (m>n) \\ \epsilon\to 0}} \left\{ \sup_{\|\eta^{(m)}\|_\infty\leq\epsilon} \left|\tilde{J}_{n,m}(x, f+\eta) - f(x)\right| \right\} \quad = \quad 0\,. \tag{13}$$

$$\lim_{\substack{n\to\infty \\ (m>2n) \\ \epsilon\to 0}} \left\{ \sup_{\|\eta^{(m)}\|_\infty \leq \epsilon} \left| \tilde{V}_{n,m}(x, f+\eta) - f(x) \right| \right\} \quad = \quad 0\,. \tag{14}$$

Furthermore, this convergence is uniform in every closed interval $[\alpha, \beta] \subset I$ *of continuity of* f.

The remaining of section 2 is dedicated to the proof of the previous main theorem and some related results. We first define the convolution $\overset{\varphi}{\circledast}$ and establish that the Fejér kernel K_n is an approximate identity for the convolution $\overset{\varphi}{\circledast}$. Then, we show that the sum $\tilde{J}_{n,m}$ can be regarded as a discretization over the m points $\{x_1^{(m)}, \ldots, x_m^{(m)}\}$ of the continuous expression $f \overset{\varphi}{\circledast} K_n$, the convolution of f with the Fejér kernel K_n, and that the first part of theorem 5 can be considered as the discrete version of proposition 4.

2.1 The convolution $\overset{\varphi}{\circledast}$

Let Φ be the set of continuous piecewise linear and strictly increasing functions φ on $[0, 2\pi]$, with $\varphi(0) = 0$ and $\varphi(2\pi) = 2\pi$. For every function φ in Φ, let $\dot{\varphi}$ be the function defined on $[0, 2\pi]$ and equal everywhere to the right derivative of φ except for 2π where $\dot{\varphi}(2\pi)$ is arbitrarily defined.

$$\forall\, \varphi \in \Phi,\ \forall\, x \in [0, 2\pi[\ : \qquad \dot{\varphi}(x) \quad = \quad \lim_{\epsilon \to 0^+} \frac{\varphi(x+\epsilon) - \varphi(x)}{\epsilon}$$

For every pair of integrable functions f and g, we define the convolution product $f \overset{\varphi}{\circledast} g$ by :

$$\forall x \in [0, 2\pi] : \quad f \overset{\varphi}{\circledast} g(x) \quad = \quad \frac{1}{2\pi} \int_0^{2\pi} f(t)\, g(\varphi(x) - \varphi(t))\, \dot{\varphi}(t)\, dt. \tag{15}$$

Remark 1 In the special case where $\varphi \equiv Id_{[0,2\,\pi]}$, the right derivative $\dot{\varphi}$ is equal to $1_{[0,2\,\pi[}$ and we get in (15) the classical convolution product $f * g$ [33, chap.II], otherwise we have :

$$\forall x \in [0, 2\pi] : \quad f \overset{\varphi}{\circledast} g(x) \quad = \quad ((f \circ \varphi^{-1} * g) \circ \varphi)(x). \tag{16}$$

Recall [6, 3.2.1] that a positive sequence $(K_n)_{n=1}^{\infty}$ of integrable functions is an approximate identity for convolution $*$ (a positive kernel for short) iff :

$$\begin{array}{ll} (i) & \displaystyle\lim_{n\to\infty} \frac{1}{2\pi} \int_0^{2\pi} K_n(t)\, dt = 1 \\ (ii) & \displaystyle\lim_{n\to\infty} \int_\delta^{2\pi-\delta} K_n(t)\, dt = 0\,, \quad \forall\, \delta \in]0, \pi[. \end{array} \tag{17}$$

The name *approximate identity* is justified by the following well-known property [6, §3.2.2].

Proposition 3 *Let $(K_n)_{n\geq 1}$ be an approximate identity for the convolution $*$. Then for every continuous function f on $[0,2\pi]$, the sequence $(f*K_n)_{n\geq 1}$ converges uniformly to f on $[0,2\pi]$:*

$$\lim_{n\to\infty} \|f - f*K_n\|_{L^\infty([0,2\pi])} \quad = \quad 0\,. \tag{18}$$

A property similar to (18) holds for the convolution $\overset{\varphi}{\circledast}$ for every function φ in the set Φ. This is due to the relationship (16) between the convolutions $\overset{\varphi}{\circledast}$ and $*$. Let φ be any function in Φ. Since the function φ^{-1} is continuous on $[0,2\pi]$, then for every continuous function f on $[0,2\pi]$, the function $f\circ\varphi^{-1}$ is continuous on $[0,2\pi]$ and by (18), the sequence $f\circ\varphi^{-1} - f\circ\varphi^{-1}*K_n$ converges uniformly to zero as n tends to ∞. The sequence $(f\circ\varphi^{-1} - f\circ\varphi^{-1}*K_n)\circ\varphi$ thus converges uniformly to zero as n tends to ∞. Using (16) we get :

Proposition 4 *For every function φ in Φ , if f is a continuous function on $[0,2\pi]$ and $(K_n)_{n\geq 0}$ is a positive kernel, then*

$$\lim_{n\to\infty} \|f - f \overset{\varphi}{\circledast} K_n\|_{L^\infty([0,2\pi])} \quad = \quad 0\,. \tag{19}$$

We could have introduced in a direct manner the notion of approximate identity for the convolution $\overset{\varphi}{\circledast}$: a sequence $(K_n)_{n=1}^{\infty}$ of positive integrable functions is an approximate identity for the convolution $\overset{\varphi}{\circledast}$ (or simply φ-positive kernel) if and only if, this sequence verifies :

$$\begin{array}{ll} (i) & \displaystyle\lim_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi} K_n(\varphi(t))\,\dot{\varphi}(t)\,d\,t = 1 \\ (ii) & \displaystyle\lim_{n\to\infty}\int_{\delta}^{2\pi-\delta} K_n(\varphi(t))\,\dot{\varphi}(t)\,dt = 0,\ \forall\,\delta\ \in]0,\pi[\,. \end{array} \tag{20}$$

However, the following lemma shows that this notion is nothing new :

Lemma 3 *A sequence $(K_n)_{n=1}^{\infty}$ of positive integrable functions is an approximate identity for the convolution $*$ if and only if the sequence $(K_n)_{n=1}^{\infty}$ is an approximate identity for the convolution $\overset{\varphi}{\circledast}$ for every element φ of Φ.*

Proof of lemma 3
Let φ be any element of the set Φ. Because of the equalities :

$$\frac{1}{2\pi}\int_0^{2\pi} K_n(\varphi(t))\,\dot{\varphi}(t)\,d\,t \quad = \quad \frac{1}{2\pi}\int_{\varphi(0)}^{\varphi(2\pi)} K_n(\theta)\,d\,\theta \quad = \quad \frac{1}{2\pi}\int_0^{2\pi} K_n(\theta)\,d\,\theta\,,$$

parts (i) in the definitions (17) and (20) are equivalents.

In addition, since for every $\delta \in]\,0\,,\,2\pi\,[$ and with

$$\zeta \quad = \quad \min\left(\delta, \varphi(\delta), 2\pi - \varphi(2\pi-\delta)\right)\ , \qquad \eta \quad = \quad \max\left(\delta, \varphi(\delta), 2\pi - \varphi(2\pi-\delta)\right)$$

we have

$$\int_{\eta}^{2\pi-\eta} K_n(\theta)\,dt \quad \leq \quad \int_{\delta}^{2\pi-\delta} K_n(\varphi(t))\,\dot{\varphi}(t)\,dt \quad = \quad \int_{\varphi(\delta)}^{\varphi(2\pi-\delta)} K_n(\theta)\,d\theta \quad \leq \quad \int_{\zeta}^{2\pi-\zeta} K_n(\theta)\,d\theta\,,$$

parts (ii) in the definitions (17) and (20) are also equivalent. ∎

2.2 Dirichlet and Fejér Kernels for $\overset{\varphi}{\circledast}$

Let φ be any function in Φ. Let $L^2([0,2\,\pi],\dot{\varphi})$ be the space of square-integrable functions on $[0,2\pi]$ with respect to the measure $\dot{\varphi}(t)\,dt$ (see [33, chap.I.3]). For every integer n, we define the complex-valued function ω_n in the interval $[0,2\pi]$ by

$$\forall\ t\ \in\ [\,0,\ 2\,\pi\,]\ : \qquad \omega_n(t)\ =\ e^{in\varphi(t)}\,.$$

Due to the equality $\frac{1}{2\pi}\int_0^{2\pi}\omega_n(t)\,\overline{\omega_m(t)}\dot{\varphi}(t)\,dt = \delta_n^m$, where δ_n^m is the Kronecker's symbol, the system $\{\omega_n\}_{n\geq 0}$ is orthonormal in $L^2([0,2\,\pi],\dot{\varphi})$.

Let D_n stands for the Dirichlet kernel of order n defined by

$$D_n(\theta) \quad = \quad \sum_{k=-n}^{n} e^{ik\theta} \quad = \quad \frac{\sin((n+1/2)\theta)}{\sin(\theta/2)} \tag{21}$$

and let K_n be the Fejér kernel of order n as defined by :

$$K_n(\theta) \quad = \quad \frac{1}{n+1}\sum_{k=0}^{n} D_k \quad = \quad \frac{1}{n+1}\left\{\frac{\sin(\frac{n+1}{2}\theta)}{\sin(\frac{1}{2}\theta)}\right\}^2 . \tag{22}$$

Let f be an integrable function on $[0,2\pi]$. Then, for every integer n, we define the Fourier coefficient $\hat{f}(n)$ of f, with respect to the system $\{\omega_n\}_{n\geq 0}$:

$$\hat{f}(n) \quad = \quad \frac{1}{2\pi}\int_0^{2\pi} f(t)\,e^{-in\varphi(t)}\,\dot{\varphi}(t)\,dt\,. \tag{23}$$

If $S_nf(t)$ denotes the sum $\displaystyle\sum_{k=-n}^{n}\hat{f}(k)e^{ik\varphi(t)}$, then by (23) and (21), we have

$$S_nf \quad = \quad f \overset{\varphi}{\circledast} D_n\,. \tag{24}$$

Now, if $\sigma_nf(x)$ stands for the sum $\displaystyle\frac{1}{n+1}\sum_{k=-n}^{n} S_kf(x)$, then using (23) and (22), the following equality holds :

$$\sigma_nf \quad = \quad f \overset{\varphi}{\circledast} F_n\,. \tag{25}$$

Remark 2 When the function $\varphi \equiv Id_{[0,2\pi]}$, then from (24) and (25), we get the well-known relations [6, chap.5] :

$$S_n f = f * D_n \qquad \text{and} \qquad \sigma_n f = f * F_n\,.$$

Recall from (22) the expression of K_n, the Fejér kernel of order n :

$$K_n(\theta) = \frac{1}{n+1}\sum_{k=0}^{n}\sum_{l=-k}^{k} e^{il\theta} = \frac{1}{n+1}\left\{\frac{\sin(\frac{n+1}{2}\theta)}{\sin(\frac{1}{2}\theta)}\right\}^2 .$$

The trigonometric polynomial K_n is of degree n and vanishes n times at the points $\frac{2\pi k}{n+1}$, for $k = 1,\dots,n$. Moreover, the polynomial K_n is differentiable everywhere and its derivative $K_n^{'}$ vanishes $n+1$ times at the points $\frac{2\pi k}{n+1}$, for $k = 0,\dots,n$. Observe also that for any constant τ, the constant term of the polynomial $K_n(\theta+\tau)$ is equal to 1 and using the following lemma 4, we get for every constant τ and for every integers m and n, m such that $m > n$:

$$\frac{1}{2\pi}\int_0^{2\pi} K_n(\theta+\tau)\,d\theta = \frac{1}{m}\sum_{k=0}^{m-1} K_n(\frac{2\pi\,k}{m}+\tau) = 1\,. \tag{26}$$

Finally, due to (26) and since $K_n > 0$ and

$$\forall\ \delta\ \in\ ]0,2\pi[\,,\ \forall\ \theta\ \in\ ]\delta,2\pi-\delta[\ \ :\ \ K_n(\theta) \le \frac{1}{n+1}\frac{1}{\sin^2(\delta/2)}\,,$$

the Fejér kernel is an approximate identity for the usual convolution $*$.

Lemma 4 *For any polynomial P of order n strictly less than m, and for any constant τ we have :*

$$\frac{1}{2\pi}\int_0^{2\pi} P(\theta+\tau)\,d\theta = \frac{1}{m}\sum_{k=0}^{m-1} P(\frac{2\pi\,k}{m}+\tau)\,. \tag{27}$$

Proof of lemma 4 :
Let $P(\theta) = \sum_{k=-n}^{n}\alpha_k\, e^{i\,k\theta}$ be a trigonometric polynomial of degree n and let τ be any constant. Observe first that the constant term α_0 is given by : $\alpha_0 = \frac{1}{2\pi}\int_0^{2\pi} P(\theta+\tau)d\theta$. Then, for every integer $m > n$, the following computation achieves the proof :

$$\sum_{k=0}^{m-1} P(\frac{2\pi\,k}{m}) = m\,\alpha_0 + \sum_{|r|=1}^{n}\alpha_r e^{ir\tau}\sum_{k=0}^{m-1}\left\{e^{i\frac{2\pi r}{m}}\right\}^k = m\,\alpha_0\,. \blacksquare$$

2.3 Jackson and de la Vallée Poussin polynomials

Recall from (12) the expression of the Jackson polynomial $\tilde{J}_{n,m}(x,f)$ of order n associated to the integrable function f at the m points $\{x_1^m,\ldots,x_m^m\}$, where m is any integer greater or equal to n, and $\varphi_m \in \Phi$ and satisfying (11) :

$$\tilde{J}_{n,m}(x) \;=\; \tilde{J}_{n,m}(x,f) \;=\; \frac{1}{m}\sum_{k=1}^{m} f(x_k^m)\,K_n(\varphi_m(x)-t_k^m)\,.$$

Remark 3 This modified definition extends the use of Jackson polynomial $\tilde{J}$ to the cases of non-equally spaced points and to a subset I of $[0,2\pi]$.

Remark 4 The Jackson polynomial $\tilde{J}$ associated to the function f with the non-equally spaced points $\{x_1^m,\ldots,x_m^m\}$ is related to the classical Jackson polynomial J associated with the function $f\circ\varphi_m^{-1}$ at the equally spaced points $\{t_1^m,\ldots,t_m^m\}$ by the relation :

$$\tilde{J}_{n,m}(x,f) \;=\; J_{n,m}(t,f\circ\varphi_m^{-1}), \qquad \text{where} \quad \varphi_m(x)=t \quad \text{for all} \quad t\in[0,2\pi]\,. \tag{28}$$

Note that, when $m=n+1$, $\tilde{J}_{n,n+1}(x,f)$ coincides with f at the points $\{x_1^{n+1},\ldots,x_{n+1}^{n+1}\}$ and that the latter do not need to be equally spaced as required in the the definition of a classical Jackson polynomial [33, chap.X.6]. On $I\setminus\{x_1^{n+1},\ldots,x_{n+1}^{n+1}\}$, the derivative $\tilde{J}'_{n,n+1}(x,f)$ of the Jackson polynomial is given by :

$$\tilde{J}'_{n,n+1}(x,f) \;=\; \frac{1}{n+1}\sum_{k=1}^{n+1} f(x_k^{n+1})\,K'_n(\varphi_{n+1}(x)-t_k^{n+1})\,\varphi'_{n+1}(x).$$

Since K'_n vanishes at the points $\frac{2\pi k}{n+1}$, for $k=0,\ldots,n$ and φ_{n+1} is a continuous piecewise linear function, the derivative $\tilde{J}'_{n,n+1}(x,f)$ can be extended continuously to the set $\{x_1^{n+1},\ldots,x_{n+1}^{n+1}\}$ with the zero value. The vanishing of the derivative $\tilde{J}'_{n,n+1}(x,f)$ at the points $\{x_1^{n+1},\ldots,x_{n+1}^{n+1}\}$ ensures a certain smoothness of the interpolating polynomial $\tilde{J}_{n,n+1}(x,f)$, but we will consider the more general polynomial $\tilde{J}_{n,m}(x,f)$.

Moreover

$$\frac{1}{m} \;=\; \frac{1}{2\pi}\,\frac{t_{k+1}^{(m)}-t_k^{(m)}}{x_{k+1}^{(m)}-x_k^{(m)}}\,(x_{k+1}^{(m)}-x_k^{(m)}) \;=\; \frac{1}{2\pi}\dot{\varphi}(x_k^{(m)})\,(x_{k+1}^{(m)}-x_k^{(m)})$$

then the Jackson polynomial $\tilde{J}_{n,m}(x,f)$ is also equal to :

$$\tilde{J}_{n,m}(x,f) \;=\; \frac{1}{2\pi}\sum_{k=1}^{(m)} f(x_k^m)\;K_n(\varphi_m(x)-\varphi_m(x_k^m))\;\dot{\varphi}(x_k^m)\;(x_{k+1}^m-x_k^m).$$

Hence, the sum $\tilde{J}_{n,m}$ can be regarded as a discretization of the continuous expression $f\overset{\varphi}{\circledast}K_n$ over the m points $\{x_1^m,\ldots,x_m^m\}$ and theorem 5 can be considered as the discrete version of proposition 4. The next proposition summarizes the main properties of Jackson and de la Vallée Poussin polynomials when approximating functions :

Proposition 5 *Let f be a bounded function on I. Then,*

1. *If $m > n$ and $M_1 \leq f \leq M_2$ then $M_1 \leq \tilde{J}_{n,m}(x, f) \leq M_2$.*
2. *Suppose now that $m > 2\,n$. The trigonometric polynomial $\tilde{V}_{n,m}(x, f)$ verifies :*

$$\|\tilde{V}_{n,m}(f)\|_{L^\infty(I)} \quad \leq \quad (3 + 2/n)\ \|f\|_{L^\infty(I)}\,. \tag{29}$$

In addition, if $m > 4\,n$, $\tilde{V}_{n,m}(p \circ \varphi_m) = p \circ \varphi_m$ for all trigonometric polynomials p of degree at most n.
Moreover,

$$\|\tilde{V}_{n,m}(f) - f\|_{L^\infty(I)} \quad \leq \quad (4 + 2/n)\ E_n(f, \varphi_m)\,,$$

where $E_n(f, \varphi_m) = \inf\Big\{\|f - p \circ \varphi_m\|_{L^\infty(I)}\ ;\ p\ :\ \text{ a polynomial of degree } n\Big\}$.

Proof of proposition 5 :
To prove the first part of the proposition 5, observe that for $m > n$ and by (26), if $M_1 \leq f \leq M_2$, then we have $M_1 \leq \tilde{J}_{n,m}(x, f) \leq M_2$. The second part of the proposition, which is [25, thm.3.1] in the case $\varphi_m \equiv Id_{[0,2\pi]}$, can be derived from the expression of $\tilde{J}_{n,m}(f)$. Using (27) and (26), we have $\|\tilde{J}_{n,m}(f)\|_{L^\infty(I)} \leq \|f\|_{L^\infty(I)}$, whence

$$\|\tilde{V}_{n,m}(f)\|_{L^\infty(I)} \quad \leq \quad (3 + 2/n)\|f\|_{L^\infty(I)}\,.$$

Now, if we replace in the definition (12) of the Jackson polynomial, the Fejér kernel and then the Dirichlet kernel by their equivalent expressions (22) and (21), we get :

$$\tilde{J}_{n,m}(x, f) \quad = \quad \frac{1}{m\,(n+1)} \sum_{k=0}^{n} \sum_{p=-k}^{k} e^{i\,p\,\varphi_m(x)} \sum_{r=1}^{m} f(x_r)\, e^{-i\,p\,\varphi_m(x_r)}.$$

In the special case where $f(\theta) \equiv e^{i\,\alpha\,\varphi_m(\theta)}$ with $|\alpha| \leq n$ and $t_1 = 0$, we have for $m > 4\,n$:

$$\tilde{J}_{n,m}(x, f) \quad = \quad \frac{1}{m\,(n+1)} \sum_{k=0}^{n} \sum_{p=-k}^{k} e^{i\,p\,\varphi_m(x)} \sum_{r=0}^{m-1} \Big\{e^{i\,\frac{2\pi}{m}\,(\alpha - p)}\Big\}^r \quad = \quad \frac{1}{(n+1)} \sum_{k=|\alpha|}^{n} e^{i\,\alpha\,\varphi_m(x)}$$

Hence,

$$\tilde{J}_{n,m}(x, f) = \frac{n + 1 - |\alpha|}{n+1}\ e^{i\,\alpha\,\varphi_m(x)} \qquad \text{and} \qquad \tilde{J}_{2\,n,m}(x, f) = \frac{2\,n + 1 - |\alpha|}{2\,n+1}\ e^{i\,\alpha\,\varphi_m(x)}\,.$$

Thus $\tilde{V}_{n,m}(e^{i\,\alpha\,\varphi_m}) = e^{i\,\alpha\,\varphi_m}$ and so by linearity

$$\forall\ p\ \in\ \mathcal{P}_n\ : \qquad \tilde{V}_{n,m}(p \circ \varphi_m) \quad = \quad p \circ \varphi_m\,.$$

Moreover,

$$\|\tilde{V}_{n,m}(f) - f\|_{L^\infty(I)} = \|\tilde{V}_{n,m}(f - p \circ \varphi_m) - (f - p \circ \varphi_m)\|_{L^\infty(I)} \leq (4 + 2/n)\ \|f - p \circ \varphi_m\|_{L^\infty(I)}. \blacksquare$$

2.4 Robust convergence of $\tilde{J}_{n,m}$ and $\tilde{V}_{n,m}$

Suppose now that $|f| \leq M$ on I and f is continuous at $x \in I$ and that $\|\eta^{(m)}\|_\infty < \epsilon$. Given $\zeta > 0$, there exists a $\kappa > 0$ depending on x and ζ such that $|f(y) - f(x)| \leq \zeta/2$ for all $y \in I$ such that $|y - x| \leq \kappa$. Also, there is an integer m_0 such that for all $m > m_0$, $x \in I_m = [x_1^m, x_m^m]$. By (26), we have :

$$\tilde{J}_{n,m}(x) - f(x) \quad = \quad \frac{1}{m} \sum_{k=1}^{(m)} (f(x_k^{(m)}) + \eta_k^m - f(x))\ K_n(\varphi_m(x) - t_k^{(m)}) \quad = \quad A + B + C,$$

where A, B and C stand respectively for :

$$A \quad = \quad \frac{1}{m} \sum_{\substack{1 \leq k \leq m \\ |x - x_k^{(m)}| \leq \kappa}} \left\{ (f(x_k^{(m)}) - f(x))\ K_n(\varphi_m(x) - t_k^{(m)}) \right\},$$

$$B \quad = \quad \frac{1}{m} \sum_{\substack{1 \leq k \leq m \\ |x - x_k^{(m)}| > \kappa}} \left\{ (f(x_k^{(m)}) - f(x))\ K_n(\varphi_m(x) - t_k^{(m)}) \right\}$$

and

$$C \quad = \quad \frac{1}{m} \sum_{1 \leq k \leq m} \left\{ \eta_k^{(m)}\ K_n(\varphi_m(x) - t_k^{(m)}) \right\}.$$

First, due to the definition of K_n and its positivity, we have by (26) :

$$|A| \quad \leq \quad \frac{\zeta}{2} \frac{1}{m} \sum_{k=1}^{m} K_n(\varphi_m(x) - t_k^{(m)}) \quad = \quad \zeta/2\,.$$

In a similar way, since $\|\eta^{(m)}\|_\infty < \epsilon$, we also have

$$|C| \quad \leq \quad \epsilon\,.$$

On the other hand, for the continuous function φ_m, define Δ_m by :

$$\Delta_m \quad = \quad \min_{\substack{y_1, y_2 \in I_m \\ |y_1 - y_2| > \kappa}} |\varphi_m(y_1) - \varphi_m(y_2)|\,.$$

Since the function φ_m is piecewise linear and monotonic, then

$$\Delta_m \quad \geq \quad \kappa \min_{1 \leq k \leq m-1} \frac{\varphi_m(x_{k+1}^{(m)}) - \varphi_m(x_k^{(m)})}{x_{k+1}^{(m)} - x_k^{(m)}} \quad = \quad \kappa\, \frac{1}{m\, \delta_m}\,.$$

In addition, given that $\delta_m = O(m^{-1})$, there exists an integer $m_1 > m_0$ such that :

$$\forall m > m_1, \qquad \Delta_m \quad > \quad \gamma$$

for some real $\gamma > 0$.

Because $(K_n)_{n>0}$ is a positive kernel, then for every $\rho > 0$

$$\lim_{n\to\infty}\left\{\sup_{|t|\geq\rho} K_n(t)\right\} \quad = \quad 0$$

and there exists an integer n_0 such that for all $n > n_0$ and for all $y \geq \Delta_m > \gamma$, we have $K_n(y) \leq \frac{\zeta}{4M}$. Hence for all $n > n_0$ and $m > m_1$,

$$|B| \quad \leq \quad 2\,M\,\frac{1}{m}\sum_{1\leq k\leq m} K_n(\varphi_m(x) - t_k^{(m)}) \quad \leq \quad \zeta/2.$$

Finally,

$$|\tilde{J}_{n,m}(x) - f(x)| \quad \leq \quad |A| + |B| + |C| \quad < \quad \zeta + \epsilon.$$

To achieve the proof of the first part of the theorem, remark that if f is continuous on $[\alpha, \beta]$, κ does not depend on x (and then neither γ nor m_1 do) and the above proof shows that $\tilde{J}_{n,m}$ converges uniformly to f in $[\alpha, \beta]$.

To prove the last part of the theorem, notice that :

$$f - \tilde{V}_{n,m} \quad = \quad \frac{2n+1}{n}(f - \tilde{J}_{2n,m}) - \frac{n+1}{n}(f - \tilde{J}_{n,m}),$$

and then we conclude using the first part.∎

2.5 Numerical evaluations of $\tilde{J}_{n,m}$ and $\tilde{V}_{n,m}$

For numerical purposes, we will derive the two matrix expressions (see (32) and (33)below) which will be used to compute easily the Jackson and de la Vallée Poussin polynomials.

In the expression of the Jackson polynomial, expand the Fejèr and then the Dirichlet kernels by (22) and (21), we get

$$\tilde{J}_{n,m}(x) \quad = \quad \frac{1}{(n+1)\,m} \quad \sum_{r=1}^{m} y_r^{(m)} \sum_{k=0}^{n}\sum_{p=-k}^{k} e^{i\,p\,(\varphi_m(x) - t_r^{(m)})}$$

Then, using the relation :

$$\sum_{k=0}^{n}\sum_{p=-k}^{k} a^p \quad = \quad (n+1) + \sum_{k=1}^{n}(n+1-k)(a^k + a^{-k})$$

we have :

$$\tilde{J}_{n,m}(x) \quad = \quad \sum_{k=-n}^{n} e^{i\,k\,\varphi_m(x)}\left(\frac{n+1-|k|}{(n+1)\,m} \quad \sum_{r=1}^{m} y_r^{(m)}\, e^{-i\,k\,t_r^{(m)}}\right).$$

In the same way, the de la Vallée Poussin polynomial is expanded to :

$$\tilde{V}_{n,m}(x) = \frac{1}{n\,m}\sum_{r=1}^{m} y_r^{(m)} \sum_{k=n+1}^{2n}\sum_{p=-k}^{k} e^{i\,p\,(\varphi_m(x)-t_r^{(m)})},$$

and making use of the relation :

$$\sum_{k=n+1}^{2n}\sum_{p=-k}^{k} a^p = \sum_{k=n+2}^{2\,n}(2n+1-k)(a^k+a^{-k}) + n\sum_{k=1}^{n+1}(a^k+a^{-k}) + n\,,$$

we have :

$$\tilde{V}_{n,m}(x) = \sum_{k=-2n}^{2n} e^{i\,k\,\varphi_m(x)}\left(\sum_{r=1}^{m} \upsilon_{n,m}^{k} \quad y_r^{(m)}\ e^{-i\,t_r^{(m)}\,k}\right),$$

where

$$\upsilon_{n,m}^{k} = \begin{cases} \frac{1}{m} & \text{si} \quad 0 \le |k| \le n+1 \\ \frac{2n+1-|k|}{nm} & \text{si} \quad n+2 \le |k| \le 2n\,. \end{cases}$$

These expressions set $\tilde{J}_{n,m}$ and $\tilde{V}_{n,m}$ as trigonometric polynomials with respect to the orthonormal system $\{\omega_n\}_{n\geq 0}$.

Let F be the column vector of length m whose r^{th} element is $f(x_r^m)$. Define the $(n+1)\times m$-matrix B^+ and the $n\times m$-matrix B^- respectively by :

$$B^+ = \left(\frac{n+1-(k-1)}{(n+1)\,m}\, e^{-i\,(k-1)\,t_r^{(m)}}\right)_{\substack{1\le k\le n+1\\ 1\le r\le m}} \quad \text{and} \quad B^- = \left(\frac{n+1-k}{(n+1)\,m}\, e^{i\,k\,t_r^{(m)}}\right)_{\substack{1\le k\le n\\ 1\le r\le m}}$$

Also define the $(2n+1)\times m$-matrix V^+ and the $2n\times m$-matrix V^- respectively by :

$$V^+ = \left(\upsilon_{n,m}^{k-1}\ e^{-i\,(k-1)t_r^{(m)}}\right)_{\substack{1\le k\le 2n+1\\ 1\le r\le m}} \quad \text{and} \quad V^- = \left(\upsilon_{n,m}^{k}\ e^{i\,k\,t_r^{(m)}}\right)_{\substack{1\le k\le 2n\\ 1\le r\le m}}$$

If $A^+(x)$ and $A^-(x)$ stand for the row vectors of functions, of lengths respectively $n+1$ and n given respectively by :

$$A^+(x) = \left(1,\dots,e^{i\,n\,\varphi_m(x)}\right) \qquad \text{and} \qquad A^-(x) = \left(e^{-i\,\varphi_m(x)},\dots,e^{-i\,n\,\varphi_m(x)}\right)$$

and if $W^+(x)$ and $W^-(x)$ stand for the row vectors of functions, of lengths respectively $2n+1$ and $2n$ given by :

$$W^+(x) = \left(1,\dots,e^{i\,2n\,\varphi_m(x)}\right) \quad \text{and} \quad W^-(x) = \left(e^{-i\,\varphi_m(x)},\dots,e^{-i\,2n\,\varphi_m(x)}\right)$$

then the polynomials $\tilde{J}_{n,m}(x)$ and $\tilde{V}_{n,m}(x)$ can be regarded as matrix products :

$$\tilde{J}_{n,m}(x) = \left(A^+(x)\ B^+ + A^-(x)\ B^-\right)\ F \tag{30}$$

and

$$\tilde{V}_{n,m}(x) \quad = \quad \Big(W^+(x)\ V^+ + W^-(x)\ V^-\Big)\ F \tag{31}$$

Hence, evaluating $\tilde{J}_{n,m}$ and $\tilde{V}_{n,m}$ at a set of points $\xi_1, \dots, \xi_p$, can be written as matrix formulae which is accurate for numerical computation mainly when a massively parallelized computer is used [30] :

$$\begin{pmatrix} \tilde{J}_{n,m}(\xi_1) \\ \vdots \\ \tilde{J}_{n,m}(\xi_p) \end{pmatrix} \quad = \quad \begin{pmatrix} A^+(\xi_1) \\ \vdots \\ A^+(\xi_p) \end{pmatrix} B^+\,F + \begin{pmatrix} A^-(\xi_1) \\ \vdots \\ A^-(\xi_p) \end{pmatrix} B^-\,F \tag{32}$$

and

$$\begin{pmatrix} \tilde{V}_{n,m}(\xi_1) \\ \vdots \\ \tilde{V}_{n,m}(\xi_p) \end{pmatrix} \quad = \quad \begin{pmatrix} W^+(\xi_1) \\ \vdots \\ W^+(\xi_p) \end{pmatrix} V^+\,F + \begin{pmatrix} W^-(\xi_1) \\ \vdots \\ W^-(\xi_p) \end{pmatrix} V^-\,F \tag{33}$$

Notice in expression (30) that the column vectors B^+F and B^-F contain the coefficients of the Jackson polynomial $\tilde{J}_{n,m}(x)$ in the basis $\{1, \dots, e^{i\,n\,\varphi_m(x)}\}$ and $\{e^{-i\,\varphi_m(x)}, \dots, e^{-i\,n\,\varphi_m(x)}\}$ while the expression (31) gives with V^+F and V^-F, the coefficients of $\tilde{V}_{n,m}(x)$ in the basis $\{1, \dots, e^{i\,2n\,\varphi_m(x)}\}$ and $\{e^{-i\,\varphi_m(x)}, \dots, e^{-i\,2n\,\varphi_m(x)}\}$.

When the points $\{x_1^m, \dots, x_m^m\}$ are equally spaced over $[0, 2\pi]$ that is when $\varphi_m \equiv Id_{[0,2\pi]}$ then the computation of these coefficients gives directly the expression of the Jackson and de la Vallée Poussin polynomials in the standard basis $\{1, e^{i\,x}\, e^{2i\,x}, \dots\}$ and $\{e^{-i\,x}, e^{-2\,i\,x}, \dots\}$.

Using the attenuation factors ([12, chap13]), these coefficients can be related to the Fourier coefficients of the function f at the m sampling values $\{f(t_1^m), \dots, f(t_m^m)\}$.

However, when the sampling points $\{x_1^m, \dots, x_m^m\}$ are not equally spaced over $[0, 2\pi]$, an auxiliary step to sample the approximating polynomials will be needed before aliasing the approximation operator.

2.6 Attenuation factors

Define the discrete Fourier coefficient $\hat{a}_p$ of order p associated to the values $\{y_1^{(m)}, \dots, y_m^{(m)}\}$ at the m equally spaced points $\{t_1^{(m)}, \dots, t_m^{(m)}\}$ by :

$$\forall\ p\ \in\ \mathbb{Z}\ :\quad \hat{a}_p \quad = \quad \frac{1}{m}\ \sum_{r=1}^{m} y_r^{(m)}\ e^{-i\,p\,t_r^{(m)}}\ . \tag{34}$$

Let $J_{n,m}(x)$ and $V_{n,m}(x)$ be the Jackson and de la Vallée Poussin polynomials associated to $\{y_1^{(m)}, \ldots, y_m^{(m)}\}$ at $\{t_1^{(m)}, \ldots, t_m^{(m)}\}$:

$$J_{n,m}(x) = \sum_{k=-n}^{n} e^{i\,k\,x}\, \frac{n+1-|k|}{(n+1)\,m} \sum_{r=1}^{m} y_r^{(m)}\, e^{-i\,k\,t_r^{(m)}} = \sum_{k=-n}^{n} \jmath_n^k\ \hat{a}_k\ e^{i\,k\,x}$$

and

$$V_{n,m}(x) = \sum_{k=-2n}^{2n} e^{i\,k\,x}\ \upsilon_n^k\ \frac{1}{m} \sum_{r=1}^{m} y_r^{(m)}\ e^{-i\,t_r^{(m)}\,k} = \sum_{k=-2n}^{2n} e^{i\,k\,x}\ \upsilon_n^k\ \hat{a}_k$$

with

$$\jmath_n^k = \frac{n+1-|k|}{n+1} \text{ for } -n \le k \le n \qquad \text{and} \qquad \upsilon_n^k = \begin{cases} 1 & 0 \le |k| \le n+1 \\ \frac{2n+1-|k|}{n} & n+2 \le |k| \le 2n\,. \end{cases}$$

These coefficients $\jmath_n^k$ and υ_n^k can be considered as attenuation factors [1] since they allow to compute the discrete Fourier coefficient $\hat{a}_k$ of order k, for $-n \le k \le n$ (respectively for $-2\,n \le k \le 2\,n$), from the coefficient $\jmath_n^k\,\hat{a}_k$ (respectively $\upsilon_n^k\,\hat{a}_k$) of the polynomial $J_{n,m}$ (respectively $V_{n,m}$).

2.7 Back to equally spaced points

When the values $\{y_1^{(m)}, \ldots, y_m^{(m)}\}$ are given at non-equally spaced points $\{x_1^{(m)}, \ldots, x_m^{(m)}\}$ over $[0, 2\pi]$, Jackson and de la Vallée Poussin polynomials are defined with respect to the basis $\mathcal{B}_{\varphi_m} = \{e^{i\,n\,\varphi_m}\}_{n\in\mathbb{Z}}$ where $\varphi_m \in \Phi$. To use these polynomials in identification (in particular to build Hankel or Toeplitz matrices as detailed in the next section), we will need their expressions in the standard basis $\{e^{i\,n\,t}\}_{n\in\mathbb{Z}}$. One simple way to come back to the standard basis is to construct the classical Jackson and de la Vallée Poussin polynomials associated respectively to $\tilde{J}_{n,m}$ and $\tilde{V}_{n,m}$ at the equally-spaced points $\{t_1, ..., t_m\}$. Let $\tilde{\tilde{J}}_{n,m}$ and $\tilde{\tilde{V}}_{n,m}$ be these polynomials. Recall the expressions of $\tilde{J}_{n,m}$:

$$\tilde{J}_{n,m}(x) = \sum_{k=-n}^{n} \left(\frac{n+1-|k|}{(n+1)\,m} \sum_{r=1}^{m} y_r^{(m)}\ e^{-i\,t_r^{(m)}\,k} \right) e^{i\,\varphi_m(x)\,k}$$

and that of $\tilde{V}_{n,m}$:

$$\tilde{V}_{n,m}(x) = \sum_{|k|=0}^{2n} \sum_{r=1}^{m} \upsilon_{n,m}^k\, y_r^{(m)}\ e^{i\,(\varphi_m(x) - t_r^{(m)})\,k} \quad \text{with} \quad \upsilon_{n,m}^k = \begin{cases} \frac{1}{m} & 0 \le |k| \le n+1 \\ \frac{2n+1-|k|}{nm} & n+2 \le |k| \le 2n\,. \end{cases}$$

[1] see [12, §13.2-III] for the definition of the attenuation factors and their use in linear and cubic interpolating functions (splines).

The trigonometric polynomials $\tilde{\tilde{J}}_{n,m}$ and $\tilde{\tilde{V}}_{n,m}$ defined respectively by :

$$\tilde{\tilde{J}}_{n,m}(x) \quad = \quad \sum_{k=-n}^{n} \left(\frac{n+1-|k|}{(n+1)\,m} \sum_{r=1}^{m} \tilde{J}_{n,m}(t_r^{(m)}) e^{-i\,k\,t_r^{(m)}} \right) e^{i\,x\,k} \tag{35}$$

$$\tilde{\tilde{V}}_{n,m}(x) \quad = \quad \sum_{p=-2n}^{2n} \left(v_{n,m}^{p} \sum_{\rho=1}^{m} \tilde{V}_{n,m}(t_\rho^{(m)}) e^{-i\,p\,t_\rho^{(m)}} \right) e^{i\,p\,x} \tag{36}$$

are then expressed in the standard basis. The next theorem states the convergence properties of these polynomials :

Theorem 6 *Let f be a bounded function on I. Let $\tilde{\tilde{J}}_{n,m}$ and $\tilde{\tilde{V}}_{n,m}$ be defined as above by the expressions* (35) *and* (36)*.*
If $\delta_m = O(m^{-1})$, then the trigonometric polynomials $\tilde{\tilde{J}}_{n,m}$ (respectively $\tilde{\tilde{V}}_{n,m}$) converges robustly to $f(x)$ at every point $x \in I$ of continuity of f as the noise level goes to zero, and as n tends to infinity, m remaining always greater than n (respectively $2n$). The convergence is uniform in every closed interval $[\alpha, \beta] \subset I$ of continuity of f.

Proof of the theorem 6 :
First, notice that :

$$f - \tilde{\tilde{V}}_{n,m} \quad = \quad \frac{2n+1}{n}\,(f - \tilde{\tilde{J}}_{2n,m}) - \frac{n+1}{n}\,(f - \tilde{\tilde{J}}_{n,m})\,.$$

Hence the proof is reduced to the part with the Jackson polynomial.
Let $J_{n,m}$ be the classical Jackson polynomial computed at equally-spaced points. Since

$$f - \tilde{\tilde{J}}_{n,m} \quad = \quad (f - J_{n,m}) \; + \; \left(J_{n,m} - \tilde{\tilde{J}}_{n,m} \right)\,,$$

the proof of theorem 6 is reduced to establishing the uniform convergence of $f - J_{n,m}$ and $J_{n,m} - \tilde{\tilde{J}}_{n,m}$ to zero as n tends to infinity and m remaining greater than n. The uniform convergence to zero of $f - J_{n,m}$ results obviously from the uniform convergence of $J_{n,m}$ to f while that of $J_{n,m} - \tilde{\tilde{J}}_{n,m}$ is due to the fact that Jackson polynomial does not increase the norm of the associated function, namely $f - \tilde{J}_{n,m}$ in this case, and so converges uniformly to zero. The theorem is thus proved.∎

Remark 5 This theorem allows us to derive a two-pass interpolating or approximating algorithm using the same procedure. This two-pass scheme can be useful when points of measurement are not equally spaced or simply inaccurately measured, these points being then considered as not “enough” equally spaced since this condition is necessary for the classical Jackson and de la Vallée Poussin polynomials to be well behaved. this is not surprising if we have in mind the shape of the graph of K_n.

Furthermore, as will be detailed in the next section 3, we need to compute, for a given function u, the Fourier coefficients associated to its restrictions $\chi_I\, u$ on I or $\chi_{\mathbb{T}\setminus I}\, u$ on $\mathbb{T}\setminus I$. The uniform convergence to u of Jackson and de la Vallée Poussin polynomials associated to u permits us to go from the Fourier coefficients associated to u to those associated to $\chi_I\, u$ or $\chi_{\mathbb{T}\setminus I}\, u$.

2.8 Restriction of functions

Suppose that the infinite series $\sum_{n\in\mathbb{Z}} \alpha_n\, e^{i\,n\,\theta}$ converges uniformly to the function $u(\theta)$ on the compact $I \subset [0, 2\pi]$. Then the Fourier series associated to the function $\chi_I\, u(\theta)$ is given by $\sum_{n\in\mathbb{Z}} \beta_n\, e^{i\,n\,\theta}$ where the coefficient β_k is defined for every integer $k \in \mathbb{Z}$ by :

$$\beta_k \;=\; \frac{1}{2\pi}\int_0^{2\pi} \chi_I\, u(\theta)\, e^{-i\,k\,\theta}\, d\theta \;=\; \sum_{n\in\mathbb{Z}} \alpha_n\, r_n^k \quad \text{with} \quad r_n^k \;=\; \frac{1}{2\pi}\int_I e^{i\,(n-k)\,\theta}\, d\theta$$

or equivalently :

$$\begin{pmatrix} \vdots \\ \beta_k \\ \vdots \end{pmatrix} = \mathcal{R} \begin{pmatrix} \vdots \\ \alpha_n \\ \vdots \end{pmatrix} \quad \text{where} \quad \mathcal{R} = \begin{pmatrix} & \vdots & \\ \cdots & r_n^k & \cdots \\ & \vdots & \end{pmatrix}$$

In the same way, if the infinite series $\sum_{n\in\mathbb{Z}} \tilde{\alpha}_n\, e^{i\,n\,\theta}$ converges uniformly to the function $\tilde{u}(\theta)$ on $\mathbb{T}\setminus I$, then the Fourier series associated to the function $\chi_{\mathbb{T}\setminus I}\, \tilde{u}(\theta)$ is given by $\sum_{n\in\mathbb{Z}} \tilde{\beta}_n\, e^{i\,n\,\theta}$ where, for every integer $k \in \mathbb{Z}$, the coefficient $\tilde{\beta}_k$ is defined by :

$$\tilde{\beta}_k \;=\; \frac{1}{2\pi}\int_0^{2\pi} \chi_{\mathbb{T}\setminus I}\, \tilde{u}(\theta)\, e^{-i\,k\,\theta}\, d\theta \;=\; \sum_{n\in\mathbb{Z}} \tilde{\alpha}_n\, \tilde{r}_n^k \qquad \text{where} \qquad \tilde{r}_n^k \;=\; \frac{1}{2\pi}\int_{\mathbb{T}\setminus I} e^{i\,(n-k)\,\theta}\, d\theta$$

or equivalently :

$$\begin{pmatrix} \vdots \\ \tilde{\beta}_k \\ \vdots \end{pmatrix} \;=\; \tilde{\mathcal{R}} \begin{pmatrix} \vdots \\ \tilde{\alpha}_n \\ \vdots \end{pmatrix} \qquad \text{where} \qquad \tilde{\mathcal{R}} \;=\; \begin{pmatrix} & \vdots & \\ \cdots & \tilde{r}_n^k & \cdots \\ & \vdots & \end{pmatrix}$$

Remark that, since :

$$\forall n \in \mathbb{Z}\,,\ \forall k \in \mathbb{Z}\ :\quad r_n^k + \tilde{r}_n^k \;=\; \delta_n^k\,,$$

where δ_n^k is the Kronecker's symbol, then the sum $\mathcal{R} + \tilde{\mathcal{R}}$ is equal to the identity matrix.

What's more, if the interval I is symmetric, that is if,

$$I \;=\; \{e^{i\theta}\ :\ -a \le \theta \le a\} \qquad \text{or} \qquad I \;=\; \{e^{i\theta}\ :\ a \le \theta \le 2\pi - a\}\,,$$

then r_n^k and $\tilde{r}_n^k$ are given by :

$$\frac{1}{2\pi}\int_{-a}^{a} e^{i\,p\,\theta}\, d\theta \;=\; \frac{a}{\pi}\,\mathrm{sinc}(p\,a) \qquad \text{and} \qquad \frac{1}{2\pi}\int_{a}^{2\pi-a} e^{i\,p\,\theta}\, d\theta \;=\; \frac{\pi - a}{\pi}\,\mathrm{sinc}\,(p\ (\pi - a))\ .$$

Moreover $\mathcal{R}$ and $\tilde{\mathcal{R}}$ are symmetric Toeplitz matrices.

3 Application to identification

As an application of robust interpolation and approximation in identification, we consider now the robust H^∞ identification scheme as described in [4], and the analytic H^2 completion problem studied in [3] for the classical example $f(z) = 3(z^2+1)/(z^2+2z+5)$ (see [24]).

First, we fix a few notations. Let $L^\infty(\mathbb{T})$ and H^∞ stand respectively for the space of essentially bounded Lebesgue measurable complex-valued functions on the unit circle $\mathbb{T}$ and the space of bounded analytic functions in the open unit disc $\mathbb{D}$. Let $L^2(\mathbb{T})$ be the space of square-modulus-integrable functions on $\mathbb{T}$ and let H^2 be the Hardy space on the unit disc $\mathbb{D}$, subspace of $L^2(\mathbb{T})$.

3.1 Robust H^∞ identification

Consider the problem of robust H^∞ identification of functions in the disc algebra that consist in recovering a scalar BIBO-stable transfer function from a series of experimental pointwise measurements at a set of frequencies lying within a bandwidth. In their article [4], the authors propose an algorithm to build a sequence of maps from data to BIBO-stable models which uniformly converges to the seeked transfer function on the bandwidth when the number of measurements goes to infinity and the noise level goes to zero, while asymptotically meeting some gauge constraint outside.

As in the classical case where data are available at frequencies distributed on the whole circle [15, 8, 25, 26], a two-stage algorithm has been found useful to approach this robust band-limited H^∞ identification problem. The first stage consists in determining a robustly convergent polynomial interpolation (or simply approximation) of the given data while the second stage can be reduced to a constraint (nonlinear) Nehari extension to obtain the best approximation to the polynomial just being computed (see [4] for the study of the robust convergence of this algorithm). The robust convergence of the interpolation step is necessary to ensure that this property globally holds for the map from the data set to the identified model above provide sufficient condition to build such a procedure.

In realistic identification applications, the transfer function is only given by its pointwise values at a set of non-equally spaced points. The good-behavior of the Jackson and de la Vallée Poussin polynomials to approximate functions from their pointwise values and mainly the extension of their use to the case of non-equally spaced points, justify the use of $\tilde{J}$ and $\tilde{V}$ (or $\tilde{\tilde{J}}$ and $\tilde{\tilde{V}}$) in the first step of such algorithms (see §2).

For the second stage of these algorithms, consider here the classical extremal problem (see [31] and the references therein for a complete study of bounded extremal problems). Let f be any function in $L^\infty(\mathbb{T})$ (f is generally the result of the first interpolating step). We seek for $g_0 \in H^\infty$, such that :

$$\|f - g_0\|_{L^\infty(\mathbb{T})} = \min_{g \in H^\infty} \|f - g\|_{L^\infty(\mathbb{T})}$$

This is a Nehari problem and the Hankel operator Γ_f will be used to give a best approximation g_0 in terms of singular values and vectors as stated by a theorem of Adamyan, Arov and Krein [1] (basic properties of Hankel operators can be found in [23, Chap.3]). The Hankel operator Γ_f with symbol f, $\Gamma_f : H^2 \to L^2 \ominus H^2$ is defined by : $\Gamma_f(x) = P_-(f\,x)$ where P_- is the orthogonal projection operator from L^2 on $L^2 \ominus H^2$. The well-known Nehari theorem giving the solution of the Nehari problem states that if the Hankel operator Γ_f attains its norm at a non-zero vector x (i.e. $\|\Gamma_f x\| = \|\Gamma_f\|\,\|x\|$), then there is a unique best approximation g_0 to f given by :

$$g_0 \quad = \quad f - \frac{\Gamma_f x}{x}. \tag{37}$$

Such a non-zero vector x is called a maximizing vector and it exists at least when the Hankel operator Γ_f is compact that is when $f \in H^\infty + C(\mathbb{T})$ by Hartman's theorem. More details about the solution of Nehari's problem can be found in [32, Chap.15].

A standard computational method to get the solution of the best H^∞ approximation to an $L^\infty(\mathbb{T})$ function is to solve numerically a Nehari problem with the aid of Hankel singular values and vectors. If $\sum_{n=-\infty}^{\infty} a_n z^n$ is the Fourier series associated to the function f, then the matrix H_f of the Hankel operator Γ_f with respect to the standard bases $\{1, z, z^2, ...\}$ of H^2 and $\{z^{-1}, z^{-2}, ...\}$ of $L^2 \ominus H^2$ is the Hankel matrix :

$$H_f \quad = \quad \begin{pmatrix} a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_{-2} & a_{-3} & a_{-4} & \cdots \\ a_{-3} & a_{-4} & a_{-5} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

and its singular values are the square roots of the eigenvalues of $H_f^* H_f$.

The effective computation of g_0 in (37) imposes to handle only with finite dimensional matrices so that truncations of accurate order of the matrix H_f are operated. These truncated matrices are constructed from a finite set of Fourier coefficients associated to the interpolant or the approximant computed during the first step. Note that the continuity in the Wiener algebra $\mathcal{W}$ of the approximation operator defined by the Nehari problem (see [33, (6.5)-p.22]) and the convergence in $\mathcal{W}$ of Jackson and de la Vallée polynomials, considered as Césaro sums (see [31]), motivate the use of $\tilde{J}$ and $\tilde{V}$ (or $\tilde{\tilde{J}}$ and $\tilde{\tilde{V}}$) in the first step of the algorithm. Finally, a Singular Value Decomposition, for example, of a truncated Hankel matrix H of H_f can then also be used to numerically compute the solution g_0.

3.2 Bounded H^2 completion

This problem also arises in band-limited frequency domain identification of stable linear dynamical causal systems when one wants to get an H^2 approximation of the transfer function. In harmonic H^2 identification of a transfer function f, when the experiments are only available in some range of frequencies I, subset of the unit circle $\mathbb{T}$ and corresponding to the bandwidth of the system, one is interested to find a bounded H^2 extension g to f on the whole circle, that is a function g which coincides with f in the bandwidth and which is as close as possible to the Hardy space H^2 while its norm remains bounded. This constraint on the norm of g is justified by the density of the Hardy space H^2 in the Lebesgue space L^2 restricted to a strict subset of the unit circle (see e.g. [2, thm.1]).

In realistic identification applications, the transfer function is only given by its pointwise values at a set of non-equally spaced points on a subset I of the unit circle. The above problem of bounded extension (or completion) which is detailed in [3], can then be solved by mean of a two-stage algorithm. The first stage consists of determining a robustly convergent polynomial interpolation (or simply approximation) of the given data on I while the second stage can be reduced to a constraint least-square approximation in $L^2(I)$ to obtain the best analytic H^2 approximation to the polynomial just being computed.

The generalization to subsets of the unit circle of the standard properties of Jackson and de la Vallée Poussin polynomials justifies their use in the first stage of such algorithms (see §2) .

For the second stage of such algorithms, let I be a subset of $\mathbb{T}$ such that both I and $\mathbb{T}\setminus I$ are of positive Lebesgue measures, and let C be a positive constant. Let the function f be in $L^2(I)$ (f is generally the result of the first interpolating step) and let B_C denote the convex subset of $L^2(\mathbb{T}\setminus I)$ defined by :

$$B_C \quad = \quad \{h \in L^2(\mathbb{T}\setminus I) : \|h\|_{L^2(\mathbb{T}\setminus I)} \leq C\}$$

We seek $h_0 \in B_C$ such that :

$$\mathrm{dist}(f \vee h_0, H^2) \quad = \quad \min_{h\in B_C} \mathrm{dist}(f\vee h, H^2)$$

or equivalently such that :

$$\|P_{\overline{H}_0^2}(f\vee h_0)\|_{L^2(\mathbb{T})} \quad = \quad \min_{h\in B_C} \|P_{\overline{H}_0^2}(f\vee h)\|_{L^2(\mathbb{T})}$$

Let $\Phi\ : H^2 \to H^2$ be the Toeplitz operator with symbol $\chi_{\mathbb{T}\setminus I}$ defined by :

$$\Phi(h) \quad = \quad P_{H^2}\left(\chi_{\mathbb{T}\setminus I}\, h\right).$$

Then the unique solution h_0 is given by [3] :

$$h_0 \quad = \quad -\,\lambda\,(I+\lambda\,\Phi)^{-1}\,P_{H^2}(\chi_I\, f)$$

for some $\lambda > -1$ such that $\|h_0\|_{L^2(\mathbb{T}\setminus I)} = C$.

To compute the solution h_0 of the above formulated completion problem, we have to solve, iteratively with respect to λ, the linear and infinite dimensional system :

$$(I + \lambda\, \Phi)\, h_0 \quad = \quad -\, \lambda\, P_{H^2}(\chi_I\, f) \tag{38}$$

Hence, if $\sum_{k\in\mathbb{N}} b_k\, e^{i\,k\,\theta}$ and $\sum_{n\in\mathbb{N}} a_n\, e^{i\,n\,\theta}$ are the Fourier series associated to the H^2 functions respectively h_0 and $P_{H^2}(\chi_I\, f)$, then the equation (38) can be stated under the following matrix equivalent form :

$$(I + \lambda\, T) \begin{pmatrix} b_0 \\ \vdots \\ b_k \\ \vdots \end{pmatrix} \quad = \quad -\, \lambda \begin{pmatrix} a_0 \\ \vdots \\ a_n \\ \vdots \end{pmatrix} \tag{39}$$

where T is the matrix associated to the Toeplitz operator Φ. Recall that the Fourier coefficients associated to an $L^2(\mathbb{T})$ function converge to that function in the L^2-norm (see [29, 4.26]. Hence $\sum_{k\in\mathbb{N}} b_k\, e^{i\,k\,\theta}$ converges to g_0 in the L^2-norm.

The uniform convergence of Jackson and de la Vallée Poussin polynomials for continuous function allows, as detailed in §2.8, to get the coefficients $a_0, a_1, \ldots$ from those of the Jackson or de la Vallée Poussin polynomials associated to any continuous function whose restriction on I is f.

The numerical computation of h_0 in (39) imposes to manipulate only finite dimensional matrices so that truncations of accurate order of the matrix T is used. The resolution of the so truncated, linear and finite dimension system, can then be solved using for example Levinson or Durbin algorithms [7]. Finally due to boundedness of the operator $(I + \lambda\, T)^{-1}$ for every $\lambda > -1$ (see [31, Chap.3]), the solution of the truncated system converges to h_0 in the L^2-norm.

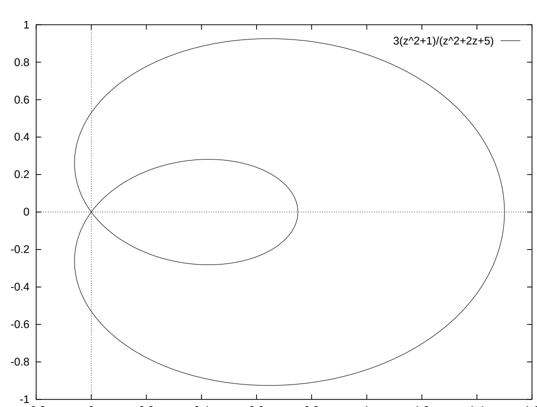
3.3 Numerical examples

We consider an example discussed in [24] namely $f(z) = 3(z^2+1)/(z^2+2z+5)$ given by its values on a set of non-equally spaced points on symmetric arcs

$$I = [e^i, e^{i(2\pi-1)}] \quad \text{in figure 1}$$

and

$$I = [e^{i3/2}, e^{i(2\pi-3/2)}] \quad \text{in figure 2}$$



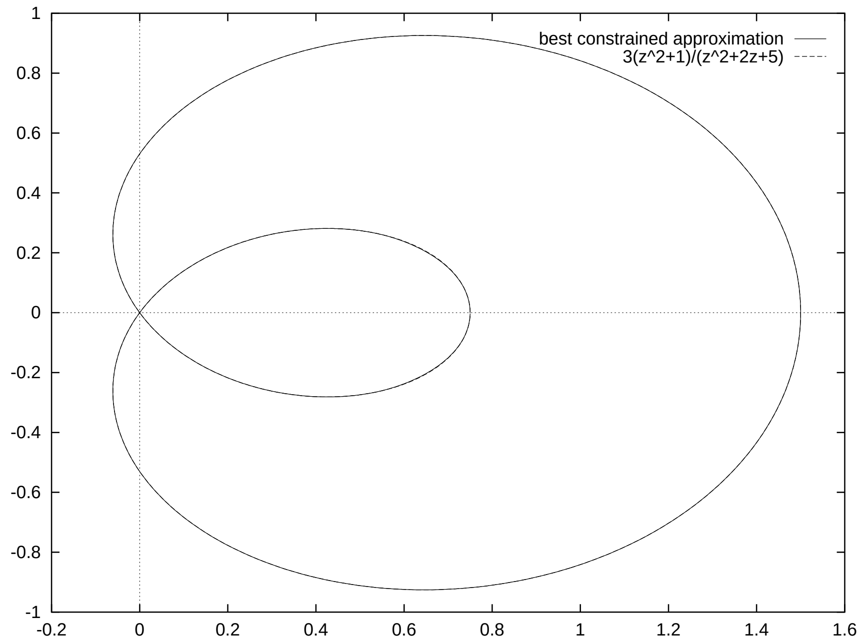


Figure 1: best analytic bounded extension recovering the function in H^∞

An extremal bounded problem is solved in [4] for this problem. In the ideal case when the constraint on the norm of the approximant is precisely the norm of f on $\mathbb{T} \setminus I$, then the approach taken in [4] allow us to recover the missing information on $\mathbb{T} \setminus I$ and to get the best approximation of f on $\mathbb{T} \setminus I$ as shown in the figure 1.

For the same example, a bounded H^2 completion problem is considered in [3]. In the best situation when the norm imposed to the extension is equal to that of f, then the function f is entirely recovered from its partial pointwise values as shown in the figure 2.

A more realistic application (hyper-frequencies filter for the French CNES) is also considered in [4]. It consists in identifying an hyper-frequencies filter given by noisy and partial measurements on $I = [e^{i\pi/2}, e^{i(3\pi/2)}]$. We refer the reader to the articles [4] and [3] for details on the two identification problems considered here.

4 Conclusion

The robustness and uniform convergence of the interpolation or the approximation step are shown to be critical in classical two stage identification algorithms. The two methods presented in this paper with an illustrated example, are simple and find their motivation in robust H^∞ identification and analytic H^2 completion from non-equally spaced points on a subset of the unit circle. We refer the reader to [26, 27, 34] and the references therein for other approximation methods from non-equally spaced points.

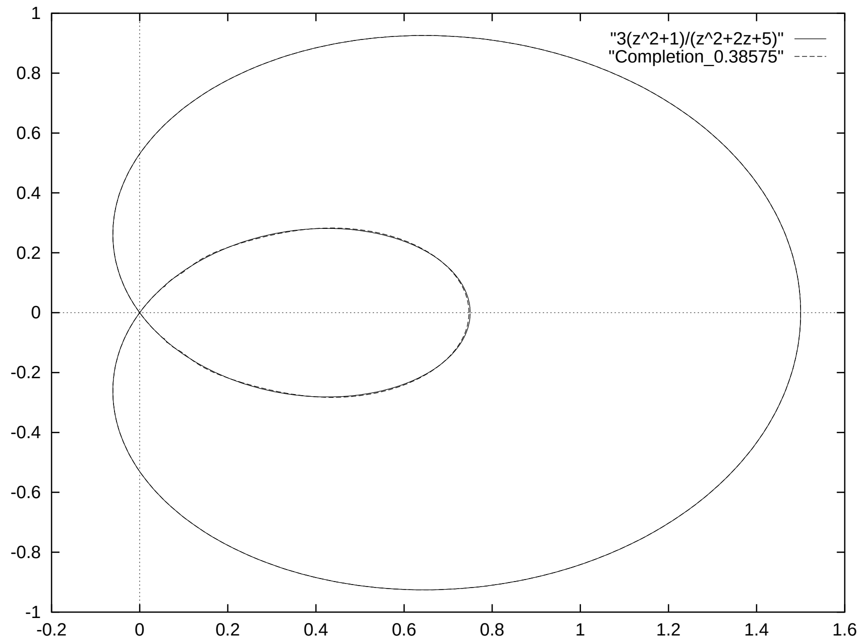


Figure 2: analytic H^2 completion with $\lambda = -1 + 0.25e - 11$.

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