Some Properties of Clothoids

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Abstract: In the present paper we prove some geometric properties of clothoids. A clothoid (or Cornu's spiral) is a plane curve whose curvature is a linear function of its length. In suitably chosen coordinates it is given by Frenel's integrals $x(t) = \int_0^t \cos(B\tau^2/2)d\tau$, $y(t) = \int_0^t \sin(B\tau^2/2)d\tau$.

The clothoid appears in the problem to find (a) shortest plane curve(s) joining two given points with given at them tangent angles and curvatures and with a bounded derivative of the curvature. It turns out that a regular (i.e. of the class C^3) point of a shortest curve is either a piece of a clothoid or a line segment.

Some of the properties exposed in the paper are used in [5] to show that when the distance between the initial and final points is sufficiently long, then, in general, shortest paths have infinitely many points of discontinuity of the curvature's derivative. Thus we are led to the problem to describe a procedure of constructing paths (called "suboptimal") longer no more than a fixed constant than the shortest one(s) and which can be constructed explicitly. They are constructed from pieces of clothoids and a line segment (see [3], [4]). The other properties of clothoids exposed in the paper are used in [3], [4] to prove the suboptimality of the constructed paths.

Key-words: car-like robot, (sub)optimal path, clothoid, Maximum Principle of Pontryagin

(Résumé : tsvp)

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Quelques propriétés des clothoïdes

Résumé : Dans cet article on prouve certaines propriétés des clothoïdes. Une clothoïde (ou spirale de Cornu) est une courbe plane dont la courbure est une fonction linéaire de sa longueur. Dans un système de coordonnées commode une clothoïde est donnée par les intégrales de Fresnel: $x(t) = \int_0^t \cos(B\tau^2/2)d\tau$, $y(t) = \int_0^t \sin(B\tau^2/2)d\tau$.

La clothoïde apparaît dans le problème de trouver les trajectoires les plus courtes joignant deux points dans $\mathbf{R}^2$, la dérivée de la courbure étant bornée par une constante B, les tangentes et les courbures du départ et de l'arrivée étant donnés, la tangente et la courbure de la trajectoire étant continues. Il se trouve qu'en tout point régulier (i.e. de la classe C^3) la trajectoire optimale est un arc de clothoïde ou un segment de droite.

Certaines propriétés démontrées dans l'article sont appliquées dans [5] pour montrer que lorsque la distance entre les points de départ et d'arrivée est assez grande, les trajectoires optimales ont, en général, une infinité de points de discontinuité de la dérivée de la courbure. Ainsi on est amené au problème de décrire une procédure pour construire des courbes "sous-optimales" qui sont plus longues que les courbes optimales d'au plus une constante et qui peuvent être construites explicitement. On les construit de morceaux de clothoïde et d'un segment de droite (voir [3], [4]). Les autres propriétés montrées ici sont utilisées pour démontrer leur sousoptimalité, voir [3], [4].

Mots-clé : robot mobile, chemin (sous)optimal, clothoïde, principe du maximum de Pontryagine

1 Introduction.

In the present paper we study some geometric properties of clothoids. A clothoid (or Cornu's spiral) is a smooth curve whose curvature is a linear function of its length. In suitably chosen coordinates it can be represented by the formulas (4). Here t is the natural parameter and the curvature at the point $(x(t), y(t))$ equals $\pm Bt$. A clothoid is a curve symmetric with respect to its point of zero curvature (see an example of a half-clothoid on Figure 1).

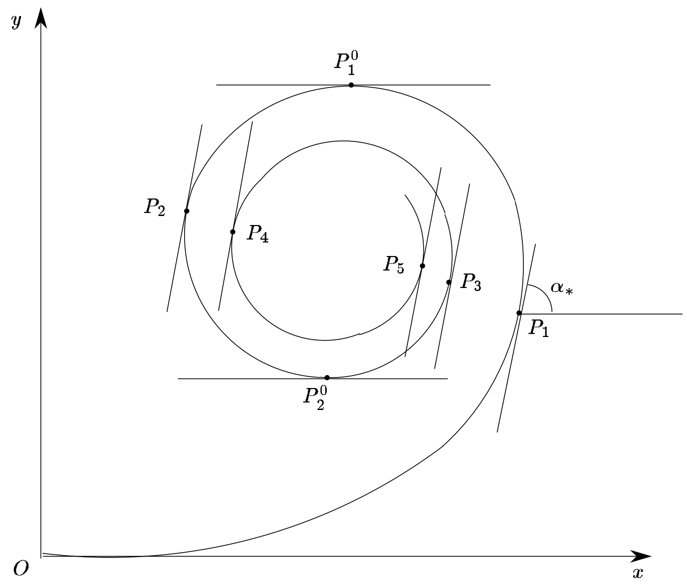

Figure 1

The interest in the study comes from the fact that the clothoid appears in the optimal control problem to find the shortest path(s) joining two given points on the plane with given initial and final tangent angles and curvatures and with bounded derivative of the curvature (see [1], [3], [4], [5]). It turns out that if (locally) the shortest path is C^3, then it is (locally) a piece of a clothoid or a straight line segment (see [1], [3], [4], [5]).

Some of the properties exposed in this paper are used in [5] to show that when the distance between the initial and final points is sufficiently long (see the precise definition in [5]), then, in general, optimal paths have infinitely many points of discontinuity of the curvature's derivative. Thus we are led to the problem to describe a procedure of constructing paths (called "suboptimal") longer by no more than a fixed constant than the shortest one(s) and which can be constructed explicitly. They are constructed from pieces of clothoids and a line segment (see [3], [4]).

The paper is organized as follows. Next section explains in details the problem to find the optimal (i.e. the shortest) paths in the planar motion with bounded derivative of the curvature. We discuss the irregularity of the optimal paths and we explain how suboptimal paths are constructed. In each of the sections $3-8$ where properties of the clothoids are exposed we explain where exactly they find application (in the optimal control problem).

2 Formulation of the optimal control problem.

We consider the problem to find the shortest path connecting two given points of $\mathbf{R}^2$ with given initial and final tangent angles and curvatures. The tangent angle and the curvature vary continuously, the speed of changing the curvature is bounded by some constant B. The real background of the problem is to find the shortest paths for a car-like robot to go from one given point to another with the above mentioned initial and final conditions. One can turn the wheels of a car with a bounded speed. Hence, the speed of changing the curvature of the path of a real car is bounded.

We consider two problems – with and without cusps. In the problem with cusps we consider the planar motion of a car which can equally move forward and backward. One of the couples of wheels of the car is considered mobile, the other is considered fixed. Call "front" the gear of the car which corresponds to the direction of the motion at the initial moment; the other gear is called "rear" (being front or rear doesn't depend on being mobile or fixed).

The angle between the mobile wheels and the axis of the car defines the curvature of its trajectory at the given moment if the direction of motion is fixed; if the direction is reversed, then the curvature changes sign. We assume that during the motion this angle can be changed continuously, with a bounded speed.

We allow cusps of such a type that at a cusp the tangent angle has a discontinuity (it changes by π) and the curvature is continuous. We assume that a point moves with constant speed equal to 1; that is why the variable t is both the arc length and the time.

In the two cases (the cusps case is to the right) we have the following systems:

$$\dot{X}(t)=\begin{cases}\dot{x}(t)=\cos\alpha(t)\\ \dot{y}(t)=\sin\alpha(t)\\ \dot{\alpha}(t)=u(t)\\ \dot{u}(t)=u'(t)\end{cases}\qquad \dot{X}(t)=\begin{cases}\dot{x}(t)=\cos(\alpha(t)+\varepsilon(t)\pi)\\ \dot{y}(t)=\sin(\alpha(t)+\varepsilon(t)\pi)\\ \dot{\alpha}(t)=u(t)\\ \dot{u}(t)=u'(t)\end{cases}\tag{1}$$

$$|u'(t)|\le B \qquad\qquad |u'(t)|\le B \qquad \varepsilon(t)\in\{0,1\}$$

with initial and final conditions

$$X(0)=(x^0,y^0,\alpha^0,u^0),\qquad X(T)=(x^T,y^T,\alpha^T,u^T).\tag{2}$$

Here $x(t)$ and $y(t)$ are the planar coordinates of a point, $u(t)$ is the curvature, $\alpha(t)$ is the tangent angle between the axis Ox and the tangent vector to the path in the without cusp case but in the cusps case instead of the tangent angle we introduce a variable $\alpha(t)$ which changes continuously along the path: it is equal to the tangent angle to the path after an even number of cusps and to the tangent angle $+\pi$ after an odd number of cusps. In the cusps case the tangent angle is equal to $\alpha(t)+\varepsilon(t)\pi$. We control the derivative of the curvature by $u'(t)$ (it is a measurable real-valued function and $u'(t)$ belongs to the interval $[-B,+B]$) and the control $\varepsilon(t)$ defines the positions of the cusps (it takes only two values 0 and 1 and ensures the continuity of the variable α at cusps).

We want to find functions $x(t)$, $y(t)$ $\alpha(t)$, $u(t)$ satisfying the system of equations and the initial and final conditions and such that the associated control function should minimize the total length of the path:

$$J(\varepsilon,u') = T = \int_0^T dt.$$

We prove the controlability for the system without cusps. To prove the controlability for the system with cusps we use the same method.

If the distance between the initial and final points is much greater than the parameter B (see inequality (5)), then one can construct a suboptimal path from $(x^0, y^0, \alpha^0, u^0)$ to $(x^T, y^T, \alpha^T, u^T)$ (see below). If not, then one can construct a suboptimal path from $(x^0, y^0, \alpha^0, u^0)$ to a point $(x^*, y^*, \alpha^0, u^0)$ such that inequality (5) holds for the points (x^0, y^0), (x^*, y^*) and (x^*, y^*), (x^T, y^T), then a suboptimal path from $(x^*, y^*, \alpha^0, u^0)$ to $(x^T, y^T, \alpha^T, u^T)$. Both suboptimal paths belong to the class of paths under consideration. So, the controllability is proved.

We prove the existence of an optimal solution and an optimal control in both cases using Filippov's existence theorem (see [1], [3]). To obtain necessary conditions for the control function and for the trajectory to be optimal we apply in both cases the Maximum Principle of Pontryagin. A measurable control and the associated trajectory satisfying the given system and all conditions of the the Maximum Principle of Pontryagin will be called *extremal control* and *extremal trajectory.* After applying the Principle we obtain the following result (see [1], [3]):

Theorem 2.1 *If an extremal path is regular (i.e. the control functions have finitely many points of discontinuity), then it is the closure of a union of open arcs of clothoids, corresponding to $u'(t) = \pm B$, and line segments in one and the same direction φ, corresponding to $u'(t) = 0$.*

A clothoid is a curve along which the curvature $u(t)$ depends linearly on the arc length t and varies continuously from $-\infty$ to $+\infty$. In our case we consider only clothoids which satisfy the following equation (see Theorem 2.1):

$$u(t) = \pm Bt, \qquad t \in (-\infty, +\infty) \tag{3}$$

We can also define the clothoid by its parametrized form (setting $x(0) = y(0) = 0$, $\alpha(0) = 0$, $u(0) = 0$)

$$\begin{cases} x(t) = \sqrt{2/B}\int_0^{t\sqrt{B/2}} \cos(\tau^2)d\tau \\ \\ y(t) = \pm\sqrt{2/B}\int_0^{t\sqrt{B/2}} \sin(\tau^2)d\tau \end{cases} \tag{4}$$

The two possible choices of the sign correspond to the two possible orientations of the clothoid.

Call B the *parameter of the clothoid.* The sign $\pm$ defines the orientation of the clothoid, the variable t is the natural parameter and the curvature equals $\pm Bt$. For $t = 0$ the clothoid has a (unique) inflexion point which is its centre of symmetry. Call *half-clothoid* its part corresponding to $t \in [0, +\infty)$ or to $t \in (-\infty, 0]$.

In the without cusps case we have the following result (see [5]):

Theorem 2.2 *If we assume that the distance between the initial and final points is greater than the constant $C = 144\sqrt{2\pi/B}$, then, in general, there exists no regular optimal path, i.e. all optimal paths have infinitely many switching points.*

In the cusps case it is not clear whether there exists a regular optimal path. So, in both cases we construct explicitly "suboptimal paths" and we prove their suboptimality. "Suboptimal" means longer than the optimal by no more than a constant depending only on the bound B of the curvature's derivative (see [3], [4]).

In the without cusps case (see [4]) we construct a suboptimal path when the distance between the initial and the final point is much greater than the parameter B. It means that there exist constants $a > 0$, $c \geq 0$ such that the following inequality holds:

$$\mathrm{dist}((x^0, y^0), (x^T, y^T)) \geq a/\sqrt{B} + c. \tag{5}$$

In the cusps case (see [3]) we construct a suboptimal path when the distance between the initial and the final point is much greater than $\max(B, |u^0|, |u^T|)$. It means that there exist constants $a > 0$, $b > 0$, $c \geq 0$ such that the following inequality holds:

$$\mathrm{dist}((x^0, y^0), (x^T, y^T)) \geq a/\sqrt{B} + b\max(|u^0|, |u^T|) + c.$$

The difference in the definition of the two cases comes from the fact that the cusps allow to move essentially from the initial point to the final one while in the without cusps case this possibility is absent.

We construct the suboptimal path whose graph of the curvature as a function of the path length in both cases looks like the graph on Figure 2. This graph is piecewise linear, the part of the graph between the points A and W corresponds to the line segment, the other parts correspond to the arcs of clothoids. In the cusps case it is impossible to "read off" the form of the path from the graph of the curvature without knowing the positions of the cusps. We assume that the tangent lines at all cusps are vertical and the path is the graph of a continuous function if we choose for x-axis the line passing through the initial and the final points.

Regard ξ' as a parameter (see Figure 2). Increasing ξ' monotonously leads to increasing of the absolute value of the variable α at the point A. Hence, if ξ' varies in some interval $[0, d']$, then the variable α at the point A will assume continuously all the values from some interval of the kind $[\beta_0, \beta_0 + 2\pi]$ or $[\beta_0, \beta_0 - 2\pi]$, $\beta_0 \in \mathbf{R}$, depending on the sign of u^0. For d' one can take the maximal length of an arc of halh-clothoid on which the tangent angle makes a full turn (i.e. 2π).

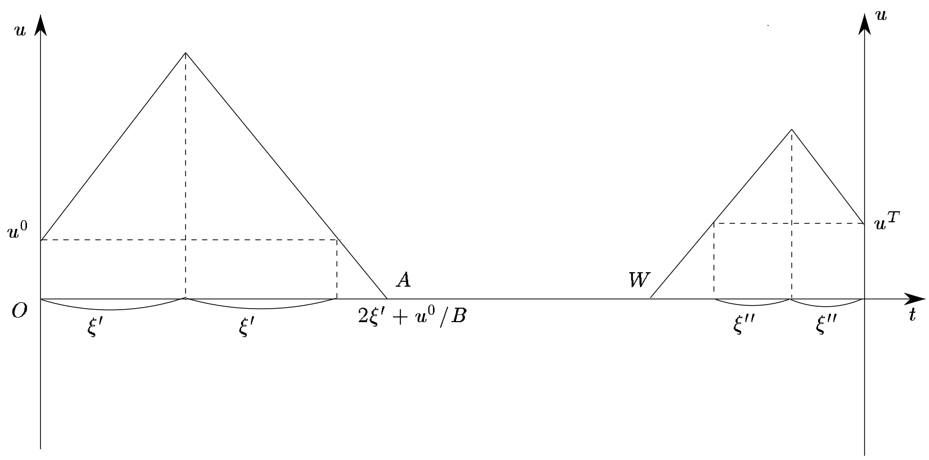

Figure 2

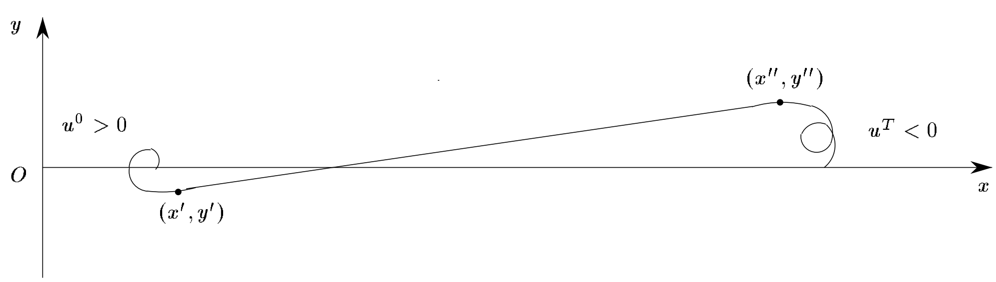

Figure 3

Denote by (x', y') and (x'', y'') the points of the path which correspond to the points A and W on the graph. Vary ξ' and ξ'' so that the tangent lines at (x', y') and (x'', y'') should be parallel. For some values of ξ', ξ'' which vary in the intervals $[0, d']$, $[0, d'']$ respectively these lines coincide. This gives the suboptimal path (see examples on Figures 3–4).

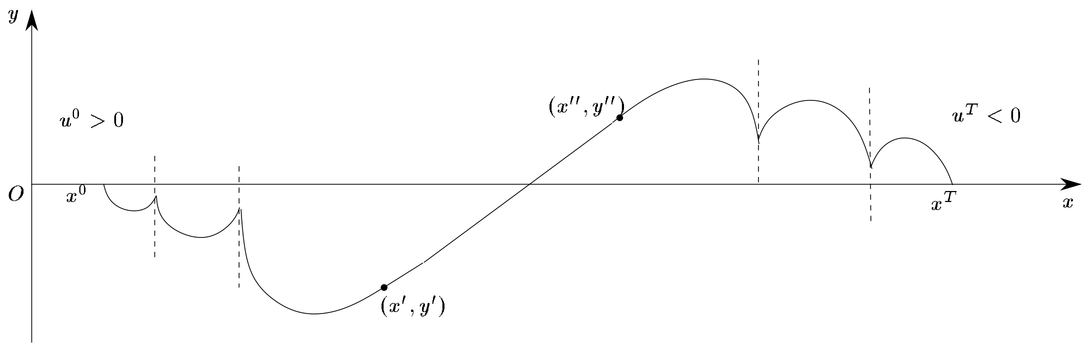

Figure 4

In order to prove the suboptimality of the constructed paths and to prove that in the general case the optimal path is irregular (i.e. it has an infinite number of switching points, see the proof in [5]) we prove some properties of clothoids. We consider these properties in the paper.

In Section 5, Subsections 8.1 and 8.2 we give various estimations which depend on the parameter B (in most cases they depend directly on r_B, which is defined by formula (14)). If the estimations happen not to depend on B, then we set $B = 2$ in the computations (for simplicity).

3 Monotonicity and asymptotic estimations of the lengths of arcs of clothoids.

We use these properties in the case with cusp to prove the suboptimality of the constructed paths.

Consider a half-clothoid (i.e. $t \geq 0$) described by the equations :

$$\begin{cases} x(t) = \int_0^t \cos\left(B\tau^2/2\right)\, d\tau \\ \\ y(t) = \int_0^t \sin\left(B\tau^2/2\right)\, d\tau \end{cases} \tag{6}$$

We set $B = 2$.

Fix a direction $\alpha_*(\bmod \pi, \text{ not } \bmod 2\pi)$ in $\mathbf{R}^2$ and let $P_1, P_2, \ldots$ denote the consecutive points on the half-clothoid with a tangent line at them of the chosen direction (with $t_1 < t_2 < \ldots$). Set $P_i = (x_i, y_i),\ x_i = x(t_i),\ y_i = y(t_i)$ (see Figure 1).

In the section we prove three lemmas, the proofs are to be found at the end of the section.

Lemma 3.1 *$\widehat{P_1P_2}$ is the longest among the arcs $\widehat{P_iP}_{i+1}$. Its length depends continuously and monotonously on the choice of the direction α_*. We have*

$$|\widehat{P_iP}_{i+1}| = \frac{\sqrt{\pi}}{2\sqrt{i}} + O\left(\frac{1}{i\sqrt{i}}\right)$$

and

$$|\ |\widehat{P_iP}_{i+1}| - |\widehat{P_{i-1}P_i}|\ | = -\frac{\sqrt{\pi}}{4i\sqrt{i}} + O\left(\frac{1}{i^2\sqrt{i}}\right)\ .$$

Lemma 3.2 *Denote by $|\mathrm{proj}_{\alpha_*}\ \overrightarrow{P_iP}_{i+1}\ |$ the absolute value of the projection of the vector $\overrightarrow{P_iP}_{i+1}$ on a line parallel to the chosen direction α_*. Then the sum*

$$S(\alpha_*) = \sum_{i=1}^{\infty} |\mathrm{proj}_{\alpha_*}\ \overrightarrow{P_iP}_{i+1}\ |$$

is finite and there exists a constant $\tilde{C} > 0$ such that $S(\alpha_*) < \tilde{C}$ for every α_*.

Lemma 3.3 *Denote by $|\mathrm{proj}_{\alpha_*^\perp}\ \overrightarrow{P_iP}_{i+1}\ |$ the absolute value of the projection of the vector $\overrightarrow{P_iP}_{i+1}$ on a line perpendicular to the chosen direction α_*. Then*

$$|\mathrm{proj}_{\alpha_*^\perp}\ \overrightarrow{P_iP}_{i+1}\ | = \frac{1}{\sqrt{\pi i}} + O(1/i\sqrt{i})$$

and it is a monotonous function of α_* if $\alpha_* \in [0, +\infty)$ is understood as a usual angle. Also

$$|\ |\mathrm{proj}_{\alpha_*^\perp}\ \overrightarrow{P_{i+1}P}_{i+2}\ | - |\mathrm{proj}_{\alpha_*^\perp}\ \overrightarrow{P_iP}_{i+1}\ |\ | = O(1/(i\sqrt{i}))\ .$$

Both estimations are uniform for $\alpha_* \in [0, +\infty)$.

Proof of Lemma 3.1

$$|\widehat{P_iP}_{i+1}| = \int_{\sqrt{\alpha_*+(i-1)\pi}}^{\sqrt{\alpha_*+i\pi}} \sqrt{\cos^2 t^2 + \sin^2 t^2}\,dt = \sqrt{\alpha_* + i\pi} - \sqrt{\alpha_* + (i-1)\pi} =$$

$$= \frac{\pi}{\sqrt{\alpha_* + i\pi} + \sqrt{\alpha_* + (i-1)\pi}} = \frac{\pi}{\sqrt{i\pi}}\frac{1}{\sqrt{1+\alpha_*/i\pi} + \sqrt{1+(\alpha_*-\pi)/i\pi}} = \frac{\sqrt{\pi}}{2\sqrt{i}} + O\left(\frac{1}{i\sqrt{i}}\right)\ ,$$

$$|\ |\widehat{P_iP}_{i+1}| - |\widehat{P_{i-1}P_i}|\ | = \left|\frac{\pi}{\sqrt{\alpha_* + i\pi} + \sqrt{\alpha_* + (i-1)\pi}} - \frac{\pi}{\sqrt{\alpha_* + (i-1)\pi} + \sqrt{\alpha_* + (i-2)\pi}}\right| =$$

$$= \frac{\pi}{\sqrt{i\pi}} * \left|\frac{1}{\sqrt{1+\alpha_*/i\pi} + \sqrt{1+(\alpha_*-\pi)/i\pi}} - \frac{1}{\sqrt{1+(\alpha_*-\pi)/i\pi} + \sqrt{1+(\alpha_*-2\pi)/i\pi}}\right| =$$

$$= \frac{\sqrt{\pi}}{\sqrt{i}} * \left| \frac{1}{1+\frac{\alpha_*}{2i\pi}+O(\frac{1}{i^2})+1+\frac{(\alpha_*-\pi)}{2i\pi}+O(\frac{1}{i^2})} - \frac{1}{1+\frac{\alpha_*-\pi}{2i\pi}+O(\frac{1}{i^2})+1+\frac{\alpha_*-2\pi}{2i\pi}+O(\frac{1}{i^2})} \right| =$$

$$= \frac{\sqrt{\pi}}{\sqrt{i}} * \left| \frac{1}{2+\frac{2\alpha_*-\pi}{2i\pi}+O(1/i^2)} - \frac{1}{2+\frac{2\alpha_*-3\pi}{2i\pi}+O(1/i^2)} \right| =$$

$$= \frac{\sqrt{\pi}}{2\sqrt{i}} * \left| \frac{1}{1+\frac{2\alpha_*-\pi}{4i\pi}+O(1/i^2)} - \frac{1}{1+\frac{2\alpha_*-3\pi}{4i\pi}+O(1/i^2)} \right| =$$

$$= \frac{\sqrt{\pi}}{2\sqrt{i}} * \left| 1 - \frac{2\alpha_*-\pi}{4i\pi} - 1 + \frac{2\alpha_*-3\pi}{4i\pi} + O(1/i^2) \right| =$$

$$= \frac{\sqrt{\pi}}{2\sqrt{i}} \left(-1/2i + O(1/i^2) \right) = -\frac{\sqrt{\pi}}{4i\sqrt{i}} + O\left(\frac{1}{i^2\sqrt{i}} \right) \ .$$

The first two statements of the lemma follow directly from the first chain of equalities. The lemma is proved. □

Proof of Lemma 3.2

$$|\mathrm{proj}_{\alpha_*} \ \overrightarrow{P_i P_{i+1}} \ | = \left| \int_{\sqrt{\alpha_*+(i-1)\pi}}^{\sqrt{\alpha_*+i\pi}} (\dot{x}\cos\alpha_* + \dot{y}\sin\alpha_*)dt \right| = I_i(\alpha_*) \ .$$

For convenience, we consider further α_* to be a usual angle, $\alpha_* \geq 0$. Hence,

$$I_i(\alpha_*) = \left| \int_{\sqrt{\alpha_*+(i-1)\pi}}^{\sqrt{\alpha_*+i\pi}} \cos\left(t^2-\alpha_*\right)dt \right| = \left| \int_{\alpha_*+(i-1)\pi}^{\alpha_*+i\pi} \frac{\cos\left(\tau-\alpha_*\right)}{2\sqrt{\tau}} d\tau \right| =$$

$$= \left| \int_0^{\pi} \frac{\cos\left(\tau'+(i-1)\pi\right)}{2\sqrt{\tau'+\alpha_*+(i-1)\pi}} d\tau' \right| = \left| \int_0^{\pi} \frac{\cos\tau'}{2\sqrt{\alpha_*+(i-1)\pi}} * \frac{d\tau'}{\sqrt{1+\tau'/(\alpha_*+(i-1)\pi)}} \right| =$$

$$= \left| \int_0^{\pi} \frac{\cos\tau'}{2\sqrt{\alpha_*+(i-1)\pi}} * \left(1 - \frac{\tau'}{2(\alpha_*+(i-1)\pi)} + f\left(\frac{\tau'}{\alpha_*+(i-1)\pi} \right) \right) d\tau' \right| \ ,$$

where

$$f(w) = \frac{1}{\sqrt{1+w}} - 1 + \frac{w}{2} \ .$$

We have

$$\frac{df(w)}{dw} = -\frac{1}{2(\sqrt{1+w})^3} + \frac{1}{2} > 0 \qquad \text{if } w > 0.$$

Hence, $f(w)$ is an increasing function of the variable w, i.e. it takes its maximal value if $\tau' = \pi$ and $\alpha_* = 0$ (because $\tau' \in [0, \pi]$, $\alpha_* \geq 0$). Thus

$$f(w) \leq \frac{1}{\sqrt{1 + 1/(i-1)}} - 1 + \frac{1}{2(i-1)} = O\left(\frac{1}{i^2}\right)$$

and

$$I_i(\alpha_*) \leq \left| \int_0^{\pi} \frac{\cos\tau' d\tau'}{2\sqrt{\alpha_* + (i-1)\pi}} \right| + \left| \int_0^{\pi} \frac{\tau' \cos\tau' d\tau'}{4(\alpha_* + (i-1)\pi)\sqrt{\alpha_* + (i-1)\pi}} \right| +$$

$$+ \left| \int_0^{\pi} \frac{\cos\tau'}{2\sqrt{\alpha_* + (i-1)\pi}} f\left(\frac{\tau'}{\alpha_* + (i-1)\pi}\right) d\tau' \right| \leq$$

$$\leq 0 + \frac{1}{4(\alpha_* + (i-1)\pi)\sqrt{\alpha_* + (i-1)\pi}} \left| \int_0^{\pi} \tau' \cos\tau' d\tau' \right| + O\left(\frac{1}{i^2}\right) = O\left(\frac{1}{i\sqrt{i}}\right) .$$

The series $\sum_{i=1}^{\infty} 1/(i\sqrt{i})$ is convergent. The lemma is proved. □

Proof of Lemma 3.3

Similarly to the proof of Lemma 3.2 we obtain the formula

$$|\text{proj}_{\alpha_*^{\perp}} \overrightarrow{P_i P_{i+1}}| = \left| \int_{\alpha_* + (i-1)\pi + \pi/2}^{\alpha_* + i\pi + \pi/2} \frac{\cos(\tau - \alpha_*)}{2\sqrt{\tau}} d\tau \right| = \left| \int_{\alpha_* - \pi/2}^{\alpha_* + \pi/2} \frac{\cos(\tau' - \alpha_* + i\pi)}{2\sqrt{\tau' + i\pi}} d\tau' \right| =$$

$$= \frac{1}{2\sqrt{i\pi}} \left| \int_{\alpha_* - \pi/2}^{\alpha_* + \pi/2} \cos(\tau' - \alpha_*) \left(1 - \frac{\tau'}{2i\pi} + O(1/i^2)\right) d\tau' \right| =$$

$$= \frac{1}{\sqrt{i\pi}} \left(1 - \left| \int_{\alpha_* - \pi/2}^{\alpha_* + \pi/2} \cos(\tau' - \alpha_*) \left(\frac{\tau'}{4i\pi} + O(1/i^2)\right) d\tau' \right| \right) = \frac{1}{\sqrt{\pi i}} + O\left(\frac{1}{i\sqrt{i}}\right) .$$

Then we obtain the second assertion of the lemma:

$$\left| \, |\text{proj}_{\alpha_*^{\perp}} \overrightarrow{P_{i+1} P_{i+2}}| - |\text{proj}_{\alpha_*^{\perp}} \overrightarrow{P_i P_{i+1}}| \, \right| =$$

$$= \left| \, \left| \int_{\alpha_* - \pi/2}^{\alpha_* + \pi/2} \frac{\cos(\tau' - \alpha_*)}{2\sqrt{(i+1)\pi}} \left(1 - \frac{\tau'}{2(i+1)\pi} + O(1/i^2)\right) d\tau' \right| - \right.$$

$$-\left|\int_{\alpha_*-\pi/2}^{\alpha_*+\pi/2}\frac{\cos(\tau'-\alpha_*)}{2\sqrt{i\pi}}\left(1-\frac{\tau'}{2i\pi}+O(1/i^2)\right)d\tau'\right|\Bigg| =$$

$$=\left|\,\left|\frac{1}{\sqrt{(i+1)\pi}}+O(1/i\sqrt{i})\right|-\left|\frac{1}{\sqrt{i\pi}}+O(1/i\sqrt{i})\right|\,\right|=\left|\frac{1}{\sqrt{(i+1)\pi}}-\frac{1}{\sqrt{i\pi}}+O(1/i\sqrt{i})\right|=$$

$$=\left|\frac{1}{\sqrt{\pi i(i+1)}(\sqrt{i+1}+\sqrt{i})}+O(1/(i\sqrt{i}))\right|=\frac{1}{\sqrt{\pi i(i+1)}(\sqrt{i+1}+\sqrt{i})}+O(1/i\sqrt{i})\ .$$

The lemma is proved. □

4 Fundamental properties of an individual clothoid.

We use them in the case without cusp to prove the suboptimality of the constructed paths. In the section we set $B=2$.

Define as "the centre of the half-clothoid" the point O_c with coordinates $(x_{O_c},\, y_{O_c})$ defined as follows:

$$\begin{cases} x_{O_c}=\int_0^\infty \cos\tau^2 d\tau=\sqrt{\pi}/(2\sqrt{2}) \\ \\ y_{O_c}=\int_0^\infty \sin\tau^2 d\tau=\sqrt{\pi}/(2\sqrt{2}) \end{cases}$$

Consider the coordinate system with the centre at the centre O_c of the half-clothoid and with the axes O_cx_c, O_cy_c parallel to the corresponding axes of the coordinate system Oxy (see Figure 5).

In the coordinate system $O_cx_cy_c$ the coordinates of the point (x_c, y_c) of the half-clothoid (6) are defined by the formulas:

$$\begin{cases} x_c(t)=x(t)-x_{O_c}=-\int_t^\infty \cos\tau^2 d\tau \\ \\ y_c(t)=y(t)-y_{O_c}=-\int_t^\infty \sin\tau^2 d\tau \end{cases} \tag{7}$$

Denote by $\vec{\rho}$ the radius-vector of a point of the half-clothoid in the coordinate system $O_cx_cy_c$.

Then

$$\rho^2={x_c}^2+{y_c}^2$$

and

$$\dot\rho(t)=\frac{1}{\rho}(x_c\dot x_c+y_c\dot y_c)=$$

$$=\frac{1}{\rho}(-\cos t^2\int_t^\infty\cos\tau^2 d\tau-\sin t^2\int_t^\infty \sin\tau^2 d\tau)=$$

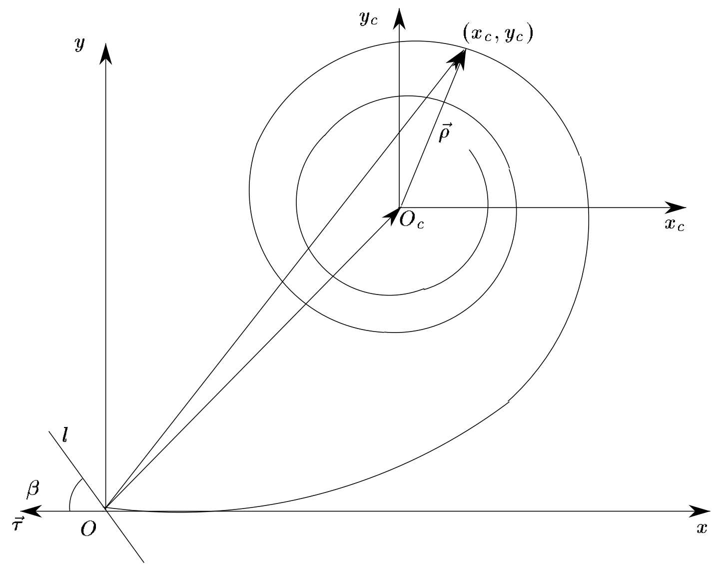

Figure 5

$$= -\frac{1}{\rho}\int_t^\infty \cos(\tau^2 - t^2)d\tau = -\frac{1}{\rho}\int_{t^2}^\infty \frac{\cos(\eta - t^2)d\eta}{2\sqrt{\eta}} = -\frac{1}{\rho}\int_0^\infty \frac{\cos\nu d\nu}{2\sqrt{\nu + t^2}} \ .$$

Thus

$$\dot{\rho}(t) = -\frac{1}{2\rho}\int_0^\infty \frac{\cos\tau d\tau}{\sqrt{\tau + t^2}} \tag{8}$$

Lemma 4.1 *The length of the radius-vector $\vec{\rho}(t)$ of the half-clothoid defined by system (6) is a monotonously decreasing function of t:*

$$\dot{\rho} < 0 \ .$$

See the proof of the lemma at the end of the section.

Lemma 4.2 *The derivative of the length of the radius-vector $\vec{\rho}(t)$ of the clothoid defined by system (6) is a monotonously increasing function of t, i.e.*

$$\ddot{\rho} > 0 \ . \tag{9}$$

See the proof of the lemma in Appendix A.

We give a geometric interpretation of the inequality $\ddot{\rho} > 0$. Denote by $\gamma(t)$ the angle between the radius-vector $\vec{\rho}(t)$ and the tangent vector $\vec{\tau}(t)$ of the point of the clothoid (6). We have

$$\dot{\rho} = \cos\gamma \ . \tag{10}$$

The angle γ is in the interval $(\pi/2, \pi)(\mod 2\pi)$ (because $\dot{\rho} < 0$, see Lemma 4.1). Hence, the function $\sin\gamma$ is positive. We have

$$\ddot{\rho} = -\dot{\gamma}\sin\gamma \tag{11}$$

and obtain, from (9), that

$$\dot{\gamma} < 0 \ . \tag{12}$$

So we obtain the geometric interpretation of Lemma 4.2:

Remark 4.3 *The angle $\gamma(t)$ between the radius-vector $\vec{\rho}(t)$ and the tangent vector $\vec{\tau}(t)$ is a monotonously decreasing function of t; $\gamma(t) \to 3\pi/4$ for $t \to 0$, $\gamma(t) \to \pi/2$ for $t \to +\infty$.*

Corollary 4.4 *If we have an "unwinding" half-clothoid (i.e. half-clothoid with decreasing absolute value of the curvature) defined by the equations:*

$$\begin{cases} x(t) = \int_0^t \cos(\tau^2 + u_0\tau + \alpha_0)d\tau & x(0) = x_0 \quad u_0 < 0 \\ y(t) = \int_0^t \sin(\tau^2 + u_0\tau + \alpha_0)d\tau & y(0) = y_0 \quad t \geq 0 \end{cases}$$

then for such a clothoid we have the following conditions:

$$\dot{\rho} > 0 \ ,$$

$$\ddot{\rho} > 0 \ .$$

Corollary 4.5 *If two half-clothoids clA and clB have the same centre O_c, the same orientation and the same parameter B then either they coincide or they have no common point.*

Consider the circle $\mathcal{C}$ with centre at the centre O_c of clA and with radius equal to the distance between the centre of clA and its point of zero curvature. Denote by $\partial\mathcal{C}$ the circumference with centre at O_c and with the same radius. Then $\mathcal{C} \setminus O_c$ is the union of non-intersecting half-clothoids. The mapping which maps each half-clothoid on its point of zero curvature (lying on $\partial\mathcal{C}$) is a bijection from the set of half-clothoids onto $\partial\mathcal{C}$.

Proof

If clA and clB intersect, then at the intersection point they have equal radius-vectors, hence, equal curvatures (see Lemma 4.1), hence, equal values of $\dot{\rho}$ (see Lemma 4.2), hence, they must coincide, because they are obtained by integrating the equations $\dot{x} = \cos(t-t_0)^2$, $\dot{y} = \sin(t-t_0)^2$ with equal initial data (x_0, y_0, t_0).

The corollary is proved. □

Proof of Lemma 4.1

Set

$$t^2 = a, \qquad \int_0^\infty \frac{\cos\tau d\tau}{\sqrt{\tau + a}} = I(a) \ .$$

The function $\cos\tau$ is periodic with period 2π. So using the property of the symmetry of the function $\cos\tau$ ($\cos(\pi-\tau) = \cos(\pi+\tau) = -\cos\tau$, $\cos(2\pi-\tau) = \cos\tau$) we can consider instead of the integral $I(a)$ the following integral:

$$\int_0^{\pi/2} \Sigma \cos\tau d\tau \ ,$$

where

$$\Sigma = \sum_{k=0}^{\infty} \left(\frac{1}{\sqrt{a+\tau+2k\pi}} - \frac{1}{\sqrt{\pi-\tau+a+2k\pi}} - \frac{1}{\sqrt{\pi+\tau+a+2k\pi}} + \frac{1}{\sqrt{2\pi-\tau+a+2k\pi}} \right)$$

This series is convergent because

$$\frac{1}{\sqrt{a+\tau+2k\pi}} - \frac{1}{\sqrt{\pi-\tau+a+2k\pi}} =$$

$$= \frac{\pi-2\tau}{\sqrt{a+\tau+2k\pi}\sqrt{\pi-\tau+a+2k\pi}(\sqrt{a+\tau+2k\pi}+\sqrt{\pi-\tau+a+2k\pi})} = O\left(\frac{1}{k\sqrt{k}}\right)$$

and

$$-\frac{1}{\sqrt{\pi+\tau+a+2k\pi}} + \frac{1}{\sqrt{2\pi-\tau+a+2k\pi}} =$$

$$\frac{-\pi+2\tau}{\sqrt{\pi+\tau+a+2k\pi}\sqrt{2\pi-\tau+a+2k\pi}(\sqrt{\pi+\tau+a+2k\pi}+\sqrt{2\pi-\tau+a+2k\pi})} =$$

$$= O\left(\frac{1}{k\sqrt{k}}\right) \ .$$

Consider the first four terms of the series. The function $f(\xi) = \frac{1}{\sqrt{\xi}}$ is convex and monotonously decreasing, see Figure 6.

For the middle lines KM and LM of the trapezoids $EABF$ and $GCDH$ respectively we have $LM \subset KM$. We have the followings formulas:

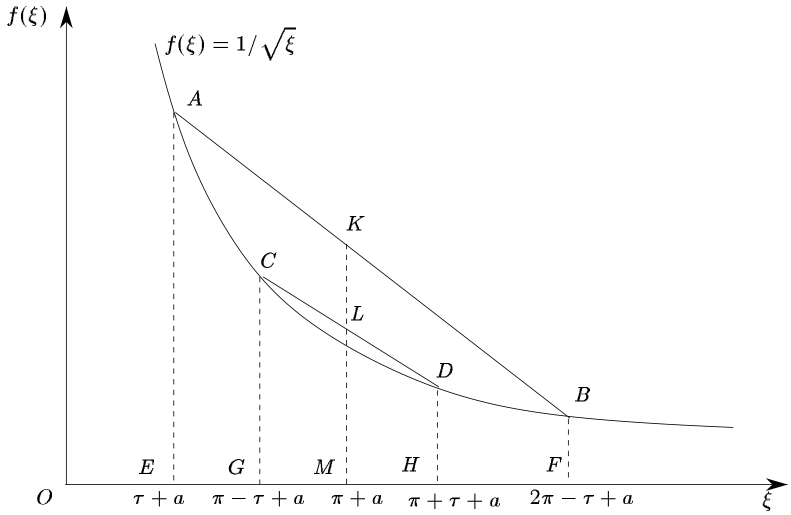

Figure 6

$$\frac{1}{\sqrt{\tau + a}} + \frac{1}{\sqrt{2\pi - \tau + a}} = 2|KM| \ ,$$

$$\frac{1}{\sqrt{\pi - \tau + a}} + \frac{1}{\sqrt{\pi + \tau + a}} = 2|LM| \ ,$$

$$|LM| < |KM| \ .$$

Hence,

$$\frac{1}{\sqrt{\tau + a}} - \frac{1}{\sqrt{\pi - \tau + a}} - \frac{1}{\sqrt{\pi + \tau + a}} + \frac{1}{\sqrt{2\pi - \tau + a}} > 0 \ .$$

Every following sum of four terms in the series can be considered analogously. This proves that the sum of the series under consideration is positive. The function $\cos \tau, \tau \in [0, \pi/2]$ is non-negative. Hence, the integral $I(a)$ is positive and the derivative of the length of the radius-vector $\vec{\rho}(t)$ is negative.

The lemma is proved. □

5 Estimation of the maximal possible distance between two points of a half-clothoid.

We use these estimations in the case without cusps to prove the irregularity of optimal paths in the general case. The estimations depend on the parameter B, so, we consider arbitrary B.

Consider a half-clothoid (6). The length of the chord OE for some point E belonging to the half-clothoid (see Figure 7) is defined as follows:

$$|OE| = \sqrt{x^2(t) + y^2(t)} = \sqrt{\left(\int_0^t \cos\left(B\tau^2/2\right) d\tau\right)^2 + \left(\int_0^t \sin\left(B\tau^2/2\right) d\tau\right)^2} . \qquad (13)$$

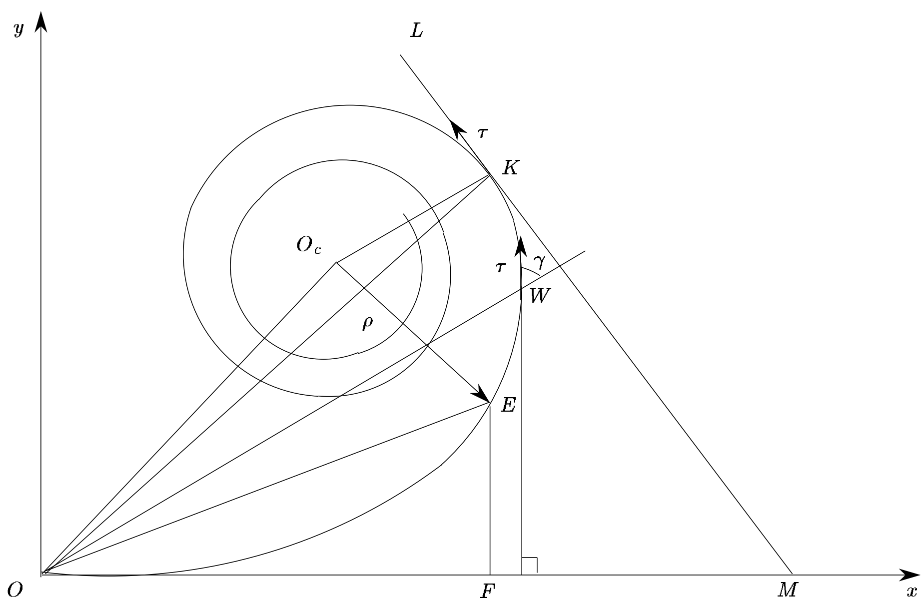

Figure 7

Denote by K the point of a half-clothoid where the chord has maximal length. Denote by r_B the length of the line segment OO_c, i.e.

$$r_B = \sqrt{\pi/(2B)} \qquad (14)$$

In the section we prove three propositions, the proofs are to be found at the end of the section.

Proposition 5.1 *The tangent angle α at the point K belongs to the interval $(\pi/2, 3\pi/4)$.*

Proposition 5.2 *The maximal possible length of the chord $|OK|$ is smaller than $3r_B/2$.*

Proposition 5.3 *The maximal possible distance between two points of a half-clothoid is smaller than $3r_B/2$.*

Proof of Proposition 5.1

At the point K the tangent vector $\vec{\tau}$ is perpendicular to the chord OK (because at K the derivative of the length of the chord is equal to zero). Denote by W the point of a half-clothoid where the tangent angle is equal to $\pi/2$. Evidently, $\alpha_K > \pi/2$, because $\frac{d|OW|}{dt} = \cos\gamma$, and $\gamma \in (0, \pi/2)$ at the point W, hence, $\frac{d|OW|}{dt} > 0$. On Figure 7 we denote by O_c the centre of a half-clothoid. Recall that the point O_c has the coordinates (x_{O_c}, y_{O_c}) defined by the following formulas:

$$\begin{cases} x_{O_c}(t) = \int_0^\infty \cos\left(B\tau^2/2\right) d\tau = \sqrt{\pi}/(2\sqrt{B}) \\ \\ y_{O_c}(t) = \int_0^\infty \sin\left(B\tau^2/2\right) d\tau = \sqrt{\pi}/(2\sqrt{B}) \end{cases}$$

The angle OKL is equal to $\pi/2$. The angle O_cKL is smaller than $\pi/2$ (because $\dot{\rho} < 0$, see Lemma 4.1). Hence, the angle MOK is smaller than the angle MOO_c. But the angle MOO_c is equal to $\pi/4$, hence, the angle MOK is smaller than $\pi/4$ and the angle OMK is greater than $\pi/4$, i.e. the tangent angle α_K at the point K is smaller than $3\pi/4$.

The proposition is proved. □

Proof of Proposition 5.2

In Lemma 4.1 we have proved the following property of the half-clothoid: the radius-vector $\rho(t)$ of the half-clothoid (6) is a monotonously decreasing function. Hence (see Figure 7),

$$|O_cK| < |O_cW| \ , \quad \text{i.e.} \quad |OK| < |OO_c| + |O_cW| = r_B + |O_cW| \ .$$

But for $|O_cW|$ we have the following formulas (see (13)):

$$|O_cW| =$$

$$\sqrt{\left(\int_0^{\sqrt{\pi/B}} \cos\left(B\tau^2/2\right) d\tau - \sqrt{\pi}\Big/\left(2\sqrt{B}\right)\right)^2 + \left(\int_0^{\sqrt{\pi/B}} \sin\left(B\tau^2/2\right) d\tau - \sqrt{\pi}\Big/\left(2\sqrt{B}\right)\right)^2} =$$

$$\sqrt{\left(\sqrt{2/B}\int_0^{\sqrt{\pi/2}} \cos\tau^2 d\tau - \sqrt{\pi}\Big/\left(2\sqrt{B}\right)\right)^2 + \left(\sqrt{2/B}\int_0^{\sqrt{\pi/2}} \sin\tau^2 d\tau - \sqrt{\pi}\Big/\left(2\sqrt{B}\right)\right)^2} =$$

$$2r_B/\sqrt{\pi}\sqrt{\left(\int_0^{\sqrt{\pi/2}} \cos\tau^2 d\tau - \sqrt{\pi}\Big/\left(2\sqrt{2}\right)\right)^2 + \left(\int_0^{\sqrt{\pi/2}} \sin\tau^2 d\tau - \sqrt{\pi}\Big/\left(2\sqrt{2}\right)\right)^2} \approx$$

$$2r_B/\sqrt{\pi}\sqrt{(0,98-0,62)^2+(0,55-0,62)^2} \approx 2r_B\sqrt{0,14}/\sqrt{\pi} \approx 0,42r_B < r_B/2\ .$$

Hence, we obtain the desired inequality:

$$|OK| < r_B + r_B/2 = 3r_B/2\ .$$

The proposition is proved. $\square$

Proof of Proposition 5.3

Consider two points P and Q of a half-clothoid (6) (see Figure 8). We prove that for any points P and Q

$$|PQ| < |OK|\ ,$$

where K is defined as in the beginning of the subsection.

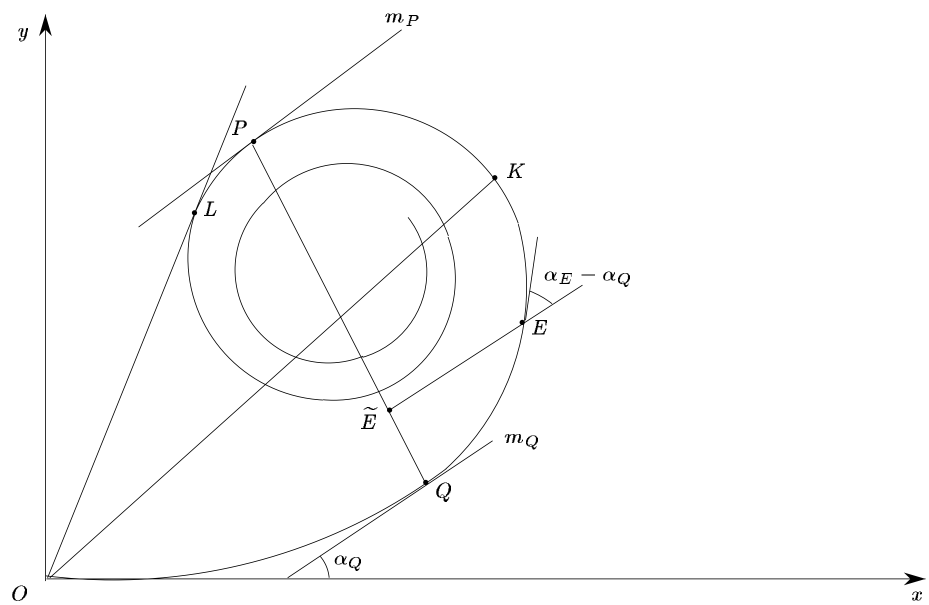

Figure 8

Evidently, we must consider the case when if only one point (P or Q) belongs to the arc $\widehat{OL}$ (the tangent line at the point L passes through the point O). If the chord PQ has the maximal possible length then the tangent lines at the points P, Q (denote them by m_P, m_Q respectively) are perpendicular to the chord PQ.

Consider a point $E \in \widehat{QP}$. Denote by α_Q the tangent angle at the point Q, by α_E – the tangent angle at the point E. The line $E\widetilde{E}$ is parallel to the lines m_P and m_Q.

The length of the straight line segment $E\widetilde{E}$ can be defined by the following formula:

$$|E\widetilde{E}| = \int_{\sqrt{2\alpha_Q/B}}^{\sqrt{2\alpha_E/B}} \cos\left(B\tau^2/2 - \alpha_Q\right) d\tau$$

(because $\alpha_E = Bt_E^2/2$).

Assume that the point E coincides with the point P, i.e. $\alpha_E = \alpha_Q + \pi$. Hence, we have the following equality:

$$0 = \int_{\sqrt{2\alpha_Q/B}}^{\sqrt{(2\alpha_Q+2\pi)/B}} \cos\left(B\tau^2/2 - \alpha_Q\right) d\tau \ . \tag{15}$$

We can rewrite equality (15) as follows:

$$0 = \int_0^\pi \frac{\cos\tau\, d\tau}{\sqrt{2B(\alpha_Q + \tau)}} \ .$$

But

$$\int_0^\pi \frac{\cos\tau\, d\tau}{\sqrt{2B(\alpha_Q + \tau)}} > 0 \ ,$$

because $\cos\tau = \cos(\pi - \tau)$ for any $\tau \in [0, \pi/2)$ and

$$\frac{1}{\sqrt{2B(\alpha_Q + \tau)}} > \frac{1}{\sqrt{2B(\alpha_Q + \pi - \tau)}} \ .$$

Hence, equality (15) isn't correct and, hence, the chord PQ can't have the maximal possible length, i.e.

$$|PQ| < |OK| \ .$$

From Proposition 5.2 we have

$$|OK| < 3r_B/2 \ .$$

Hence, the maximal possible distance between two points of a half-clothoid is smaller than $3r_B/2$.

The proposition is proved. □

6 Properties of two arcs of clothoids at their concatenation point.

We use them in the case without cusp to prove the suboptimality of the constructed paths. In the section we set $B = 2$.

Consider two clothoids $cl1$ and $cl2$ (see Figure 9) which for $t = 0$ have the same initial conditions $(x_0, y_0, \alpha_0, u_0)$, $u_0 < 0$, the absolute value of the curvature of $cl1$ is decreasing with t, the one of $cl2$ is increasing with t; $cl1$ and $cl2$ are defined by equations:

$$cl1: \begin{cases} x(t) = \int_0^t \cos(\tau^2 + u_0\tau + \alpha_0)d\tau + x_0 \\ \\ y(t) = \int_0^t \sin(\tau^2 + u_0\tau + \alpha_0)d\tau + y_0 \end{cases} \tag{16}$$

$$cl2: \begin{cases} x(t) = \int_0^t \cos(-\tau^2 + u_0\tau + \alpha_0)d\tau + x_0 \\ \\ y(t) = \int_0^t \sin(-\tau^2 + u_0\tau + \alpha_0)d\tau + y_0 \end{cases} \tag{17}$$

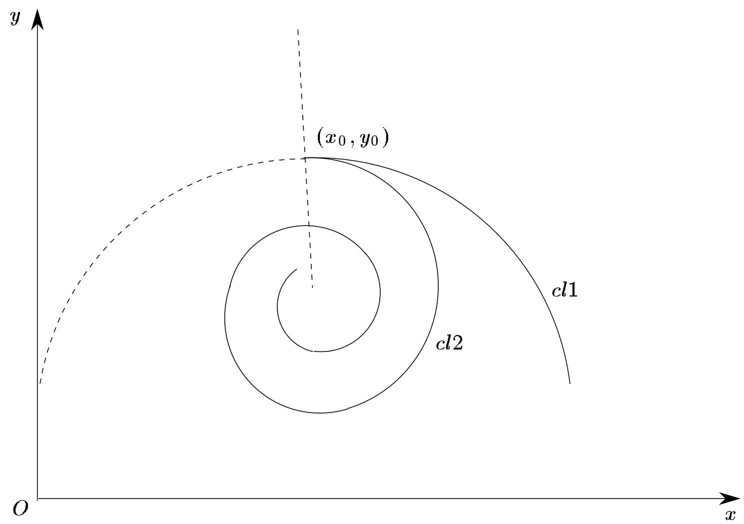

Figure 9

Consider clothoids $cl1$ and $cl2$ on a small interval $t \in [0, s]$ (see Figure 10).

On this figure the point O is the centre of $cl1$, the point A is the initial point, the points B and C belong to the clothoids $cl1$ and $cl2$ respectively and $|\widehat{AB}| = |\widehat{AC}| = s$. The angle between the tangent vector to $cl1$ and $cl2$ at point A and the vector equal to $(-\vec{\rho}_A)$ ($\vec{\rho}_A$ is the radius-vector at point A) is denoted by θ_0. The angles between the tangent vectors to $cl1$ and $cl2$ at the points B and C and the vector equal to $(-\vec{\rho}_A)$ are denoted θ_1 and θ_2 respectively. The angles between the tangent vector at the point A and the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are denoted ψ_1 and ψ_2 respectively. And the angles between the radius-vector $\vec{\rho}_A$ and the radius-vectors $\vec{\rho}_B$ and $\vec{\rho}_C$ at the points B and C are denoted φ_1 and φ_2 respectively. Denote by δ_i $(i = 1, 2)$ the angles between the tangent lines at the points B and C and their radius-vectors $(\delta_i = \theta_i + \varphi_i, \;\; i = 1, 2)$.

Lemma 6.1 *For the clothoids $cl1$ and $cl2$ on a small interval $t \in [0, s]$ the following equalities hold:*

$$\rho_B^2 - \rho_C^2 = \frac{4}{3}\rho_A \sin\theta_0 s^3 + O(s^4) \ , \tag{18}$$

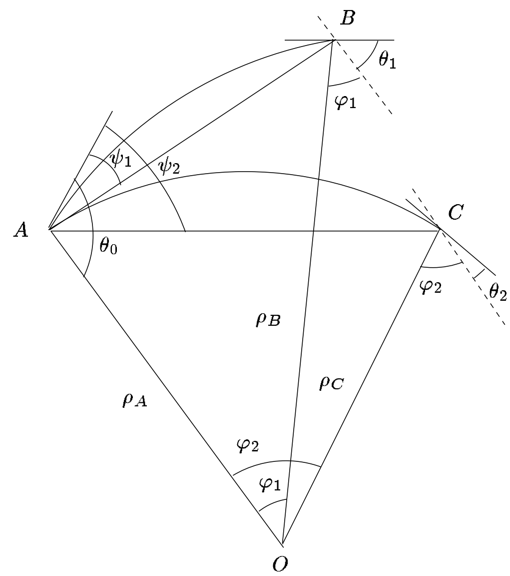

Figure 10

$$\delta_1 - \delta_2 = 2s^2 + \frac{2\cos\theta_0}{3\rho_A}s^3 + O(s^4)\ . \tag{19}$$

See the proof of the lemma in Appendix B.

Corollary 6.2 *Denote by C_c the point of the clothoid $cl1$ with the same curvature as the point C belonging to clothoid $cl2$. Denote by C_ρ the point of the clothoid $cl1$ with the same length of the radius-vector $\vec{\rho}_C$ as the point C of the clothoid $cl2$; and denote by C_γ the point of the clothoid $cl1$ with the same angle γ between the radius-vector and the tangent vector as the point C of $cl2$. Denote by γ_A, γ_B, γ_C the angles γ at the points A, B, C. Then the points C_c, A, C_γ, C_ρ, B on a small interval $[0, s]$ are encountered in their order along $cl1$.*

See the proof of the corollary in Appendix B.

7 Two properties of concatenated arcs of half-clothoids.

We use these properties in the case without cusp. In the section we set $B = 2$.

1) At first we consider a property used to prove the suboptimality of the constructed path.

Consider two paths with the same initial conditions $(x_0, y_0, \alpha_0, u_0)$ and whose graphs of the curvature as a function of the path length are shown on Figure 11.

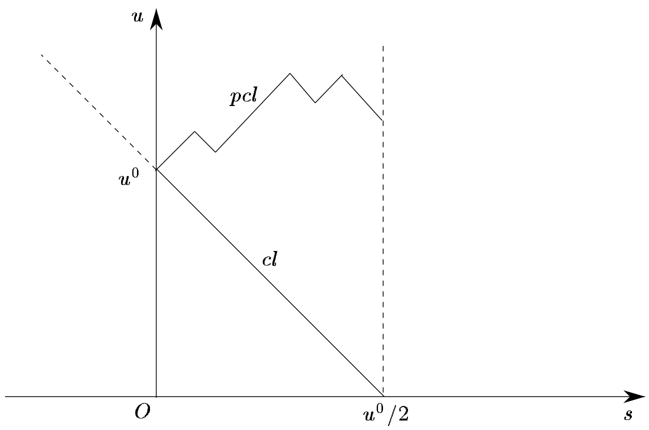

Figure 11

The path cl is a piece of a half-clothoid whose curvature is defined by the equation $u = -2s + u_0$ ($u_0 > 0$). The path pcl consists of several pieces of clothoids whose curvatures are defined by equations of the kind $u = -2s + \tilde{u}^0$ or $u = 2s + \tilde{\tilde{u}}^0$ ($\tilde{u}^0 > 0$ and $\tilde{\tilde{u}}^0 > 0$), the sum of their lengths is equal to $u_0/2$. Denote by O_{cl} the centre of cl, by $\vec{\rho}_{cl}(t)$ the radius-vector of a point of cl in the coordinate system with centre at O_{cl}. Denote by $\vec{\rho}_{pcl}(t)$ the radius-vector of a point of the path pcl in this coordinate system. For $t = 0$ we have $\vec{\rho}_{cl}(0) = \vec{\rho}_{pcl}(0)$.

Lemma 7.1 *For any path pcl (defined as above) and for the path cl (both paths are defined on the interval $s \in [0, u_0/2]$) we have the following inequality:*

$$\rho_{cl}(s) > \rho_{pcl}(s), \quad \text{for every } s \in (0, u_0/2] \ . \tag{20}$$

See the proof of the lemma in Appendix C .

Denote by $\mathcal{D}$ the class of the paths with initial conditions $(x_0, y_0, \alpha_0, u_0)$, of length $u_0/2$ and whose graphs of the curvature u as a function of the path length s belong to the class Lip(2). Denote by $\vec{\rho}_L(t)$ the radius-vector of the point of some path L from the class $\mathcal{D}$ in the coordinate system with centre at O_{cl}. Then we have

Corollary 7.2 *For any path L from the class $\mathcal{D}$ and for the path cl from Lemma 7.1 (both paths are defined on the interval $s \in [0, u_0/2]$) we have the following inequality:*

$$\rho_{cl}(s) > \rho_L(s), \quad \text{for every } s \in (0, u_0/2] \ .$$

Really, the class of paths L belongs to the closure of the class of all paths *pcl* defined at the beginning of the subsection.

2) We use the second property to prove the irregularity of optimal paths in the general case.

Consider the graphs of the curvatures as functions of the path length t (see Figure 12) of two paths: the first path $\mathcal{R}_1$ consists of two arcs of half-clothoids (the arcs correspond to the pieces MN and NP of the graph, the switching point of $\mathcal{R}_1$ corresponds to the point N on the graph) and the second path $\mathcal{R}_2$ consists of two arcs of half-clothoids corresponding to the pieces ML and LQ of the graph and a line segment corresponding to the piece QP of the graph. The path $\mathcal{R}_2$ has two switching points corresponding to the points L and Q of the graph. The lengths of $\mathcal{R}_1$ and $\mathcal{R}_2$ are equal to $2a$ (here a and b are lengths of the arcs of half-clothoids and $0 < b < a$).

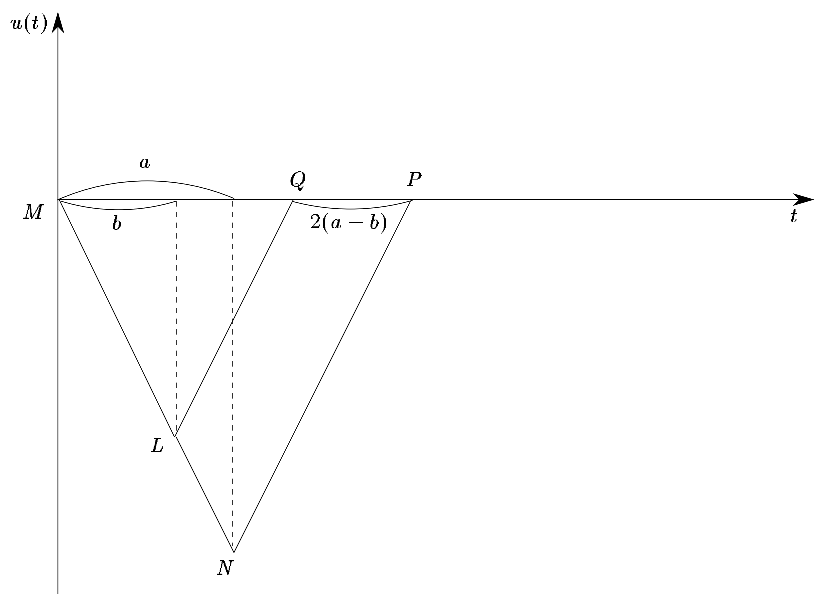

Figure 12

Construct the two paths $\mathcal{R}_1$, $\mathcal{R}_2$ by means of the graph $u(t)$ and with the following initial conditions:

$$\begin{cases} \alpha_M = 2b^2 \\ x_M = \int_0^b [\cos(-\tau^2 + 2b\tau + b^2) + \cos\tau^2] d\tau \\ y_M = \int_0^b [\sin(-\tau^2 + 2b\tau + b^2) + \sin\tau^2] d\tau \end{cases}$$

Denote the points of the paths $\mathcal{R}_1$ and $\mathcal{R}_2$, corresponding to the points M, L, N, Q of the graph by M', L', N', Q' and denote by P_1' the point of $\mathcal{R}_1$ corresponding to P and by P_2' the point of $\mathcal{R}_2$ corresponding to P. The point Q' coincides with the origin of the coordinate system Oxy and the line segment $Q'P_2'$ belongs to the axis Ox (see Figure 13).

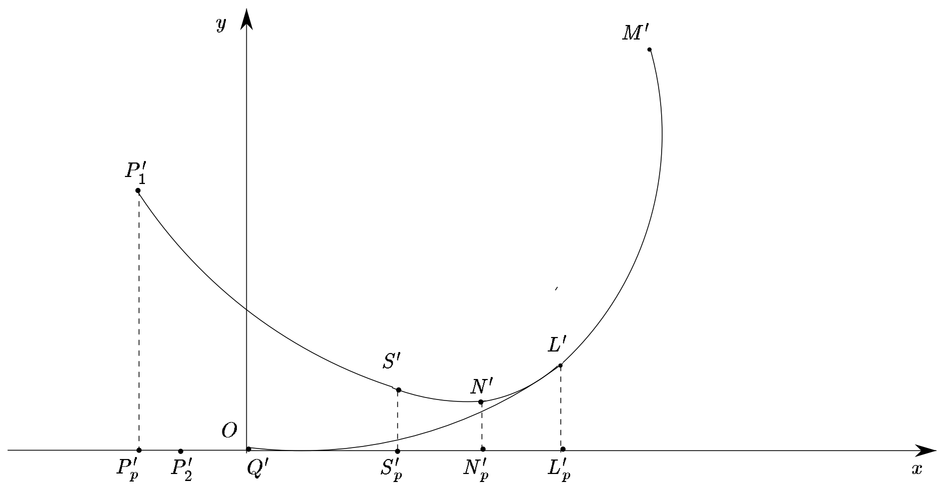

Figure 13

Denote by S' a point of $\mathcal{R}_1$ such that $|\widehat{NS}| = a - b$ and denote by P_p', S_p', N_p' L_p' the projections of the points P_1', S', N', L' on the x-axis. Consider the case when $|OP_p'| > |OP_2'|$. Estimate the maximal possible length $|P_p'P_2'|$.

Proposition 7.3 *If* $|OP_p'| > |OP_2'|$ *then* $|P_p'P_2'| < |OL'| - |OL_p'|$.

Proof

To estimate the maximal possible length of $P_p'P_2'$ consider that $|P_1'S'| = |P_p'S_p'|$ and $|S'L'| = |S_p'L_p'|$. We have the following equalities and inequalities:

$$|P_1'S'| = |Q'L'| \ ,$$

$$|S'L'| < |\widehat{S'L'}| = 2(a-b) \ ,$$

$$|P_p'L_p'| = |P_p'S_p'| + |S_p'L_p'| = |P_1'S'| + |S'L'| = |Q'L'| + |S'L'| < |Q'L'| + 2(a-b) \ ,$$

$$|P_2'L_p'| = |P_2'Q'| + |Q'L_p'| = 2(a-b) + |Q'L_p'| \ .$$

Hence

$$|P_p'P_2'| = |P_p'L_p'| - |P_2'L_p'| <$$

$$< \left[2(a-b)+|Q'L'|\right]-\left[2(a-b)+|Q'L'_p|\right]=|Q'L'|-|Q'L'_p| \ .$$

Thus

$$\max_{t\in[0,C_{arc}]}|P'_pP'_2|<|Q'L'|-|Q'L'_p| \ .$$

The proposition is proved. □

8 Some properties of half-clothoids used in the without cusp case to prove the irregularity of optimal paths in the general case.

Plan of Section 8.

1. In subsection 8.1 we obtain the following result: if the optimal path is a finite concatenation of arcs of clothoids, then the length of any piece between two neighbouring points with zero curvature is no longer than some constant C_{arc} depending only on the parameter B (see Corollary 8.3).

2. In subsection 8.2 we obtain the following result: if we denote by E some point belonging to a half-clothoid and by F its projection on the axis Ox (see Figure 7), then the maximal value of the function $f(t)=\frac{|OE|-|OF|}{|\widehat{OE}|-|OE|}$ is smaller than 3 (see Corollary 8.8).

3. In subsection 8.3 we obtain two important properties of the function $h(t)=|\widehat{OE}|-|OE|$ and of its inverse function h^{-1}:

1) the function $h(t)$ is monotonously increasing for $t\in[0,+\infty)$ and it is concave for $t\in[0,t_K]$ (see Proposition 8.9); here by K we denote the point of a half-clothoid whose chord has a maximal possible length and by t_K the value of t corresponding to the point K,

2) for every point $E\in\widehat{OK}$ we have the following inequality:

$$\frac{h^{-1}(k\Delta)}{h^{-1}(\Delta)}<1-(1-k)/64$$

(see Lemma 8.10).

8.1 Some estimations of the length of an optimal path.

These estimations depend on B, hence, we consider arbitrary B in the subsection.

Consider an arbitrary C^2 and piecewise C^3 path Ψ defined on the interval $[0,T]$ which is a finite concatenation of arcs of clothoids defined by equation (3) and which satisfies the initial and final conditions (2). Suppose that this path is optimal. Denote its length by l_{opt}. Denote by d the distance between the initial and the final points. Consider a suboptimal path Ψ_{sopt} described in [4]. Denote its length by l_{sopt}. This path consists of five pieces: the first, second, forth and the fifth pieces are arcs of half-clothoids and the third piece is a line segment. The example of the graph of the curvature u as a function of the arc length t for the first and the second pieces of Ψ_{sopt} is shown on Figure 14. On this figure ξ' and η' are arc lengths, $\eta' = u^0/B$; we consider ξ' as a parameter.

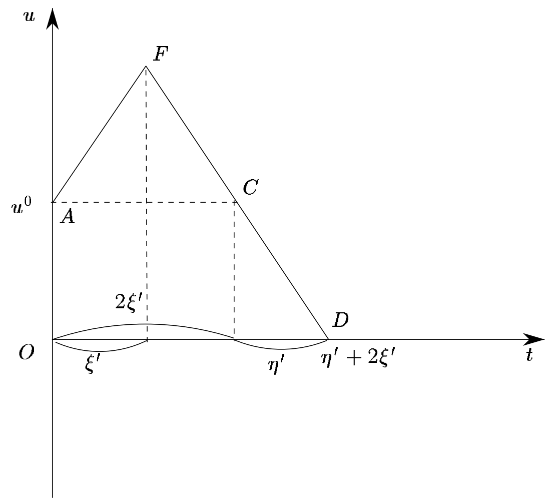

Figure 14

In [4] we have obtained the following result:

$$\xi' \leq 2\sqrt{2} r_B \ , \qquad \xi'' \leq 2\sqrt{2} r_B \ , \tag{21}$$

where r_B is defined by formula (14) and ξ'' is, respectively, the parameter corresponding to the forth and to the fifth pieces of Ψ_{sopt}.

We prove two propositions and a corollary (see the proofs at the end of the subsection).

Proposition 8.1 *The suboptimal path is longer than the distance d between the initial and final points by no more than the sum of some constant C_1 , which depends only on the parameter B of the clothoid, and of a constant C_2, which is equal to the sum of the absolute values of the initial and final curvatures devided by the parameter B, i.e.*

$$l_{sopt} \leq d + C_1 + C_2, \tag{22}$$

where

$$C_1 = C_1(B), \qquad C_2 = \left(|u^0| + |u^T|\right)/B \ .$$

Proposition 8.2 *For the length of an optimal path we have the following inequality:*

$$l_{opt} \leq d + C_1 + C_2,$$

where

$$C_1 = (6 + 8\sqrt{2})r_B = C_1(B), \qquad\qquad C_2 = \left(|u^0| + |u^T|\right)/B \; .$$

Proposition 8.2 follows directly from Proposition 8.1.

Consider an optimal path Ψ. Denote by X' (X'') the first (the last) point of Ψ of zero curvature. Consider the path Ψ from the point X' to the point X'' (denote it by $\widetilde{\Psi}$). We have the following

Corollary 8.3 *The length of any piece of* $\widetilde{\Psi}$ *between two neighbouring points with zero curvature (denote it by* l_p*) is no greater than some constant* C_{arc} *depending only on the parameter* B *(more precisely, on* r_B*):* $C_{arc} = (15 + 8\sqrt{2})r_B$.

Proof of Proposition 8.1

The piece AF of the graph $u(t)$ (see Figure 14) corresponds to some arc of a half-clothoid, the piece FD corresponds to another arc of a half-clothoid. From Proposition 5.3 we know that the maximal possible distance between two points of a half-clothoid is smaller than $3r_B/2$. Hence, the largest possible distance between the initial point and the point of Ψ_{sopt} corresponding to the point D on the graph $u(t)$ is equal to $3r_B$.

On Figure 15 we denote by X^0 (X^T) the initial (the final) point, by X'_D the point of Ψ_{sopt} corresponding to the point D on the graph $u(t)$ and by X''_D the point belonging to the corresponding part of Ψ_{sopt} from the final point. On this figure we show a disposition of the points X^0, X^T, X'_D, X''_D such that the distance between the points X'_D and X''_D is the largest possible.

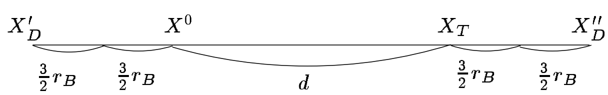

Figure 15

The length of Ψ_{sopt} from the point X^0 to the point X'_D is equal to $2\xi' + |u^0|/B$ (respectively, the length of Ψ_{sopt} from the point X^T to the point X''_D is equal to $2\xi'' + |u^T|/B$). Hence, from (21) we obtain that this length is no longer than $4\sqrt{2}r_B + |u^0|/B$ (respectively, than $4\sqrt{2}r_B + |u^T|/B$).

The largest possible distance between the points X'_D and X''_D is equal to $d + 6r_B$. Hence, we obtain the following inequality:

$$l_{sopt} \le d + 6r_B + 4\sqrt{2}r_B + |u^0|/B + 4\sqrt{2}r_B + |u^T|/B = d + (6 + 8\sqrt{2})r_B + \left(|u^0| + |u^T|\right)/B,$$

i.e. we obtain the inequality (22) where

$$C_1 = (6 + 8\sqrt{2})r_B = C_1(B), \qquad\qquad C_2 = \left(|u^0| + |u^T|\right)/B \; .$$

The proposition is proved. □

Proof of Corollary 8.3

Denote the length of the part of Ψ between the points X^0 and X' by l^0, denote, respectively, the length of the part of Ψ between the points X^T and X'' by l^T. We have

$$l^0 \geq \frac{|u^0|}{B}, \qquad l^T \geq \frac{|u^T|}{B} \ .$$

Denote the length of $\widetilde{\Psi}$ by $\widetilde{l}$ and the length of Ψ by l. Then we have

$$l = l^0 + l^T + \widetilde{l} \geq \frac{|u^0| + |u^T|}{B} + \widetilde{l} \tag{23}$$

and

$$l = l^0 + l^T + \widetilde{l} \leq d + (6 + 8\sqrt{2}) r_B + \left(|u^0| + |u^T|\right) / B \tag{24}$$

(from Proposition 8.2).

Hence, from (23) and (24) we obtain

$$\widetilde{l} \leq d + (6 + 8\sqrt{2}) r_B \ . \tag{25}$$

The minimal possible distance between the points X' and X'' is equal to $d - 6r_B$ (see Figure 16).

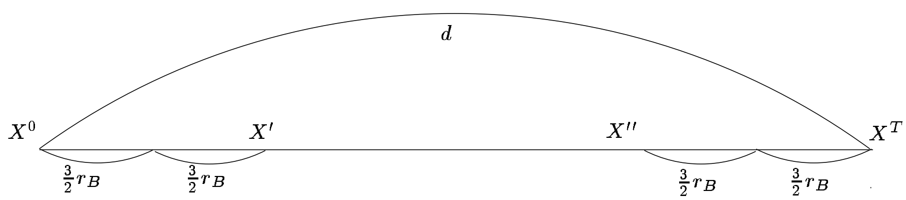

Figure 16

Denote by d_p the distance between the initial and the final points of the piece of $\widetilde{\Psi}$ with the maximal possible length l_p. By Proposition 5.3 we have $d_p < (3/2 r_B) \times 2 = 3r_B$. Hence, we obtain the following inequality:

$$\widetilde{l} \geq l_p + (d - 6r_B) - d_p \geq l_p + (d - 6r_B) - 3r_B = l_p + d - 9r_B \ . \tag{26}$$

Comparing (26) with (25) we obtain

$$l_p \leq (15 + 8\sqrt{2}) r_B \ .$$

The corollary is proved.

8.2 Estimation of the maximal possible value of the function $f(t) = \frac{|OE|-|OF|}{|\widehat{OE}|-|OE|}$.

In the subsection we consider an arbitrary B in Lemma 8.4 and we set $B = 2$ in the others lemmas.

Consider a half-clothoid (6). The length of the projection of the chord OE on the axis Ox (denote it by $|OF|$, see Figure 7) is defined as

$$|OF| = \int_0^t \cos\left(B\tau^2/2\right) d\tau \ .$$

The length of the projection of the chord OE on the axis Oy (denote it by $|EF|$, see Figure 7) is defined as

$$|EF| = \int_0^t \sin\left(B\tau^2/2\right) d\tau \ .$$

The length of the arc $\widehat{OE}$ is equal to t.

Consider the function

$$f(t) = \frac{|OE| - |OF|}{|\widehat{OE}| - |OE|}$$

on the interval $[0, C_{arc}]$, where the constant C_{arc} is defined in Corollary 8.3. We have the following

Lemma 8.4 *The function $f(t)$ defined on $[0, C_{arc}]$ is a positive-valued continuous function and, hence,*

$$\sup_{t\in[0,C_{arc}]} f(t) = M > 0 \ , \qquad \inf_{t\in[0,C_{arc}]} f(t) = m > 0 \ .$$

See the proof of the lemma at the end of the subsection.

In order to estimate $\sup_{t\in[0,C_{arc}]} f(t)$ (see the estimation in Corollary 8.8) we prove axiliary lemmas (Lemmas 8.5–8.7).

Denote by γ the angle between the vectors $\overrightarrow{OE}$ and $\vec{\tau}$, by β the angle between the vectors $\overrightarrow{OF}$ and $\overrightarrow{OE}$ (see Figure 17).

Recall that by α we denote the tangent angle to the path. For these angles we have the following equality:

$$\alpha = \beta + \gamma \ . \tag{27}$$

Lemma 8.5 *The angle γ as a function of t is a monotonously increasing function for $t \in \left[0, \sqrt{3\pi/(2B)}\right]$, i.e.*

$$\dot{\gamma}(t) > 0 \ .$$

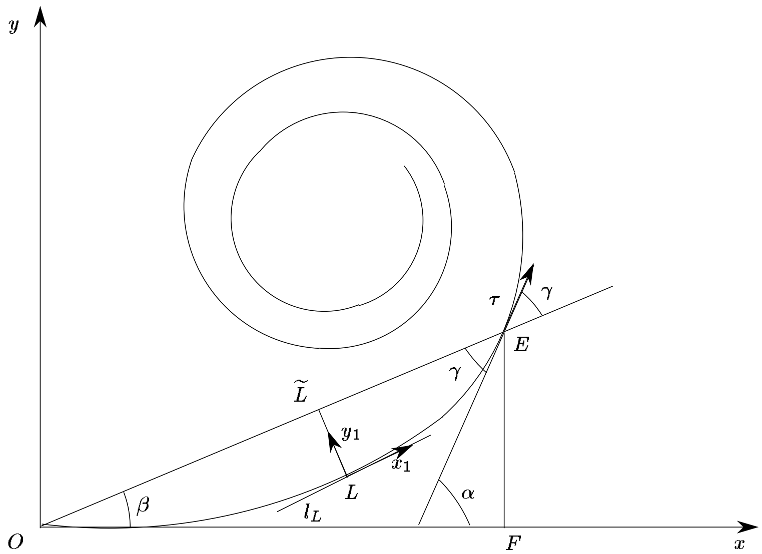

Figure 17

See the proof of Lemma 8.5 in Appendix D.

On Figure 17 denote by L the point of the arc $\widehat{OE}$ where the tangent line (denoted by l_L) is parallel to the chord OE. Denote by $\widetilde{L}$ the orthogonal projection of the point L on the chord OE.

Lemma 8.6 *For $t \in \left(0, \sqrt{3\pi/2}\right]$ we have the following inequality:*

$$\gamma < \alpha < 2\gamma \ . \tag{28}$$

For $t \in (0, \sqrt{\pi}]$ we have the following inequalities:

$$|O\widetilde{L}| > |\widetilde{L}E| \ , \tag{29}$$

$$|\widehat{OL}| > |\widehat{LE}| \ . \tag{30}$$

See the proof of Lemma 8.6 in Appendix E.

Consider the function $q(t) = 1 + f(t)$, i.e.

$$q(t) = \frac{|\widehat{OE}| - |OF|}{|\widehat{OE}| - |OE|} = \frac{t - \int_0^t \cos\alpha(\tau)d\tau}{t - \int_0^t \cos\gamma(\tau)d\tau} \ . \tag{31}$$

For this function we have the following

Lemma 8.7 *On the interval* $t \in [0, C_{arc}]$

$$\max_{t\in[0,C_{arc}]} q(t) < 4 \ .$$

See the proof of Lemma 8.7 in Appendix **F**.

Corollary 8.8 *For* $t \in [0, C_{arc}]$ *one has* $f(t) = \frac{|OE|-|OF|}{|\widehat{OE}|-|OE|} < 3$.

Corollary 8.8 follows directly from Lemma 8.7.

Proof of Lemma 8.4

Obviously, on $[0, C_{arc}]$ and outside a small half-neighbourhood of zero the function $f(t)$ is a positive-valued continuous function.

Consider the function $f(t)$ on a small half-neighbourhood of zero. In this small half-neighbourhood of zero for the functions $|\widehat{OE}|$, $|OE|$, $|OF|$ and $|EF|$ we have the following formulas:

$$|\widehat{OE}| = t \ ,$$

$$|OF| = \int_0^t \cos\left(B\tau^2/2\right) d\tau = \sqrt{2/B}\int_0^{\sqrt{B/2}t} \cos\tau^2 d\tau =$$

$$= \sqrt{2/B}\int_0^{\sqrt{B/2}t}\left(1-\tau^4/2+O(\tau^8)\right)d\tau = \sqrt{2/B}\left(\sqrt{B/2}t-\left(\sqrt{B/2}t\right)^5/10+O(t^9)\right) =$$

$$= t - B^2t^5/40 + O(t^9) \ , \tag{32}$$

$$|EF| = \int_0^t \sin\left(B\tau^2/2\right) d\tau = \sqrt{2/B}\int_0^{\sqrt{B/2}t} \sin\tau^2 d\tau =$$

$$= \sqrt{2/B}\int_0^{\sqrt{B/2}t}(\tau^2+O(\tau^6))d\tau = \sqrt{2/B}\left(\left(\sqrt{B/2}t\right)^3/3+O(t^7)\right) =$$

$$= Bt^3/6 + O(t^7) \ ,$$

$$|OE| = \sqrt{|EF|^2+|OF|^2} = \sqrt{B^2t^6/36+O(t^{10})+t^2-B^2t^6/20} =$$

$$= t\sqrt{1-B^2t^4/45+O(t^8)} = t\left(1-B^2t^4/90+O(t^8)\right) =$$

$$= t - B^2t^5/90 + O(t^9)\ . \tag{33}$$

Hence, from formulas (32), (33) we obtain

$$f(t) = \frac{t - B^2t^5/90 + O(t^9) - (t - B^2t^5/40 + O(t^9))}{t - (t - B^2t^5/90 + O(t^9))} =$$

$$= \frac{B^2t^5/72 + O(t^9)}{B^2t^5/90 + O(t^9)} = 5/4 + O(t^4)\ .$$

Thus

$$\lim_{t \mapsto +0} f(t) = 5/4 > 0\ .$$

The lemma is proved. □

8.3 Some properties of the function $h(t) = |\widehat{OE}| - |OE|$ and the inverse function h^{-1}.

In the subsection we prove four propositions, the proofs are to be found at the end of the subsection and in Appendix F. We set $B = 2$ in the subsection.

Proposition 8.9 *The function $h(t) = |\widehat{OE}| - |OE|$ (here E is an arbitrary point of a half-clothoid) is a monotonously increasing function for $t \in [0, +\infty)$ and it is a concave function on the interval $[0, t_K]$ (recall that by K we denote the point of a half-clothoid (6) whose chord has the maximal possible length and by t_K the value of t corresponding to the point K).*

Consider the function $h(t) = |\widehat{OE}| - |OE|$ on the interval $[0, t_K]$. Consider the inverse function $h^{-1}(\omega)$ on the corresponding interval $[0, h(t_K)]$ (see Figure 18).

Consider some point $E \in h^{-1}(\omega)$ for $\omega \in [0, h(t_K)]$. The line EB is tangent to the graph of the function $h^{-1}(\omega)$. The point C is a projection of the point E on the argument-axis. Denote $|OC|$ by Δ. Fix some number k, $0 < k < 1$. The point $R \in \widehat{OE}$ is such point that $|OR'| = k|OC| = k\Delta$ (here by R' we denote a projection of the point R on the argument-axis).

The line ER intersects the argument axis at the point P. Denote $|OP|$ by l and denote $|BO|$ by c. We have the following estimation of the ratio

$$\frac{h^{-1}(k\Delta)}{h^{-1}(\Delta)}\ .$$

Lemma 8.10 *For every point $E \in \widehat{OK}$ we have the inequality:*

$$\frac{h^{-1}(k\Delta)}{h^{-1}(\Delta)} < 1 - (1-k)/64\ .$$

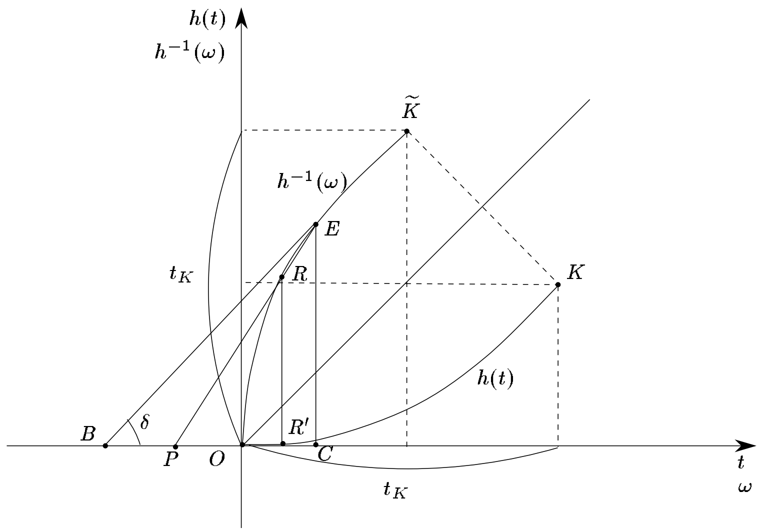

Figure 18

The proof of Lemma 8.10 is in Appendix F.

In order to prove the lemma we prove some auxiliary propositions.

Denote by L the point of the arc $\widehat{OE}$ where the tangent line is parallel to the chord OE (see Figure 19). The lines $L\widetilde{L}$ and ED are perpendicular to the chord OE. Denote by χ the angle $LO\widetilde{L}$ and by ε the angle $\widetilde{L}EL$.

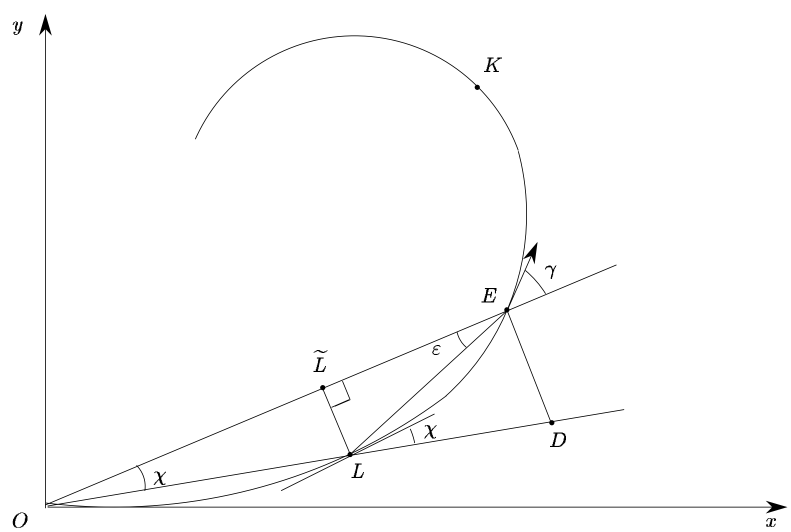

Figure 19

Proposition 8.11 *For every point $E \in \widehat{OK}$ we have the following inequalities:*

$$|OD| < |\widehat{OLE}| \ , \tag{34}$$

$$H(E)/D(E) < \cos\chi \ . \tag{35}$$

Proposition 8.12 *For every point $E \in \widehat{OK}$ and corresponding t*

$$H(E)/D(E) = H(t)/D(t) = H'(\theta t)/D'(\theta t) \ ,$$

where $1/2 < \theta < 1$.

Proof of Proposition 8.9

The length of the projection of the unitary tangent vector $\vec{\tau}$ at the point E on the chord OE is smaller than 1. Hence, $|\widehat{OE}|$ grows more quickly than $|OE|$ and the function $h(t)$ is a monotonously increasing function for $t \in [0, +\infty)$. Consider the function $h(t) = |\widehat{OE}| - |OE|$ on the interval $[0, t_K]$. We have

$$h'(t) = 1 - \frac{\cos t^2 \int_0^t \cos\tau^2 d\tau + \sin t^2 \int_0^t \sin\tau^2 d\tau}{\sqrt{\left(\int_0^t \cos\tau^2 d\tau\right)^2 + \left(\int_0^t \sin\tau^2 d\tau\right)^2}} =$$

$$= 1 - \frac{|OF|\cos t^2 + |EF|\sin t^2}{|OE|} = 1 - \left(\frac{|OF|}{|OE|}\cos t^2 + \frac{|EF|}{|OE|}\sin t^2\right) \ .$$

Recall that by β we denote the angle between the vectors $\overrightarrow{OF}$ and $\overrightarrow{OE}$, by γ the angle between the vector $\overrightarrow{OE}$ and the tangent vector $\vec{\tau}$ (see Figure 17) and that $\alpha = t^2$ for $B = 2$. Hence,

$$h'(t) = 1 - (\cos\beta\cos\alpha + \sin\beta\sin\alpha) = 1 - \cos(\alpha - \beta) = 1 - \cos\gamma \ .$$

So

$$h''(t) = \dot{\gamma}\sin\gamma \ .$$

Recall that $t_K \in (\sqrt{\pi/B}, \sqrt{3\pi/(2B)})$ by Proposition 5.1. Hence, by Lemma 8.5 the angle γ is a monotonously increasing function for $t \in [0, t_K]$ and, hence, $\dot{\gamma} > 0$, $\sin\gamma > 0$.

Thus we obtain $h''(t) > 0$, i.e. the function $h(t)$ is a concave function on the interval $[0, t_K]$.

The proposition is proved. □

Proof of Proposition 8.11

For the angles χ and ε we have the following formulas:

$$\cos\chi = \frac{|O\widetilde{L}|}{|OL|} = \frac{|O\widetilde{L}|}{\sqrt{|O\widetilde{L}|^2 + |\widetilde{L}L|^2}} \ ,$$

$$\cos\varepsilon = \frac{|\widetilde{L}E|}{|LE|} = \frac{|\widetilde{L}E|}{\sqrt{|\widetilde{L}E|^2 + |\widetilde{L}L|^2}} \ .$$

From Lemma 8.6 we have the inequality $|O\widetilde{L}| > |\widetilde{L}E|$. Hence,

$$\cos\chi > \cos\varepsilon \ , \tag{36}$$

because this inequality is equivalent to the following inequality:

$$\frac{|O\widetilde{L}|^2}{|O\widetilde{L}|^2 + |\widetilde{L}L|^2} > \frac{|\widetilde{L}E|^2}{|\widetilde{L}E|^2 + |\widetilde{L}L|^2} \ ,$$

i.e.

$$|O\widetilde{L}|^2|\widetilde{L}E|^2 + |O\widetilde{L}|^2|\widetilde{L}L|^2 > |O\widetilde{L}|^2|\widetilde{L}E|^2 + |\widetilde{L}L|^2|\widetilde{L}E|^2 \ ,$$

hence,

$$|O\widetilde{L}| > |\widetilde{L}E| \ .$$

For the lengths of the line segments LD and LE we have the following formulas:

$$|LD| = |\widetilde{L}E|/\cos\chi \ , \qquad\qquad |LE| = |\widetilde{L}E|/\cos\varepsilon \ .$$

So, using inequality (36), we obtain

$$|LD| < |LE| \ . \tag{37}$$

Now, using (37) we have

$$|OD| = |OL| + |LD| < |\widehat{OL}| + |LE| < |\widehat{OL}| + |\widehat{LE}| = |\widehat{OLE}| \ .$$

Inequality (34) is proved. To prove inequality (35) we estimate the ratio $H(E)/D(E)$, using inequality (35), as follows:

$$H(E)/D(E) = |OE|/|\widehat{OLE}| < |OE|/|OD| = \cos\chi \ .$$

Thus, we obtain inequality (35).

The proposition is proved. □

Proof of Proposition 8.12

Consider the two functions $H(t)$ and $D(t)$. Apply Roll's Theorem for every point $E \in \widehat{OK}$. We have

$$\frac{H(t_E) - H(0)}{D(t_E) - D(0)} = \frac{H'(t_M)}{D'(t_M)} \qquad \text{where } t_M \in [0, t_E] \ . \tag{38}$$

But $H(0) = D(0) = 0$, hence, we can rewrite (38) as follows:

$$\frac{H(t_E)}{D(t_E)} = \frac{H'(t_M)}{D'(t_M)} \ ,$$

and, using (35), we obtain the following inequality:

$$\frac{H'(t_M)}{D'(t_M)} < \cos\chi \ . \tag{39}$$

Denote by $\gamma(t_M)$ the angle between the chord OM and the tangent vector at the point M. Then

$$\frac{H'(t_M)}{D'(t_M)} = \cos\gamma(t_M) \tag{40}$$

and, from (39) and (40) we obtain that $\cos\gamma(t_M) < \cos\chi$, i.e. $\gamma(t_M) > \chi$.

The angle χ is the angle between the chord OL and the tangent vector at the point L (we can denote χ by $\gamma(t_L)$). The angle γ is a monotonously increasing function of t on the interval $[0, \sqrt{3\pi/4}]$ (by Lemma 8.5), hence, γ is a monotonously increasing function of t on the interval $[0, t_K]$ (by Proposition 5.1). Hence, if $\gamma(t_M) > \gamma(t_L)$, then $t_M > t_L$, i.e. $M \in \widehat{LE}$. From Lemma 8.6 we know that $|\widehat{OL}| > |\widehat{LE}|$, i.e. $t_L > t_E/2$, and, hence, $t_M > t_E/2$.

Thus we have proved that

$$\frac{H(t_E)}{D(t_E)} = \frac{H'(\theta t_E)}{D'(\theta t_E)}, \qquad \text{where } 1/2 < \theta < 1 \ .$$

The proposition is proved. □

9 Acknowledgement

J.D.Boissonnat stated the problem and together with A.Cérézo and J.Leblond participated in numerous and helpful discussions. To all of them we express our most sincere gratitude.

A Appendix: Proof of Lemma 4.2.

An arbitrary point A of the clothoid (6) has a tangent vector $\vec{\tau}(t)$ with coordinates ($\cos t^2$, $\sin t^2$) (see Figure 20).

Consider a point D of the clothoid (6) with tangent vector $\vec{\tau}_n = (0, 1)$. The point A is mapped onto the point D by means of the rotation on angle θ defined by the rotation matrix

$$\begin{pmatrix} \sin t^2 & -\cos t^2 \\ \cos t^2 & \sin t^2 \end{pmatrix} .$$

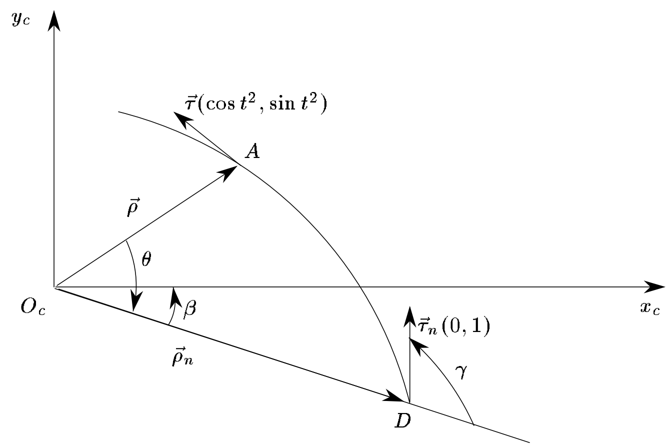

Figure 20

Hence, th e radius-vector $\vec{\rho} = (-\int_t^\infty \cos\tau^2 d\tau, -\int_t^\infty \sin\tau^2 d\tau)$ (see (7)) is mapped into the radius-vector

$$\vec{\rho}_n = \left(-\sin t^2 \int_t^\infty \cos\tau^2 d\tau + \cos t^2 \int_t^\infty \sin\tau^2 d\tau\ ,\right.$$

$$\left. -\cos t^2 \int_t^\infty \cos\tau^2 d\tau - \sin t^2 \int_t^\infty \sin\tau^2 d\tau\right) =$$

$$= \left(\int_t^\infty \sin(\tau^2 - t^2) d\tau\ ,\ -\int_t^\infty \cos(\tau^2 - t^2) d\tau\right) =$$

$$\left(\int_0^\infty \frac{\sin\nu d\nu}{2\sqrt{\nu + t^2}}\ ,\ -\int_0^\infty \frac{\cos\nu d\nu}{2\sqrt{\nu + t^2}}\right)\ .$$

We want to investigate the function $d\rho/dt$. Instead of it we can investigate the function $d\gamma/dt$ (see formula (10)). Denote by β the angle between the vector $\vec{\rho}_n$ and the axis $O_c x_c$. At the point D we have the following relations between the angles γ, β and the coordinates x_n, y_n of the vector $\vec{\rho}_n$:

$$\cot\gamma = -\tan\beta = -\frac{y_n}{x_n} = \int_0^\infty \frac{\cos\nu d\nu}{2\sqrt{\nu + t^2}} \Big/ \int_0^\infty \frac{\sin\nu d\nu}{2\sqrt{\nu + t^2}}\ .$$

Compute the derivative $d(\tan\beta)/dt$:

$$\frac{d(\tan\beta)}{dt} = -\frac{t}{4{x_n}^2}\left[\int_0^\infty \frac{\cos\tau d\tau}{(\sqrt{\tau + t^2})^3} \int_0^\infty \frac{\sin\tau d\tau}{\sqrt{\tau + t^2}} - \int_0^\infty \frac{\sin\tau d\tau}{(\sqrt{\tau + t^2})^3} \int_0^\infty \frac{\cos\tau d\tau}{\sqrt{\tau + t^2}}\right] =$$

$$= -\frac{t}{4{x_n}^2}\left[\left\{\frac{3}{2}\int_0^\infty \frac{\sin\tau\, d\tau}{\left(\sqrt{\tau+t^2}\right)^5} - \left.\frac{\sin\tau}{\left(\sqrt{\tau+t^2}\right)^3}\right|_0^\infty\right\}\int_0^\infty \frac{\sin\tau\, d\tau}{\sqrt{\tau+t^2}} - \right.$$

$$\left. - \left\{\frac{1}{2}\int_0^\infty \frac{\sin\tau\, d\tau}{\left(\sqrt{\tau+t^2}\right)^3} - \left.\frac{\sin\tau}{\sqrt{\tau+t^2}}\right|_0^\infty\right\}\int_0^\infty \frac{\sin\tau\, d\tau}{\left(\sqrt{\tau+t^2}\right)^3}\right] =$$

$$= -\frac{t}{8{x_n}^2}\left[3\int_0^\infty \frac{\sin\tau\, d\tau}{\left(\sqrt{\tau+t^2}\right)^5}\int_0^\infty \frac{\sin\tau\, d\tau}{\sqrt{\tau+t^2}} - \left(\int_0^\infty \frac{\sin\tau\, d\tau}{\left(\sqrt{\tau+t^2}\right)^3}\right)^2\right]$$

(We use integration by parts).

Denote the expression in the brackets as $J(t^2)$. Consider $J(t^2)$ with ∞ changed to $2\pi p$ $(p \in \mathbf{N}, p > 1)$. Consider the corresponding Riemann sums with step $\Delta = \pi/n$ instead of the integrals:

$$\int_0^{2\pi p} \frac{\sin\tau\, d\tau}{\left(\sqrt{\tau+t^2}\right)^i} \cong \sum_{k=1}^{2np} \frac{\sin\tau_k}{\left(\sqrt{\tau_k+t^2}\right)^i}\Delta + O(\Delta), \quad \tau_k = \pi k/n, \quad i = \{1,3,5\} \tag{41}$$

The function $\sin\tau$ is periodic with period 2π and $\sin(\pi+\tau) = -\sin\tau$.

Denote the three Riemann sums (corresponding to the three integrals) by

$$d_1 + \ldots + d_{np}\ , \qquad g_1 + \ldots + g_{np}\ , \qquad h_1 + \ldots + h_{np}\ ,$$

where if $j = s + \nu n, \quad s = 1, \ldots, n, \quad \nu = 0, \quad \ldots, p-1$, then

$$d_j = \frac{\sin\tau_s}{\sqrt{\tau_s + 2\nu\pi + t^2}} - \frac{\sin\tau_s}{\sqrt{\tau_s + 2\nu\pi + \pi + t^2}}\ ,$$

$$g_j = \frac{\sin\tau_s}{\left(\sqrt{\tau_s + 2\nu\pi + t^2}\right)^3} - \frac{\sin\tau_s}{\left(\sqrt{\tau_s + 2\nu\pi + \pi + t^2}\right)^3}\ ,$$

$$h_j = \frac{\sin\tau_s}{\left(\sqrt{\tau_s + 2\nu\pi + t^2}\right)^5} - \frac{\sin\tau_s}{\left(\sqrt{\tau_s + 2\nu\pi + \pi + t^2}\right)^5}\ .$$

Show that

$$I \equiv 3d_j h_j - {g_j}^2 > 0\ . \tag{42}$$

Set $\tau_s + 2\nu\pi = a$. Then rewrite I as follows:

$$3\left(\frac{\sin a}{\left(\sqrt{a+t^2}\right)^5} - \frac{\sin a}{\left(\sqrt{a+\pi+t^2}\right)^5}\right)\left(\frac{\sin a}{\sqrt{a+t^2}} - \frac{\sin a}{\sqrt{a+\pi+t^2}}\right) -$$

$$-\left(\frac{\sin a}{(\sqrt{a+t^2})^3}-\frac{\sin a}{(\sqrt{a+\pi+t^2})^3}\right)^2 .$$

Denote $\sqrt{a+t^2}$ by α, $\sqrt{a+\pi+t^2}$ by β. Then

$$I=3\left(\frac{1}{\alpha^5}-\frac{1}{\beta^5}\right)\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)-\left(\frac{1}{\alpha^3}-\frac{1}{\beta^3}\right)^2=3\frac{(\beta^5-\alpha^5)(\beta-\alpha)}{\alpha^6\beta^6}-\frac{(\beta^3-\alpha^3)^2}{\alpha^6\beta^6}=$$

$$=\frac{3(\beta-\alpha)^2(\beta^4+\beta^3\alpha+\beta^2\alpha^2+\beta\alpha^3+\alpha^4)-(\beta-\alpha)^2(\beta^2+\beta\alpha+\alpha^2)^2}{\alpha^6\beta^6}=$$

$$=\frac{(\beta^2-\alpha^2)^2[3(\beta^4+\beta^3\alpha+\beta^2\alpha^2+\beta\alpha^3+\alpha^4)-(\beta^2+\beta\alpha+\alpha^2)^2]}{\alpha^6\beta^6(\beta+\alpha)^2}=$$

$$=\frac{\pi^2(2\beta^4+2\alpha^4+\beta^3\alpha+\beta\alpha^3)}{\alpha^6\beta^6(\beta+\alpha)^2}>0 .$$

Thus we prove (42). Show that

$$K\equiv 3(d_ih_j+d_jh_i)-2g_ig_j>0 . \tag{43}$$

Set

$$\tau_s+2\nu\pi=a_i , \qquad \tau_w+2\nu\pi=a_j ,$$

$$\sqrt{a_i+t^2}=\alpha , \qquad \sqrt{a_i+\pi+t^2}=\beta ,$$

$$\sqrt{a_j+t^2}=\gamma , \qquad \sqrt{a_j+\pi+t^2}=\delta .$$

Rewrite K as follows:

$$K=3\left[\left(\frac{\sin a_i}{\alpha^5}-\frac{\sin a_i}{\beta^5}\right)\left(\frac{\sin a_j}{\gamma}-\frac{\sin a_j}{\delta}\right)+\left(\frac{\sin a_j}{\gamma^5}-\frac{\sin a_j}{\delta^5}\right)\left(\frac{\sin a_i}{\alpha}-\frac{\sin a_i}{\beta}\right)\right]-$$

$$-2\left(\frac{\sin a_i}{\alpha^3}-\frac{\sin a_i}{\beta^3}\right)\left(\frac{\sin a_j}{\gamma^3}-\frac{\sin a_j}{\delta^3}\right)=$$

$$=\frac{\pi^2\sin a_i\sin a_j}{(\beta+\alpha)(\gamma+\delta)\alpha\beta\gamma\delta}\left[\frac{3(\beta^4+\beta^3\alpha+\beta^2\alpha^2+\beta\alpha^3+\alpha^4)}{\alpha^4\beta^4}+\right.$$

$$+\frac{3(\delta^4+\delta^3\gamma+\delta^2\gamma^2+\delta\gamma^3+\gamma^4)}{\gamma^4\delta^4}-$$

$$-\left.\frac{2(\beta^2\delta^2+\beta^2\delta\gamma+\beta^2\gamma^2+\delta^2\beta\alpha+\beta\alpha\delta\gamma+\gamma^2\beta\alpha+\alpha^2\delta^2+\alpha^2\delta\gamma+\alpha^2\gamma^2)}{\alpha^2\beta^2\gamma^2\delta^2}\right] \ . \qquad (44)$$

Estimate the expression in the brackets (denote it by L).

$$L=\left\{\frac{\alpha^4+2\alpha^2\beta^2+\beta^4}{\alpha^4\beta^4}+\frac{\gamma^4+2\gamma^2\delta^2+\delta^4}{\gamma^4\delta^4}-\frac{2(\beta^2\delta^2+\alpha^2\delta^2+\beta^2\gamma^2+\beta^2\delta^2)}{\alpha^2\beta^2\gamma^2\delta^2}\right\}+$$

$$+\left\{\frac{1}{2}\left(\frac{1}{\alpha^3\beta}+\frac{1}{\alpha\beta^3}+\frac{1}{\gamma^3\delta}+\frac{1}{\gamma\delta^3}\right)-\frac{2}{\alpha\beta\gamma\delta}\right\}+$$

$$+\left\{2\left(\frac{1}{\alpha^4}+\frac{1}{\beta^4}+\frac{1}{\gamma^4}+\frac{1}{\delta^4}\right)-\frac{2(\beta^2\delta\gamma+\delta^2\beta\alpha+\gamma^2\beta\alpha+\alpha^2\delta\gamma)}{\alpha^2\beta^2\gamma^2\delta^2}\right\}+$$

$$+\left\{\frac{1}{\alpha^2\beta^2}+\frac{1}{2}\left(\frac{1}{\alpha^3\beta}+\frac{1}{\alpha\beta^3}+\frac{1}{\gamma^3\delta}+\frac{1}{\gamma\delta^3}\right)+\frac{1}{\gamma^2\delta^2}\right\} \ . \qquad (45)$$

The expression in the first parentheses is positive. Really,

$$\frac{\alpha^4+2\alpha^2\beta^2+\beta^4}{\alpha^4\beta^4}+\frac{\gamma^4+2\gamma^2\delta^2+\delta^4}{\gamma^4\delta^4}=\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)^2+\left(\frac{1}{\gamma^2}+\frac{1}{\delta^2}\right)^2>$$

$$>2\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)\left(\frac{1}{\gamma^2}+\frac{1}{\delta^2}\right)=\frac{2(\beta^2\delta^2+\alpha^2\delta^2+\beta^2\gamma^2+\alpha^2\gamma^2)}{\alpha^2\beta^2\gamma^2\delta^2} \ .$$

The expression within the second parentheses is also positive because we have the following inequality:

$$\frac{1}{4}\left\{\frac{1}{\alpha^3\beta}+\frac{1}{\alpha\beta^3}+\frac{1}{\gamma^3\delta}+\frac{1}{\gamma\delta^3}\right\}>\left(\frac{1}{\alpha^3\beta\alpha\beta^3\gamma^3\delta\gamma\delta^3}\right)^{1/4}=\frac{1}{\alpha\beta\gamma\delta} \ .$$

Hence

$$\frac{1}{2}\left\{\frac{1}{\alpha^3\beta}+\frac{1}{\alpha\beta^3}+\frac{1}{\gamma^3\delta}+\frac{1}{\gamma\delta^3}\right\}>\frac{2}{\alpha\beta\gamma\delta} \ .$$

Estimate the expression within the third parentheses. Denote by M the following fraction:

$$M=\frac{2(\beta^2\delta\gamma+\delta^2\beta\alpha+\gamma^2\beta\alpha+\alpha^2\delta\gamma)}{\alpha^2\beta^2\gamma^2\delta^2}=\frac{2}{\gamma\delta}\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)+\frac{2}{\alpha\beta}\left(\frac{1}{\gamma^2}+\frac{1}{\delta^2}\right) \ .$$

Using the inequalities

$$\frac{2}{\alpha\beta} < \frac{1}{\alpha^2} + \frac{1}{\beta^2}$$

and

$$\frac{2}{\gamma\delta} < \frac{1}{\gamma^2} + \frac{1}{\delta^2}$$

we obtain

$$M < 2\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right)\left(\frac{1}{\gamma^2} + \frac{1}{\delta^2}\right) < \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right)^2 + \left(\frac{1}{\gamma^2} + \frac{1}{\delta^2}\right)^2 =$$

$$= \frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{2}{\alpha^2\beta^2} + \frac{1}{\gamma^4} + \frac{1}{\delta^4} + \frac{2}{\gamma^2\delta^2} < 2\left(\frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4} + \frac{1}{\delta^4}\right) .$$

So the expression within the third parentheses is positive and L (see (45)) is positive. Hence the expression K (see (44)) is positive because the points a_i, a_j belong to the interval $(0, \pi]$ and, hence, the functions $\sin a_i$, $\sin a_j$ are non-negative. So we prove (43).

From (42) and (43) when $n \to \infty$ it follows that

$$3\int_0^{2\pi p} \frac{\sin\tau\, d\tau}{(\sqrt{\tau + t^2})^5} \int_0^{2\pi p} \frac{\sin\tau\, d\tau}{\sqrt{\tau + t^2}} - \left(\int_0^{2\pi p} \frac{\sin\tau\, d\tau}{(\sqrt{\tau + t^2})^3}\right)^2 > 0 .$$

If $2\pi p \to \infty$ and $n \to \infty$ we obtain that $J(t^2) > 0$ and, hence, $d(\tan\beta)/dt < 0$. Remember that $\tan\beta = -\cot\gamma$ and $\ddot{\rho} = -\dot{\gamma}\sin\gamma$ (see (11)). Hence,

$$\frac{d(\cot\gamma)}{dt} = -\frac{\dot{\gamma}}{\sin^2\gamma} > 0 , \qquad \dot{\gamma} < 0$$

and

$$\ddot{\rho} > 0 .$$

The lemma is proved. □

B Appendix: Proofs of Lemma 6.1 and Corollary 6.2.

B.1 Proof of Lemma 6.1.

Consider a coordinate system $A\xi\eta$ (see Figure 21), the axis η coincides with the tangent vector to $cl1$ and $cl2$ at the point A, the axis ξ is a perpendicular to the axis η.

In this coordinate system $cl1$ and $cl2$ are defined by the following equations:

$$cl1 : \quad \begin{cases} \xi(t) = \int_0^t \cos(\tau^2 + u_0\tau + \pi/2)d\tau \\ \\ \eta(t) = \int_0^t \sin(\tau^2 + u_0\tau + \pi/2)d\tau \end{cases}$$

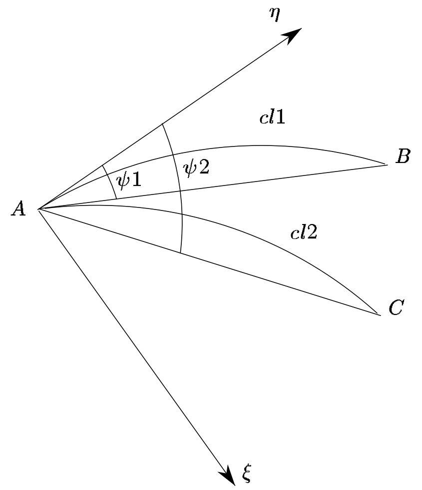

Figure 21

$$cl2 : \quad \begin{cases} \xi(t) = \int_0^t \cos\left(-\tau^2 + u_0\tau + \pi/2\right) d\tau \\ \\ \eta(t) = \int_0^t \sin\left(-\tau^2 + u_0\tau + \pi/2\right) d\tau \end{cases}$$

So for the coordinates of the points B and C we have the following formulas:

$$\xi_B(s) = -\int_0^s \sin\left(\tau^2 + u_0\tau\right) d\tau \ , \qquad \eta_B(s) = \int_0^s \cos\left(\tau^2 + u_0\tau\right) d\tau \ ,$$

$$\xi_C(s) = -\int_0^s \sin\left(-\tau^2 + u_0\tau\right) d\tau \ , \qquad \eta_C(s) = \int_0^s \cos\left(-\tau^2 + u_0\tau\right) d\tau \ .$$

Then, using the Taylor series at 0 for the functions $\sin x, \cos x$ we obtain:

$$\xi_B(s) = -\frac{u_0}{2}s^2 - \frac{1}{3}s^3 + O(s^4) \ , \qquad \eta_B(s) = s - \frac{u_0^2}{6}s^3 + O(s^4) \ ,$$

$$\xi_C(s) = -\frac{u_0}{2}s^2 + \frac{1}{3}s^3 + O(s^4) \ , \qquad \eta_C(s) = s - \frac{u_0^2}{6}s^3 + O(s^4) \ .$$

But $\tan\psi_1 = \xi_B(s)/\eta_B(s)$ and $\tan\psi_2 = \xi_C(s)/\eta_C(s)$. Now we use Taylor series again and obtain the following formulas for $\tan\psi_1$ and $\tan\psi_2$:

$$\tan\psi_1 = -\frac{u_0}{2}s - \frac{1}{3}s^2 + O(s^3) \ , \qquad \tan\psi_2 = -\frac{u_0}{2}s + \frac{1}{3}s^2 + O(s^3) \ .$$

Since we consider the clothoids $cl1$ and $cl2$ on a small interval $[0, s]$, we can use for the angles ψ_1 and ψ_2 the following formulas:

$$\psi_1 = -\frac{u_0}{2}s - \frac{1}{3}s^2 + O(s^3) \ , \qquad \psi_2 = -\frac{u_0}{2}s + \frac{1}{3}s^2 + O(s^3) \ . \tag{46}$$

Compute the values of ρ_B^2 and ρ_C^2. For this purpose we use the cosine theorem, the Taylor series and formulas (46):

$$\rho_B^2 = \rho_A^2 + s^2 - 2\rho_A s \cos(\theta_0 - \psi_1) =$$

$$= \rho_A^2 + s^2 - 2\rho_A s \left[\cos\theta_0 \cos\left(-\frac{u_0}{2}s - \frac{1}{3}s^2 + O(s^3)\right) + \sin\theta_0 \sin\left(-\frac{u^0}{2}s - \frac{1}{3}s^2 + 0(s^3)\right)\right] =$$

$$= \rho_A^2 + s^2 - 2\rho_A s \left[\cos\theta_0 \left(1 - \frac{1}{2}\left(\frac{u_0}{2}s + \frac{1}{3}s^2\right)^2\right) - \sin\theta_0 \left(\frac{u_0}{2}s + \frac{1}{3}s^2\right)\right] =$$

$$= \rho_A^2 - 2\rho_A \cos\theta_0 s + (1 + \rho_A u_0 \sin\theta_0)s^2 + \left(\rho_A \frac{u_0^2}{4}\cos\theta_0 + \frac{2}{3}\rho_A \sin\theta_0\right)s^3 + O(s^4) \ ,$$

$$\rho_C^2 = \rho_A^2 + s^2 - 2\rho_A s \cos(\theta_0 - \psi_2) =$$

$$= \rho_A^2 + s^2 - 2\rho_A s \left[\cos\theta_0 \cos\left(-\frac{u_0}{2}s + \frac{1}{3}s^2 + O(s^3)\right) + \sin\theta_0 \sin\left(-\frac{u_0}{2}s + \frac{1}{3}s^2 + 0(s^3)\right)\right] =$$

$$= \rho_A^2 + s^2 - 2\rho_A s \left[\cos\theta_0 \left(1 - \frac{1}{2}\left(-\frac{u_0}{2}s + \frac{1}{3}s^2\right)^2\right) + \sin\theta_0 \left(-\frac{u_0}{2}s + \frac{1}{3}s^2\right)\right] =$$

$$= \rho_A^2 - 2\rho_A \cos\theta_0 s + (1 + \rho_A u_0 \sin\theta_0)s^2 + \left(\rho_A \frac{u_0^2}{4}\cos\theta_0 - \frac{2}{3}\rho_A \sin\theta_0\right)s^3 + O(s^4) \ .$$

Thus we obtain formula (18):

$$\rho_B^2 - \rho_C^2 = \frac{4}{3}\rho_A \sin\theta_0 s^3 + O(s^4) \ .$$

Compute the values of the angles δ_1 and δ_2. From formulas (16) and (17) for the angles θ_1 and θ_2 we obtain:

$$\begin{cases} \theta_1 = \theta_0 + u_0 s + s^2 \\ \theta_2 = \theta_0 + u_0 s - s^2 \end{cases} \tag{47}$$

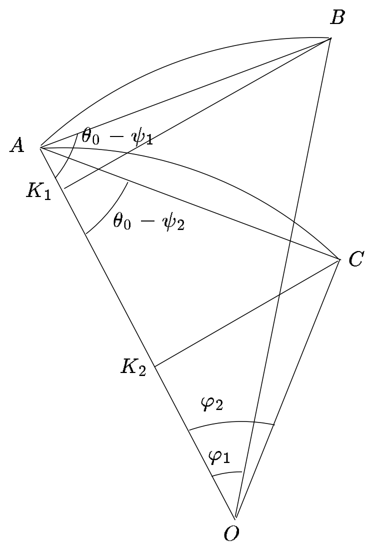

Figure 22

To compute the angles φ_1 and φ_2 make the additional construction (see Figure 22): the segments BK_1 and CK_2 are perpendicular to the line OA.

We have

$$|K_1B| = |AB|\sin(\theta_0 - \psi_1) = |OK_1|\tan\varphi_1 \ ,$$

$$|K_2C| = |AC|\sin(\theta_0 - \psi_2) = |OK_2|\tan\varphi_2 \ .$$

Hence,

$$\tan\varphi_1 = \frac{|AB|}{|OK_1|}\sin(\theta_0 - \psi_1) \ ,$$

$$\tan\varphi_2 = \frac{|AC|}{|OK_2|}\sin(\theta_0 - \psi_2) \ .$$

But

$$|AB| = s + O(s^2) \ , \qquad |AC| = s + O(s^2) \ ,$$

$$|OK_1| = |OA| - |AK_1| = \rho_A - |AB|\cos(\theta_0 - \psi_1) = \rho_A - s\cos(\theta_0 - \psi_1) \ ,$$

$$|OK_2| = |OA| - |AK_2| = \rho_A - |AC|\cos(\theta_0 - \psi_2) = \rho_A - s\cos(\theta_0 - \psi_2) \ .$$

Thus we have that

$$\tan\varphi_1 = \frac{s\sin(\theta_0-\psi_1)}{\rho_A - s\cos(\theta_0-\psi_1)}\ , \qquad \tan\varphi_2 = \frac{s\sin(\theta_0-\psi_2)}{\rho_A - s\cos(\theta_0-\psi_2)}\ .$$

Now using formulas (47) and Taylor series for the functions $\cos x$, $\sin x$ and $f(x) = 1/(1+x)$ at 0 we obtain the following expressions:

$$\sin(\theta_0-\psi_1) = \sin\theta_0\cos\psi_1 - \cos\theta_0\sin\psi_1 =$$

$$= \sin\theta_0\left(1-\frac{1}{2}\left(\frac{u_0}{2}s+\frac{1}{3}s^2\right)^2\right) + \cos\theta_0\left(\frac{u_0}{2}s+\frac{1}{3}s^2\right) =$$

$$= \sin\theta_0 + \frac{u_0}{2}\cos\theta_0 s + \left(\frac{\cos\theta_0}{3} - \frac{u_0^2}{8}\sin\theta_0\right)s^2 + O(s^3)\ ,$$

$$\cos(\theta_0-\psi_1) = \cos\theta_0\cos\psi_1 + \sin\theta_0\sin\psi_1 =$$

$$= \cos\theta_0\left(1-\frac{1}{2}\left(\frac{u_0}{2}s+\frac{1}{3}s^2\right)^2\right) - \sin\theta_0\left(\frac{u_0}{2}s+\frac{1}{3}s^2\right) =$$

$$= \cos\theta_0 - \frac{u_0}{2}\sin\theta_0 s - \left(\frac{\sin\theta_0}{3} + \frac{u_0^2}{8}\cos\theta_0\right)s^2 + O(s^3)\ ,$$

$$\tan\varphi_1 = \frac{s\sin(\theta_0-\psi_1)}{\rho_A}\frac{1}{1-\frac{s}{\rho_A}\cos(\theta_0-\psi_1)} =$$

$$= \frac{s}{\rho_A}\sin(\theta_0-\psi_1)\left(1+\frac{s}{\rho_A}\cos(\theta_0-\psi_1)+\frac{s^2}{{\rho_A}^2}\cos^2(\theta_0-\psi_1)\right)\ .$$

Hence after this series of transformations we obtain the formula for $\tan\varphi_1$:

$$\tan\varphi_1 = \frac{\sin\theta_0}{\rho_A}s + \left(\frac{\sin 2\theta_0}{2\rho_A^2} + \frac{u_0\cos\theta_0}{2\rho_A}\right)s^2 +$$

$$+\left(\frac{\cos\theta_0}{3\rho_A} - \frac{u_0^2\sin\theta_0}{8\rho_A} + \frac{u_0\cos 2\theta_0}{2\rho_A^2} + \frac{\sin 2\theta_0\cos\theta_0}{2\rho_A^3}\right)s^3 + O(s^4)\ . \tag{48}$$

After analogous transformations we obtain the formula for $\tan\varphi_2$:

$$\tan\varphi_2 = \frac{\sin\theta_0}{\rho_A}s + \left(\frac{\sin 2\theta_0}{2\rho_A^2} + \frac{u_0\cos\theta_0}{2\rho_A}\right)s^2 +$$

$$+\left(-\frac{\cos\theta_0}{3\rho_A} - \frac{u_0^2\sin\theta_0}{8\rho_A} + \frac{u_0\cos 2\theta_0}{2\rho_A^2} + \frac{\sin 2\theta_0\cos\theta_0}{2\rho_A^3}\right)s^3 + O(s^4)\ . \qquad (49)$$

In a small neighbourhood of the initial point A $\tan\varphi_i = \varphi_i + O(\varphi_i^3)$ $(i = 1, 2)$. Hence, from the definitions of the angles δ_1 and δ_2 and from formulas (48)– (49) we obtain equality (19):

$$\delta_1 - \delta_2 = 2s^2 + \frac{2\cos\theta_0}{3\rho_A}s^3 + O(s^4)\ .$$

The lemma is proved. □

B.2 Proof of Corollary 6.2.

It follows from (17) that the absolute value of the curvature at the point C is greater than the one at the point A. That is why the point C_c is located before the point A.

Note that the angles γ_i and δ_i are connected by the following equations: $\gamma_i = \pi - \delta_i$ $(i = 1, 2)$. Hence from formulas (47) and (48) we obtain that

$$\gamma_A - \gamma_B = \delta_1 - \theta_0 = \left(u_0 + \frac{\sin\theta_0}{\rho_A}\right)s + \left(1 + \frac{\sin 2\theta_0}{2\rho_A^2} + \frac{u_0\cos\theta_0}{2\rho_A}\right)s^2 + O(s^3)\ .$$

From Remark 4.3 we obtain that the angle γ is a monotonously decreasing function, hence,

$$\gamma_B < \gamma_A\ .$$

From (19) we have

$$\gamma_C - \gamma_B = \delta_1 - \delta_2 = 2s^2 + \frac{2\cos\theta_0}{3\rho_A}s^3 + O(s^4)\ .$$

So, $\gamma_B < \gamma_C$ and $\gamma_B < \gamma_A$. But the difference between γ_B and γ_A is of order s, and the difference between γ_B and γ_C is of order s^2. Hence, we obtain the following inequalities

$$\gamma_A > \gamma_C > \gamma_B$$

and the point C_γ is located between the points A and B.

The difference between ρ_B^2 and ρ_C^2 is of order s^3 (see (18)). The difference between γ_C and γ_B is of order s^2. Hence, the point C_ρ is located between the points C_γ and B.

The corollary is proved. □

C Appendix: Proof of Lemma 7.1.

a) Consider the path cl. We parametrise it by the natural parameter s, setting $s = 0$ for the point $(x_0, y_0, \alpha_0, u_0)$. Hence, the graph of the curvature u as a function of the path length s looks like the one shown on Figure 11 (for $s < 0$ it is given by the dotted line). In the proof we consider the path cl only on $[-u_0/2, u_0/2]$.

Denote by O the point of the path cl with zero curvature (i.e. $s = u_0/2$), by A – the point with curvature $2u_0$ (i.e. $s = -u_0/2$), by S – the point with curvature u_0 (i.e. $s = 0$) and by P – an arbitrary point corresponding to some value of the parameter $s \in (-u_0/2, u_0/2)$ $(u_P(s) \in (0, 2u_0))$, see Figure 23.

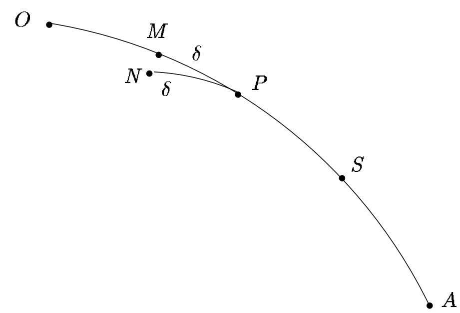

Figure 23

Consider a small δ-half-neighbourhood $(s, s+\delta)$ of the point P and consider a path beginning at the point P which is piecewise clothoid ($u = -2s + \tilde{u}^0$ or $u = 2s + \tilde{\tilde{u}}^0$, $\tilde{u}^0 > 0$, $\tilde{\tilde{u}}^0 > 0$), of length δ and with the same values of x, y, α, u at the point P as the ones of the point P of cl. Denote the final point of this path by N, the final point of the corresponding piece of the clothoid cl by M (the lengths of the arcs $\widehat{PM}$ and $\widehat{PN}$ are equal to δ, the curvature of cl is decreasing from P to M). Denote by N_c the point of the clothoid cl with the same curvature as the point N, by N_ρ – the point of the clothoid cl with the same length of the radius-vector $\vec{\rho}(t)$ as the point N, by N_γ – the point of the clothoid cl with the same angle $\gamma(t)$ between the radius-vector $\vec{\rho}(t)$ and the tangent vector $\vec{\tau}(t)$ as the point N. Then for every point P there exists a small δ-half-neighbourhood where the points $N_c, P, N_\gamma, N_\rho, M$ are encountered in this order along cl (see Corollary 6.2). Denote this disposition of the points $N_c, P, N_\gamma, N_\rho, M$ by *disposition*$(*)$. The number δ can be chosen the same for all values of $s \in [-u_0/2, u_0/2]$; assume that δ is fixed.

b) Consider some path $\mathcal{P}$ of the class $\mathcal{A}$ of all paths beginning at the point P, piecewise clothoid ($u = -2s + \tilde{u}^0$ or $u = 2s + \tilde{\tilde{u}}^0$, $\tilde{u}^0 > 0$, $\tilde{\tilde{u}}^0 > 0$), of length $\leq \nu(s) = u_0/2 - |s|$ and consisting of n pieces ($n > 1/\delta$, each piece being of length $1/n$ except the first one which is of length $\leq 1/n$).

We prove the lemma for paths $\mathcal{P} \in \mathcal{A}$ first, by induction on n. For paths pcl defined at the beginning of the subsection the lemma will be proved in **c)**.

For the first piece of the path $\mathcal{P}$ we have *disposition*$(*)$ (because the length of this piece is $\leq \delta$ and for the δ-half-neighbourhood of the point P we have this disposition). Suppose that *disposition*$(*)$ doesn't hold at some moment s'. If s' is the very first moment when it happens, then 3 cases can occur:

1) *If at the moment s' the point N_γ coincides with the point N_ρ.* Then at the next moment we shall have *disposition*$(*)$. Really, using the Taylor series, as in Lemma 6.1, we shall obtain the result of Corollary 6.2, because at the moment s' both paths cl and $\mathcal{P}$ have the same value of the radius-vector $\vec{\rho}(t)$ and the same angle $\gamma(t)$ between the radius-vector $\vec{\rho}(t)$ and the tangent vector $\vec{\tau}(t)$, and the curvature at the point N_γ of the path $\mathcal{P}$ is greater than the curvature at the point N_γ of the path cl.

2) *If at some moment s' the points N_γ, N_ρ and N_c coincide.* Then this means that we move along a half-clothoid cl but with a delay; hence, we either continue like that and come with a delay, or at some moment we have again *disposition*$(*)$.

3) *If at some moment s' the point N_γ coincides with the point N_c and the point N_ρ is situated after them (see Figure 24).* Then it doesn't happen in the first piece of the path $\mathcal{P}$ (see the definition of δ). Hence, if it happens in the k-th piece of the path $\mathcal{P}$ then for the $(k-1)$-st piece of the path $\mathcal{P}$ *disposition*$(*)$ holds.

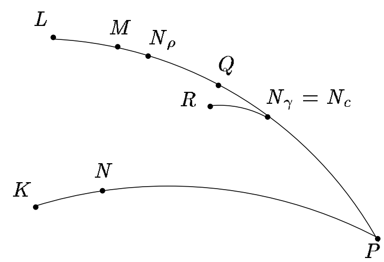

Figure 24

Prove that in this case

$$\rho_{cl}(s) - \rho_{cl}(s') \geq \rho_{\mathcal{P}}(s) - \rho_{\mathcal{P}}(s') \qquad \text{for } s \geq s' \ .$$

We denote by M the point belonging to the path cl and corresponding to the moment s', by N – the point belonging to the path $\mathcal{P}$ and corresponding to the moment s' (see Figure 24). Note that the notation is the same as the one of Figure 23. Denote by $\widehat{ML}$ an arc of the path cl corresponding to the interval $[s', s'+s^*]$ for some $s^* > 0$ and by $\widehat{NK}$ – an arc of the path $\mathcal{P}$ corresponding to the same interval $[s', s'+s^*]$. Denote by s_{pc} $(s_{pc} < s')$ the moment to which the point $N_\gamma = N_c$ corresponds and denote by $\widehat{N_\gamma Q}$ an arc of the path cl corresponding to the interval $[s_{pc}, s_{pc}+s^*]$. Translate the arc $\widehat{NK}$ so that the point N should coincide with the point N_γ, then rotate the image so that the tangent vector to the image at the point N_γ should coincide with the tangent vector to the arc $\widehat{N_\gamma Q}$ at the point N_γ. Denote the obtained arc by $\widehat{N_\gamma R}$.

For the lengths of the radius-vectors $\vec{\rho}(s)$ at the points N_γ, N_ρ and M we have the following inequalities:

$$\rho_{N_\gamma} < \rho_{N_\rho} < \rho_M \ .$$

This follows from Corollary 4.4 ($\dot{\rho}(s) > 0$).

Rotate the arcs $\widehat{N_\gamma Q}$, $\widehat{N_\gamma R}$ and $\widehat{NK}$ around O_{cl} on different angles so that the points M, N_γ and N should be on the line $O_{cl}M$, see Figure 25 a).

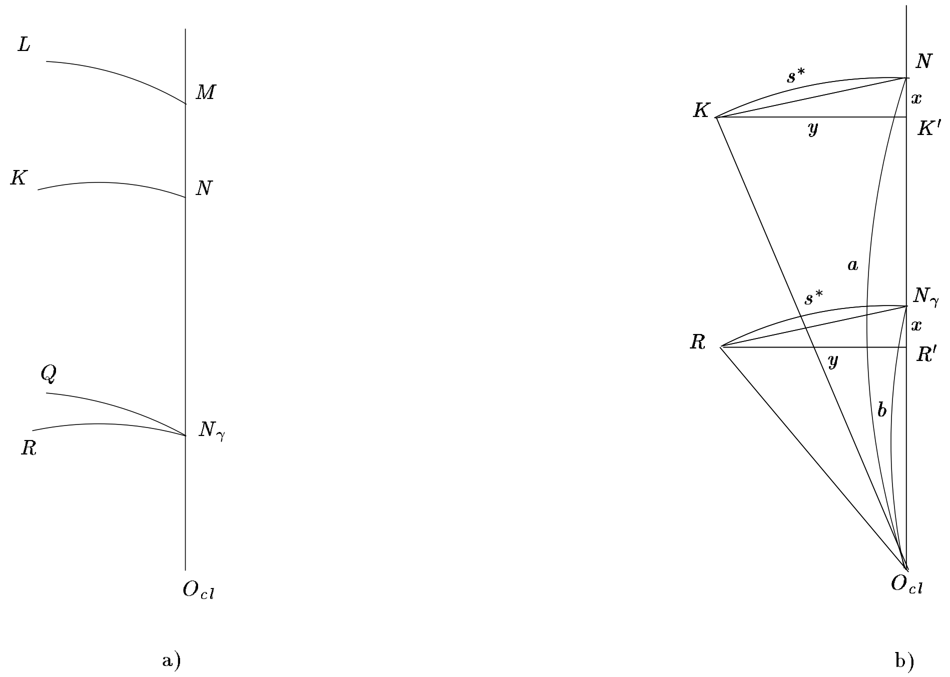

Figure 25

Denote

$$\Delta\rho_{\mathcal{P}} = |\ \overrightarrow{O_{cl}K}\ | - |\ \overrightarrow{O_{cl}N}\ | \ , \qquad \Delta\rho_{\mathcal{P}_{tr}} = |\ \overrightarrow{O_{cl}R}\ | - |\ \overrightarrow{O_{cl}N_\gamma}\ | \ ,$$

$$\Delta\rho_{clpr} = |\ \overrightarrow{O_{cl}Q}\ | - |\ \overrightarrow{O_{cl}N_\gamma}\ | \ , \qquad \Delta\rho_{cl} = |\ \overrightarrow{O_{cl}L}\ | - |\ \overrightarrow{O_{cl}M}\ | \ .$$

We know that for the $(k-1)$-st piece of the path $\mathcal{P}$ *disposition*$(*)$ holds. Hence,

$$\Delta\rho_{\mathcal{P}_{tr}} < \Delta\rho_{clpr} \tag{50}$$

(by the inductive assumption, as $k < n$).

Using Corollary 4.4 ($\ddot{\rho} > 0$) we obtain

$$\Delta\rho_{clpr} < \Delta\rho_{cl} \ . \tag{51}$$

Prove that

$$\Delta\rho_{\mathcal{P}} < \Delta\rho_{\mathcal{P}_{tr}} \ . \tag{52}$$

The tangent angles at the points N and N_γ are the same, the curvatures – too. Hence (see Figure 25 b)),

$$|KK'| = |RR'| = y, \qquad |NK'| = |N_\gamma R'| = x \ .$$

Denote

$$|O_{cl}N| = a, \qquad |O_{cl}N_\gamma| = b \ .$$

We have the following equalities:

$$\Delta\rho_{\mathcal{P}} = \sqrt{(a \pm x)^2 + y^2} - a \ , \qquad \Delta\rho_{\mathcal{P}_{tr}} = \sqrt{(b \pm x)^2 + y^2} - b \ .$$

Inequality (52) is equivalent to

$$\sqrt{(a \pm x)^2 + y^2} - a < \sqrt{(b \pm x)^2 + y^2} - b \ ,$$

i.e.

$$(a \pm x)^2 - (b \pm x)^2 < (a - b)\left(\sqrt{(a \pm x)^2 + y^2} + \sqrt{(b \pm x)^2 + y^2}\right) \ ,$$

$$(a + b \pm 2x) < \left(\sqrt{(a \pm x)^2 + y^2} + \sqrt{(b \pm x)^2 + y^2}\right) .$$

Thus we have

$$a + b \pm 2x = (a \pm x) + (b \pm x) \le |a \pm x| + |b \pm x| < \left(\sqrt{(a \pm x)^2 + y^2} + \sqrt{(b \pm x)^2 + y^2}\right) \ .$$

This chain of inequalities is correct, hence, inequality (52) is also correct. Thus, from inequalities (50)–(52) we obtain the desired inequality:

$$\Delta\rho_{\mathcal{P}} < \Delta\rho_{cl} \ ,$$

i.e. $\rho_{cl}(s) - \rho_{cl}(s') > \rho_{\mathcal{P}}(s) - \rho_{\mathcal{P}}(s')$ for $s > s'$.

Thus we proved that if at some moment s' *disposition*$(*)$ doesn't hold then for the moments $s > s'$ the length of the radius-vector $\vec{\rho}_{cl}(s)$ for the point belonging to *cl* is greater than the length of the radius-vector $\vec{\rho}_{\mathcal{P}}(s)$ for the point belonging to $\mathcal{P}$. This holds for any path of the class $\mathcal{A}$ for any point P corresponding to some value of the parameter $s \in (-u_0/2, u_0/2)$.

c) Assume that the point P coincides with the point S (see Figure 23). The curvature of the path *pcl* and the curvature of any path of the class $\mathcal{A}$ are continuous functions. Hence, if $n \to \infty$, then we can uniformly approximate the path *pcl* by a sequence of paths of the class $\mathcal{A}$. Hence, for $s \in [0, u_0/2]$ the length of the radius-vector $\vec{\rho}_{cl}(s)$ is greater than the length of the radius-vector $\vec{\rho}_{pcl}(s)$, i.e. inequality (20) is proved.

The lemma is proved. □

D Appendix: Proof of Lemma 8.5.

From equality (27) we obtain

$$\dot\gamma = \dot\alpha - \dot\beta.$$

For the angles α and β we have the folowing formulas

$$\alpha = t^2, \qquad \tan\beta = \frac{|EF|}{|OF|} = \frac{\int_0^t \sin\tau^2 d\tau}{\int_0^t \cos\tau^2 d\tau} \ .$$

Hence,

$$\dot\alpha = 2t \ ,$$

$$\dot\beta = \frac{1}{\left[1 + \left(\int_0^t \sin\tau^2 d\tau / \int_0^t \cos\tau^2 d\tau\right)^2\right]} \frac{\sin t^2 \int_0^t \cos\tau^2 d\tau - \cos t^2 \int_0^t \sin\tau^2 d\tau}{\left(\int_0^t \cos\tau^2 d\tau\right)^2} =$$

$$= \frac{\int_0^t \sin(t^2-\tau^2) d\tau}{\left(\int_0^t \cos\tau^2 d\tau\right)^2 + \left(\int_0^t \sin\tau^2 d\tau\right)^2} \ .$$

Thus we obtain the following equality:

$$\dot\alpha - \dot\beta = 2t - \frac{\int_0^t \sin(t^2-\tau^2) d\tau}{\left(\int_0^t \cos\tau^2 d\tau\right)^2 + \left(\int_0^t \sin\tau^2 d\tau\right)^2} \ .$$

We want to prove the following inequality:

$$2t\left[\left(\int_0^t \cos\tau^2 d\tau\right)^2 + \left(\int_0^t \sin\tau^2 d\tau\right)^2\right] > \int_0^t \sin(t^2-\tau^2) d\tau \ \text{ for } t \in \left[0, \sqrt{3\pi/4}\right] \ . \tag{53}$$

We prove inequality (53) at first for $t \in \left[0, \sqrt{\pi/4}\right]$. For this interval the function $\cos\tau^2$ is monotonously decreasing function, hence,

$$\left(\int_0^t \cos\tau^2 d\tau\right)^2 > (t\cos(\pi/4))^2 = t^2/2 \ . \tag{54}$$

For the function $\sin(t^2-\tau^2)$ we have the following inequality:

$$\sin(t^2-\tau^2) < t^2 - \tau^2 < t^2 \ ,$$

i.e.

$$\int_0^t \sin(t^2-\tau^2) d\tau < t * t^2 = t^3 \ . \tag{55}$$

From inequalities (54) and (55) we obtain that inequality (53) is true for $t \in \left[0, \sqrt{\pi/4}\right]$.

Consider now the interval $\left[\sqrt{\pi/4}, \sqrt{\pi/2}\right]$. For the functions $\sin(t^2 - \tau^2)$ and $\cos \tau^2$ restricted to this interval we have the following estimations:

$$\int_0^t \sin(t^2 - \tau^2) d\tau < t \sin(\pi/2) = t \ ,$$

$$\left(\int_0^t \cos \tau^2 d\tau\right)^2 > \left(\int_0^{\sqrt{\pi/6}} \cos \tau^2 d\tau + \int_{\sqrt{\pi/6}}^{J\sqrt{\pi/4}} \cos \tau^2 d\tau\right)^2 >$$

$$> \left(\sqrt{\pi/6} \cos \pi/6 + \left(\sqrt{\pi/4} - \sqrt{\pi/6}\right) \cos \pi/4\right)^2 =$$

$$= \left(\sqrt{\pi}/(2\sqrt{2}) + \sqrt{\pi}(\sqrt{3} - \sqrt{2})/(2\sqrt{6})\right)^2 =$$

$$= \pi \left(1/(2\sqrt{2}) + 1/(2\sqrt{2}) - 1/(2\sqrt{3})\right)^2 =$$

$$= \pi \left(1/\sqrt{2} - 1/(2\sqrt{3})\right)^2 = \pi \left(\sqrt{6} - 1\right)^2 /12 \ . \tag{56}$$

Hence, from inequality (56) we obtain

$$\left(\int_0^t \cos \tau^2 d\tau\right)^2 + \left(\int_0^t \sin \tau^2 d\tau\right)^2 > \pi \left(\sqrt{6} - 1\right)^2 /12 + \left(\int_0^t \sin \tau^2 d\tau\right)^2 >$$

$$> \pi \left(\sqrt{6} - 1\right)^2 /12 + \left(\int_{\sqrt{\pi/4}}^{\sqrt{\pi/2}} \sin \tau^2 d\tau\right)^2 >$$

$$> \pi \left(\sqrt{6} - 1\right)^2 /12 + \left(\sqrt{\pi/2} - \sqrt{\pi/4}\right)^2 \sin^2 \pi/4 =$$

$$= \pi \left(\sqrt{6} - 1\right)^2 /12 + \pi \left(\sqrt{2} - 1\right)^2 /8 = \pi \left(7 - 2\sqrt{6}\right) /12 + \pi \left(3 - 2\sqrt{2}\right) /8 =$$

$$= \pi \left(23 - 4\sqrt{6} + 6\sqrt{2}\right) /24 > \left(23 - 4\sqrt{6} + 6\sqrt{2}\right) /8 \ . \tag{57}$$

Also, we know that

$$23 - 4\sqrt{6} + 6\sqrt{2} > 23 - 0,98 > 22 \ .$$

Hence, from last ineqiality and inequality (57) we obtain that

$$2t\left[\left(\int_0^t \cos\tau^2 d\tau\right)^2 + \left(\int_0^t \sin\tau^2 d\tau\right)^2\right] > 2t \times \frac{22}{8} = \frac{11}{2}t > 5t\ .$$

But

$$\int_0^t \sin(t^2 - \tau^2)d\tau < t,$$

hence, we have proved that inequality (53) is true for $t \in \left[\sqrt{\pi/4}, \sqrt{\pi/2}\right]$.

Consider the interval $\left[\sqrt{\pi/2}, \sqrt{3\pi/4}\right]$. For the function $\cos\tau^2$ restricted to this interval we have the following inequality:

$$\left(\int_0^t \cos\tau^2 d\tau\right)^2 = \left(\int_0^{t^2} \frac{\cos\tau}{2\sqrt{\tau}} d\tau\right)^2 > \left(\int_0^{\pi/4} \frac{\cos\tau}{2\sqrt{\tau}} d\tau\right)^2\ ,$$

because

$$\int_{\pi/4}^{\pi/2} \frac{\cos\tau}{2\sqrt{\tau}} d\tau > 0 > \int_{\pi/2}^{3\pi/4} \frac{\cos\tau}{2\sqrt{\tau}} d\tau\ .$$

Hence,

$$\left(\int_0^t \cos\tau^2 d\tau\right)^2 > \left(\int_0^{\sqrt{\pi}/2} \cos\tau^2 d\tau\right) > \left(\sqrt{\pi}/2\cos(\pi/4)\right)^2 = \pi/8\ . \tag{58}$$

For the function $\sin\tau^2$ we have the following inequality:

$$\left(\int_0^t \sin\tau^2 d\tau\right)^2 > \left(\int_0^{\sqrt{\pi/2}} \sin\tau^2 d\tau\right)^2 = \left(\int_0^{\pi/2} \frac{\sin\tau}{2\sqrt{\tau}} d\tau\right)^2 >$$

$$> \left(\int_0^{\pi/2} \sin\tau\, d\tau\right)^2 /2\pi = 1/(2\pi)\ . \tag{59}$$

Now from inequalities (58) – (59) we obtain the following inequality:

$$2t\left[\left(\int_0^t \cos\tau^2 d\tau\right)^2 + \left(\int_0^t \sin\tau^2 d\tau\right)^2\right] > 2t\left(\pi/8 + 1/(2\pi)\right) = t\left(\pi/4 + 1/\pi\right) > t\ .$$

But

$$\int_0^t \sin(t^2 - \tau^2)d\tau < t,$$

hence, we have proved that inequality (53) is true for $t \in \left[\sqrt{\pi/2}, \sqrt{3\pi/4}\right]$.

The lemma is proved. □

E Appendix: Proof of Lemma 8.6.

From equality (27) we obtain that $\gamma < \alpha$. If we prove that $\beta < \gamma$, then using equality (27) we prove that $\alpha < 2\gamma$.

Consider the function $J(t)$ which is equal to the distance between the line l_L and the point of the arc $\widehat{OE}$.

In order to prove inequality (28) we can consider only the interval $(0, \sqrt{\pi})$, because on the interval $\left[\sqrt{\pi}, \sqrt{3\pi/2}\right]$ for the angles γ, α and β we have the following inequalities:

$$\beta < \pi/2, \quad \alpha = \beta + \gamma > \pi, \quad \text{hence}, \quad \gamma > \pi/2 \ ,$$

and, hence, evidently, $\beta < \gamma$ and inequality (28) holds.

Consider the coordinate system Lx_1y_1 with the origin at the point L and with the x_1-axis coinciding with the tangent vector to the half-clothoid at the point L (see Figure 17). In this coordinate system we can define the function $J(t)$ on the interval $[-t_L, t_E - t_L]$ as follows:

$$J(t) = \int_0^t \sin(\tau^2 + 2t_L\tau)d\tau, \quad t \in [0, t_E - t_L]$$

and

$$J(t) = \int_0^t \sin(-\tau^2 - 2t_L\tau + \pi)d\tau = \int_0^t \sin(\tau^2 + 2t_L\tau)d\tau, \quad t \in [-t_L, 0]$$

(we have $J(0) = 0$ at the point L).

Consider the graph of the function $f(t) = t^2 + 2t_Lt$ (see Figure 26).

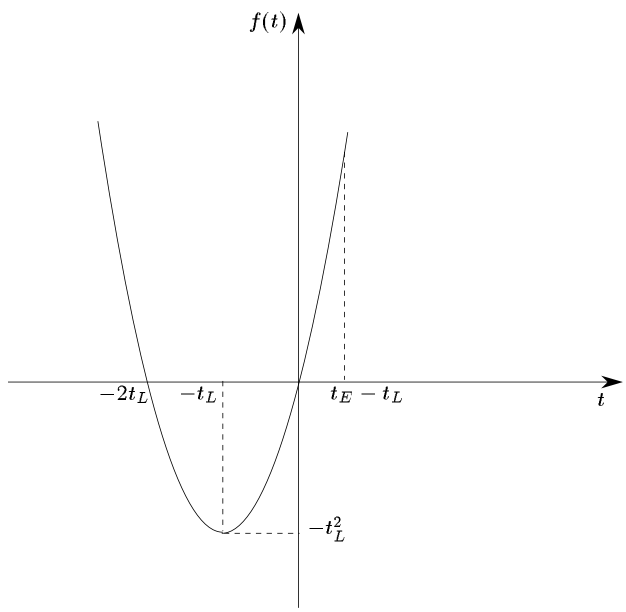

Figure 26

The function $f(t) = t^2+2t_Lt$ is a monotonously increasing quadratic function on the interval $[-t_L, t_E - t_L]$. Hence, the graph of the function $\sin f(t)$ on the interval $[-t_L, t_E - t_L]$ is of the kind as on Figure 27 (recall that $t_E \leq \sqrt{\pi}$, hence, $t_L < \sqrt{\pi}$).

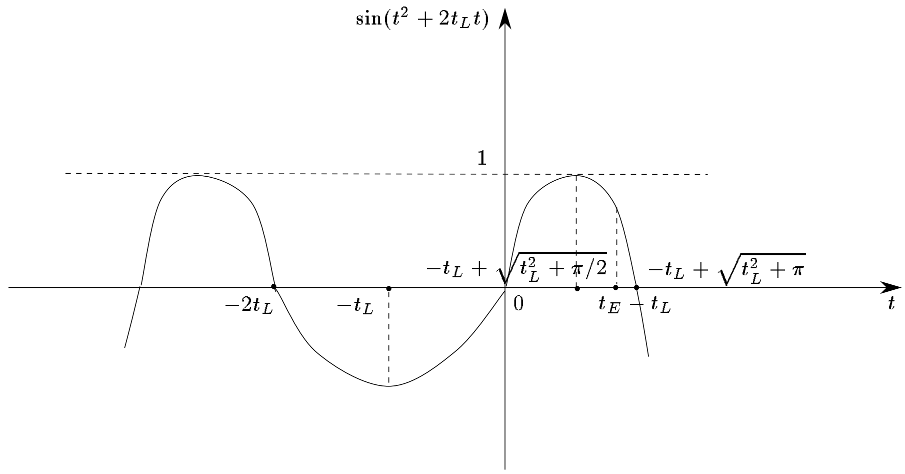

Figure 27

From the inequality $t < \sqrt{\pi}$ follows that $t_E - t_L < -t_L + \sqrt{t_L^2 + \pi}$. Indeed, $t_E < \sqrt{\pi}$, i.e. $t_E - t_L < \sqrt{\pi} - t_L$, but $-t_L + \sqrt{t_L^2 + \pi} > \sqrt{\pi} - t_L$, hence, $t_E - t_L < -t_L + \sqrt{t_L^2 + \pi}$.

Consider a pair of points $M \in \widehat{OL}$, $N \in \widehat{LE}$ such that $J(M) = J(N)$. From the graph of the function $\sin(t^2 + 2t_Lt)$ we see that

1) $|t_M| > t_N$,

2) the angle $t^2 + 2t_Lt$ for the point M is smaller than this angle for the point N.

For the point M coinciding with the point O and for the point N coinciding with the point E we obtain from 1) inequality (30), i.e.

$$|\widehat{OL}| > |\widehat{LE}| \ ,$$

and from 2) we obtain the following inequality:

$$\beta < \gamma \ .$$

Thus, we have proved inequalities (28) and (30). In order to prove inequality (29) remark that from 2) we obtain that $\cos(t^2 + 2t_Lt)$ at M is larger than $\cos(t^2 + 2t_Lt)$ at N. But

$$|\widetilde{OL}| = \int_0^{-t_L} \cos(\tau^2 + 2t_L\tau)d\tau \ ,$$

$$|\widetilde{L}E| = \int_0^{t_E - t_L} \cos(\tau^2 + 2t_L\tau)d\tau \ ,$$

hence, using inequality (30) ($t_L > t_E - t_L$), we obtain inequality (29):

$$|O\widetilde{L}| > |\widetilde{L}E| \ .$$

The lemma is proved. □

F Appendix: Proof of Lemma 8.7.

For the function $q(t)$ (see formula (31)) we can apply Cauchy Theorem on the interval $[0, C_{arc}]$ because functions $t - \int_0^t \cos\alpha(\tau)d\tau$ and $t - \int_0^t \cos\gamma(\tau)d\tau$ satisfy all the conditions of the theorem. So, we obtain that

$$q(t) = \frac{1 - \cos\alpha(t^*)}{1 - \cos\gamma(t^*)} \ ,$$

where $t^* \in [0, t]$. Hence,

$$q(t) = \frac{\sin^2(\alpha(t^*)/2)}{\sin^2(\gamma(t^*)/2)} \ .$$

From Lemma 8.6 we have the following inequalities:

$$\alpha/2 < \gamma < \alpha < 2\gamma \ .$$

Consider the function $q(t)$ on the interval $(0, \sqrt{\pi}]$. Hence, $\alpha(t^*)/2 \in (0, \pi/2]$ and, hence, by Lemma 8.6, we have $\gamma(t^*)/2 \in (0, \pi/2)$. On this interval the function $\sin(t^2/2)$ is a monotonously increasing function and $\gamma > \alpha/2$. Hence

$$\frac{\sin^2(\alpha(t^*)/2)}{\sin^2(\gamma(t^*)/2)} < \frac{\sin^2(\alpha(t^*)/2)}{\sin^2(\gamma(t^*)/4)} = 4\cos^2(\alpha(t^*)/4) \leq 4 \ ,$$

i.e.

$$q(t) < 4 \qquad \text{for } t^2 \in (0, \pi] \ .$$

When the function $q(t)$ is defined on the interval $[\sqrt{\pi}, \sqrt{3\pi/2}]$, then $\alpha(t^*)/2 \in [\pi/2, 3\pi/4]$ and $\gamma(t^*)/2 \in (\pi/4, 3\pi/4)$. Hence, the maximal value of the ratio

$$\frac{\sin^2(\alpha(t^*)/2)}{\sin^2(\gamma(t^*)/2)}$$

will be obtained for $\alpha(t^*)/2 = \pi/2$ and $\gamma(t^*)/2 = \pi/4$, i.e.

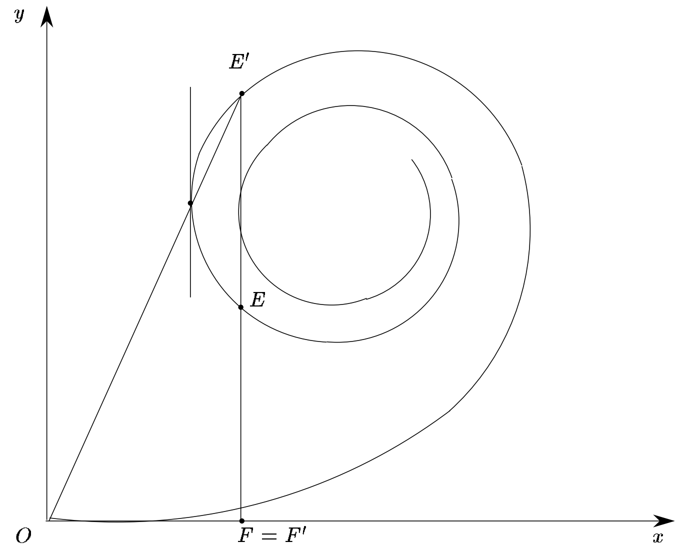

Figure 28

$$q(t) = \frac{\sin^2(\alpha(t^*)/2)}{\sin^2(\gamma(t^*)/2)} < \frac{1}{\left(\sqrt{2}/2\right)^2} = 2 \qquad \text{for } t^2 \in [\pi, 3\pi/2] \ .$$

Consider the function $q(t)$ on the interval $[\sqrt{3\pi/2}, C_{arc}]$. Consider for every point E of an arc of half-clothoid defined on this interval the point E' of an arc of half-clothoid defined on the interval $[0, \sqrt{3\pi/2}]$ such that $F' = F$ (see Figure 28).

We prove that $q(E) < q(E')$.

$$q(E) = 1 + f(E) = 1 + \frac{|OE| - |OF|}{|\widehat{OE}| - |OE|} \ ,$$

$$q(E') = 1 + f(E') = 1 + \frac{|OE'| - |OF|}{|\widehat{OE}'| - |OE'|} \ .$$

But

$$|OE'| - |OF| > |OE| - |OF|$$

and

$$|\widehat{OE}'| - |OE'| < |\widehat{OE}| - |OE|$$

(because $|\widehat{OE}'| < |\widehat{OE}|$ and $|OE'| > |OE|$). Hence, $f(E') > f(E)$ and $q(E') > q(E)$. Thus, we prove that

$$\max_{t\in[0,C_{arc}]} q(t) < 4 \ .$$

The lemma is proved. $\square$

G Appendix: Proof of Lemma 8.10.

We want to estimate the ratio

$$\frac{h^{-1}(k\Delta)}{h^{-1}(\Delta)} \ .$$

We can rewrite it in the following form (see Figure 18):

$$\frac{h^{-1}(k\Delta)}{h^{-1}(\Delta)} = \frac{|RR'|}{|EC|} = \frac{|PR'|}{|PC|} = \frac{|PO|+|OR'|}{|PO|+|OC|} = \frac{l+k\Delta}{l+\Delta} = \frac{l/\Delta+k}{l/\Delta+1} \ .$$

On the interval $[0,h(t_K)]$ the function $h^{-1}(\omega)$ is convex (because by Proposition 8.9 the function $h(t)$ is concave on this interval). Hence,

$$l < c$$

or

$$\frac{l/\Delta+k}{l/\Delta+1} < \frac{c/\Delta+k}{c/\Delta+1}$$

because this inequality is equivalent to the following inequality:

$$lc/\Delta^2 + l/\Delta + kc/\Delta + k < lc/\Delta^2 + c/\Delta + kl/\Delta + k \ ,$$

i.e.

$$(1-k)l/\Delta < (1-k)c/\Delta \ .$$

Thus, we can estimate the ratio

$$\frac{h^{-1}(k\Delta)}{h^{-1}(\Delta)}$$

in the following way:

$$\frac{h^{-1}(k\Delta)}{h^{-1}(\Delta)} < \frac{c/\Delta+k}{c/\Delta+1} = 1 - \frac{1-k}{c/\Delta+1} \ . \tag{60}$$

Denote by $(h^{-1}(\Delta))'$ the derivative of the function $h^{-1}(\Delta)$. So for the function $h^{-1}(\Delta)$ we have the following formula (see Figure 18):

$$h^{-1}(\Delta) = (c+\Delta)\tan\delta = (c+\Delta)(h^{-1}(\Delta))' \ ,$$

i.e.

$$c/\Delta = \frac{h^{-1}(\Delta)}{(h^{-1}(\Delta))'\Delta} - 1$$

and we can rewrite inequality (60) as follows:

$$\frac{h^{-1}(k\Delta)}{h^{-1}(\Delta)} < 1 - \frac{1-k}{h^{-1}(\Delta)}(h^{-1}(\Delta))'\Delta \ . \tag{61}$$

Denote by $D(t)$ the length of the arc, by $H(t)$ the length of the chord corresponding to some t and by $D'(t)$, $H'(t)$ we denote their derivatives. Recall that

$$h(t) = D(t) - H(t), \qquad h^{-1}(h(t)) = D(t) \ .$$

Then,

$$h^{-1}(\Delta) = D(t), \quad (h^{-1}(\Delta))' = 1/(1-H'(t)), \quad \Delta = D(t) - H(t) \ ,$$

and we can rewrite inequality (61):

$$\frac{h^{-1}(k\Delta)}{h^{-1}(\Delta)} < 1 - \frac{1-k}{D(t)} \times \frac{D(t)-H(t)}{1-H'(t)} = 1 - (1-k) \times \frac{1-H(t)/D(t)}{1-H'(t)} \ . \tag{62}$$

We estimate the right hand-side of inequality (62) using Proposition 8.12. So we can rewrite inequality (62) as follows:

$$\frac{h^{-1}(k\Delta)}{h^{-1}(\Delta)} < 1 - (1-k) \times \frac{1-H(t)/D(t)}{1-H'(t)} = 1 - (1-k) \times \frac{1-H'(\theta t)/D'(\theta t)}{1-H'(t)} \ ,$$

where $1/2 < \theta < 0$.

But

$$\frac{1-H'(\theta t)/D'(\theta t)}{1-H'(t)} = \frac{1-\cos\gamma(\theta t)}{1-\cos\gamma(t)} = \frac{\sin^2(\gamma(\theta t)/2)}{\sin^2(\gamma(t)/2)} \ .$$

Hence

$$\frac{h^{-1}(k\Delta)}{h^{-1}(\Delta)} < 1 - (1-k) \times \frac{\sin^2(\gamma(\theta t)/2)}{\sin^2(\gamma(t)/2)} \quad \text{for some} \quad 1/2 < \theta < 1 \ . \tag{63}$$

From Lemma 8.6 we have the following inequalities for the angle $\gamma(t)$:

$$\alpha(t)/2 < \gamma(t) < \alpha(t) \qquad \text{for } t \in \left(0, \sqrt{3\pi/2}\right] \ ,$$

where $\alpha(t)$ is the tangent angle, i.e. $\alpha(t) = t^2$. Hence, for the angles $\gamma(\theta t)/2$ and $\gamma(t)/2$ we have two inequalities:

$$(\theta t)^2/4 < \gamma(\theta t)/2 < (\theta t)^2/2, \tag{64}$$

$$t^2/4 < \gamma(t)/2 < t^2/2 \ . \tag{65}$$

Recall that $1/2 < \theta < 1$, hence, we can rewrite inequality (64) as follows:

$$t^2/16 < \gamma(\theta t)/2 < t^2/2 \ . \tag{66}$$

Now using inequalities (65), (66) we obtain the following estimation:

$$\frac{\sin(\gamma(\theta t)/2)}{\sin(\gamma(t)/2)} > \frac{\sin(t^2/16)}{\sin(t^2/2)} \ .$$

But

$$\sin(t^2/2) = 8\sin(t^2/16)\cos(t^2/16)\cos(t^2/8)\cos(t^2/4) \ ,$$

hence,

$$\frac{\sin(t^2/16)}{\sin(t^2/2)} = \frac{1}{8\cos(t^2/16)\cos(t^2/8)\cos(t^2/4)} > 1/8,$$

i.e.

$$\frac{\sin(\gamma(\theta t)/2)}{\sin(\gamma(t)/2)} > 1/8 \ . \tag{67}$$

Thus, using inequality (67) we can rewrite inequality (63):

$$\frac{h^{-1}(k\Delta)}{h^{-1}(\Delta)} < 1 - (1-k)/64 \ .$$

The lemma is proved. □

References

[1] J.D.Boissonnat, A.Cérézo and J.Leblond. *A note on shortest paths in the plane subject to a constraint on the derivative of the curvature.* INRIA, Nice – Sophia Antipolis, Research Report N^0 2160, January 1994.

[2] L.Cesari. *Optimization theory and applications.* Number 17 in Applications of mathematics. Springer-Verlag, 1983.

[3] V.P.Kostov and E.V.Degtiariova-Kostova. *Suboptimal paths in the problem of a planar motion with bounded derivative of the curvature.* INRIA, Nice – Sophia Antipolis, Research Report N^0 2051, September 1993.

[4] V.P.Kostov and E.V.Degtiariova-Kostova. *The planar motion with bounded derivative of the curvature and its suboptimal paths.* INRIA, Nice – Sophia Antipolis, Research Report N^0 2189, January 1994.

[5] V.P.Kostov and E.V.Degtiariova-Kostova. *Irregularity of optimal trajectories in a control problem for a car-like robot* INRIA, Nice – Sophia Antipolis, Research Report (to appear).

[6] L.S.Pontryagin, V.R.Boltyanskii, R.V.Gamkrelidze and E.F.Mishchenko. *The mathematical theory of optimal processes.* Interscience, 1962.

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