On Dynamic Feedback Linearization of Four-dimensional Affine Control Systems with Two Inputs

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N° 2751

Décembre 1995

PROGRAMME 5

Rapport de recherche

ISSN 0249-6399

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Programme 5 — Traitement du signal, automatique et productique

Projet MIAOU

Rapport de recherche n° 2751 — Décembre 1995 — 98 pages

Abstract: This paper considers control affine systems in $\mathbb{R}^4$ with two inputs, and gives necessary and sufficient conditions for dynamic feedback linearization of these systems with the restriction that the "linearizing outputs" must be some functions of the original state and inputs only. This also gives conditions for non-affine systems in $\mathbb{R}^3$.

Key-words: Dynamic feedback linearization, Linearizing outputs, small dimensions, differential flatness.

(Résumé : tsvp)

This manuscript was submited on March 25, 1996, in its present form, to EUROPEAN SERIES IN APPLIED AND INDUSTRIAL MATHEMATICS : COCV (CONTRÔLE, OPTIMISATION ET CALCUL DES VARIATIONS). It is the revision of a manuscript submitted to ANNALES DE L'I.H.P. – ANAYLSE NON-LINÉAIRE on August 6, 1993. The length of the revision was the reason for changing journal.

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Linéarisation dynamique des systèmes à deux commandes et quatre états, affines en les commandes

Résumé : Dans cet article, on considère un système commandé affine en la commande dont l'état vit dans $\mathbb{R}^4$ et la commande dans $\mathbb{R}^2$. On donne des conditions nécessaires et suffisantes pour que ce système soit linéarisable par retour d'état dynamique avec la restriction que les "sorties linéarisantes" doivent ne dépendre que de l'état et de la commande. Ceci fournit aussi des conditions pour les systèmes non-affines dans $\mathbb{R}^3$.

Mots-clé : Linéarisation par feedback dynamique, sorties linéarisantes, petites dimensions, platitude différentielle.

1 Introduction

A deterministic finite dimensional nonlinear control system

$$\dot{x} \;=\; f(x,u) \tag{1}$$

where the state x lives in $\mathbb{R}^n$, the control u lives in $\mathbb{R}^m$, and f is smooth —smooth means $\mathcal{C}^\infty$ in this article— is said to be locally *static feedback equivalent* around $(\bar{x},\bar{u})$ to another system

$$\dot{z} \;=\; \widetilde{f}(z,v) \tag{2}$$

around $(\bar{z},\bar{v})$ if there exists a *nonsingular feedback transformation*, i.e. two maps

$$\begin{array}{rcl} u & = & \alpha(x,v) \\ x & = & \phi(z) \end{array} \tag{3}$$

such that $(z,v) \mapsto (\phi(z),\alpha(z,v))$ is a local diffeomorphism sending $(\bar{z},\bar{v})$ to $(\bar{x},\bar{u})$, that transforms (1) into (2). The interest of feedback equivalence is that the transformation (3) allows one to convert the solution to a certain control problem for system (1) to the solution of a similar control problem for system (2). It is clear that (germs of) static feedback transformations form a group acting on (germs of) systems, and that static feedback equivalence *is* an equivalence relation. This feedback equivalence has been very much studied, see for instance [5, 3, 15]. Classification of control systems modulo this equivalence is of course a very ambitious and difficult program, almost out of reach. A more restricted problem is the one of describing the orbits of controllable linear systems, i.e. systems of the form $\dot{z} = Az + Bv$ with (controllability) the columns of $B, AB, A^2B, A^3B, \ldots$ having full rank. This problem is known as static feedback linearization, and has been completely solved : in [17, 13], some explicit conditions are given for a general nonlinear system to be locally static feedback equivalent to a controllable linear system.

A dynamic feedback, or dynamic compensator, as opposed to static, is one where, the "new" and "old" controls u and v are not computed from one another by simply static functions, as in (3), but through a dynamic system which has a certain state ξ :

$$\begin{array}{rcl} u & = & \alpha(x,\xi,v) \\ \dot{\xi} & = & \gamma(x,\xi,v) \\ z & = & \phi(x,\xi)\ , \end{array} \tag{4}$$

where ξ lives in $\mathbb{R}^\ell$, $\ell > 0$, and ϕ is a (local) diffeomorphism of $\mathbb{R}^{n+\ell}$. (x,v) may be viewed as the "input" of the control system, and (u,x,ξ), or (u,X) a its "output".

Clearly, (4) allows one to transform system (1) into a system like (2). However, contrary to the case of static feedback, the dimension of the state of the transformed system (2) is strictly larger than the dimension of the state of the original system (1), and for this reason, it a priori difficult to say what an "invertible" dynamic feedback "transformation" can be. One may however, following [7], state the problem of dynamic feedback linearization as the one of deciding when a system (1) can be transformed via a dynamic feedback (4) into a linear controllable system. The problem of deciding if a given system is dynamic feedback linearizable is much more difficult than the one for static feedback and is still open... In [7], where a general panorama and further references on dynamic feedback linearization from the point of view of compensators (4) can be found, some interesting results are given : a single input system ($u \in I\!R$), at a regular point in a certain sense, is dynamic feedback linearizable if and only if it is static feedback linearizable, a necessary condition for dynamic feedback linearizability at a point $(x,u) = (\bar{x},0)$ is that the linear approximation of the system be controllable, a controllable system which is affine in the control —i.e. the right-hand side of (1) is affine with respect to u— and such that the dimension of the state is larger than the dimension of the control by at most one —if they are equal, the system is of the form $\dot{x} = u$ modulo a static feedback— is dynamic feedback linearizable, and some sufficient conditions are also given, but they have the annoying drawback of not being invariant by static feedback...

As seen above, the case of systems with one control is completely understood (outside singularities), so that the nontrivial cases have at least two controls, and then the state must have dimension at least 3 —if it is 2, the system is $\dot{x} = u$ modulo static feedback— but if the system is affine the case of 3 states is already known because it exceeds the number of controls by one only. The case of non-affine systems with a three dimensional state is considered in section 6, but most of the paper is devoted to the case of affine systems with two controls and a state of dimension 4, i.e. systems

$$\dot{x} \;=\; X_0(x) \;+\; u_1 X_1(x) \;+\; u_2 X_2(x) \tag{5}$$

where $x \in I\!R^4$ and u_1 and u_2 are in $I\!R$ ($u = (u_1, u_2)$). X_0, X_1 and X_2 are smooth vector fields in $I\!R^4$. Smooth means $\mathcal{C}^\infty$ in this article.

Of course, since it is the simplest non-trivial case, the problem of dynamic feedback linearization for the four dimensional system (5) has already been studied. In [18], based on the results from [7], some sufficient conditions on X_0, X_1 and X_2 are given. The main drawback of these results is that they are not invariant by static feedback, and that they are only sufficient conditions. They are contained in the results of the present paper.

Rather recently, some conceptual progress has been made on dynamic equivalence and dynamic linearization, initiated in [19, 8, 9]. In [19], a restricted class of compensators (4) is studied, called *endogenous* dynamic feedbacks, they are exactly these that should be called "invertible". They are the compensators (4) such that, by differentiating relations (1) and (4), it is possible to express ξ and v as functions of x, u, $\dot{u}$, and a finite number of time-derivatives of u. The compensator (4) may then be replaced by some formulas giving z and v as functions of $(x, u, \dot{u}, \ddot{u}, \dots)$, which is "invertible" by some formulas giving x and u as functions of $(z, v, \dot{v}, \ddot{v}, \dots)$. On the other hand, the notion of *differential flatness* for control systems is introduced in [19, 8, 9], as roughly speaking, existence of m —two for system (5)— functions of x, u, $\dot{u}$ and a finite number of time-derivatives of u which are differentially independent (the jacobian of any finite number of these functions and their time derivatives has maximum rank) and such that both x and u can be expressed as functions of these m functions and a finite number of their time-derivatives. These functions are called *linearizing outputs*, or "flat outputs". It is proved there that differential flatness is equivalent to equivalence by *endogenous* dynamic feedback to a controllable linear system. In the differential algebraic framework of [8, 9], flatness is defined as the differential field representing the system being *non-differentially* algebraic over a purely transcendental differential extension of the base field, and the linearizing output is a transcendance basis. Of course, the linearizing outputs are then "restricted" to be algebraic. With a suitable definition of endogenous dynamic equivalence between differential fields, it is proved that differential flatness is equivalent to equivalence by *endogenous* dynamic feedback to a controllable linear system.

In [16], a notion of dynamic equivalence in terms of transformations on "trajectories" of the system is studied; different types of transformations are defined there in terms of infinite jets of trajectories, for smooth systems, one of them is proved there to be exactly the one studied here. propoerty of "freedom" is introduced that is close to differential flatness and is proved to be equivalent to equivalence to a linear system.

Of course, there are some recent and interesting results and points of view on dynamic feedback equivalence and dynamic feedback linearization, like [27] and subsequent works that make a link between dynamic feedback linearization and the notion of absolute equivalence defined by E. Cartan for pfaffian systems, that we do not develop here. See [1, 2], and also [10, 7], for a more complete panorama and list of references.

There was a need to develop a geometric framework for the invertible transformations that represent dynamic feedback. This was done recently by the author in [23] and independently by the authors of [8, 9, 19] in [11, 12]. In these papers, an (infinite dimensional) differential geometric approach, based on infinite jet spaces, is used, and the transformations described above may be seen as a particular case of infinite order contact transformations, or Lie-Bäcklund transformations used in the "geometric" study of differential systems and partial differential relations. [23] and [11, 12] are quite similar in spirit although [11, 12] is more general and more formal, and tends to give as a conclusion that systems (1) is not a general enough class of system for control theory, whereas [23] defines everything in coordinates with the aim of developing the "sufficient" framework to use classical tools from differential calculus for the study of dynamic feedback. Here, we shall adopt the notations from [23], which are summed up in section 2. Roughly speaking, to a system (1), one associates the "manifold" where some coordinates are $x, u, \dot{u}, \ddot{u}, u^{(3)}, \ldots$, and the vector field $F = f\frac{\partial}{\partial x} + \dot{u}\frac{\partial}{\partial u} + \ddot{u}\frac{\partial}{\partial \dot{u}} + \cdots$ on this infinite dimensional manifold. The Lie derivative along this vector fields is the "time-derivative" along the system (5), and we often write $\frac{\mathrm{d}}{\mathrm{dt}}$ instead of L_F. An m-uple of smooth functions from this manifold to $\mathbb{R}$ (smooth means depending only on a finite number of variables) is a system of linearizing outputs if and only if these functions, together with their Lie derivatives of all order along F form a system of local coordinates. Endogenous dynamic feedback transformations are local "diffeomorphisms" between two such manifolds (they mimic the transformations defined in [19]). In this framework, a linearizing output is a m-uple of functions (of $x, u, \dot{u}, \ddot{u}, u^{(3)}, \ldots$) such that these and all their derivatives along the vector field representing the system are a set of local coordinates (this is recalled more precisely in section 2), and their existence is necessary and sufficient for linearizability by endogenous dynamic feedback, see [23] or section 2 fore details.

The problem of deciding endogenous —actually, a proof of necessity for general dynamic feedback is announced in [11, 12]— dynamic linearizability is then the one of deciding existence of a system of linearizing outputs.

The first difficulty is that there is no known a priori bound on the number of time-derivatives of the input the linearizing outputs should depend upon (similarly, in the "dynamic compensator" approach, there was no a priori bound on the dimension of ξ in a compensator (4) that would transform a given nonlinear system into a linear system if such a compensator exists). Even for four-dimensional systems (5), no such bound is known.

If such a bound K were known, some PDEs might be written for a m-uple of functions of $(x, u, \dot{u}, \ldots, u^{(K)})$ to be linearizing outputs, and it would then be in principle possible to decide, via formal integrability algorithms, if these PDEs have solutions, and this would provide necessary and sufficient conditions for existence of linearizing outputs depending on $(x, u, \dot{u}, \ldots, u^{(K)})$, i.e. for existence of general linearizing outputs, i.e. for endogenous dynamic feedback linearizability. Even if the bound K is not known, one may look for the conditions under which there exists some linearizing outputs depending on $(x, u, \dot{u}, \ldots, u^{(K)})$ for some arbitrarily fixed K... when such linearizing outputs exist, the system is of course linearizable, but these PDEs have no solutions, it only means that there is no linearizing outputs depending on $(x, u, \dot{u}, \ldots, u^{(K)})$... but there might exist some depending on $(x, u, \dot{u}, \ldots, u^{(K)}), u^{(K+1)})$.

In this paper, we do not address the difficulty of finding a bound K, so that we do not obtain necessary conditions for endogenous dynamic linearizability of (5). We "only" (but it is not so easy technically) give necessary and sufficient for existence of linearizing outputs depending on x and u for the small dimensional system (5). We call x-dynamic and (x, u)-dynamic linearizability existence of linearizing outputs depending on u or on (x, u), so that we given necessary and sufficient conditions for x-dynamic linearizability and (x, u)-dynamic linearizability.

Of course, this could in principle be done according to the above mentioned method : write the PDEs that two functions of x (resp. of (x, u)) must satisfy in order to be linearizing outputs, and use formal integrability conditions. However, the PDEs for x-dynamic linearizability or (x, u)-dynamic linearizability in arbitrary coordinates may be written, but are very complicated and huge, involving many cases, and applying formal integrability criteria (differential elimination) to these, even using computer algebra on a big computer is out of question seen the size of the equations.

In the case of x-dynamic linearizability however, using some normal forms modulo static feedback and working in appropriate coordinates, it is possible —and we do it here— to treat all the simple cases, and to find some necessary conditions for integrability in the remaining case that will suffice here.

For (x, u)-dynamic linearizability, even using some normal forms under static feedback to write the PDEs are huge, and even using symbolic calculation computer programs, they are not tractable. What we do in this case is writing some different equations : we use the fact that there exists two differential forms (these from the *infinitesimal Brunovský form* introduced in [1, 2, 24], not necessarily exact, that, except exactness, have all the properties the differentials of some linearizing outputs must have, and we do not write PDEs on the linearizing outputs, but —following [1, 2, 24]— on the coefficients of transformations on pairs of forms that would preserve these properties and would render the forms integrable. It turns out that doing so and keeping in mind the meaning of the equations, we are able to derive conditions using only first order integrability conditions, i.e. Frobenius theorem. Note however that the computations are still heavy and require the use of computer algebra. It would be interesting to know whether it is general that [2, 1] provides a method to write the equations for linearizing outputs in a more tractable manner.

The paper is organized as follows. Section 2 precises the above definitions of what is intended here by feedback linearization, and then proceeds to recall some results from [23, 2]. In section 3, we apply these results to derive some necessary and sufficient conditions for x-dynamic linearization, and in section 4 for (x, u)-dynamic linearization. Section 5 presents an illustrative example. Section 6 shows that, thanks to a necessary condition given in [25], all non affine systems in $\mathbb{R}^3$ which are dynamic feedback linearizable may be transformed to an affine system (5) by a simple dynamic extension. Section 7 is devoted to the proofs ; the Appendix is devoted to some basic facts on pfaffian systems.

2 Statement of the problem

2.1 Static Feedback

A static feedback transformation followed by a change of coordinates on x may be seen as a local transformation on (x, u) of the form $(z, v) = (\phi_1(x), \phi_2(x, u))$ where ϕ_1 is a (local) diffeomorphism and $\frac{\partial \phi_2}{\partial u}$ is invertible.

Since we are only concerned with systems like (5) where the controls appear linearly, we shall only consider *affine* static feedback. A local affine static feedback transformation is defined locally by :

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \alpha(x) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \beta(x) \tag{6}$$

with $\alpha(x)$ an invertible 2×2 matrix and $\beta(x)$ a vector, both depending smoothly in x. It transforms system (5) into

$$\dot{x} = \widetilde{X}_0(x) + u_1 \widetilde{X}_1(x) + u_2 \widetilde{X}_2(x)$$
$$\text{with} \quad \left\{ \begin{array}{rcl} \widetilde{X}_0 &=& X_0 + \beta_1 X_1(x) + \beta_2 X_2 \\ \widetilde{X}_1 &=& \alpha_{11} X_1 + \alpha_{21} X_2 \\ \widetilde{X}_2 &=& \alpha_{12} X_1 + \alpha_{22} X_2 \end{array} \right.$$

A system is **static feedback linearizable** if and only if it may be transformed by such a transformation into a system which, in some coordinates, reads like a controllable linear system $\dot{z} = Az + Bv$ in $I\!R^4$ with two inputs; these linear systems are all, modulo a linear feedback —like (6) with α and β constant— and a linear change of coordinates, of the form (a) or (b) below

$$\text{(a)} \left\{ \begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = u_1 \\ \dot{x}_4 = u_2 \end{array} \right. \qquad \text{(b)} \left\{ \begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = u_1 \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = u_2 \end{array} \right. \tag{7}$$

which are the two Brunovský canonical forms for controllable linear systems with two inputs and four states, see [4]. Static feedback linearizable systems are a particular case of x-dynamic linearizable (and hence (x,u)-dynamic linearizable) systems because (x_1, x_4) for the form (a), and (x_1, x_3) for the form (b) may be chosen as a pair of linearizing outputs.

Static feedback will also be used in the present paper to give some simple "normal" forms modulo this transformation and a change of coordinates on x of the systems considered for each case, or set of conditions, see (33), (34), (36), (37), (39), (45), (61), (67).

2.2 Linearizing Outputs and Endogenous Dynamic Linearization

Here we sum up some notations from [23, 2], and explain quickly dynamic linearization in this infinite dimensional geometric framework, the aim being to define linearizing outputs properly. The infinite dimensional framework is only a rather convenient way of manipulating functions which depend on a finite but a priori unknown functions, and it allows to say the transformation by dynamic feedback are "diffeomorphisms".

We call *generalized state manifold* for system (1) with n states and m inputs the "infinite dimensional manifold" $\mathcal{M}_\infty^{m,n}$ where some coordinates are $(x_1,\ldots,x_n,$ $u_1,\ldots,u_m,\ \dot u_1,\ldots,\dot u_m,\ \ddot u_1,\ldots,\ddot u_m,\ldots\ldots)$. It is the projective limit of the finite dimensional manifolds $\mathcal{M}_K^{m,n}$, $K\geq -1$ with coordinates $(x_1,\ldots,x_n,\ u_1,\ldots,u_m,$ $\dot u_1,\ldots,\dot u_m,\ \ldots,u_1^{(K)},\ldots,u_m^{(K)})$ —when $K=-1$, this means $(x_1,\ldots,x_n)$— and we have the obvious projections π_K from $\mathcal{M}_\infty^{m,n}$ to $\mathcal{M}_K^{m,n}$:

$$\pi_K(x_1\ldots x_n,u_1\ldots u_m,\ \ldots\ \ldots)\ =\ (\,x_1\ldots x_n,u_1\ldots u_m,\ \ldots\ u_1^{(K)}\ldots u_m^{(K)}\,)\,. \qquad (8)$$

The topology is the product topology, the least fine such that all these projections are continuous, i.e. an open set is always of the form $\pi_K^{-1}(O)$ with O a (finite-dimensional) open subset of $\mathcal{M}_K^{m,n}$. In particular when a property holds *locally around a point* $(x,u,\dot u,\ddot u,u^{(3)},\ldots\ldots)$, it means that it holds on a neighborhood of this point, i.e. for points whose first coordinates (an unknown a priori but finite number) are close to these of the original point, but with no restriction on the remaining coordinates. Actually, we will often say "in a neighborhood of $(x,u,\ldots,u^{(K)})$" to indicate that the value of $(u^{(K+1)},u^{(K+2)},\ldots\ldots)$ does not matter, i.e. the neighborhood is of the form $\pi_K^{-1}(O)$ with O a neighborhood of $(x,u,\ldots,u^{(K)})$ in $\mathcal{M}_K^{m,n}$.

Smooth functions are functions of a finite number of coordinates which are smooth in the usual sense. Differential forms of degree 1 are *finite* linear combinations : $a_{-1}^1\mathrm{d}x_1+\cdots+a_{-1}^n\mathrm{d}x_n+a_0^1\mathrm{d}u_1+\cdots+a_0^m\mathrm{d}u_m+\ldots+a_J^1\mathrm{d}u_1^{(J)}+\cdots+a_J^m\mathrm{d}u_m^{(J)}$ where the a_i^j's are smooth function. Forms of any degree may be defined similarly. Vector fields are (possibly infinite) linear combinations $b_{-1}^1\frac{\partial}{\partial x_1}+\cdots+b_{-1}^n\frac{\partial}{\partial x_n}+b_0^1\frac{\partial}{\partial u_1}+\cdots+$ $b_0^m\frac{\partial}{\partial u_m}+b_1^1\frac{\partial}{\partial \dot u_1}+\cdots+b_1^m\frac{\partial}{\partial \dot u_m}+\cdots$. Note that this infinite sum is only symbolic. There is no notion of "convergence" here since a vector filed may be defined as a derivation on smooth functions, which, by definition depend only on a finite number of variables, so that the sum becomes finite when computing the (Lie) derivative of a smooth function along this vector field. Lie Brackets, exterior derivative, Lie derivatives and all objects from usual differential calculus may be defined because they (or each of their components) may all be computed finitely and depend on a finite number of variables; all identities from differential calculus are valid (any given such identity really involves only a finite number of variable).

A diffeomorphism is a mapping φ from $\mathcal{M}_\infty^{m,n}$ to $\mathcal{M}_\infty^{\tilde m,\tilde n}$ which is invertible and such that φ and φ^{-1} are smooth mappings, in the sense that, for any smooth function h from $\mathcal{M}_\infty^{\tilde m,\tilde n}$ to $I\!R$, $h\circ\varphi$ is a smooth function from $\mathcal{M}_\infty^{m,n}$ to $I\!R$, and for any smooth function k from $\mathcal{M}_\infty^{m,n}$ to $I\!R$, $k\circ\varphi^{-1}$ is a smooth function from $\mathcal{M}_\infty^{\tilde m,\tilde n}$ to $I\!R$.

Let us come back to the small dimensional system (5). We associate to it the following vector field on $\mathcal{M}_\infty^{2,4}$:

$$F \;=\; X_0 \,+\, u_1 X_1 \,+\, u_2 X_2 \,+\, \dot{u}_1 \frac{\partial}{\partial u_1} \,+\, \dot{u}_2 \frac{\partial}{\partial u_2} \,+\, \ddot{u}_1 \frac{\partial}{\partial \dot{u}_1} \,+\, \dots\;. \tag{9}$$

The Lie derivative along this vector field it the "time-derivative" along system (5). It is the derivation of the differential algebraic approach in [8, 9, 19, 10]. It will often be denoted $\frac{\mathrm{d}}{\mathrm{dt}}$ instead of L_F.

Here is the definition of **linearizing outputs** that we shall use. It is the one from [2], and is equivalent to the notion originally given in [8, 9, 19]. Existence of some linearizing outputs is called "differential flatness" in [8, 9, 19].

Definition 2.1 ([2]) *A pair of functions (h_1, h_2) on M is called a pair of* **linearizing outputs** *on an open subset U of M if the functions $\left(L_F^j h_k\right)_{k\in\{1,2\}\,,\,j\geq 0}$ are a set of coordinates on U, i.e. if $\mathcal{X} \mapsto \left(L_F^j h_k(\mathcal{X})\right)_{k\in\{1,2\}\,,\,j\geq 0}$ is a diffeomorphism from U to an open subset of $\mathbb{R}^{2\mathbb{N}} = \mathcal{M}_\infty^{2,0}$. It is said to be a pair of linearizing output at point $(\bar{x}, \bar{u}, \dot{\bar{u}}, \dots, \bar{u}^{(J)})$ with $J \geq -1$ (when $J = -1$, this stands for $\bar{x}$) if it is a pair of linearizing output on an open set U of the form $\pi_J^{-1}(U_J)$ (see (8)) where U_J is a neighborhood of $(\bar{x}, \bar{u}, \dot{\bar{u}}, \dots, \bar{u}^{(J)})$ in $\mathcal{M}_J^{2,4}$, i.e. $\mathbb{R}^{2J+6}$.*

We do not ignore singularities here, and this is the reason for the open set U. Some functions h_1 and h_2, defined all over $\mathbb{R}^4 \times \mathbb{R}^2$ might very well satisfy the above property around some points and not around some others. It is even not true that one may take in general $J = K$: we shall see, for example (section 5), that for two functions $h_1(x,u)$ and $h_2(x,u)$ depending on x and u only (i.e. $K = 0$), which are linearizing outputs "generically" (say around any points in an open dense set), it is possible (it is even the general case) that for all pair (x,u), there is a value of $\dot{u}$ such that these functions are not linearizing outputs at this point –because the functions $h_k^{(j)}$, $k = 1, 2$, $j \geq 0$ are not independent for these values of x, u and $\dot{u}$– and therefore J has to be taken at least equal to 1.

Let us explain very quickly (it is not really necessary for the sequel, and can be found in [23, 2] for instance) what is endogenous feedback in this framework, and why existence of linearizing outputs is necessary and sufficient for linearizability by endogenous feedback. We call **canonical linear system with two inputs** the vector field

$$C = \dot{v}_1 \frac{\partial}{\partial v_1} + \dot{v}_2 \frac{\partial}{\partial v_2} + \ddot{v}_1 \frac{\partial}{\partial \dot{v}_1} + \ddot{v}_2 \frac{\partial}{\partial \dot{v}_2} + v_1^{(3)} \frac{\partial}{\partial \ddot{v}_1} + \dots$$

on the manifold $\mathcal{M}_\infty^{2,0}$ where some coordinates are $v_1, v_2, \dot{v}_1, \dot{v}_2, \ddot{v}_1, \ddot{v}_2, \dots$.

Any controllable linear system with 2 inputs can be (globally) transformed a diffeomorphism into the "canonical" linear system on $\mathcal{M}_\infty^{2,0}$, see [2]. For instance, for the first case in (7), the diffeomorphism is given by $v_1 = x_1$, $\dot{v}_1 = x_2$, $\ddot{v}_1 = x_3$, $v_1^{(3)} = u_1$, $v_1^{(4)} = \dot{u}_1, \dots, v_2 = x_4$, $\dot{v}_2 = u_2$, $\ddot{v}_2 = \dot{u}_2, \dots$. Hence, system (5) is said to be locally linearizable by endogenous dynamic feedback, or simply **endogenous dynamic linearizable** at $\mathcal{X} \in \mathcal{M}_\infty^{2,4}$ if and only if there is a diffeomorphism φ from an open neighborhood of $\mathcal{X}$ in $\mathcal{M}_\infty^{2,4}$ to an open set of $\mathcal{M}_\infty^{2,0}$ which transforms the vector field F defined in (9) into the vector field C on $\mathcal{M}_\infty^{2,0}$. Such a diffeomorphism defines two functions $h_1 = v_1 \circ \varphi$ and $h_2 = v_2 \circ \varphi$ on $\mathcal{M}_\infty^{2,4}$ which have the property that all their Lie derivatives $L_F^j h_k$ are transformed by the diffeomorphism into the coordinate $v_k^{(j)}$, which implies that the functions $L_F^j h_k$ are locally a set of coordinates on $\mathcal{M}_\infty^{2,4}$; conversely, if two functions exists which have this property, it is very easy to build a diffeomorphism from $\mathcal{M}_\infty^{2,4}$ to $\mathcal{M}_\infty^{2,0}$ which transforms F into C.

We now focus on deciding whether some linearizing outputs exists. Actually, we shall only be able to decide when there exists some depending only on x, or only on x and u, and we call the corresponding properties "x-dynamic linearizability" and "(x,u)-dynamic linearizability" :

Definition 2.2 *System (5) is said to be* $\boldsymbol{(x,u)}$**-dynamically linearizable** *at the point* $\overline{\mathcal{X}} = (\bar{x}, \bar{u}, \dots, \bar{u}^{(J)})$ *if and only if there exists a pair of linearizing outputs* (h_1, h_2) *that depend on x and u only on an open set* $\pi_K^{-1}(\overline{\mathcal{X}})$*, a pair of linearizing outputs depending on x and u only. It is said to be* $\boldsymbol{x}$**-dynamically linearizable** *if these linearizing outputs depend on x only.*

One might also define some less and less restrictive properties : "$(x,u,\dot{u})$-dynamic linearizability", "$(x,u,\dot{u},\ddot{u})$-dynamic linearizability"... which would allow the linearizing outputs to depend on more and more time-derivatives of u, and whose "union" is endogenous dynamic linearizability. Of course, a very interesting question is to know whether, for a given system, there exists a bound K, depending on some simple characteristics of this system such that if it is dynamic linearizable at all, then it is $(x,u,\dot{u},\ddot{u},\dots,u^{(K)})$-dynamic linearizable. Even for systems of the form (5), this is the subject of ongoing research.

2.3 (Non-) accessibility

In some cases, we shall conclude that there exists no linearizing outputs for system (5) because the vector fields X_i are such that is is not accessible. Since we shall only work at regular points, non accessibility means (with the dimensions as in (5)) that there exists one function χ, or two functions χ_1 and χ_2, such that $\dot{\chi} = \varphi(\chi)$ for some function φ, or $\dot{\chi}_i = \varphi_i(\chi_1, \chi_2)$, $i = 1, 2$ for some functions φ_1 and φ_2.

In that case, there may not exist a pair of linearizing outputs. Indeed, if there are some linearizing outputs (h_1, h_2), $\mathrm{d}\chi_1$ and $\mathrm{d}\chi_2$ are independent linear combinations of the $\mathrm{d}h_k^{(j)}$, $k = 1, 2$, $0 \leq j \leq J$ for a certain $J \geq 0$ —or $\mathrm{d}\chi$ is a nonzero linear combination of these— and then $\dot{\chi}_i = \varphi_i(\chi_1, \chi_2)$, $i = 1, 2$ —or $\dot{\chi} = \varphi(\chi)$— would imply that the forms $\{\mathrm{d}h_k^{(j)}, k \in \{1, 2\}, 0 \leq j \leq J + 1\}$ would not be linearly independent, which would contradict the fact that (h_1, h_2) is a pair of linearizing outputs.

2.4 The infinitesimal Brunovský form

The following sequence of modules (over smooth functions) of 1-forms and of vector fields is defined in [2] or in [1]. Recall that $\dot{\omega}$ stand for $L_F\omega$, the Lie derivative along the vector filed F defined in (9) of the differential form ω.

$$\left.\begin{array}{rcl} \mathcal{H}_0 & = & \mathrm{Span}\,\{\,\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3, \mathrm{d}x_4\,,\,\mathrm{d}u_1, \mathrm{d}u_2\,\} \\ \mathcal{H}_1 & = & \mathrm{Span}\,\{\,\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3, \mathrm{d}x_4\,\} \\ & \vdots & \\ \mathcal{H}_{k+1} & = & \{\,\omega \in \mathcal{H}_k\,,\ \dot{\omega} = L_F\omega \in \mathcal{H}_k\,\} \\ & \vdots & \\ \mathcal{H}_\infty & = & \bigcap\limits_k \mathcal{H}_k\ , \end{array}\right\} \tag{10}$$

$$\left.\begin{array}{rcl} \mathcal{D}_0 & = & \mathrm{Span}\,\{\,\frac{\partial}{\partial \dot{u}}\,,\ \frac{\partial}{\partial \ddot{u}}\,,\ \frac{\partial}{\partial u^{(3)}}\,,\ \dots\,\} \\ \mathcal{D}_1 & = & \mathrm{Span}\,\{\,\frac{\partial}{\partial u}\,,\ \frac{\partial}{\partial \dot{u}}\,,\ \frac{\partial}{\partial \ddot{u}}\,,\ \frac{\partial}{\partial u^{(3)}}\,,\ \dots\,\} \\ & \vdots & \\ \mathcal{D}_{k+1} & = & \mathcal{D}_k\ +\ [\,F\,,\,\mathcal{D}_k\,] \\ & \vdots & \\ \mathcal{D}_\infty & = & \sum\limits_k \mathcal{D}_k \end{array}\right\} \tag{11}$$

and, since the $\mathcal{D}_k$'s are "infinite-dimensional", we define for each of them its "$\frac{\partial}{\partial x}$ part" :

$$\widehat{\mathcal{D}}_k \;=\; \mathcal{D}_k \,\cap\, \mathrm{Span}\,\{\,\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3},\frac{\partial}{\partial x_4}\,\}\;,\quad k\geq 0 \tag{12}$$

which makes the vector space spanned at a point by $\widehat{\mathcal{D}}_k$ finite-dimensional, and also yields

$$\mathcal{D}_k \;=\; \widehat{\mathcal{D}}_k \,\oplus\, \mathcal{D}_1\;. \tag{13}$$

A point where the rank of all the modules $\widehat{\mathcal{D}}_k$, or more precisely the rank of the corresponding distributions all the distributions are constant in a neighborhood is called **Brunovský regular**.

We have the following relation between the $\mathcal{H}_k$'s and the $\mathcal{D}_k$'s, where $\mathcal{D}^\perp$ stands for the annihilator of the module of vector fields $\mathcal{D}$, and vice-versa : $\mathcal{H}^\perp = \{\,X,\,\forall\omega\in\mathcal{H},\,\langle\omega,X\rangle=0\}$ and $\mathcal{D}^\perp = \{\,\omega,\,\forall X\in\mathcal{D},\,\langle\omega,X\rangle=0\}$.

Proposition 2.1 ([2]) *All the modules $\mathcal{D}_k$ and $\mathcal{H}_k$ are invariant by static feedback, and for all k, around Brunovský-regular points, i.e. points where the rank of the distribution spanned by $\widehat{\mathcal{D}}_k$ is constant,*

$$\mathcal{D}_k^\perp \;=\; \mathcal{H}_k \quad \textit{and} \quad \mathcal{H}_k^\perp \;=\; \mathcal{D}_k, \tag{14}$$

The proof is in [2] and is a simple application of the identity

$$\begin{aligned}\langle\dot\omega\,,\,X\rangle &= \langle L_F\omega\,,\,X\rangle\\ &= L_F\langle\omega\,,\,X\rangle \;-\; \langle\omega\,,\,[F,X]\rangle\\ &= \frac{\mathrm{d}}{\mathrm{dt}}\langle\omega\,,\,X\rangle \;-\; \langle\omega\,,\,[F,X]\rangle\;.\end{aligned} \tag{15}$$

The first distributions in $\mathcal{D}_k$'s are given by

$$\begin{aligned}\widehat{\mathcal{D}}_1 &= \{0\}\;,\\ \widehat{\mathcal{D}}_2 &= \mathrm{Span}\,\{\,X_1\,,\,X_2\,\}\\ \widehat{\mathcal{D}}_3 &= \widehat{\mathcal{D}}_2 \;+\; [\,X_0+u_1X_1+u_2X_2\,,\,\widehat{\mathcal{D}}_2\,]\\ &= \mathrm{Span}\,\{\,X_1\,,\,X_2\,,\,[X_0,X_1]-u_1[X_1,X_2]\,,\,[X_0,X_2]+u_2[X_1,X_2]\,\}\\ \widehat{\mathcal{D}}_4 &= \widehat{\mathcal{D}}_3 \;+\; [\,X_0+u_1X_1+u_2X_2\,,\,\widehat{\mathcal{D}}_3\,]\;.\end{aligned} \tag{16}$$

If X_1 and X_2 are linearly independent, a point is Brunovský-regular if and only if these four first distributions have constant rank locally. At such a point, the ranks of $\widehat{\mathcal{D}}_1$, $\widehat{\mathcal{D}}_2$, $\widehat{\mathcal{D}}_3$, $\widehat{\mathcal{D}}_4$ may only be (0,2,2,2), (0,2,3,4) or (0,2,4,4), and the sequence $\mathcal{D}_k$ is constant afterwards, i.e. $\mathcal{D}_\infty = \mathcal{D}_4$.

The following proposition is, modulo some detail, a particular case of [2, theorem 2], we however give a (simple) proof.

Proposition 2.2 (Infinitesimal Brunovský Form) *Around a point where $\widehat{\mathcal{D}}_2$, $\widehat{\mathcal{D}}_3$ and $\widehat{\mathcal{D}}_4$ have constant rank,*

- *If $\widehat{\mathcal{D}}_2$, $\widehat{\mathcal{D}}_3$ and $\widehat{\mathcal{D}}_4$ have rank (2,2,2) or (2,3,3), system (5) is locally non-accessible (see section 2.3).*
- *If $\widehat{\mathcal{D}}_2$, $\widehat{\mathcal{D}}_3$ and $\widehat{\mathcal{D}}_4$ have rank (2,3,4), by taking ω_1 a nonzero 1-form such that*

$$\widehat{\mathcal{D}}_3 = \{\omega_1\}^\perp , \tag{17}$$

and ω_2 a 1-form which is not a linear combination of ω_1, $\dot\omega_1$ and $\ddot\omega_1$, $(L_F^{(j)}\omega_k)_{k\in\{1,2\},\, j\geq 0}$ is a basis of the module of differential forms on U, and more precisely $\{\omega_1, \dot\omega_1\}$ is a basis of $\mathcal{H}_2$, $\{\omega_1, \dot\omega_1, \ddot\omega_1, \omega_2\}$ is a basis of $\mathcal{H}_1$, $\{\omega_1, \dot\omega_1, \ddot\omega_1, \omega_1^{(3)}, \omega_2, \dot\omega_2\}$ is a basis of $\mathcal{H}_0$.

- *If $\widehat{\mathcal{D}}_2$, $\widehat{\mathcal{D}}_3$ and $\widehat{\mathcal{D}}_4$ have rank (2,4,4), by taking for ω_1 and ω_2 two linearly independent 1-forms such that*

$$\widehat{\mathcal{D}}_2 = \{\omega_1, \omega_2\}^\perp , \tag{18}$$

$(L_F^{(j)}\omega_k)_{k\in\{1,2\},\, j\geq 0}$ is a basis of the module of differential forms on U, and more precisely $\{\omega_1, \omega_2, \dot\omega_1, \dot\omega_2\}$ is a basis of $\mathcal{H}_1$, $\{\omega_1, \omega_2, \dot\omega_1, \dot\omega_2, \ddot\omega_1, \ddot\omega_2\}$ is a basis of $\mathcal{H}_0$.

- *In the above two cases, the 1-forms ω_1 and ω_2 may be chosen involving x only.*

The term "infinitesimal Brunovský form" refers to the fact that, with the above choices of the 1-forms ω_1 and ω_2, system (5) implies :

$$\left.\begin{array}{ll}
\text{In case "}(2,3,4)\text{"}, & \left\{\begin{array}{lcl}
\frac{d}{dt}\omega_1 &=& \dot\omega_1\\
\frac{d}{dt}\dot\omega_1 &=& \ddot\omega_1\\
\frac{d}{dt}\ddot\omega_1 &=& \sum_1^4 \alpha_{1,i}\mathrm{d}x_i \;+\; \beta_{1,1}\mathrm{d}u_1 \;+\; \beta_{1,2}\mathrm{d}u_2\\
\frac{d}{dt}\omega_2 &=& \sum_1^4 \alpha_{2,i}\mathrm{d}x_i \;+\; \beta_{2,1}\mathrm{d}u_1 \;+\; \beta_{2,2}\mathrm{d}u_2
\end{array}\right.\\[2em]
\text{In case "}(2,4,4)\text{"}, & \left\{\begin{array}{lcl}
\frac{d}{dt}\omega_1 &=& \dot\omega_1\\
\frac{d}{dt}\dot\omega_1 &=& \sum_1^4 \alpha_{1,i}\mathrm{d}x_i \;+\; \beta_{1,1}\mathrm{d}u_1 \;+\; \beta_{1,2}\mathrm{d}u_2\\
\frac{d}{dt}\omega_2 &=& \dot\omega_2\\
\frac{d}{dt}\dot\omega_2 &=& \sum_1^4 \alpha_{2,i}\mathrm{d}x_i \;+\; \beta_{2,1}\mathrm{d}u_1 \;+\; \beta_{2,2}\mathrm{d}u_2
\end{array}\right.
\end{array}\right\} \tag{19}$$

where the functions $\beta_{i,j}$ are such that the 2×2 matrix $[\beta_{i,j}]$ is invertible on a neighborhood of $(\bar{x}, \bar{u})$. If the forms ω_1 and ω_2 were integrable, one might define z function of x and v function of x, u (static feedback transformation) by $\mathrm{d}z_1 = \omega_1$, $\mathrm{d}z_2 = \dot{\omega}_1$, $\mathrm{d}z_3 = \ddot{\omega}_1$, $\mathrm{d}u_1 = \omega_1^{(3)}$, $\mathrm{d}z_4 = \omega_2$, $\mathrm{d}u_2 = \dot{\omega}_2$ in case "(2,3,4)" and $\mathrm{d}z_1 = \omega_1$, $\mathrm{d}z_2 = \dot{\omega}_1$, $\mathrm{d}u_1 = \ddot{\omega}_1$, $\mathrm{d}z_3 = \omega_2$, $\mathrm{d}z_4 = \dot{\omega}_2$, $\mathrm{d}u_2 = \ddot{\omega}_2$ in case "(2,3,4)", such that (5) reads $\dot{z}_1 = z_2$, $\dot{z}_2 = z_3$, $\dot{z}_3 = v_1$, $\dot{z}_4 = v_2$ in case "(2,3,4)" or $\dot{z}_1 = z_2$, $\dot{z}_2 = v_1$, $\dot{z}_3 = z_4$, $\dot{z}_4 = v_2$. in case "(2,3,4)" ; these are the two Brunovský canonical forms [4] for controllable linear systems with 4 states and two inputs.

Proof of proposition 2.2 : Point 1 is a consequence of [2, theorem 1] but is almost obvious : it is simple to see that distribution $\widetilde{\mathcal{D}}_2$ in case (2,2,2) is integrable because $\widetilde{\mathcal{D}}_3 = \widetilde{\mathcal{D}}_2$ for u in an open set implies that $[X_1, X_2]$ is a linear combination of X_1 and X_2 and that distribution $\widetilde{\mathcal{D}}_3$ is integrable in case (2,3,3) because if it has rank 3 on an open set, a basis of it is made of X_1, X_2 and one of $[X_0, X_1]$, $[X_0, X_2]$ or $[X_1, X_2]$, and the fact that, for u in an open set, $\widetilde{\mathcal{D}}_{43} = \widetilde{\mathcal{D}}_3$ implies that it is involutive; define then χ or χ_1, χ_2 (see section 2.3) to be some first integrals of the distribution $\widetilde{\mathcal{D}}_\infty$.

Points 2 and 3 are very similar to [2, theorem 2], except we build explicitly the forms ω_1 and ω_2. Here, it is not difficult, in the various cases, to deduce from the constant rank of $\widehat{\mathcal{D}}_2$, $\widehat{\mathcal{D}}_3$ and $\widehat{\mathcal{D}}_4$ that the rank of $\{\omega_1, \dot{\omega}_1, \ddot{\omega}_1, \omega_2\}$ or $\{\omega_1, \omega_2, \dot{\omega}_1, \dot{\omega}_2\}$ is constant equal to 4, and that the matrix α remains invertible on a neighborhood. Let us just prove in detail that, in case (2,4,4), the rank of $\{\omega_1, \omega_2, \dot{\omega}_1, \dot{\omega}_2\}$ does not drop : if it would drop at a certain point, then there would exist a linear combination $\lambda_1\dot{\omega}_1 + \lambda_2\dot{\omega}_2 + \mu_1\omega_1 + \mu_2\omega_2$ would vanish *at this point* without all the coefficients vanishing; this implies that, for $i = 1, 2$, $\langle \lambda_1\dot{\omega}_1 + \lambda_2\dot{\omega}_2, X_i \rangle$ would also vanish at this point (because $\langle \omega_1, X_i \rangle = \langle \omega_2, X_i \rangle = 0$), which implies, from identity (15), that

$$\langle \lambda_1\omega_1 + \lambda_2\omega_2 \, , \, [X_0 + u_1 X_1 + u_2 X_2, X_i] \rangle$$

vanishes at this point, which in turn implies that for all X in $\widehat{\mathcal{D}}_3$, $\langle \lambda_1\omega_1 + \lambda_2\omega_2, X \rangle$ vanishes at the point under consideration and hence that the rank of $\widehat{\mathcal{D}}_3$ cannot be 4 at this point at this point (λ_1 and λ_2 do not vanish together at this point because ω_1 and ω_2 are independent).

Finally, the forms ω_1 and ω_2 may always be chosen so that they involve x only because they are defined from distributions which have this property. ∎

2.5 The link with dynamic linearization

In [2] or [1], a necessary and sufficient condition for (endogenous) dynamic linearizability is given in terms of the 1-forms which yield the "infinitesimal Brunovský Form", and we shall use this result here, in the restricted form for x-dynamic linearizability and (x, u)-dynamic linearizability. Let us first give some definitions:

Let $\mathcal{A}(U)$ be the $\mathcal{C}^\infty(U)$ algebra :

$$\mathcal{A}(U) \;\stackrel{\Delta}{=}\; \mathcal{M}_{m\times m}\left(\, \mathcal{C}^\infty(U)[L_F]\,\right) \;. \tag{20}$$

of 2×2 matrices whose entries are differential operators, polynomial in the derivation along F, i.e. of the form

$$p_0 \;+\; p_1\frac{\mathrm{d}}{\mathrm{dt}} \;+\; p_2\frac{\mathrm{d}}{\mathrm{dt}}^2 \;+\;\ldots\;+\; p_K\frac{\mathrm{d}}{\mathrm{dt}}^K \quad ,$$

where the p_i's are smooth functions from U to $I\!R$ (recall it means they depend only on x and a finite number of time-derivatives of u). Elements of $\mathcal{A}(U)$ act in an obvious manner on pairs of functions, or on pairs of differential forms.

A pair of linearizing outputs, as defined in definition 2.1, is obviously such that $(\mathrm{d}L_F^{(j)}h_k)_{k\in\{1,2\},\,j\geq 0}$ is a basis of the module of differential forms on U. Since $(L_F^{(j)}\omega_k)_{k\in\{1,2\},\,j\geq 0}$ enjoys the same property and all the pairs of forms enjoying this property are transformed into one another, one has, as explained in [1, 2], that a pair of functions (h_1, h_2) is a pair of linearizing outputs if and only if there exists P in $\mathcal{A}(U)$, invertible in $\mathcal{A}(U)$, such that $(\mathrm{d}h_1, \mathrm{d}h_2)^T = P(\omega_1, \omega_2)^T$; also, the fact that h_1 and h_2 depend on x and u only translates in some bounds on the degree of the entries of P. This is explained into details in [1, 2], and is summed up in the following proposition :

Proposition 2.3 *Let $(\bar{x}, \bar{u})$ be a point where the ranks of $\widehat{\mathcal{D}}_2$, $\widehat{\mathcal{D}}_3$ and $\widehat{\mathcal{D}}_4$ are constant, and ω_1 and ω_2 be defined in a neighborhood of $(\bar{x}, \bar{u})$ as in proposition 2.2.*

- *If the ranks of $\widehat{\mathcal{D}}_2$, $\widehat{\mathcal{D}}_3$ and $\widehat{\mathcal{D}}_4$ are (2,2,2) or (2,3,3), system (5) is not accessible and therefore not dynamic linearizable.*
- *If the ranks of $\widehat{\mathcal{D}}_2$, $\widehat{\mathcal{D}}_3$ and $\widehat{\mathcal{D}}_4$ are (2,3,4), system (5) is x-dynamic linearizable (resp. (x, u)-dynamic linearizable) at point $(\bar{x}, \bar{u}, \ldots, \bar{u}^{(J)})$ if and only if there exists*

a neighborhood U of this point, and a 2×2 polynomial matrix $P \in \mathcal{A}(U)$ such that

$$\left.\begin{array}{l} P \textit{ has an inverse in } A(U), \\ P(\frac{\mathrm{d}}{\mathrm{dt}}) \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \mathrm{d}h_1 \\ \mathrm{d}h_2 \end{pmatrix} \end{array}\right\} \tag{21}$$

where the degree of the entries of the first column is at most 2 (resp. at most 3) and the degree of the entries of the second column is at most 0 (resp. at most 1).

- *If the ranks of $\widehat{\mathcal{D}}_2$, $\widehat{\mathcal{D}}_3$ and $\widehat{\mathcal{D}}_4$ are (2,4,4), system (5) is x-dynamic linearizable (resp. (x,u)-dynamic linearizable) at point $(\bar{x}, \bar{u}, \ldots, \bar{u}^{(J)})$ if and only if there exists a neighborhood U of this point, and a 2×2 polynomial matrix $P \in \mathcal{A}(U)$ satisfying (21) where the degree of the entries of P is at most 1 (resp. at most 2).*

These are the characterizations of x- and (x,u)- dynamic linearizability we are going to use in next sections. What makes them tractable is the following lemma, which allows one to describe all the possible invertible matrices of degree 1 or 2 as simple products of elementary matrices, at least on an open dense set.

Proposition 2.4 *Let P be a 2×2 matrix with entries polynomials in $\frac{\mathrm{d}}{\mathrm{dt}}$ with coefficients smooth functions on U, and which has an inverse Q of the same type.*

- *If the degree of P is 1 on an open dense subset of U (i.e. P has degree at most 1 at every point, and possibly zero on a closed set of empty interior), then there is an open dense subset U_0 of U such, for that all $\mathcal{X} \in U_0$, there is a neighborhood $V_{\mathcal{X}}$, a scalar a, and two invertible matrices (of degree 0) J_1 and J_2, all smooth functions defined on $V_{\mathcal{X}}$, such that, on $V_{\mathcal{X}}$,*

$$P(\frac{\mathrm{d}}{\mathrm{dt}}) = J_1 \begin{pmatrix} 1 & -a\frac{\mathrm{d}}{\mathrm{dt}} \\ 0 & 1 \end{pmatrix} J_2 \tag{22}$$

- *If the degree of P is 2 on an open dense subset of U (i.e. P has degree at most 2 at every point, and possibly 1 or zero on a closed set of empty interior), then there is an open dense subset U_0 of U such, for that all $\mathcal{X} \in U_0$, there is a neighborhood $V_{\mathcal{X}}$, scalars α, λ, a and b, and two invertible matrices (of degree 0) J_1 and J_2, all smooth functions defined on $V_{\mathcal{X}}$, such that, on $V_{\mathcal{X}}$, either*

$$P(\frac{\mathrm{d}}{\mathrm{dt}}) = J_1 \begin{pmatrix} 1 & -a\frac{\mathrm{d}}{\mathrm{dt}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b\frac{\mathrm{d}}{\mathrm{dt}} & 1 \end{pmatrix} J_2 \tag{23}$$

or

$$P(\frac{\mathrm{d}}{\mathrm{dt}}) \;=\; J_1 \begin{pmatrix} 1 & 0 \\ -a\frac{\mathrm{d}}{\mathrm{dt}} - b\frac{\mathrm{d}}{\mathrm{dt}}^2 & 1 \end{pmatrix} J_2 \tag{24}$$

with

$$\textit{either} \quad J_2 \;=\; \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix} \qquad \textit{or} \;\; J_2 \;=\; \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} . \tag{25}$$

Proof of proposition 2.4 : This is a decomposition of invertible matrices as products of elementary matrices, i.e. matrices which are
- either invertible diagonal (hence diagonal elements are of degree zero and do not vanish),
- or permutation matrices,
- or matrices which differ from the identity only by one non diagonal entry.
The decomposition being of a special kind that will be useful.

Although the ring of polynomials $\mathcal{C}^\infty(U)[\frac{\mathrm{d}}{\mathrm{dt}}]$ is less simple that the one where coefficients are constant (non commutative, it is not possible to divide by a nonzero function at a point where it vanishes), there is a right and left euclidian division by polynomials whose leading coefficient does not vanish. Since euclidian division is all that is needed to reduce an invertible matrix into a product of elementary matrices, it will possible around points where none of the leading coefficients of the polynomials one has to use as the divisor of a euclidian division vanish.

The particular decompositions given here are obtained by doing "elementary operations" on the columns of P. If both polynomials in the second column of P have degree zero, then at any point one of them at least does not vanish, and dividing by it the corresponding polynomial (degree 2) in the first column yields a degree two polynomial $-\alpha + a\frac{\mathrm{d}}{\mathrm{dt}} + b\frac{\mathrm{d}}{\mathrm{dt}}^2$ such that

$$P \begin{pmatrix} 1 & 0 \\ -\alpha + a\frac{\mathrm{d}}{\mathrm{dt}} + b\frac{\mathrm{d}}{\mathrm{dt}}^2 & 1 \end{pmatrix}$$

has all entries of degree zero. This yields (24) with the second expression for J_2 in (25). If both polynomials in the second column of P have degree at most 1 with at least one of them exactly one, then at any point the leading coefficient of this one does not vanish, euclidian division by this polynomial of the corresponding polynomial (degree 2) in the first column yields a degree one polynomial $-\alpha + b\frac{\mathrm{d}}{\mathrm{dt}}$

such that

$$P\begin{pmatrix} 1 & 0 \\ -\alpha + b\frac{\mathrm{d}}{\mathrm{dt}} & 1 \end{pmatrix}$$

has a first column of degree zero, and then dividing by a nonvanishing element of this first column yields a such that

$$P\begin{pmatrix} 1 & 0 \\ -\alpha + b\frac{\mathrm{d}}{\mathrm{dt}} & 1 \end{pmatrix}\begin{pmatrix} 1 & a\frac{\mathrm{d}}{\mathrm{dt}} \\ 0 & 1 \end{pmatrix}$$

has all entries of degree zero. This yields (23) with the second expression for J_2 in (25). If at least one of the polynomials in the second column of P has degree 2, the, at points where its leading coefficient does not vanish, dividing the corresponding polynomial in the first column by this coefficient yields a function λ such that

$$P\begin{pmatrix} 0 & 1 \\ 1 & -\lambda \end{pmatrix}$$

has both entries in its second column of degree at most 1 (λ is identically zero if the first column of P had degree 1 or 0). The first two cases considered for P now apply to $P\begin{pmatrix} 0 & 1 \\ 1 & -\lambda \end{pmatrix}$ and yields either (24) or (23), with the first expression for J_2 in (25).

We had to divide by at most three polynomials, the points where they vanish without being zero on a neighborhood —if they are zero on an open set, then the corresponding polynomial has locally a smaller degree— is closed with empty interior, the open set U_0 is its complement. ∎

2.6 Two ways of writing the equations for the linearizing outputs

Consider the problem of deciding for instance if there exists some linearizing outputs function of x only.

The direct method consists in writing the equations for two functions $h_1(x)$ and $h_2(x)$ to be linearizing outputs. It is not difficult to see that, under the assumption that X_1 and X_2 in (5) or (9) are linearly independent, they are linearizing outputs if and only if the following two equations (which may be translated into some determinants being zero) are satisfied

$$\mathrm{rank}\begin{pmatrix} \frac{\partial \dot h_1}{\partial \dot u_1} & \frac{\partial \dot h_1}{\partial \dot u_2} \\ \frac{\partial \dot h_2}{\partial \dot u_1} & \frac{\partial \dot h_2}{\partial \dot u_2} \end{pmatrix} \;\le\; 1 \tag{26}$$

$$\mathrm{rank}\begin{pmatrix} \frac{\partial \dot h_1}{\partial u_1} & \frac{\partial \dot h_1}{\partial u_2} & 0 & 0 \\ \frac{\partial \dot h_2}{\partial u_1} & \frac{\partial \dot h_2}{\partial u_2} & 0 & 0 \\ \frac{\partial \ddot h_1}{\partial u_1} & \frac{\partial \ddot h_1}{\partial u_2} & \frac{\partial \ddot h_1}{\partial \dot u_1} & \frac{\partial \ddot h_1}{\partial \dot u_2} \\ \frac{\partial \ddot h_2}{\partial u_1} & \frac{\partial \ddot h_2}{\partial u_2} & \frac{\partial \ddot h_2}{\partial \dot u_1} & \frac{\partial \ddot h_2}{\partial \dot u_2} \end{pmatrix} \;\le\; 2 \tag{27}$$

and also the following inequality (some determinants being nonzero) :

$$\mathrm{rank}\{\, \mathrm{d}h_1 \,,\, \mathrm{d}h_2 \,,\, \mathrm{d}\dot h_1 \,,\, \mathrm{d}\dot h_2 \,,\, \mathrm{d}\ddot h_1 \,,\, \mathrm{d}\ddot h_2 \,\} \;=\; 6\;. \tag{28}$$

Let us check that these are necessary and sufficient. If (h_1, h_2) is a pair of linearizing outputs, the forms in (28) have to be equivalent by definition, furthermore if the rank in (26) would be 2, it is clear that the only linear combinations of the $\mathrm{d}h_k^{(j)}$'s which are also linear combinations of $\mathrm{d}x_1$, $\mathrm{d}x_2$, $\mathrm{d}x_3$, $\mathrm{d}x_4$, would have all their coefficients zero except the coefficients of $\mathrm{d}h_1$ and $\mathrm{d}h_2$, which would contradict the fact that $\mathrm{d}x_1$, $\mathrm{d}x_2$, $\mathrm{d}x_3$ and $\mathrm{d}x_4$ are linear combinations of the $\mathrm{d}h_k^{(j)}$'s, and similarly, it may be shown that if the rank in (27) is at least 3, there cannot be more than three independent linear combinations of the $\mathrm{d}h_k^{(j)}$'s which are also linear combinations of $\mathrm{d}x_1$, $\mathrm{d}x_2$, $\mathrm{d}x_3$, $\mathrm{d}x_4$. This proves that the above conditions are necessary. We shall only use the necessary part ; to see that the conditions are sufficient, one simply proves that they endure that $\mathrm{d}x_1$, $\mathrm{d}x_2$, $\mathrm{d}x_3$ and $\mathrm{d}x_4$ are linear combinations of the $\mathrm{d}h_k^{(j)}$'s.

The first two equations give some PDEs in h_1 and h_2, and the last one an inequality. These have solutions of and only if the system is x-dynamic linearizable. The equations (26)-(27) are more explicit using :

$$\frac{\partial \ddot h_i}{\partial \dot u_k} \;=\; \frac{\partial \dot h_i}{\partial u_k} \;=\; L_{X_k} h_i \tag{29}$$

and

$$\frac{\partial \ddot h_i}{\partial u_k} \;=\; L_{X_0} L_{X_k} h_i \,+\, L_{X_k} L_{X_0} h_i \,+\, 2\, u_k L_{X_k}^2 h_i \,+\, u_{k'} \left(L_{X_{k'}} L_{X_k} h_i \,+\, L_{X_k} L_{X_{k'}} h_i \right) \tag{30}$$

where k' is 2 if $k = 1$ and 1 if $k = 2$.

Of course the same thing may be done for linearizing outputs depending on x and u.

We shall use this direct method only when looking for conditions for x-dynamic linearizability (existence of linearizing outputs depending only on x).

The second solution is described in the previous sub-section : under some nonsingularity conditions (being at a "Brunovský regular" point), there exists two differential forms such that $\{\omega_1, \omega_2, \dot{\omega}_1, \dot{\omega}_2\}$ (or $\{\omega_1, \dot{\omega}_1, \ddot{\omega}_1, \omega_2\}$ but let us consider the first case only), these forms may be constructed explicitly, and the system is x-linearizable or (x, u)-dynamic linearizable if and only if there exists some invertible polynomial matrix such that

$$P(\frac{\mathrm{d}}{\mathrm{dt}}) \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

is made of two exact one-forms, with some bounds on the degree of the entries of P, this is proposition 2.3. We then write some PDEs that the coefficients of the matrix (we use the decompositions described above) have to satisfy for the above to be true; the system is x-dynamic or (x, u)-dynamic linearizable if and only if these PDEs have solutions.

These two methods —writing directly the PDEs a pair of functions has to satisfy to be a pair of linearizing outputs or writing the PDEs the coefficients of the elementary matrices in the decomposition of P have to satisfy for the Pfaffian system $P(\frac{\mathrm{d}}{\mathrm{dt}})\,(\omega_1, \omega_2)^T$ to be integrable— are obviously equivalent, although they lead to different equations.

One drawback of the second method is that it needs existence of an "infinitesimal Brunovský form" and therefore it only works at "Brunovský-regular" points, while Brunovský-regularity is not necessary for dynamic feedback linearization, see the example in section 5. Although Brunovský-regular points form in general an open dense set, one cannot neglect this weakness. One might use the following trick to overcome it : if one may find a necessary condition which is valid at Brunovský-regular points, and if this condition is closed, it will be necessary also at other points by density, and sufficiency is usualkly exstablished by constructing a pair of linearizing outputs, it may be possible to establish that they may be prolonged to non Brunovský regular points, into functions which are linearizing outputs at these points too.

We have used the first method to give criteria for x-linearization because in the simplest cases, referring to the infinitesimal Brunovský form is useless since the linearizing outputs may be constructed, or proved not to exist, by elementary methods (cases 1 to 5 in theorem 3.1), and it avoids the problems arising from Brunovský regularity (for instance case 6 should have been splitted into two cases depending whether (53) holds or not because this corresponds to different infinitesimal Brunovský forms (second and third points in proposition 2.2). However, in the non elementary cases, the core of the proof at Brunovský regular points, i.e. if one is only interested in "generic" points for exemple, would be simpler using the infinitesimal Brunovský form, as seen in the sketch of an alternative proof of point 6 of theorem (3.1) proposed in section 7.1.

To test for (x, u)-linearizability, we were not able to use the direct method, and we use the second one based on infinitesimal Brunovský form. It turns out that the first one yield rather huge PDEs in the linearizing outputs while the second one gives some PDEs that, though heavy computations are needed, may be handled by elementary methods.

In general, this second method yields equations that may be considered more geometrically, but it is not clear that they may, in general (higher dimensions, more derivatives of the controls...), be solved in such a simple way.

3 x-dynamic linearizability

We define the following distributions

$$\begin{array}{rcl} \Delta_2 & = & \mathrm{Span}\,\{\,X_1\,,\,X_2\,\} \\ \mathcal{M}_0 & = & \Delta_2 \;+\; [\Delta_2,\Delta_2] \;\;=\;\; \mathrm{Span}\,\{\,X_1\,,\,X_2\,,\,[X_1,X_2]\,\} \\ \mathcal{M}_1 & = & \mathcal{M}_0 \;+\; [\mathcal{M}_0,\mathcal{M}_0] \\ & = & \mathrm{Span}\,\{\,X_1\,,\,X_2\,,\,[X_1,X_2]\,,\,[X_1,[X_1,X_2]]\,,\,[X_2,[X_1,X_2]]\,\} \\ \Delta_3 & = & \mathrm{Span}\,\{\,X_1\,,\,X_2\,,\,[X_1,X_2]\,,\,[X_0,X_1]\,,\,[X_0,X_2]\,\} \end{array} \tag{31}$$

We will only study the situation in the neighborhood of points where the rank these distributions are constant, and the vector fields X_1 and X_2 are linearly independent and we define the integers m_0, m_1, δ_3 by :

$$\operatorname{rank}\Delta_2 \;=\; 2 \qquad \begin{array}{rcl} \delta_3 & \triangleq & \operatorname{rank}\Delta_3 \\ m_0 & \triangleq & \operatorname{rank}\mathcal{M}_0 \\ m_1 & \triangleq & \operatorname{rank}\mathcal{M}_1 \quad . \end{array} \tag{32}$$

It is easy to see that, at a point where these ranks are constant, the only possible values for the triple (m_0, m_1, δ_3) are $(2,2,2)$, $(2,2,3)$, $(2,2,4)$, $(3,3,3)$, $(3,3,4)$, $(3,4,3)$ and $(3,4,4)$. Actually, we will not distinguish between cases $(3,4,3)$ and $(3,4,4)$, so that when $(m_0, m_1) = (3,4)$, the rank of Δ_3 need not be constant.

The following theorem allows one, in each of the cases depending on the different possible values of the above ranks, to decide whether system (5) is x-dynamic linearizable or not. Note that in some cases it is even static feedback linearizable, that in some cases where it is not x-dynamic linearizable, we are able to conclude that it is not linearizable by endogenous dynamic feedback at all —in some of these cases because it is not accessible— whereas in other cases where it is not x-dynamic linearizable, it is possibly (x,u)-dynamic linearizable —this is the subject of section 4— or dynamic linearizable with linearizing outputs depending on higher order time-derivatives of u.

In addition, for each case, we give a normal form for system (5) in some coordinates and after a nonsingular *static* feedback transformation (see (6)).

Theorem 3.1 *Let $\bar{x}$ be such that the distributions the distributions spanned by the modules Δ_2, $\mathcal{M}_0$, $\mathcal{M}_1$ and Δ_3 have constant rank in a neighborhood of $\bar{x}$, with Δ_2 of rank 2, and*

and we have the following conclusions for each of these cases :

1. **If $m_0 = m_1 = 2$ and $\delta_3 = 2$**, *system (5) is locally* **non accessible** *and therefore* **non linearizable by endogenous feedback**. *Locally around $\bar{x}$, in some coordinates and after a preliminary nonsingular feedback transformation, it has the following form, with a_1 and a_2 some smooth functions :*

$$\begin{array}{rcl} \dot{z}_1 & = & a_1(z_1, z_2) \\ \dot{z}_2 & = & a_2(z_1, z_2) \\ \dot{z}_3 & = & v_1 \\ \dot{z}_4 & = & v_2 \quad . \end{array} \tag{33}$$

2. **If $m_0 = m_1 = 2$ and $\delta_3 = 3$**, *there are three subcases :*

(a) **If Δ_3 is not involutive** *(i.e. if there are points x arbitrarily close to $\bar{x}$ such that $[\Delta_3, \Delta_3](x) \not\subset \Delta_3(x)$, even if $[\Delta_3, \Delta_3](\bar{x}) \subset \Delta_3(\bar{x})$), system (5) is* **not linearizable by endogenous dynamic feedback**. *It has locally, around $\bar{x}$, the following form in some coordinates and after a preliminary nonsingular feedback transformation :*

$$\begin{array}{rcl} \dot{z}_1 & = & a(z_1, z_2, z_3) \\ \dot{z}_2 & = & z_3 \\ \dot{z}_3 & = & v_1 \\ \dot{z}_4 & = & v_2 \end{array} \tag{34}$$

where a is a smooth function such that

$$\frac{\partial^2 a}{\partial z_3^2} \text{ is not identically zero on any neighborhood of } \bar{x}. \tag{35}$$

(b) **If Δ_3 involutive and the rank of $\Delta_3 + [X_0, \Delta_3]$ is 3** *in a neighborhood of $\bar{x}$, system (5) is locally* **non accessible** *and therefore* **non linearizable by endogenous feedback**. *Locally around $\bar{x}$, in some coordinates and after a preliminary nonsingular feedback transformation, it has the following form, with a some smooth function :*

$$\begin{array}{rcl} \dot{z}_1 & = & a(z_1) \\ \dot{z}_2 & = & z_3 \\ \dot{z}_3 & = & v_1 \\ \dot{z}_4 & = & v_2 \end{array} \tag{36}$$

(c) **If Δ_3 involutive and the rank of $\Delta_3 + [X_0, \Delta_3]$ is 4** *at point $\bar{x}$ (and therefore in a neighborhood), system (5) is locally* **static feedback linearizable**. *It has, in some local coordinates around $\bar{x}$, and after a preliminary nonsingular feedback transformation, the form (7.a).*

3. **If $m_0 = m_1 = 2$ and $\delta_3 = 4$**, *system (5) is locally* **static feedback linearizable**. *It has, in some local coordinates around $\bar{x}$, and after a preliminary nonsingular feedback transformation, the form (7.b).*

4. **If $m_0 = m_1 = 3$ and $\delta_3 = 3$**, *system (5) is locally* **non accessible** *and therefore* **non linearizable by endogenous feedback**. *Locally around $\bar{x}$, in* *some coordinates and after a preliminary nonsingular feedback transformation, it has the following form, with* a_1 *and* a_3 *some smooth functions :*

$$\begin{array}{rcl} \dot z_1 &=& a_1(z_1) \\ \dot z_2 &=& v_1 \\ \dot z_3 &=& a_3(z_1,z_2,z_3,z_4) \ + \ z_4 v_1 \\ \dot z_4 &=& v_2 \end{array} \tag{37}$$

where a_1 *and* a_3 *are some smooth functions.*

5. **If $\boldsymbol{m_0 = m_1 = 3}$ and $\boldsymbol{\delta_3 = 4}$**, *system (5) is locally* x-**dynamic linearizable** *at a point* $(\bar x, \bar u_1, \bar u_2, \ldots)$ *if and only if*

$$\begin{array}{l} \operatorname{rank}_{I\!R} \{ X_1(\bar x) \,,\, X_2(\bar x) \,,\, [X_0,X_1](\bar x) \,-\, \bar u_2 [X_1,X_2](\bar x) \,, \\ \qquad\qquad\qquad [X_0,X_2](\bar x) \,+\, \bar u_1 [X_1,X_2](\bar x) \,\} \;=\; 4 \;. \end{array} \tag{38}$$

In some local coordinates around $\bar x$*, the system has the following form after a preliminary nonsingular feedback transformation :*

$$\begin{array}{rcl} \dot z_1 &=& z_2 \\ \dot z_2 &=& v_1 \\ \dot z_3 &=& a_3(z_1,z_2,z_3,z_4) \ + \ z_4 v_1 \\ \dot z_4 &=& v_2 \end{array} \tag{39}$$

with a *is a smooth function. A possible choice of linearizing outputs is given, in these coordinates, by* $h_1 = z_1$*,* $h_2 = z_3$*. Condition (38) reads :*

$$v_1 \ + \ \frac{\partial a_3}{\partial z_4} \ \neq \ 0 \ . \tag{40}$$

6. **If $\boldsymbol{m_0 = 3}$ and $\boldsymbol{m_1 = 4}$**, *there exists a unique (up to a nonzero multiplicative function) linear combination of* X_1 *and* X_2 *:* $\widetilde X = \lambda_1 X_1 + \lambda_2 X_2$ *such that*

$$[\widetilde X, [X_1, X_2]] \ \in \ \operatorname{Span}\{X_1, X_2, [X_1, X_2]\} \tag{41}$$

(this is the characteristic vector field, or characteristic direction of the distribution spanned by the independent vector fields X_1*,* X_2 *and* $[X_1, X_2]$*).*

System (5) is $\boldsymbol{x}$**-dynamic linearizable at** $(\bar x, \bar u)$ **if and only if**

$$[\, \widetilde X \,,\, X_0] \ \ \in \ \ \operatorname{Span}\{\, X_1 \,,\, X_2 \,,\, [X_1, X_2] \,\} \tag{42}$$

on a neighborhood of $\bar{x}$ and

$$\operatorname{rank}_{I\!\!R}\{\, \widetilde{X}(\bar{x})\,,\ [X_0,\widetilde{X}](\bar{x}) + \bar{u}_1[X_1,\widetilde{X}](\bar{x}) + \bar{u}_2[X_2,\widetilde{X}](\bar{x})\,\} \;=\; 2 \qquad (43)$$

$$\begin{aligned}\operatorname{rank}_{I\!\!R}\{\, & X_1(\bar{x})\,,\ X_2(\bar{x})\,,\ [X_1,X_2](\bar{x})\,,\ [X_0,X_1](\bar{x})\,,\ [X_0,X_2](\bar{x})\,,\\ & [X_0,[X_1,X_2]](\bar{x}) + \bar{u}_1[X_1,[X_1,X_2]](\bar{x}) + \bar{u}_2[X_2,[X_1,X_2]](\bar{x})\,\} \;=\; 4 \qquad (44)\end{aligned}$$

Around a point where $(m_0,m_1)=(3,4)$, in some coordinates and after a possible invertible feedback transformation (6), system (5) has the form :

$$\begin{aligned}\dot{z}_1 &= v_1\\ \dot{z}_2 &= f_2(z_1,z_2,z_3,z_4) \;+\; z_3v_1\\ \dot{z}_3 &= f_3(z_1,z_2,z_3,z_4) \;+\; z_4v_1\\ \dot{z}_4 &= v_2\end{aligned} \qquad (45)$$

Condition (42) is equivalent to f_2 being independent of z_4 :

$$\frac{\partial f_2}{\partial z_4} \;=\; 0\;, \qquad (46)$$

and conditions (43) and (44) translate into :

$$v_1 \;+\; \frac{\partial f_3}{\partial z_4} \;\neq\; 0 \qquad (47)$$

and

$$\left(\; v_1 \;+\; \frac{\partial f_2}{\partial z_3}\;,\ f_3 \;-\; \frac{\partial f_2}{\partial z_1} \;-\; z_3\frac{\partial f_2}{\partial z_2} \;+\; z_4v_1 \;\right) \;\neq\; (0,0) \qquad (48)$$

at the point under consideration. A pair of linearizing outputs is, for instance, given by (z_1,z_2) at a point where $v_1+\frac{\partial f_2}{\partial z_3}$ does not vanish, and by $(z_3, z_2-z_1z_3)$ at a point where $v_1\,+\,\frac{\partial f_2}{\partial z_3}$ vanishes.

Note that this theorem does not say anything about the situation around points $\bar{x}$ where

- either one of the distributions spanned by Δ_2, $\mathcal{M}_0$ or $\mathcal{M}_1$ is singular,
- or they are regular, $(m_0,m_1)\neq(3,4)$ and the distribution spanned by Δ_3 is singular,
- or $(m_0,m_1,\delta_3)=(2,2,3)$, the distribution spanned by Δ_3 —i.e. by $\{X_1,X_2,[X_0,X_1],[X_0,X_2]\}$ since $(m_0,m_1)=(2,2)$— has rank 3 and is integrable, but the distribution spanned by $\{X_1,X_2,[X_0,[X_0,X_1]],[X_0,[X_0,X_2]]\}$ is singular.

4 (x,u)-dynamic linearizability

4.1 Problem statement

In this section 4, we shall consider the case when theorem 3.1 concludes that system (5) is not x-dynamic linearizable, i.e. there exists no pair of linearizing outputs depending on x only. Of course this does not prevent the system from being linearizable by endogenous dynamic feedback, i.e. there may exist some pairs of linearizing outputs depending not only on x but also on u and some time-derivatives of u. This section gives an answer to the question of deciding on (x,u)-dynamic linearizability, i.e. on existence of a pair of linearizing outputs depending on x and u.

We do not study here the cases where theorem 3.1 did not conclude because of singularities, i.e. we only consider the last case of theorem 3.1, and we suppose that (42) does not hold. Actually, we exclude one more singularity than in the previous section, i.e. we do not study points where (42) fails to hold on a neighborhood of the point $(\bar{x},\bar{u})$ but holds punctually at this point : we suppose that the rank of $\mathcal{M}_0 + \mathrm{Span}\{[\widetilde{X},X_0]\}$ is constant, and therefore equal to 4, on a neighborhood of $(\bar{x},\bar{u})$, this is (52) below. Furthermore, we assume that (x,u) is a Brunovský regular point, this is (53).

Let us sum up the rank assumptions we make all over the present section :

$$\begin{aligned}
\operatorname{rank}\{X_1\,,\,X_2\,\} &= 2 && (49)\\
\operatorname{rank}\{X_1\,,\,X_2\,,\,[X_1,X_2]\,\} &= 3 && (50)\\
\operatorname{rank}\{X_1\,,\,X_2\,,\,[X_1,X_2]\,,\,[X_1,[X_1,X_2]]\,,\,[X_2,[X_1,X_2]]\,\} &= 4 && (51)\\
\operatorname{rank}\{\,X_1\,,\,X_2\,,\,[X_1,X_2]\,,\,[X_0,\widetilde{X}]\} &= 4 && (52)\\
\operatorname{rank}\{X_1\,,\,X_2\,,\,[X_0,X_1]-u_2[X_1,X_2]\,,\,[X_0,X_2]+u_1[X_1,X_2]\,\} &= 4 && (53)
\end{aligned}$$

with $\widetilde{X}$ defined by (41).

In this case, the results from section 3 allow one to conclude that system (5) is not x-linearizable. We are investigating the possibility of its being (x,u)-linearizable.

4.2 Main result

Let us now proceed with some preparation for our characterization of (x,u)-dynamic linearizability. The following proposition provides a particular choice of ω_1 and ω_2 (basis of $\mathcal{H}_2$) such that the expressions of $\mathrm{d}\omega_1$ and $\mathrm{d}\omega_2$ are convenient and "canonical".

Proposition 4.1 *Let $(\bar{x},\bar{u})$ be such that the rank conditions (49)-(50)-(51)-(52)-(53) are satisfied. Let ω_1 and ω_2 to be two differential forms of degree 1, linear combinations of $\mathrm{d}x_1$, $\mathrm{d}x_2$, $\mathrm{d}x_3$, $\mathrm{d}x_4$, such that none of these forms vanish at $(\bar{x},\bar{u})$ and*

$$\begin{array}{rcl}\omega_1 & \in & \{X_1, X_2, [X_1,X_2]\}^\perp \\ \omega_2 & \in & \{X_1, X_2, [X_0+u_1X_1+u_2X_2, \widetilde{X}]\}^\perp \ .\end{array} \tag{54}$$

Then $\{\omega_1,\omega_2,\dot{\omega}_1,\dot{\omega}_2\}$ is a basis of $\mathrm{Span}\,\{\mathrm{d}x\}$ *and there exist some uniquely defined functions $\delta^k_{i,j}$ and γ such that*

$$\mathrm{d}\omega_1 \;\equiv\; \delta^2_{1,2}\,\omega_2\wedge\dot{\omega}_2 \quad \text{modulo } \omega_1 \ , \tag{55}$$

$$\mathrm{d}\omega_2 \;\equiv\; \omega_1\wedge\left(\delta^1_{2,1}\dot{\omega}_1+\delta^2_{2,1}\dot{\omega}_2-\gamma\ddot{\omega}_2\right) \;+\; \gamma\,\dot{\omega}_1\wedge\dot{\omega}_2 \quad \text{modulo } \omega_2 \ . \tag{56}$$

The functions γ and $\delta^2_{1,2}$ do not vanish at a point where conditions (51) and (52) are met.

Note that it is clear from (54) that, in general, ω_1 can be chosen so as to involve x only, but ω_2 involves x and u, i.e. it is a linear combination of $\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3, \mathrm{d}x_4$ with coefficients depending both on x and u. The functions γ and $\delta^k_{i,j}$ a priori depend on x, u and a certain number of time-derivatives of u.

Proof of proposition 4.1 : Suppose that ω_1 and ω_2 are chose according to (54). Then (52) and (41) imply that the rank of $\{X_1, X_2, [X_1,X_2], [X_0+u_1X_1+u_2X_2,\widetilde{X}]\}$ is 4, and hence that $\{\omega_1,\omega_2\}$ is a basis of the annihilator of $\{X_1,X_2\}$.

The fact that ω_1 in the orthogonal of $\{X_1,X_2,[X_1,X_2]\}$ implies that it is in the first derived system of the pfaffian system $\{\omega_1,\omega_2\}$ –see the Appendix– and hence that

$$\mathrm{d}\omega_1 \;=\; \omega_1\wedge\Gamma_{1,1} \;+\; \omega_2\wedge\Gamma_{1,2} \tag{57}$$

for some forms $\Gamma_{1,1}$ and $\Gamma_{1,2}$. Now the forms ω_1, ω_2 and $\Gamma_{1,2}$ must be linearly independent from (51), and then the Cartan characteristic system of $\{\omega_1\}$ is $\{\omega_1,\omega_2,\Gamma_{1,2}\}$

—see the Appendix (187)—, but, by definition of $\widetilde{X}$, this characteristic system is the annihilator of $\widetilde{X}$, and a basis of the annihilator of $\widetilde{X}$ is $\{\omega_1, \omega_2, \dot{\omega}_2\}$ because, from(15),

$$0 \;=\; \frac{\mathrm{d}}{\mathrm{dt}}\langle \omega_2, \widetilde{X}\rangle \;=\; \langle \dot{\omega}_2, \widetilde{X}\rangle \;+\; \langle \omega_2, [X_0 + u_1 X_1 + u_2 X_2, \widetilde{X}]\rangle$$

and hence $\langle \dot{\omega}_2, \widetilde{X}\rangle$ is zero ; this proves that $\Gamma_{1,2}$ must be a linear combination of ω_1, ω_2 and ω_2, which, substituted in (57), yields (55) with $\delta^2_{1,2}$ does not vanish because ω_1, ω_2 and $\Gamma_{1,2}$ are linearly independent.

On the other hand, $\{\omega_1, \omega_2\}$ is the annihilator of $\{X_1, X_2\}$ and therefore has a basis that can be written with the variable x only ; this implies —see (188) in the Appendix— that its characteristic system is at most $\mathrm{Span}\{\mathrm{d}x\}$; since $\{\omega_1, \omega_2, \dot{\omega}_1, \dot{\omega}_2\}$ is a basis of $\mathrm{Span}\{Dx\}$, this implies

$$\mathrm{d}\omega_2 \;\;=\;\; \omega_1 \wedge \Gamma_{2,1} \;+\; \omega_2 \wedge \Gamma_{2,2} \;+\; \gamma\, \dot{\omega}_1 \wedge \dot{\omega}_2 \tag{58}$$

for some forms $\Gamma_{2,1}$ and $\Gamma_{2,2}$. But we have seen above that $\{\omega_1, \omega_2, \dot{\omega}_2\}$ is the Cartan characteristic system of $\{\omega_1\}$; It is therefore completely integrable, and this implies that $\mathrm{d}\dot{\omega}_2 \equiv 0$ modulo $\{\omega_1, \omega_2, \dot{\omega}_2\}$; but taking the time derivative of (58) yields $\mathrm{d}\dot{\omega}_2 \equiv \dot{\omega}_1 \wedge (\Gamma_{2,1} + \gamma\ddot{\omega}_2))$ modulo $\{\omega_1, \omega_2, \dot{\omega}_2\}$; $\Gamma_{2,1} \equiv -\gamma\ddot{\omega}_2$, which does imply, together with (58), the relation (56). ∎

We are now ready to state our theorem that characterizes (x, u)-linearizability :

Theorem 4.1 *Let $(\bar{x}, \bar{u})$ be such that conditions (49)-(50)-(51)-(52)-(53) are met. Then system (5) is (x, u)-dynamically linearizable at point $\overline{\mathcal{X}} = (\bar{x}, \bar{u}, \dot{\bar{u}}, \ldots\ldots)$ if and only if the function $\delta^1_{2,1}$ —or equivalently the form of degree 5 $\mathrm{d}\omega_2 \wedge \omega_2 \wedge \dot{\omega}_2 \wedge \ddot{\omega}_2$— does not vanish at $\overline{\mathcal{X}}$ and the first derived system of the pfaffian system $\{\, \omega_1 - \frac{2\gamma}{\delta^1_{2,1}}\dot{\omega}_2 \,,\, \omega_2\}$ has rank 1 and is integrable, i.e. there exists a function α, defined on a neighborhood of $\overline{\mathcal{X}}$, such that*

$$\mathrm{d}\left(\omega_1 + \alpha\omega_2 - \frac{2\,\gamma}{\delta^1_{2,1}}\dot{\omega}_2\right) \;\wedge\; \left(\omega_1 + \alpha\omega_2 - \frac{2\,\gamma}{\delta^1_{2,1}}\dot{\omega}_2\right) \;=\; 0\;. \tag{59}$$

When these conditions are met, all the possible pairs of linearizing outputs depending on x and u may be described as follows. Let $\Omega_3 = \omega_1 + \alpha\omega_2 - \frac{2\gamma}{\delta^1_{2,1}}\dot{\omega}_2$, and $\dot{\Omega}_3$ be the time-derivative of this differential form (i.e. its Lie derivative along the dynamics F of the system). The Pfaffian system $\{\omega_2, \Omega_3, \dot{\Omega}_3\}$ is completely integrable.

A pair of functions (h_1, h_2) depending on (x, u) is a pair of linearizing outputs if and only if $\{\mathrm{d}h_1, \mathrm{d}h_2\} \subset \{\omega_2, \Omega_3, \dot{\Omega}_3\}$ with $\Omega_3 \in \{\mathrm{d}h_1, \mathrm{d}h_2\}$ and $\dot{\Omega}_3 \notin \{\mathrm{d}h_1, \mathrm{d}h_2\}$. A possible construction is as follows : since $\mathrm{d}\Omega_3 \wedge \Omega_3 = 0$, take h_1 such that $\mathrm{d}h_1$ does not vanish and $\mathrm{d}h_1 = k\Omega_3$ (k nonvanishing function) ; take for h_2 another first integral of $\{\omega_2, \Omega_3, \dot{\Omega}_3\}$ such that the coefficient of ω_2 when expressing h_2 as a linear combination of ω_2, Ω_3 and $\dot{\Omega}_3$ does not vanish (i.e. the rank of $\{\mathrm{d}h_2, \mathrm{d}h_1, \mathrm{d}\dot{h}_1\}$ does not drop to 2).

The proof is given separately in section 7.2.

This theorem is stated in terms of the forms ω_1 and ω_2. These forms are only defined up to a nonvanishing multiplicative function by relation (54). However, the condition does not depend on the particular choice of ω_1 and ω_2. In a sense this is a consequence of the theorem itself since (x, u)-dynamic linearizability is clearly static feedback invariant and does not depend on the choice of ω_1 and ω_2, but the following proposition asserts that a priori these conditions *are* static feedback invariant.

Proposition 4.2 *The conditions of theorem 4.1 are invariant by static feedback and do not depend on the particular choice of ω_1 and ω_2 in (54). Indeed the pfaffian system $\{\omega_2, \omega_1 - \frac{2\gamma}{\delta^1_{2,1}}\dot{\omega}_2\}$ does not depend on this particular choice.*

Proof : It can be checked from (56) that if one changes ω_1 into $\lambda_1\omega_1$ and ω_2 into $\lambda_2\omega_2$, where λ_1 and λ_2 are nonvanishing functions, then $\delta^1_{2,1}$ is changed into $\frac{\lambda_2}{\lambda_1^2}\delta^1_{2,1}$ and γ into $\frac{1}{\lambda_1}\gamma$. This implies the proposition since (54) defines ω_1 and ω_2 up to a nonzero multiplicative function in a feedback invariant way. ∎

Let us make a remark on "singular" points, i.e. points where the ranks considered in (49)-(50)-(51)-(52)-(53) are not constant. We do not study the situation at these points, in particular at points which are not Brunovský-regular, i.e. points where the rank in (53) drops. As illustrated by the example in section 5, this singularity is usually not a singularity of (x, u)-dynamic linearization, but only of the proofs given here : the linearizing outputs are well defined at these points too, enjoy the property of being linearizing outputs. On the contrary, points where $\delta^1_{2,1}$, or the form $\mathrm{d}\omega_2 \wedge \omega_2 \wedge \dot{\omega}_2 \wedge \ddot{\omega}_2$, vanish are, according to the theorem, actual singularities of (x, u)-dynamic linearizability : in a domain where the rank assumptions (49)-(50)-(51)-(52)-(53) hold, there exists no linearizing outputs function of x and u in the neighborhood of a point where $\delta^1_{2,1}$ vanishes. It is interesting, with this respect, to notice that, under the

—generic— assumptions (49)-(50)-(51)-(52)-(53), it is *impossible* to build an example where (x,u)-dynamic feedback linearization would be everywhere nonsingular since for any value of x and u, there is a value of $\dot{u}$ where $\delta^1_{2,1}$ vanishes.

4.3 How to check the conditions

We claim that the conditions of theorem 4.1 are completely explicit. Let us explain how to check them on a system (5) given by the expression of the vector fields X_0, X_1 and X_2 in some coordinates x_1, x_2, x_3, x_4 :

1. Compute ω_1 and ω_2 according to (54). This involves the computation of some Lie brackets, and then finding the annihilator of some families of vectors, which in coordinates is common linear algebra (Gauss elimination).

2. Compute $\dot{\omega}_1$, $\dot{\omega}_2$ and $\ddot{\omega}_2$. The time-derivatives are Lie derivatives along the vector field (9).

3. To compute $\delta^1_{2,1}$ and γ, use the following identities, consequence of (56) :

$$\begin{aligned} \mathrm{d}\omega_2 \wedge \omega_2 \wedge \dot{\omega}_2 \wedge \ddot{\omega}_2 &= \delta^1_{2,1}\, \omega_1 \wedge \dot{\omega}_1 \wedge \omega_2 \wedge \dot{\omega}_2 \wedge \ddot{\omega}_2 \\ \mathrm{d}\omega_2 \wedge \omega_2 \wedge \dot{\omega}_1 \wedge \dot{\omega}_2 &= \gamma\, \omega_1 \wedge \ddot{\omega}_2 \wedge \omega_2 \wedge \dot{\omega}_1 \wedge \dot{\omega}_2 \\ &= \gamma\, \omega_1 \wedge \dot{\omega}_1 \wedge \omega_2 \wedge \dot{\omega}_2 \wedge \ddot{\omega}_2 \\ \mathrm{d}\omega_2 \wedge \omega_1 \wedge \omega_2 &= \gamma\, \dot{\omega}_1 \wedge \dot{\omega}_2 \wedge \omega_1 \wedge \omega_2\ . \end{aligned} \tag{60}$$

   Hence one may for instance compute the forms of degree 5 $\mathrm{d}\omega_2 \wedge \omega_2 \wedge \dot{\omega}_2 \wedge \ddot{\omega}_2$ and $\mathrm{d}\omega_2 \wedge \omega_2 \wedge \dot{\omega}_1 \wedge \dot{\omega}_2$, check that the first one does not vanish, they appear to be of the form $\rho_1 \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4 \wedge \mathrm{d}u_1 + \rho_2 \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4 \wedge \mathrm{d}u_2$ and $\rho_3 \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4 \wedge \mathrm{d}u_1 + \rho_4 \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4 \wedge \mathrm{d}u_2$ respectively, with ρ_1, ρ_2, ρ_3 and ρ_4 some functions of x, u and $\dot{u}$, with $\rho_1\rho_4 - \rho_2\rho_3 = 0$, then

$$\frac{2\,\gamma}{\delta^1_{2,1}} = \frac{2\,\rho_3}{\rho_1} = \frac{2\,\rho_4}{\rho_2}\ .$$

4. The pfaffian system $\{\,\omega_1 - \frac{2\,\gamma}{\delta^1_{2,1}}\dot{\omega}_2\,,\,\omega_2\}$ is then known

5. Use usual procedure to compute its first derived system : the forms $\mathrm{d}\omega_2$ and $\mathrm{d}\left(\omega_1 - \frac{2\,\gamma}{\delta^1_{2,1}}\dot{\omega}_2\right)$ must be proportional modulo $\{\,\omega_1 - \frac{2\,\gamma}{\delta^1_{2,1}}\dot{\omega}_2\,,\,\omega_2\}$; if it

is the case, this yields α such that $\mathrm{d}\left(\omega_1 - \frac{2\gamma}{\delta^1_{2,1}}\dot{\omega}_2 + \alpha\omega_2\right)$ is zero modulo $\{\,\omega_1 - \frac{2\gamma}{\delta^1_{2,1}}\dot{\omega}_2\,,\,\omega_2\}$.

6. Check that $\mathrm{d}\left(\omega_1 - \frac{2\gamma}{\delta^1_{2,1}}\dot{\omega}_2 + \alpha\omega_2\right)$ is also zero modulo $\omega_1 - \frac{2\gamma}{\delta^1_{2,1}}\dot{\omega}_2 + \alpha\omega_2$.

4.4 The result in some particular coordinates

Let us now give a "normal form" for the systems we are studying in this section, i.e. these meeting conditions (49)-(50)-(51)-(52)-(53). It basically consists, as in "case 6" of theorem 3.1, in taking some coordinates (they exist from (49)-(50)-(51)) in which the control distribution is in "Engels normal form", and use a feedback to annihilate two components of the drift, then the coordinates are slightly changed to emphasize condition (52) :

Proposition 4.3 *If the rank conditions (49)-(50)-(51)-(52)-(53) hold around a point $(\bar{x},\bar{u})$, there exists a system of coordinates around this point, and a static feedback defined around this point which give the following form to system (5) :*

$$\left.\begin{array}{rcl} \dot{z}_1 & = & v_1 \\ \dot{z}_2 & = & z_4 \;+\; z_3 v_1 \\ \dot{z}_3 & = & f(z_1,z_2,z_3,z_4) \;+\; g(z_1,z_2,z_3,z_4)v_1 \\ \dot{z}_4 & = & v_2 \end{array}\right\} \tag{61}$$

where

$$\frac{\partial g}{\partial z_4} \tag{62}$$

and

$$\begin{array}{rcl} D_1 & = & \dfrac{\partial g}{\partial z_4}(v_2 - f v_1) \;+\; z_4\dfrac{\partial g}{\partial z_2} \;+\; f\dfrac{\partial g}{\partial z_3} \\ & & -\;\left(\dfrac{\partial f}{\partial z_1} + z_3\dfrac{\partial f}{\partial z_2} + g\dfrac{\partial f}{\partial z_3} + f\dfrac{\partial f}{\partial z_4}\right) \end{array} \tag{63}$$

do not vanish at $(\bar{x},\bar{u})$.

Proof of proposition 4.3 : The conditions that allowed us to get the normal form (45) in section 3 still hold —it is a consequence of (49)-(50)-(51) only— and condition (52) implies that $\frac{\partial f_2}{\partial z_4}$ does not vanish (see the proof of the last part of theorem 3.1). One may therefore take as new coordinates $(z_1, z_2, z_3, f_2(z_1, z_2, z_3, z_4))$ instead of (z_1, z_2, z_3, z_4), and this yields the normal form (61) (changing also v_2). Relations (62) are simply a translation of (51) and (53). ∎

Proposition 4.4 *System (61) —which is system (5) written in some coordinates— is (x, u)-dynamic linearizable around a point $\overline{\mathcal{X}}$ if and only if the functions f and g have, in a neighborhood of $\overline{\mathcal{X}}$, the form*

$$f \;=\; \frac{a_0 \,+\, a_1 z_4 \,+\, a_2 z_4^2}{c_0 \,+\, c_1 z_4} \quad ; \quad g \;=\; \frac{b_0 \,+\, b_1 z_4}{c_0 \,+\, c_1 z_4} \tag{64}$$

where a_0, a_1, a_2, b_0, b_1, c_0 and c_1 are some functions of z_1, z_2, z_3 only, which satisfy the following PDE :

$$\mathrm{d}\Gamma \wedge \Gamma \;=\; 0 \quad \textit{with} \quad \Gamma \;=\; (b_1 - z_3 a_2)\,\mathrm{d}z_1 \;+\; a_2 \mathrm{d}z_2 \;-\; c_1 \mathrm{d}z_3 \tag{65}$$

and $\delta^1_{2,1}$ does not vanish at this point ($c_0 \,+\, c_1 z_4$ should obviously not vanish either).

Remarks :

1- The system of PDEs (65) reads :

$$z_3 \left(c_1 \tfrac{\partial a_2}{\partial z_2} - a_2 \tfrac{\partial c_1}{\partial z_2} \right) + c_1 \tfrac{\partial a_2}{\partial z_1} - a_2 \tfrac{\partial c_1}{\partial z_1} + b_1 \tfrac{\partial c_1}{\partial z_2} - c_1 \tfrac{\partial b_1}{\partial z_2} - a_2 \tfrac{\partial b_1}{\partial z_3} + b_1 \tfrac{\partial a_2}{\partial z_3} + {a_2}^2 \;=\; 0 \tag{66}$$

2- There is an explicit formula for $\delta^1_{2,1}$ using the a_i, b_i and c_i but it is quite long, and does not really matter here.

This proposition gives a simple way to check whether the system is (x, u)-dynamic linearizable provided one has found some coordinates where it is in the normal form (61) —of course finding these coordinates involves solving some linear PDEs, so that the really explicit test is given by theorem 4.1 which only involves some differentiations, and some algebraic manipulations—. Actually, the coordinates in which a given system meeting conditions (49)-(50)-(51)-(52)-(53) is in the form (61) are not unique, and the expression of f and g, for the same system, may depend on the choice of coordinates, among all these that yield a form like (61)). Naturally, the fact that these f and g meet or not the conditions of the proposition does not depend on this choice. It however raises the question of finding, among all the coordinates that produce a normal form like (61), these which produce the "simplest" f and g. Let us give an answer only for the special case when the conditions of the proposition are met (i.e. in the (x,u)-linearizable case). It is obvious that if f and g are affine in z_4 (special case of (64) : $a_2 = c_1 = 0$, $c_0 = 1$), the PDE (59) is met, because Γ is simply $b_1 \mathrm{d}z_1$; it turns out that the converse is true : if f and g are not affine, but of the form (64) with $a_2 \neq 0$ or $c_1 \neq 0$, and with the PDE (59), then some "better" coordinates may be found, in which f and g are affine in the fourth coordinate :

Proposition 4.5 *There are some coordinates where the system, after a static feedback transformation, is in the form (61) with f and g satisfying the conditions of proposition 4.4, if and only if there is another set of coordinates $(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$, and another static feedback transformation which yields a normal form (61) with f and g affine with respect to the fourth coordinate :*

$$\left.\begin{array}{rcl} \dot{\zeta}_1 &=& w_1 \\ \dot{\zeta}_2 &=& \zeta_4 \;+\; \zeta_3\, w_1 \\ \dot{\zeta}_3 &=& p_0(\zeta_1,\zeta_2,\zeta_3) + \zeta_4 p_1(\zeta_1,\zeta_2,\zeta_3) \;+\; (q_0(\zeta_1,\zeta_2,\zeta_3) + \zeta_4 q_1(\zeta_1,\zeta_2,\zeta_3))\, w_1 \\ \dot{\zeta}_4 &=& w_2 \end{array}\right\} \qquad (67)$$

and $\delta^1_{2,1}$ does not vanish if and only if the following quantity does not vanish :

$$\begin{array}{l} q_1 \dot{w}_1 \;+\; w_1\,(p_1 + w_1 q_1)^2 \;+\; w_1 \frac{\partial}{\partial \zeta_1}\,(p_1 + w_1 q_1) \\ \quad -\, \frac{\partial}{\partial \zeta_2}\,[(p_0 + w_1 q_0) - \zeta_3 w_1 (p_1 + w_1 q_1)] \;-\; (p_1 + w_1 q_1)^2\, \frac{\partial}{\partial \zeta_3} \frac{p_0 + w_1 q_0}{p_1 + w_1 q_1}\;. \end{array} \qquad (68)$$

In these coordinates, a pair of linearizing outputs is given by
$h_1 = \zeta_1$, $h_2 = \zeta_3 - (p_1 - w_1 q_1)\,\zeta_2$.

Proof of proposition 4.5 : The expression (68) is obtained by computing $\mathrm{d}\omega_2 \wedge \omega_2 \wedge \dot{\omega}_2 \wedge \ddot{\omega}_2$ with the simple choice

$$\begin{array}{rcl} \omega_1 &=& \mathrm{d}\zeta_2 - \zeta_3 \mathrm{d}\zeta_1 \\ \omega_2 &=& \mathrm{d}\zeta_3 - q_1(\zeta_1,\zeta_2,\zeta_3)\mathrm{d}\zeta_1 \;-\; (p_1(\zeta_1,\zeta_2,\zeta_3) + w_1 q_1(\zeta_1,\zeta_2,\zeta_3))\,\omega_1 \end{array}$$

and checking that, at points where (53) holds, i.e. where $\omega_1 \wedge \omega_2 \wedge \dot{\omega}_2$ does not vanish, it vanishes if and only if the expression (68) vanishes. This tedious computation is left to the reader

The "if" part of the proposition is obvious because, as noticed just above the proposition, (67) is a particular case of (61)-(59), and (68) ensures that $\delta^1_{2,1} \neq 0$. Let us prove the "only if" part. we suppose that the conditions of proposition 4.4 hold, and we build an invertible transformation $(z_1,z_2,z_3,z_4)\mapsto(\zeta_1,\zeta_2,\zeta_3,\zeta_4)$, and an invertible static feedback transformation $(z_1,z_2,z_3,z_4,v_1,v_2)\mapsto(z_1,z_2,z_3,z_4,w_1,w_2)$, that transforms (61) into (67).

(65) implies that there exists a function $\psi_1(z_1,z_2,z_3)$ and a nonvanishing function $k(z_1,z_2,z_3)$ such that

$$\mathrm{d}\psi_1 \;=\; k\,\Gamma\;. \tag{69}$$

Now, ω_1 may be chosen $\omega_1=\mathrm{d}z_2-z_3\mathrm{d}z_1$ and then Γ defined in (65) is also equal to : $\Gamma=b_1\mathrm{d}z_1+a_2\omega_1-c_1\mathrm{d}z_3$. Since the rank of $\{\mathrm{d}z_1,\mathrm{d}z_3,\omega_1\}$ is 3 and b_1 and c_1 do not vanish simultaneously (this would cause $\frac{\partial g}{\partial z_4}$ to vanish), the rank of $\{\omega_1,\Gamma\}$ is locally constant, equal to 2, and this pfaffian system is therefore completely integrable, because these two forms involve only three variables (z_1, z_2, z_3); hence there exists three functions ψ_2, k', k'', such that

$$\mathrm{d}\psi_2 \;=\; k'\omega_1\;+\;k''\Gamma\;,\quad k'\neq 0\;\;. \tag{70}$$

Let us then define

$$\begin{aligned} w_1 \;&=\; \dot\psi_1 \;=\; k\,\langle\,\Gamma\;,\;X_0\;+\;u_1X_1\;+\;u_2X_2\,\rangle \\ &=\; k\,\frac{(c_0b_1-c_1b_0)v_1\;-\;c_1a_0\;+\;(c_0a_2-a_1c_1)z_4}{c_0\;+\;c_1z_4}\;. \end{aligned} \tag{71}$$

From this equation, one may express v_1 as a function of w_1. Substituting v_1 for this expression in (61)-(64), one obtains the following expressions for $\dot z_1$, $\dot z_2$, $\dot z_3$, which are now linear with respect to z_4 :

$$\dot z_1 \;=\; \frac{1}{c_0b_1-c_1b_0}\left(\frac{c_0+c_1z_4}{k}w_1\;+\;c_1a_0\;+\;(a_1c_1-c_0a_2)z_4\right) \tag{72}$$

$$\dot z_2 \;=\; \frac{1}{c_0b_1-c_1b_0}\left(\frac{z_3}{k}(c_0+c_1z_4)w_1\;+\;z_3c_1a_0\;+\;(c_0b_1-c_1b_0+a_1c_1-c_0a_2)z_4\right) \tag{73}$$

$$\dot z_3 \;=\; \frac{b_0+b_1z_4}{k\,(c_0b_1-c_1b_0)}\,w_1\;+\;a_0b_1\;+\;(a_1b_1-a_2b_0)z_4 \tag{74}$$

Let us then define

$$\zeta_1 \;=\; \psi_1(z_1,z_2,z_3) \tag{75}$$

$$\zeta_2 \;=\; \psi_2(z_1,z_2,z_3) \tag{76}$$

$$\zeta_3 \;=\; \frac{k''(z_1,z_2,z_3)}{k(z_1,z_2,z_3)} \tag{77}$$

$$\zeta_4 \;=\; k'(z_1,z_2,z_3)\,z_4 \tag{78}$$

Let us see that in these coordinates, and with w_1 given by (71), we have (67) :
- $\dot{\zeta}_1 = w_1$ is a consequence of (75) and (71),
- From (76), $\dot{\zeta}_2 = \langle\, \mathrm{d}\psi_2 \,,\, X_0 + u_1 X_1 + u_2 X_2 \,\rangle$, which is also equal, from (71) and (70), to $\frac{k''}{k} w_1 + k' \langle\, \omega_1 \,,\, X_0 + u_1 X_1 + u_2 X_2 \,\rangle$, which, since $\langle\, \omega_1 \,,\, X_0 + u_1 X_1 + u_2 X_2 \,\rangle = z_4$, and considering (77) and (78), yields $\dot{\zeta}_2 = \zeta_4 + \zeta_3 w_1$.
- In the expressions for $\dot{z}_1$, $\dot{z}_2$ and $\dot{z}_3$ given by (72), (73) and (74), all the functions of (z_1, z_2, z_3) may be expressed as functions of $(\zeta_1, \zeta_2, \zeta_3)$, and z_4 may be substituted for $\frac{\zeta_4}{k'}$ (see (78)); therefore, $\dot{z}_1$, $\dot{z}_2$ and $\dot{z}_3$ are polynomials in ζ_4 and w_1 with coefficients function of $(\zeta_1, \zeta_2, \zeta_3$ with one term of degree zero, one term of degree 1 in ζ_4, one term of degree 1 in w_1 and one term of degree 2 in $\zeta_4 w_1$; since ζ_3 is a function of (z_1, z_2, z_3), $\dot{\zeta}_3$ is also such a polynomial, which allows one to define functions p_o, p_1, q_0 and q_1 such that $\dot{\zeta}_3$ is as in (67).
- $\dot{\zeta}_4$ is equal to $k'(\zeta_1, \zeta_2, \zeta_3) v_2$ plus some terms which depend only on ζ_1, ζ_2, ζ_3, ζ_4 and v_1, which allows us, since k' does not vanish, to call all this expression w_2, thus getting the required form (and defining a nonsingular feedback). ∎

5 An example

Let us consider the following system :

$$\begin{array}{rcl} \dot{x}_1 & = & u_1 \\ \dot{x}_2 & = & x_4 + x_3 u_1 \\ \dot{x}_3 & = & x_1 + x_4 u_1 \\ \dot{x}_4 & = & u_2 \;. \end{array} \tag{79}$$

It is very similar to example 2 in [7] : by a simple nonsingular feedback and a permutation of the coordinates, the example given there is transformed into (79) with x_1 replaced by x_2 in the third line. Note that (79) is already in the form (61), and even (67), so that it is (x, u)-dynamic feedback linearizable at "generic" points. It is therefore "flat" in the "analytic" sense of [19], and we shall see that, since the linearizing outputs are algebraic, it is also flat in the sense of [8, 9].

Let us go through the method given here in the case of this system. First, it is not hard to see that Brunovský-regular points are points where

$$1 - u_2 + u_1 x_1 \neq 0 \;. \tag{80}$$

The simplest choice for ω_1 and ω_2 is (see (166)) :

$$\begin{array}{rcl}\omega_1 &=& \mathrm{d}x_2 \;-\; x_3\mathrm{d}x_1 \\ \omega_2 &=& \mathrm{d}x_3 \;-\; u_1\mathrm{d}x_2 \;+\; (u_1x_3 \;-\; x_4)\mathrm{d}x_1\end{array} \tag{81}$$

which yields

$$\begin{array}{rcl}\dot\omega_1 &=& \mathrm{d}x_4 \;+\; u_1\mathrm{d}x_3 \;-\; (x_1 \;+\; u_1x_4)\mathrm{d}x_1 \\ \dot\omega_2 &=& -\,u_1^2\mathrm{d}x_3 \;-\; \dot u_1\mathrm{d}x_2 \;+\; (-u_2+1+u_1x_1+u_1^2x_4+x_3\dot u_1)\mathrm{d}x_1\end{array} \tag{82}$$

and it can be checked that this is a basis for $\mathrm{Span}\{\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3, \mathrm{d}x_4\}$ at points where (80) holds. $\delta^1_{2,1}$ may then be computed :

$$\delta^1_{2,1} \;=\; \frac{2\,(u_1^3 \;+\; \dot u_1)}{1 \;-\; u_2 \;+\; u_1x_1} \quad ; \quad b \;=\; -\,\frac{1}{u_1^3 \;+\; \dot u_1} \tag{83}$$

so that $\delta^1_{2,1} \neq 0$ is equivalent to $u_1^3 + \dot u_1 \neq 0$. The derived system of $\{\omega_2, \omega_1 - b\dot\omega_2\}$ can be computed, and the α such that $\omega_1 - b\dot\omega_2 + \alpha\omega_2$ is in this derived system is given by

$$\alpha \;=\; \frac{u_1^2}{u_1^3 \;+\; \dot u_1}\;, \tag{84}$$

so that

$$\Omega_3 \;=\; \omega_1 \;-\; b\dot\omega_2 \;+\; \alpha\omega_2 \;=\; \frac{1 \;-\; u_2 \;+\; u_1x_1}{u_1^3 \;+\; \dot u_1}\,\mathrm{d}x_1\;. \tag{85}$$

Hence, as in the proof of theorem 4.1, the pfaffian system $\{\Omega_1, \Omega_3, \dot\Omega_3\}$, which is $\{\mathrm{d}z_3 - u_1\mathrm{d}z_2, \mathrm{d}z_1, \mathrm{d}u_1\}$ (at Brunovský-regular points), and hence one has, for example

$$\begin{array}{l}\begin{pmatrix}\mathrm{d}x_1 \\ \mathrm{d}(x_3 - x_2u_1)\end{pmatrix} \;=\; \begin{pmatrix}1 & -\,x_2\,\frac{\mathrm{d}}{\mathrm{dt}} \;-\; (u_1x_3 - x_3) \\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & \frac{u_1^3 + \dot u_1}{1 - u_2 + u_1x_1}\end{pmatrix} \\ \qquad\qquad\qquad\qquad \begin{pmatrix}1 & 0 \\ -\,b\,\frac{\mathrm{d}}{\mathrm{dt}} & 1\end{pmatrix}\begin{pmatrix}0 & 1 \\ 1 & \alpha\end{pmatrix}\begin{pmatrix}\omega_1 \\ \omega_2\end{pmatrix}\end{array} \tag{86}$$

which may be re-arranged into an expression like (128) with some scalar function a and the matrix function J_1, and with

$$h_1 = x_1 \quad ; \quad h_2 = x_3 - u_1x_2\;. \tag{87}$$

The expressions for a, b, α and J_2 are indeed singular at “singular points” –as meant here– and so is the “infinitesimal Brunovský form”, but these are only some intermediates for dynamic feedback linearization, probably due to our method, and as a matter of fact, the functions h_1 and h_2 are defined everywhere by (87), and, since

$$\begin{array}{rclrcl} \dot{h}_1 &=& u_1 & \dot{h}_2 &=& x_1 - u_1^2 x_3 - \dot{u}_1 x_2 \\ \ddot{h}_1 &=& \dot{u}_1 & \ddot{h}_2 &=& -(u_1^3 + \dot{u}_1)x_4 - u_1(x_1 u_1 - 1 + 3x_3\dot{u}_1) - x_2\ddot{u}_1 \ , \end{array} \tag{88}$$

they are linearizing outputs at all points where $\delta^1_{2,1} \neq 0$.

The conclusion for this system is :
- it is (x,u)-dynamic linearizable at all points where $\dot{u}_1 \neq -u_1^3$; this is obvious from (88), and is a consequence of theorem 4.1 at points where $1 - u_2 + u_1 x_1 \neq 0$
- it is not (x,u)-dynamic linearizable at points where $\dot{u}_1 = -u_1^3$ and $1-u_2+u_1x_1 \neq 0$, as a consequence of theorem 4.1,
- it is probably not (x,u)-dynamic linearizable at points where $\dot{u}_1 = -u_1^3$ and $1 - u_2 + u_1 x_1 = 0$, but this is not a consequence of this paper.

Note that the singularity $\dot{u}_1 = -u_1^3$ does not correspond to a singularity of the linear approximation: for instance, the linear approximation at $x = 0$ (equilibrium) is the controllable linear system $\dot{x}_3 = x_1$, $\dot{x}_1 = u_1$, $\dot{x}_2 = x_4$, $\dot{x}_1 = u_2$, but the system is not (x,u)-dynamic linearizable at $x = 0$, $u_1 = u_2 = \dot{u}_1 = 0$.

6 Non-affine systems in $\mathbb{R}^3$

Consider a system

$$\dot{\xi} \;=\; f(\xi, w_1, w_2) \tag{89}$$

where ξ lives in $\mathbb{R}^3$. A system of the form (5) can always be brought to this form at a point where one of the control vector fields does not vanish by finding some coordinates in which this control vector field is the first coordinate vector field, dropping the corresponding control and taking this first coordinate as a new control. The converse is not correct, but a recent necessary condition for dynamic feedback linearization introduced in [25] (see also [26] for a more particular case) allows one to derive the following proposition, which is a consequence of theorem 1 in [25], the only extra being the regularity of γ, but this is automatic if one wants the linearizing outputs to be smooth :

Proposition 6.1 ([25]) *At a point $(\bar{\xi}, \bar{w}_1, \bar{w}_2)$ where* $\operatorname{rank}\{\frac{\partial f}{\partial w_1}, \frac{\partial f}{\partial w_2}\}$ *is 2, a necessary condition for system (89) to be dynamic feedback linearizable is that there exist,* *locally around* $(\bar{\xi}, \bar{w}_1, \bar{w}_2)$*, a static feedback transformation* $(w_1, w_2) = \gamma(\xi, v_1, v_2)$ *such that* $f(\xi, \gamma(\xi, v_1, v_2))$ *be affine with respect to* v_1 : $f(\xi, \gamma(\xi, v_1, v_2)) = a(\xi, v_2) + v_1 b(\xi, v_2)$.

It is not very difficult, in the case of system (89), to give an explicit condition for existence of this static feedback transformation, but this outside the scope of the present paper.

On the other hand, it is clear that the necessary condition for dynamic linearization given in proposition 6.1 is exactly the condition needed to transform system (89) into an affine 4-dimensional system.

This is summed up in the following result, which clearly allows one to apply to 3-dimensional non-affine systems (89) all the results obtained in the previous sections for 4-dimensional affine systems.

Proposition 6.2 *At a point* $(\bar{\xi}, \bar{w}_1, \bar{w}_2)$ *where* $\operatorname{rank}\{\frac{\partial f}{\partial w_1}, \frac{\partial f}{\partial w_1}\}$ *is 2, either system (89) is not dynamic feedback linearizable or one may construct a static feedback transformation* $(w_1, w_2) = \gamma(\xi, v_1, v_2)$ *such that dynamic feedback linearization of (89) is equivalent to dynamic feedback linearization of*

$$\begin{array}{rcl} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} & = & a(x_1, x_2, x_3, x_4) \; + \; u_1 \, b(x_1, x_2, x_3, x_4) \\ \dot{x}_4 & = & u_2 \; . \end{array} \tag{90}$$

7 The proofs

All over these proofs, some known facts about pfaffian systems (derived systems, characteristic system...) are used. They are briefly recalled in the Appendix.

7.1 Proof of theorem 3.1

Case 1

$m_0 = 2$ means that the distribution spanned by the control vector fields X_1 and X_2 is involutive. Frobenius theorem yields some coordinates (z_1, z_2, z_3, z_4) such that

$\{\frac{\partial}{\partial z_3}, \frac{\partial}{\partial z_4}\}$ is a basis of this distribution, then

$$\begin{array}{rcl} v_1 & = & L_{X_0} z_3 \ + \ u_1 L_{X_1} z_3 \ + \ u_2 L_{X_2} z_3 \\ v_2 & = & L_{X_0} z_4 \ + \ u_1 L_{X_1} z_4 \ + \ u_2 L_{X_2} z_4 \end{array}$$

is a nonsingular static feedback because X_1 and X_2 are independent at point $\bar{x}$. System (5) reads, in the above coordinates as

$$\begin{array}{rclcrcl} \dot{z}_1 & = & a_1(z_1, z_2, z_3, z_4) & \qquad & \dot{z}_3 & = & v_1 \\ \dot{z}_2 & = & a_2(z_1, z_2, z_3, z_4) & & \dot{z}_4 & = & v_2 \ . \end{array}$$

Δ_3 is then spanned by $\frac{\partial}{\partial z_3}, \frac{\partial}{\partial z_4}, \frac{\partial a_1}{\partial z_3}\frac{\partial}{\partial z_1} + \frac{\partial a_2}{\partial z_3}\frac{\partial}{\partial z_2}$ and $\frac{\partial a_1}{\partial z_4}\frac{\partial}{\partial z_1} + \frac{\partial a_2}{\partial z_4}\frac{\partial}{\partial z_2}$. $\delta_3 = 2$ implies that a_1 and a_2 do not depend on z_3 and z_4. This yields (33).

Case 2.a

Since $\{X_1, X_2\}$ is integrable of rank 2, there exists two independent functions constant along X_1 and X_2, and one of them at least has either its Lie derivative along $[X_0, X_1]$ or its Lie derivative along $[X_0, X_2]$ that does not vanish at $\bar{x}$ because if not the rank of Δ_3 would drop to two ; let z_2 be this one, and z_1 be the other one, and define $z_3 = L_{X_0} z_2$. $L_{X_1} z_3$ or $L_{X_2} z_3$ does not vanish at $\bar{x}$ (because they are equal to $L_{[X_1,X_0]} z_2$ and $L_{[X_2,X_0]} z_2$) and hence z_3 is independent from z_1 and z_2, let z_4 be a fourth function, such that (z_1, z_2, z_3, z_4) is a system of coordinates. The nonsingular feedback

$$\begin{array}{rcl} v_1 & = & L_{X_0}^2 z_2 \ + \ u_1 L_{X_1} L_{X_0} z_2 \ + \ u_2 L_{X_2} L_{X_0} z_2 \\ v_2 & = & L_{X_0} z_4 \ + \ u_1 L_{X_1} z_4 \ + \ u_2 L_{X_2} z_4 \end{array} \qquad (91)$$

transforms system (5) into

$$\begin{array}{rcl} \dot{z}_1 & = & a(z_1, z_2, z_3, z_4) \\ \dot{z}_2 & = & z_3 \\ \dot{z}_3 & = & v_1 \\ \dot{z}_4 & = & v_2 \ , \end{array} \qquad (92)$$

with a a certain smooth function. Since Δ_3 spans a distribution of rank 3 and :

$$\Delta_3 \ = \ \text{Span}\,\{ \frac{\partial}{\partial z_3}, \frac{\partial}{\partial z_4}, \frac{\partial}{\partial z_2} + \frac{\partial a}{\partial z_3}\frac{\partial}{\partial z_1}, \frac{\partial a}{\partial z_4}\frac{\partial}{\partial z_1} \} \ ,$$

the function a cannot depend on z_4, and then

$$\Delta_3 + [\Delta_3, \Delta_3] \quad = \quad \text{Span}\,\{\, \frac{\partial}{\partial z_3}\,,\ \frac{\partial}{\partial z_4}\,,\ \frac{\partial}{\partial z_2} + \frac{\partial a}{\partial z_3}\frac{\partial}{\partial z_1}\,,\ \frac{\partial^2 a}{\partial z_3^2}\frac{\partial}{\partial z_1}\,\}$$

so that the assumption on Δ_3 is equivalent to $\frac{\partial^2 a}{\partial z_3^2}$ being identically zero on no neighborhood of $\bar{x}$. This proves that system (5) has the form (34) with the condition (35), after the change of coordinates and the nonsingular feedback transformation we just introduced. There remains to prove that system (34) cannot be linearizable by endogenous feedback under condition (35). This is a consequence of the following lemma (7.1) because if system (34) was linearizable by endogenous feedback on a neighborhood of a point $\bar{x}$, then there would exist a pair of linearizing outputs on a neighborhood of this point, and hence the system would also be linearizable by endogenous feedback around any point of that neighborhood, including these, given by condition (35), where $\frac{\partial^2 a}{\partial z_3^2}$ is non zero.

Lemma 7.1 *System (34) is not linearizable by endogenous dynamic feedback in any neighborhood of a point $\bar{\bar{z}} = (\bar{\bar{z}}_1, \bar{\bar{z}}_2, \bar{\bar{z}}_3, \bar{\bar{z}}_4)$ such that*

$$\frac{\partial^2 a}{\partial z_3^2}(\bar{\bar{z}}_1, \bar{\bar{z}}_2, \bar{\bar{z}}_3) \quad \neq \quad 0\ .$$

Proof of lemma 7.1 : Suppose that there exists two linearizing functions h_1 and h_2, smooth functions of a finite number of variables among z_1, z_2, z_3, z_4, v_1, v_2, $\dot{v}_1$, $\dot{v}_2$, $\ddot{v}_1$, $\ddot{v}_2$, ... $v_1^{(L)}$, $v_2^{(L)}$, with L a non negative integer, defined on an open subset $O \subset \mathbb{R}^{2L+6}$ containing a point $(\bar{\bar{z}}_1, \bar{\bar{z}}_2, \bar{\bar{z}}_3, \bar{\bar{z}}_4, \bar{\bar{v}}_1, \bar{\bar{v}}_2, \ldots, \bar{\bar{v}}_1^{(L)}, \bar{\bar{v}}_2^{(L)})$ for some $(\bar{\bar{v}}_1, \bar{\bar{v}}_2, \ldots, \bar{\bar{v}}_1^{(L)}, \bar{\bar{v}}_2^{(L)})$. All variables may be recovered from h_1, h_2 and all their time derivatives so that in particular there exists some smooth functions ψ_1 and ψ_2 such that

$$z_1 \quad = \quad \psi_1(h_1, \dot{h}_1, \ldots, h_1^{(K_{1,1})}, h_2, \dot{h}_2, \ldots, h_2^{(K_{1,2})}) \tag{93}$$

$$z_2 \quad = \quad \psi_2(h_1, \dot{h}_1, \ldots, h_1^{(K_{2,1})}, h_2, \dot{h}_2, \ldots, h_2^{(K_{2,2})})\ . \tag{94}$$

This holds in the open set O, which may be restricted so that $\frac{\partial^2 a}{\partial z_3^2}(z_1, z_2, z_3)$ does not vanish on O. The integers $K_{i,j}$ are such that ψ_i does depend on $h_j^{(K_{i,j})}$ on O, i.e.

$\frac{\partial \psi_i}{\partial h_j^{(K_{i,j})}}$ is not identically zero. Then, since (34) implies $\dot z_1 = a(z_1, z_2, \dot z_2)$, one has, by substitution,

$$\begin{aligned}&\frac{\partial\psi_1}{\partial h_1}\dot h_1 + \cdots + \frac{\partial\psi_1}{\partial h_1^{(K_{1,1})}} h_1^{(K_{1,1}+1)} + \frac{\partial\psi_1}{\partial h_2}\dot h_2 + \cdots + \frac{\partial\psi_1}{\partial h_2^{(K_{1,2})}} h_2^{(K_{1,2}+1)} \\ =\ & a\Big(\, \psi_1\, ,\, \psi_2\, ,\, \frac{\partial\psi_2}{\partial h_1}\dot h_1 + \cdots + \frac{\partial\psi_2}{\partial h_1^{(K_{2,1})}} h_1^{(K_{2,1}+1)} + \frac{\partial\psi_2}{\partial h_2}\dot h_2 + \cdots + \frac{\partial\psi_2}{\partial h_2^{(K_{2,2})}} h_2^{(K_{2,2}+1)} \Big)\, .\end{aligned} \tag{95}$$

One must have

$$K_{1,1} \;=\; K_{2,1} \quad , \qquad K_{1,2} \;=\; K_{2,2} \tag{96}$$

because the left-hand side in (95) depends only on $h_1, \dot h_1, \ldots, h_1^{(K_{1,1}+1)}, h_2, \dot h_2, \ldots, h_2^{(K_{1,2}+1)}$ and does depend on $h_1^{(K_{1,1}+1)}$ and $h_2^{(K_{2,1}+1)}$, and the right-hand side depends only on $h_1, \dot h_1, \ldots, h_1^{(K_{2,1}+1)}, h_2, \dot h_2, \ldots, h_2^{(K_{2,2}+1)}$ and does depend on $h_1^{(K_{2,1}+1)}$ and $h_2^{(K_{2,2}+1)}$ because, since $\frac{\partial^2 a}{\partial z_3^2}$ does not vanish on O, $\frac{\partial a}{\partial z_3}$ cannot be identically zero on any open subset of O.

Differentiating two times both sides of (95) with respect to $h_j^{(K_{1,j}+1)}$, and keeping in mind that, from (96), $K_{1,j} = K_{2,j}$, one has (note that neither ψ_1 nor ψ_2 nor the partial derivatives of them depend on $h_j^{(K_{1,j}+1)}$) :

$$\begin{aligned} 0 \;=\; \left(\frac{\partial\psi_2}{\partial h_j^{(K_{1,j})}}\right)^2 \frac{\partial^2 a}{\partial z_3^2}\Big(\, \psi_1\, ,\, \psi_2\, ,\, &\frac{\partial\psi_2}{\partial h_1}\dot h_1 + \cdots + \frac{\partial\psi_2}{\partial h_1^{(K_{1,1})}} h_1^{(K_{1,1}+1)} + \frac{\partial\psi_2}{\partial h_2}\dot h_2 + \cdots \\ &\cdots + \frac{\partial\psi_2}{\partial h_2^{(K_{1,2})}} h_2^{(K_{1,2}+1)} \Big)\, ,\end{aligned} \tag{97}$$

for $j \in \{1, 2\}$, and hence $\frac{\partial\psi_2}{\partial h_j^{(K_{1,j})}}$ is identically zero on O which contradicts the fact that it was precisely chosen (small enough) not to be identically zero on O. ∎

Case 2.b

Since Δ_3 is integrable of rank 3, and $\{X_1, X_2\}$ is integrable of rank 2, and contained in Δ_3, there are two independent functions z_1 and z_2 such that z_1 and z_2 are constant along X_1 and X_2 and z_1 constant along the vector fields of Δ_3. Let z_3 be given by $z_3 = L_{X_0} z_2$ and z_4 be such that (z_1, z_2, z_3, z_4) is a system of coordinates. The nonsingular feedback (91) transforms system (5) into a system of the form (92) above, for a certain smooth function a, and the rank assumptions clearly imply that a depends on z_1 only. $\dot z_1 = a(z_1)$ clearly implies non-accessibility.

Case 2.c

Static feedback linearization follows from classical results, see [17, 13]. Let us however describe the coordinates in which the system has the form (7.a). Since Δ_3 is integrable of rank 3, there is a function z_1 such that $\mathrm{d}z_1$ is the annihilator of Δ_3. Let z_2 and z_3 be given by $z_2 = L_{X_0}z_1$ and $z_3 = L_{X_0}^2 z_1$, the rank of $\{\mathrm{d}z_1, \mathrm{d}z_2, \mathrm{d}z_3\}$ is 3 because $\delta_3 = 3$. Let z_4 be any function such that $\{z_1, z_2, z_3, z_4\}$ is a system of coordinates. The nonsingular feedback

$$\begin{array}{rcl} v_1 & = & L_{X_0}^3 z_1 \;+\; u_1 L_{X_1} L_{X_0}^2 z_1 \;+\; u_2 L_{X_2} L_{X_0}^2 z_1 \\ v_2 & = & L_{X_0} z_4 \;+\; u_1 L_{X_1} z_4 \;+\; u_2 L_{X_2} z_4 \end{array}$$

transforms system (5) into (7.a).

Case 3

As in case 2.c, static feedback linearization follows from classical results, see [17, 13], but we however describe the coordinates in which the system has the form (7.b). Because $m_0 = 2$, X_1 and X_2 span an integrable distribution of rank 2, let z_1 and z_3 be two independent functions who annihilate X_1 and X_2, and let z_2 and z_4 be defined by $z_2 = L_{X_0}z_1$ and $z_4 = L_{X_0}z_3$. $\delta_3 = 4$ implies that (z_1, z_2, z_3, z_4) is a system of coordinates, and the following nonsingular feedback

$$\begin{array}{rcl} v_1 & = & L_{X_0}^2 z_1 \;+\; u_1 L_{X_1} L_{X_0} z_1 \;+\; u_2 L_{X_2} L_{X_0} z_1 \\ v_2 & = & L_{X_0}^2 z_3 \;+\; u_1 L_{X_1} L_{X_0} z_3 \;+\; u_2 L_{X_2} L_{X_0} z_3 \end{array}$$

transforms system (5) into (7.b).

Cases 4 and 5

Since $m_0 = m_1 = 3$, Δ_3 spans an integrable distribution of rank 3. Let z_1 be a first integral of this distribution. In case 5 ($\delta_3 = 4$), define z_2 by

$$z_2 \;=\; L_{X_0} z_1 \; . \tag{98}$$

One then has, for $i \in \{1, 2\}$, $L_{X_i}z_2 = -L_{[X_0,X_i]}z_1$ because $L_{X_i}z_1 \equiv 0$, $i = 1, 2$, and hence $\delta_3 = 4$ prevents $L_{X_1}z_2$ and $L_{X_2}z_2$ from both vanishing at $\bar{x}$. Modulo a permutation of the two controls, we may suppose that

$$L_{X_1}(\bar{x}) \;\neq\; 0 \; . \tag{99}$$

In case 4 ($\delta_3 = 3$), pick any z_2 such that (99) holds, it is possible since X_1 does not vanish. Since $L_{X_1} z_1 = 0$, the rank of $\{\mathrm{d}z_1, \mathrm{d}z_2\}$ is 2 at point $\bar{x}$. The vector field

$$(L_{X_2} z_2)\, X_1 \;-\; (L_{X_1} z_2)\, X_2 \tag{100}$$

does not vanish at point $\bar{x}$, z_1 and z_2 are two independent functions constant along it, let z_3 be a third independent first integral of this vector field, and z_4 be given by

$$z_4 \;=\; \frac{L_{X_1} z_3}{L_{X_1} z_2}\,.$$

(z_1, z_2, z_3, z_4) is a system of coordinates because z_1, z_2 and z_3 are constant along the vector field (100) while the Lie derivative of z_4 along it does not vanish at $\bar{x}$ (a simple computation shows that if it would vanish, the rank of $\mathcal{M}_0$ would drop to 2). Defining v_1 and v_2 according to the nonsingular feedback transformation

$$\begin{array}{rcl} v_1 & = & L_{X_0} z_2 \;+\; u_1 L_{X_1} L_{X_0} z_1 \;+\; u_2 L_{X_2} L_{X_0} z_1 \\ v_2 & = & L^2_{X_0} z_3 \;+\; u_1 L_{X_1} L_{X_0} z_3 \;+\; u_2 L_{X_2} L_{X_0} z_3 \end{array}$$

(with a possible permutation of the indices 1 and 2 in the right-hand sides, if needed to get (99)) yields, in the above defined coordinates, the normal form (37) in case 4, and (39) in case 5. In both cases, a_3 is given by

$$a_3 \;=\; L_{X_0} z_3 \;-\; \frac{L_{X_1} z_3}{L_{X_1} z_2} L_{X_0} z_2\;,$$

$\dot{z}_3$ is obtained because

$$L_{X_2} z_3 \;=\; \frac{L_{X_2} z_2}{L_{X_1} z_2} L_{X_1} z_3\;,$$

and (in case 4) $a_1 = L_{X_0} z_1$ depends only on z_1 because $\delta_3 = 3$ implies that $\Delta_3 = \mathcal{M}_0$ and hence that $L_{X_0} z_1$ is a first integral of the three dimensional integrable distribution spanned by $\mathcal{M}_0$.

In case 4, non-accessibility follows immediately from the normal form (37). In case 5, let us prove that system (39) is x-dynamic linearizable around $(\bar{z}, \bar{v})$ if and only if $\frac{\partial a}{\partial z_4}(\bar{z}) + \bar{v}_1 \neq 0$. Let (h_1, h_2) be a pair of linearizing functions, depending on z only.

Lemma 7.2 *Let h_1, h_2 be two functions depending on z only such that (h_1, h_2) is a pair of linearizing functions for system (39) on a neighborhood of $(\bar{z}, \bar{v})$. Then the rank of $\{\mathrm{d}z_1, \mathrm{d}h_1, \mathrm{d}h_2\}$ is 2 on a neighborhood of $\bar{z}$.*

Proof : If it was not the case, there would be points $\bar{\bar{z}}$, arbitrarily close to $\bar{z}$, where this rank would be 3, and where (h_1, h_2) would still be a pair of linearizing functions. z_1 is constant along both control vector fields, and since (h_1, h_2) would still be a pair of linearizing functions, there is, from (26), a nonzero linear combination of X_1 and X_2, say Z, along which both h_1 and h_2 are constant. It is impossible that $L_{X_i}h_j$ vanishes at $\bar{\bar{x}}$ for all $(i,j) \in \{1,2\}^2$, so that modulo a permutation, we may suppose that $L_{X_1}h_1 \neq 0$. This yields, following the same construction as above —construction of coordinates where the system has form (39)— a set of coordinates

$$(\zeta_1, \zeta_2, \zeta_3, \zeta_4) \quad = \quad (z_1, h_1, h_2, \frac{L_{X_1}h_2}{L_{X_1}h_1})$$

and a nonsingular feedback $w_1 = \dot{h}_1$, $w_2 = \zeta_4$ such that the system is also of the form (39) with ζ instead of z and w instead of v :

$$\begin{array}{rclrcl} \dot{\zeta}_1 & = & \zeta_2 & \qquad \dot{\zeta}_3 & = & a_3(\zeta_1, \zeta_2, \zeta_3, \zeta_4) \; + \; \zeta_4 w_1 \\ \dot{\zeta}_2 & = & w_1 & \qquad \dot{\zeta}_4 & = & w_2 \end{array}$$

where (ζ_2, ζ_3) should be a pair of linearizing outputs. This is impossible from (27) because

$$\begin{pmatrix} \frac{\partial \dot{\zeta}_2}{\partial w_1} & \frac{\partial \dot{\zeta}_2}{\partial w_2} & 0 & 0 \\ \frac{\partial \dot{\zeta}_3}{\partial w_1} & \frac{\partial \dot{\zeta}_3}{\partial w_2} & 0 & 0 \\ \frac{\partial \ddot{\zeta}_2}{\partial w_1} & \frac{\partial \ddot{\zeta}_2}{\partial w_2} & \frac{\partial \ddot{\zeta}_2}{\partial \dot{w}_1} & \frac{\partial \ddot{\zeta}_2}{\partial \dot{w}_2} \\ \frac{\partial \ddot{\zeta}_3}{\partial w_1} & \frac{\partial \ddot{\zeta}_3}{\partial w_2} & \frac{\partial \ddot{\zeta}_3}{\partial \dot{w}_1} & \frac{\partial \ddot{\zeta}_3}{\partial \dot{w}_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \zeta_4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\partial a_3}{\partial \zeta_2} \; + \; \zeta_4 \frac{\partial a_3}{\partial \zeta_3} \; + \; w_2 & \frac{\partial a_3}{\partial \zeta_4} \; + \; w_1 & \zeta_4 & 0 \end{pmatrix}$$

and hence $\frac{\partial a_3}{\partial \zeta_4} + w_1$ should be identically zero on an open set, which is absurd because its derivative with respect to w_1 is 1. ∎

From the lemma, z_1 is a function of the two linearizing functions, and therefore one may replace h_1 *or* h_2 by z_1 in (h_1, h_2) and still have a pair of linearizing outputs. Let for instance $h_1 = z_1$, then (26) is automatically satisfied, and (27) implies that h_2 must depend on ζ_1, ζ_2, ζ_3 only because

$$\begin{pmatrix} \frac{\partial \dot{h}_1}{\partial v_1} & \frac{\partial \dot{h}_1}{\partial v_2} & 0 & 0 \\ \frac{\partial \dot{h}_2}{\partial v_1} & \frac{\partial \dot{h}_2}{\partial v_2} & 0 & 0 \\ \frac{\partial \ddot{h}_1}{\partial v_1} & \frac{\partial \ddot{h}_1}{\partial v_2} & \frac{\partial \ddot{h}_1}{\partial \dot{v}_1} & \frac{\partial \ddot{h}_1}{\partial \dot{v}_2} \\ \frac{\partial \ddot{h}_2}{\partial v_1} & \frac{\partial \ddot{h}_2}{\partial v_2} & \frac{\partial \ddot{h}_2}{\partial \dot{v}_1} & \frac{\partial \ddot{h}_2}{\partial \dot{v}_2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ * & \frac{\partial h_2}{\partial \zeta_4} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ * & * & * & \frac{\partial h_2}{\partial \zeta_4} \end{pmatrix} ,$$

and then (28) implies (40). Conversely, if (40) is satisfied, system (39) is x-dynamic linearizable with (ζ_1, ζ_3) as a pair of linearizing outputs, because ζ_2 is $\dot{\zeta}_1$, and ζ_4 is (inverse function theorem) a function of $\dot{\zeta}_3$, ζ_1, ζ_2, ζ_3, v_1, i.e. of $\dot{\zeta}_3$, ζ_1, $\dot{\zeta}_1$, ζ_3, $\ddot{\zeta}_1$.

Case 6

Let us prove the "normal form" first. It is a consequence of the following lemma :

Lemma 7.3 ("Engels normal form") *Let X_1 and X_2 be some vector fields in $\mathbb{R}^4$ and let $\bar{x} \in \mathbb{R}^4$ be such that*

$$\begin{aligned} &\mathrm{rank}\,\{\,\{X_1(\bar{x})\,,\,X_2(\bar{x})\,\}\ = \ 2\ , \\ &\mathrm{rank}\,\{\,\{X_1(\bar{x})\,,\,X_2(\bar{x})\,,\,[X_1,X_2](\bar{x})\,\}\ = \ 3\ , \\ &\mathrm{rank}\,\{\,\{X_1(\bar{x})\,,\,X_2(\bar{x})\,,\,[X_1,X_2](\bar{x})\,,\,[X_1,[X_1,X_2]](\bar{x})\,,\,[X_2,[X_1,X_2]](\bar{x})\,\}\ = \ 4\ . \end{aligned}$$

Then there exists some functions function α_{11}, α_{12}, α_{21}, α_{22}, and some coordinates (z_1, z_2, z_3, z_4) such that the matrix $\begin{pmatrix} \alpha_{11}(\bar{x}) & \alpha_{12}(\bar{x}) \\ \alpha_{21}(\bar{x}) & \alpha_{22}(\bar{x}) \end{pmatrix}$ is invertible, and, locally around $\bar{x}$,

$$\alpha_{11}X_1 + \alpha_{21}X_2 \ = \ \frac{\partial}{\partial z_1} + z_3\frac{\partial}{\partial z_2} + z_4\frac{\partial}{\partial z_3}\ ; \quad \alpha_{12}X_1 + \alpha_{22}X_2 \ = \ \frac{\partial}{\partial z_4}\ . \tag{101}$$

The proof is very classical, see for example [6]. Now, by assumption, the vector fields X_1 and X_2 satisfy these assumptions, and the feedback

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \ = \ \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \begin{pmatrix} L_{X_0}z_1 \\ L_{X_0}z_4 \end{pmatrix}$$

yields the equations (45) in the coordinates given by lemma 7.3. The fact that the coordinate-free and feedback invariant conditions (42), (43), (44) translate into (46), (47), (48) respectively is a routine computation.

Suppose that there exists a pair of linearizing outputs (h_1, h_2) with h_1 and h_2 depending on x only. We shall proceed as explained in section 2.6 —see equations (26) and (27)— to derive some necessary conditions. We have

$$\dot{h}_i \ = \ L_{X_0}h_i \ + \ u_1 L_{X_1}h_i \ + \ u_2 L_{X_2}h_i\ . \tag{102}$$

The functions $\dot{h}_1$ and $\dot{h}_2$ cannot depend independently on the controls u_1 and u_2, or in other words equation (26) implies that the rank of

$$\begin{pmatrix} L_{X_1}h_1 & L_{X_2}h_1 \\ L_{X_1}h_2 & L_{X_2}h_2 \end{pmatrix}$$

must be one. Without loss of generality, suppose that $L_{X_1}h_1$ does not vanish at the point under consideration (since $m_1 = 4$, $L_{X_1}h_1$ and $L_{X_2}h_1$ cannot vanish together). Then, with λ the function given by $\lambda = L_{X_2}h_1/L_{X_1}h_1$, and defining the vector field Z_2 by

$$Z_2 \;=\; X_2 \;-\; \lambda\, X_1 \tag{103}$$

one has

$$L_{Z_2}h_1 \;=\; L_{Z_2}h_2 \;=\; 0\ . \tag{104}$$

and on the other hand, with

$$w_1 \;=\; \dot{h}_1 \;=\; L_{X_0}h_1 + u_1 L_{X_1}h_1 + u_2 L_{X_2}h_1 \;=\; L_{X_0}h_1 + (u_1{+}\lambda u_2) L_{X_1}h_1 \tag{105}$$

—$(u_1, u_2) \mapsto (w_1, u_2)$ is a regular static feedback— and the vector fields Z_0 and Z_1 given by

$$Z_0 \;=\; X_0 \;-\; \frac{L_{X_0}h_1}{L_{X_1}h_1}\, X_1\ \ , \qquad Z_1 \;=\; \frac{1}{L_{X_1}h_1}\ X_1 \tag{106}$$

systems (5) reads

$$\dot{x} \;=\; Z_0 \;+\; w_1\, Z_1 \;+\; u_2\, Z_2\ . \tag{107}$$

and one has

$$\begin{aligned} \dot{h}_1 &= w_1 \\ \dot{h}_2 &= L_{Z_0}h_2 \;+\; w_1\, L_{Z_1}h_2\ \ . \end{aligned} \tag{108}$$

The second time-derivatives are then given by :

$$\begin{aligned} \ddot{h}_1 &= \dot{w}_1 \\ \ddot{h}_2 &= L_{Z_0}^2 h_2 \;+\; w_1\, (L_{Z_1}L_{Z_0} \;+\; L_{Z_0}L_{Z_1})\, h_2 \;+\; w_1\, L_{Z_1}^2 h_2 \;+\; \dot{w}_1\, L_{Z_1}h_2 \\ &\quad + \; (L_{Z_2}L_{Z_0}h_2 \;+\; w_1\, L_{Z_2}L_{Z_1}h_2)\, u_2\ \ . \end{aligned} \tag{109}$$

The function $\ddot{h}_2$ must not depend on u_2 —this is (27)— and hence

$$L_{Z_2}L_{Z_0}h_2 \;=\; L_{Z_2}L_{Z_1}h_2 \;=\; 0\ . \tag{110}$$

Now, on one hand $L_{Z_1}h_1$ is identically equal to 1 from (108)-(107), and $L_{Z_2}h_1$ is identically zero from (104), and on the other hand, since $L_{Z_2}h_2$ is identically zero from (104), $L_{Z_2}L_{Z_1}h_2$ is equal to $L_{[Z_2,Z_1]}h_2$; this and (110) above implies :

$$L_{[Z_2,Z_1]}h_2 \;=\; L_{[Z_2,Z_1]}h_2 \;=\; 0 \;. \tag{111}$$

The two independent functions h_1 and h_2 are, from (111) and (104), constant along the vector fields Z_2 and $[Z_1,Z_2]$, which are linearly independent because $m_1 = 3$. This implies that the distribution spanned by these two vector fields is integrable, and therefore that the Lie Bracket $[Z_2,[Z_1,Z_2]]$ is a linear combination of Z_2 and $[Z_1,Z_2]$. From (103) and (106), the Lie bracket $[Z_2,[Z_1,Z_2]]$ is equal to $\lambda[Z_2,[X_1,X_2]] + (L_{X_1}\lambda)[X_1,X_2] + ((L_{X_1}\lambda)^2 - L_{X_2}L_{X_1}\lambda)X_1$. Hence, $[Z_2,[X_1,X_2]]$ must be a linear combination of X_1, X_2 and $[X_1,X_2]$. This implies, from the definition of the characteristic vector field $\widetilde{X}$ —see (41)—that Z_2 is collinear-linear to $\widetilde{X}$:

$$Z_2 \;=\; \alpha\,\widetilde{X} \tag{112}$$

with α a nonzero function.

Now, on one hand $L_{Z_0}h_1$ and $L_{Z_2}h_1$ are identically zero from (104)-(108)-(107), and and on the other hand, since $L_{Z_2}h_2$ is identically zero from (104), $L_{Z_2}L_{Z_0}h_2$ is equal to $L_{[Z_2,Z_0]}h_2$; this and (110) implies :

$$L_{[Z_2,Z_0]}h_2 \;=\; L_{[Z_2,Z_0]}h_2 \;=\; 0 \;.$$

Since the vector fields annihilating $\mathrm{d}h_1$ and $\mathrm{d}h_2$ are the linear combinations of Z_2 and $[Z_2,Z_1]$, this and (111) imply that $[Z_2,Z_0]$ is a linear combination of Z_2 and $[Z_2,Z_1]$, which implies in particular that it is a linear combination of X_1, X_2 and $[X_1,X_2]$. From (112), this implies condition (42).

Furthermore, the rank of $\{\mathrm{d}h_1,\mathrm{d}h_2,\mathrm{d}\dot{h}_1,\mathrm{d}\dot{h}_2\}$ must be 4, which is equivalent to

$$\mathrm{rank}_{\mathbb{R}}\,\{\,Z_2(\bar{x})\,,\,[Z_1,Z_2](\bar{x})\,,\,[F,Z_2](\bar{x})\,,\,[F,[Z_1,Z_2]](\bar{x})\,\} \;=\; 4 \;. \tag{113}$$

with F given by (9). From (106) and (112), this may be rewritten

$$\begin{aligned}\mathrm{rank}_{\mathbb{R}}\,\{\,\widetilde{X}(\bar{x})\,,\,[\tfrac{1}{L_{X_1}h_1}X_1, X_2-\lambda X_1](\bar{x})\,,\,[X_0+u_1X_1+u_2X_2,\widetilde{X}](\bar{x})\,,\\ [X_0+u_1X_1+u_2X_2,[\tfrac{1}{L_{X_1}h_1}X_1,X_2-\lambda X_1]](\bar{x})\,\} \;=\; 4 \;,\end{aligned} \tag{114}$$

which implies (43) obviously, and (44) because the four vector fields above are linear combinations of the six ones there.

Conversely, suppose that (42) holds in a neighborhood of $\bar{x}$ and (43) and (44) hold at $(\bar{x}, \bar{u}_1, \bar{u}_2)$, and hence in a neighborhood. We have proved that, in a neighborhood of such a point, there exists a static feedback and some coordinates such that the system has the form (45), with f_3 depending on z_1, z_2 and z_3 only —this is (46)— and (47) and (48) holding in a neighborhood. If $v_1 + \frac{\partial f_2}{\partial z_3}$ does not vanish, take $h_1 = z_1$ and $h_2 = z_2$. A simple computations shows that the determinant of the jacobian of the application

$$(z_1, z_2, z_3, z_4, v_1, \dot{v}_1) \quad \mapsto \quad (h_1, h_2, \dot{h}_1, \dot{h}_2, \ddot{h}_1, \ddot{h}_2)$$

is $(v_1 + \frac{\partial f_2}{\partial z_3})^2 (v_1 + \frac{\partial f_3}{\partial z_4})$, hence the inverse function theorem tells us that (h_1, h_2) is a pair of linearizing outputs. If $v_1 + \frac{\partial f_2}{\partial z_3}$ vanishes, then from (48), $f_3 - \frac{\partial f_2}{\partial z_1} - z_3 \frac{\partial f_2}{\partial z_2} + z_4 v_1$ does not vanish, and if one takes $h_1 = z_3$ and $h_2 = z_2 - z_1 z_3$, one has

$$\dot{h}_1 \;=\; f_3 + z_4 v_1 \quad , \quad \dot{h}_2 \;=\; f_2 - z_1(f_3 + z_4 v_1) \;=\; f_2 - z_1 \dot{h}_1 \; ,$$

hence the jacobian of the application

$$(z_1, z_2, z_3, \dot{h}_1) \quad \mapsto \quad (h_1, h_2, \dot{h}_1, \dot{h}_2)$$

is equal to $f_3 - \frac{\partial f_2}{\partial z_1} - z_3 \frac{\partial f_2}{\partial z_2} + z_4 v_1$. Hence, from the local inverse function theorem, there exists locally three smooth functions χ_1, $i = 1, 2, 3$, such that $z_i = \chi_i(h_1, h_2, \dot{h}_1, \dot{h}_2)$ but since $v_1 + \frac{\partial f_3}{\partial z_4}$ does not vanish, z_4 may be expressed as a function of $\dot{z}_3$ and z_1, z_2, z_3 and $v_1 = \dot{z}_1$, hence z_4 is a smooth function of $(h_1, h_2, \dot{h}_1, \dot{h}_2, \ddot{h}_1, \ddot{h}_2)$. We have proved that, depending on which function does not vanish in (48), one of the two choices (z_1, z_2) or $(z_3, z_2 - z_1 z_3)$ provides a pair of linearizing outputs.

Alternative proof of Case 6

Here we suppose in addition that we are at Brunovský-regular point, i.e. the rank condition (53) holds, and we give a proof for case 6 based on the infinitesimal Brunovský form. To give a thorough treatment of case 6, one should consider the case when the rank in (53) is three in a neighborhood —then there is a different infinitesimal brunovsky form, as in the second point of proposition 2.2, and also points where it three, while being 4 in an open dense set of a neighborhood —at such points, an infinitesimal Brunovský form does not exist but one might conclude by density.

Condition (53) implies, see proposition 2.2, that if some forms ω_1 and ω_2 make up a basis of $\widehat{\mathcal{D}}_2^{\perp}$, then $\{\omega_1, \omega_2, \dot{\omega}_1, \dot{\omega}_2\}$ is a basis of $\mathrm{Span}\{\mathrm{d}x\}$. In addition, ω_1 may be taken in $\mathcal{M}_0^\perp$ (i.e. $\{\omega_1\}$ is the first derived system of the pfaffian system $\{\omega_1, \omega_2\}$). Then we have :

$$\begin{array}{lcll} \mathrm{d}\omega_1 & \equiv & \omega_2 \wedge (\delta_1 \dot{\omega}_1 + \delta_2 \dot{\omega}_2) & \text{modulo } \omega_1 \\ \mathrm{d}\omega_2 & \equiv & \gamma\, \dot{\omega}_1 \wedge \dot{\omega}_2 & \text{modulo } \{\omega_1, \omega_2\} \ . \end{array} \tag{115}$$

Since on one hand the rank of $\mathcal{M}_1$ is constant equal to 4, and on the other hand the rank of $\mathcal{M}_0$ is constant equal to 3,

$$\left.\begin{array}{ll} \delta_1 \text{ and } \delta_2 & \text{do not vanish simultaneouly,} \\ \gamma & \text{does not vanish.} \end{array}\right\} \tag{116}$$

A computations shows that :

$$\mathrm{Span}\{\widetilde{X}\} \quad = \quad \{\omega_1\, ,\, \omega_2\, ,\, \delta_1\dot{\omega}_1 + \delta_2\dot{\omega}_2\, \}^\perp \ . \tag{117}$$

The proof of characterization (42) relies on the following lemma, which is proved further :

Lemma 7.4 *The following two properties are equivalent :*

(i) *There exist two invertible matrices J_1 and J_2 of degree zero and three functions a, h_1 and h_2, all defined on a neighborhood of the point $\mathcal{X}$, such that*

$$\left(\begin{array}{c} \mathrm{d}h_1 \\ \mathrm{d}h_2 \end{array}\right) \quad = \quad J_1 \left(\begin{array}{cc} 1 & -\,a\,\frac{\mathrm{d}}{\mathrm{dt}} \\ 0 & 1 \end{array}\right) J_2 \left(\begin{array}{c} \omega_1 \\ \omega_2 \end{array}\right) \tag{118}$$

(ii) $\delta_2 \;=\; 0$ *on a neighborhood of $\mathcal{X}$.*

(iii) *(42) hold on a neighborhood of $\mathcal{X}$.*

This is enough to conclude. Indeed, sufficiency in case 6 of theorem 3.1 is obvious because, from proposition 2.3, point (i) implies x-dynamic linearizability. Let us prove necessity : if system (5) is x-dynamic linearizable in a neighborhood of a point $\overline{\mathcal{X}}$, then from propositions 2.3 and 2.4, there is an open set U_0, dense in a neighborhood of $\overline{\mathcal{X}}$, such that point (i) holds for all $\mathcal{X} \in U_0$. From the lemma, this implies that δ_2 is zero on U_0. Hence it is zero on a neighborhood of $\overline{\mathcal{X}}$. This completes the proof of case 6 of theorem 3.1, the normal form being proved the same way as in the first proof.

Proof of lemma 7.4 :

(ii)$\Leftrightarrow$(iii) : We have, from (117) and identity (15),

$$\begin{array}{rcl} 0 &=& \frac{\mathrm{d}}{\mathrm{dt}}\langle \delta_1\omega_1+\delta_2\omega_2\,,\,\widetilde{X}\rangle \;=\; \langle \dot\delta_1\omega_1+\dot\delta_2\omega_2+\delta_1\dot\omega_1+\delta_2\dot\omega_2\,,\,\widetilde{X}\rangle \\ && \qquad +\,\langle \delta_1\omega_1+\delta_2\omega_2\,,\,[X_0+u_1X_1+u_2X_2,\widetilde{X}]\rangle\;, \end{array}$$

which, from the fact that $\langle \omega_i, \widetilde{X}\rangle$ and $\langle \omega_1, [X_i,\widetilde{X}]\rangle$ are identically zero for $i=1,2$, yields

$$\delta_1\,\langle\omega_1\,,\,[X_0,\widetilde{X}]\rangle \;+\; \delta_2\,\langle\omega_2\,,\,[X_0+u_1X_1+u_2X_2,\widetilde{X}]\rangle \;=\; 0 \tag{119}$$

which implies, since δ_1 and δ_2 do not vanish simultaneously and $[X_0{+}u_1X_1{+}u_2X_2,\widetilde{X}]$ does not vanish, that $\delta_2=0$ is equivalent to $\langle\omega_1,[X_0,\widetilde{X}]\rangle=0$, i.e. to (42).

(ii)$\Rightarrow$(i) : Since $\delta_2=0$, (115) implies that $\{\omega_1,\omega_2,\dot\omega_1\}$ is the characteristic system of ω_1 and therefore is integrable. In particular, there exists a function h_2 such that

$$\mathrm{d}h_2 \;=\; \lambda_0\omega_1 \;+\; \lambda_1\dot\omega_1 \;+\; \lambda_2\omega_2$$

with a nonvanishing λ_2; then

$$\mathrm{d}\omega_1 \;\equiv\; \widetilde{\delta}_1\,\mathrm{d}h_2\;\wedge\dot\omega \quad \text{modulo}\;\omega_1$$

which implies that $\{\omega_1,\mathrm{d}h_2\}$ is integrable and in particular that there exists a function h_1 such that

$$\begin{pmatrix}\mathrm{d}h_1\\ \mathrm{d}h_2\end{pmatrix} = \begin{pmatrix}\mu_1 & \mu_2\\ 0 & 1\end{pmatrix}\begin{pmatrix}\omega_1\\ \mathrm{d}h_2\end{pmatrix} = \begin{pmatrix}\mu_1 & \mu_2\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \lambda_0+\lambda_1\frac{\mathrm{d}}{\mathrm{dt}} & \lambda_2\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}$$

where $\mu_1\lambda_1$ does not vanish. This is point (i).

(i)$\Rightarrow$(ii) : Let Ω_1, Ω_2 and Ω_3 be defined by

$$\begin{pmatrix}\Omega_1\\ \Omega_2\end{pmatrix} \;=\; J_2\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}\;, \tag{120}$$

$$\begin{pmatrix}\Omega_3\\ \Omega_2\end{pmatrix} \;=\; \begin{pmatrix}1 & -a\frac{\mathrm{d}}{\mathrm{dt}}\\ 0 & 1\end{pmatrix}\begin{pmatrix}\Omega_1\\ \Omega_2\end{pmatrix} \;=\; \begin{pmatrix}\Omega_1 \,-\, a\dot\Omega_2\\ \Omega_2\end{pmatrix}\,. \tag{121}$$

then (118) implies that $\{\Omega_3,\Omega_2\}$ is integrable and hence, for some 1-forms $\Gamma_{i,j}$,

$$\begin{array}{rcl} \mathrm{d}\Omega_2 &=& \Omega_2\wedge\Gamma_{2,2} \;+\; \Omega_3\wedge\Gamma_{2,3} \;=\; \Omega_1\wedge\Gamma_{2,3} \;+\; \Omega_2\wedge\Gamma_{2,2} \;-\; a\,\dot\Omega_2\wedge\Gamma_{2,3}\\ \mathrm{d}\Omega_3 &=& \Omega_2\wedge\Gamma_{3,2} \;+\; \Omega_3\wedge\Gamma_{3,3} \;=\; \Omega_1\wedge\Gamma_{2,3} \;+\; \Omega_2\wedge\Gamma_{2,2} \;-\; a\,\dot\Omega_2\wedge\Gamma_{2,3} \end{array} \tag{122}$$

Taking the time-derivative of the second equation yields

$$\begin{array}{rcl} \mathrm{d}\dot{\Omega}_2 & = & \Omega_1 \wedge \dot{\Gamma}_{2,3} \;+\; \dot{\Omega}_1 \wedge \Gamma_{2,3} \\ & & +\, \Omega_2 \wedge \dot{\Gamma}_{2,2} \;+\; \dot{\Omega}_2 \wedge \left(\Gamma_{2,2} \,-\, a\dot{\Gamma}_{2,3} \,-\, \dot{a}\Gamma_{2,3}\right) \;-\; a\,\ddot{\Omega}_2 \wedge \Gamma_{2,3} \end{array} \tag{123}$$

and finally, since $\mathrm{d}\Omega_1 = \mathrm{d}(\Omega_3 + a\dot{\Omega}_2) = \mathrm{d}\Omega_3 + a\mathrm{d}\dot{\Omega}_2 - \dot{\Omega}_2 \wedge \mathrm{d}a$,

$$\begin{array}{rcl} \mathrm{d}\Omega_1 & = & \Omega_1 \wedge \left(\Gamma_{2,3} \,+\, a\dot{\Gamma}_{2,3}\right) \;+\; \Omega_2 \wedge \left(\Gamma_{2,2} \,+\, a\dot{\Gamma}_{2,2}\right) \\ & & +\, a\,\dot{\Omega}_1 \wedge \Gamma_{2,3} \;+\; \dot{\Omega}_2 \wedge \left(-\,a\Gamma_{2,3} \,+\, a\Gamma_{2,2} \,-\, a^2\dot{\Gamma}_{2,3} \,-\, a\dot{a}\Gamma_{2,3} \,-\, \mathrm{d}a\right) \\ & & -\, a^2\,\ddot{\Omega}_2 \wedge \Gamma_{2,3} \\ \mathrm{d}\Omega_2 & = & \Omega_1 \wedge \Gamma_{2,3} \;+\; \Omega_2 \wedge \Gamma_{2,2} \;-\; a\,\dot{\Omega}_2 \wedge \Gamma_{2,3} \end{array} \tag{124}$$

From (120), $\{\Omega_1, \Omega_2\}$ is the same differential system as $\{\omega_1, \omega_2\}$ and therefore, from (115), $\mathrm{d}\Omega_i \equiv \lambda_i\dot{\Omega}_1 \wedge \dot{\Omega}_2$ modulo $\{\Omega_1, \Omega_2\}$ for $i = 1, 2$ and λ_i certain functions; from the second equation in (124), this implies that $\Gamma_{2,3}$ is a linear combination of $\Omega_1, \Omega_2, \dot{\Omega}_1, \dot{\Omega}_2$; from the first equation in (124), it is actually a linear combination of $\Omega_1, \Omega_2, \dot{\Omega}_2$ because the $\dot{\Omega}_1$-term would produce a $\ddot{\Omega}_2 \wedge \dot{\Omega}_1$-term in the last term of $\mathrm{d}\Omega_1$ (it cannot be canceled by another term because there is no $\ddot{\Omega}_2$ in $\gamma_{2,3}$); this implies, if $\Gamma_{2,3} = \lambda_1\Omega_1 + \lambda_1\Omega_2 + \lambda_0\dot{\Omega}_2$,

$$\mathrm{d}\Omega_2 \;=\; \Omega_2 \wedge \widetilde{\Gamma}_{2,2} \;+\; (\lambda_0 + a\lambda_1)\Omega_1 \wedge \dot{\Omega}_2 \tag{125}$$

where $\widetilde{\Gamma}_{2,2}$ contains $\Gamma_{2,2}$ plus other terms. This implies in particular that $\mathrm{d}\Omega_2 \equiv 0$ modulo $\{\Omega_1, \Omega_2\}$ which implies that Ω_2 is in the first derived system of $\{\Omega_1, \Omega_2\}$ (i.e. in the annihilator of $\{X_1, X_2, [X_1, X_2]\}$) and therefore that it is collinear to ω_1, or in other terms that matrix J_2 is triangular :

$$\begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix} \;=\; \begin{pmatrix} \beta_1 & \beta_2 \\ \alpha & 0 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \tag{126}$$

where $\alpha\beta_2$ does not vanish. Then (115) yields

$$\mathrm{d}\Omega_2 \;\equiv\; \frac{1}{\beta_2}\Omega_1 \;\wedge \left((\delta_1 - \frac{\beta_1\delta_2}{\beta_2})\dot{\Omega}_2 \,+\, \delta_2\frac{\alpha}{\beta_2}\dot{\Omega}_1\right) \quad \text{modulo } \Omega_2 \tag{127}$$

By comparing this and (125), we see that $\delta_2\alpha = 0$ which implies that δ_2 is identically zero because α does not vanish.

7.2 Proof of the results on (x,u)-dynamic linearizability

In this section, we shall prove theorem 4.1 and proposition 4.4. They are proved together because we were not able to prove the intrinsic condition of theorem 4.1 without the help of the coordinates of the normal form (61). Basically, we shall prove that (x,u)-dynamic linearizability implies the conditions of proposition 4.4 in the coordinates of the normal form, and then that these conditions imply the condition of theorem 4.1 that is itself sufficient for dynamic linearizability.

The following is the key technical lemma :

Lemma 7.5 *The following four assertions are equivalent :*

1. *There exists an invertible matrix J_1 of degree zero and six functions α, λ, a, b, h_1 and h_2, all defined on a neighborhood of the point $\mathcal{Y}$, such that b does not vanish on this neighborhood and*

$$\begin{pmatrix} \mathrm{d}h_1 \\ \mathrm{d}h_2 \end{pmatrix} = J_1 \begin{pmatrix} 1 & -a\frac{\mathrm{d}}{\mathrm{dt}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b\frac{\mathrm{d}}{\mathrm{dt}} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \tag{128}$$

2. *There exist three functions α, λ and b, all defined on a neighborhood of $\mathcal{Y}$, such that b does not vanish on this neighborhood and, with*

$$\begin{pmatrix} \Omega_1 \\ \Omega_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -b\frac{\mathrm{d}}{\mathrm{dt}} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} , \tag{129}$$

   one has

$$\mathrm{d}\Omega_1 \equiv 0 \quad \textit{modulo } \{ \Omega_1 , \Omega_3 , \dot{\Omega}_3 \} \tag{130}$$

$$\mathrm{d}\Omega_3 \equiv 0 \quad \textit{modulo } \{ \Omega_3 , \Omega_1 \wedge \dot{\Omega}_3 \} \tag{131}$$

3. *$\delta^1_{2,1}$ does not vanish at $\mathcal{Y}$ and the first derived system of the pfaffian system $\{ \omega_1 - \frac{2\gamma}{\delta^1_{2,1}}\dot{\omega}_2 , \omega_2\}$ has rank 1 and is integrable, i.e. there exists a (unique) function α such that (59) is satisfied.*

4. *The function $\delta^1_{2,1}$ does not vanish at $\mathcal{Y}$ and, in the normal form (61), the functions f and g are, on a neighborhood of $\mathcal{Y}$, of the form (64) where a_0, a_1, a_2, b_0, b_1, c_0 and c_1 are some functions of z_1, z_2, z_3 only, which satisfy (65).*

If one of these conditions is met (and therefore all of them), λ, α and b in (128) and (129) are uniquely defined :

$$\lambda \;=\; 0\;, \quad b \;=\; \frac{2\gamma}{\delta^1_{2,1}}\;, \quad \alpha \textit{ is uniquely defined by (59)}\,. \tag{132}$$

We shall prove lemma 7.5 further. Since the matrix (with entries polynomials in $\frac{\mathrm{d}}{\mathrm{dt}}$) that is applied to $(\omega_1, \omega_2)^T$ in (128) is obviously invertible and of degree 2, it is clear from proposition 2.3 that point 1 implies (x, u)-dynamic linearizability, the converse is less clear because not all 2×2 invertible matrices are of the form displayed in (128), especially at points where some leading coefficients vanish —see proposition 2.4. The above lemma may be viewed as the nonsingular case, in a sense. However, it is possible to deduce from it the following result that implies both theorem 4.1 and proposition 4.4.

Lemma 7.6 *The system (5) is (x,u)-dynamic linearizable at a point $\overline{\mathcal{X}}$ if and only if one of the four (and hence all four) equivalent conditions of lemma 7.5 is satisfied in a neighborhood of point $\overline{\mathcal{X}}$. Then all the possible pairs of linearizing outputs depending on x and u are these described in theorem 4.1.*

It is clear that theorem 4.1 and proposition 4.4 are consequences of these two lemmae because the condition given by theorem 4.1 for (x, u)-linearizability is item 3 in lemma 7.5 and the condition in proposition 4.4 is item 4.

Let us now state, and prove, four more technical lemmae (7.7, 7.8, 7.9 and 7.10). We shall then proceed to prove lemma 7.6, and the end of the section will be devoted to the computational proof of lemma 7.5.

Lemma 7.7 *Let b and J_2 be respectively a scalar and a 2×2 invertible matrix both smooth functions defined on a neighborhood of a point $\mathcal{Y}$ and let Ω_1 and Ω_3 be defined by*

$$\begin{pmatrix} \Omega_1 \\ \Omega_3 \end{pmatrix} \;=\; \begin{pmatrix} 1 & 0 \\ -b\frac{\mathrm{d}}{\mathrm{dt}} & 1 \end{pmatrix} J_2 \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \quad . \tag{133}$$

The forms Ω_1 and Ω_3 satisfy the identities (130)-(131) on a neighborhood of $\mathcal{Y}$ if and only if there exist functions h_1, h_2 and a, and a 2×2 invertible matrix J_1, all *smooth functions defined on a neighborhood of $\mathcal{Y}$, such that*

$$\begin{aligned}\begin{pmatrix} \mathrm{d}h_1 \\ \mathrm{d}h_2 \end{pmatrix} &= J_1 \begin{pmatrix} 1 & -a\frac{\mathrm{d}}{\mathrm{dt}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b\frac{\mathrm{d}}{\mathrm{dt}} & 1 \end{pmatrix} J_2 \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \\ &= J_1 \begin{pmatrix} 1 & -a\frac{\mathrm{d}}{\mathrm{dt}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Omega_1 \\ \Omega_3 \end{pmatrix} . \end{aligned} \tag{134}$$

Proof of lemma 7.7 : Suppose that Ω_1 and Ω_3 satisfy the identities (130)-(131) on a neighborhood of $\mathcal{Y}$. Then the Pfaffian system $\{\Omega_1, \Omega_3, \dot{\Omega}_3\}$ is completely integrable because (130)-(131) obviously imply that $\mathrm{d}\Omega_1$ and $\mathrm{d}\Omega_3$ are zero modulo $\{\Omega_1, \Omega_3, \dot{\Omega}_3\}$, and (131) implies that, for a certain 1-form Γ_3 and a certain function k, $\mathrm{d}\Omega_3 = \Omega_3 \wedge \Gamma_3 + k\Omega_1 \wedge \dot{\Omega}_3$, but taking the time-derivative of both sides yields

$$\mathrm{d}\dot{\Omega}_3 = \dot{\Omega}_3 \wedge \Gamma_3 + \Omega_3 \wedge \dot{\Gamma}_3 + \dot{k}\Omega_1 \wedge \dot{\Omega}_3 + k\dot{\Omega}_1 \wedge \dot{\Omega}_3 + k\Omega_1 \wedge \ddot{\Omega}_3$$

which obviously implies that $\mathrm{d}\dot{\Omega}_3$ is zero modulo $\{\Omega_1, \Omega_3, \dot{\Omega}_3\}$. Integrability of this pfaffian system implies that there exists a function h_1 defined on a neighborhood of $\mathcal{Y}$ such that

$$\mathrm{d}h_1 = \lambda_1\Omega_1 + \lambda_2\Omega_3 + \lambda_3\dot{\Omega}_3$$

with λ_1, λ_2, λ_3 some functions, λ_1 nonzero at $\mathcal{Y}$. Then $\{\Omega_3, \mathrm{d}h_1\}$ is integrable because (131) implies that $\mathrm{d}\Omega_3$ is zero modulo $\{\Omega_3, \mathrm{d}h_1 \wedge \dot{\Omega}_3\}$, and hence modulo $\{\Omega_3, \mathrm{d}h_1\}$. Hence there is a second function h_2 such that

$$\mathrm{d}h_2 = \mu_1\mathrm{d}h_1 + \mu_2\Omega_3$$

with μ_1, μ_2 some functions, μ_2 nonzero at $\mathcal{Y}$. The functions h_1, h_2 built above, together with $a = -\lambda_3/\lambda_2$ and $J_1 = \begin{pmatrix} 1 & 0 \\ \mu_1 & \mu_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & \lambda_2 \\ 0 & 1 \end{pmatrix}$ satisfy (134).

Conversely, suppose that (134) holds. Let us define $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ by

$$\begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix} = J_2 \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} , \tag{135}$$

$$\begin{pmatrix} \Omega_4 \\ \Omega_3 \end{pmatrix} = \begin{pmatrix} 1 & -a\frac{\mathrm{d}}{\mathrm{dt}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b\frac{\mathrm{d}}{\mathrm{dt}} & 1 \end{pmatrix} \begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix} \tag{136}$$

i.e.

$$\Omega_3 \;=\; \Omega_2 \,-\, b\,\dot{\Omega}_1 \tag{137}$$

$$\Omega_4 \;=\; \Omega_1 \,-\, a\dot{\Omega}_3 \;=\; \Omega_1 \,-\, a\left(\dot{\Omega}_2 \,-\, \dot{b}\,\dot{\Omega}_1 \,-\, b\,\ddot{\Omega}_1\right) \tag{138}$$

We shall use the following basis (over smooth functions) for the space of all 1-forms :

$$\begin{array}{l} \{\Omega_1, \Omega_2, \Omega_3, \dot{\Omega}_2, \Omega_4, \dot{\Omega}_4, \\ \qquad \ddot{\Omega}_2, \Omega_2^{(3)}, \Omega_2^{(4)}, \ldots\ldots \quad , \\ \qquad \Omega_1^{(4)}, \Omega_1^{(5)}, \Omega_1^{(6)}, \ldots\ldots \quad \} \end{array} \tag{139}$$

where, in addition, $\{\Omega_1, \Omega_2, \Omega_3, \dot{\Omega}_2\}$ is a basis of Span$\{\mathrm{d}x\}$.

Then (128) implies that the pfaffian system (Ω_3, Ω_4) is completely integrable and therefore that there exists some 1-forms $\Gamma_{i,j}$ such that

$$\left.\begin{array}{rcl} \mathrm{d}\Omega_3 &=& \Omega_3 \wedge \Gamma_{3,3} \,+\, \Omega_4 \wedge \Gamma_{3,4} \\ \mathrm{d}\Omega_4 &=& \Omega_3 \wedge \Gamma_{4,3} \,+\, \Omega_4 \wedge \Gamma_{4,4} \end{array}\right\} \tag{140}$$

It is possible to express the 1-forms $\Gamma_{i,j}$ in (140) as (finite) linear combinations of the forms in (139), and it is always possible to chose them such that, for $i = 3, 4$,

$$\left.\begin{array}{l} \Gamma_{i,3} \text{ has no } \Omega_3 \text{ term,} \\ \Gamma_{i,4} \text{ has no } \Omega_3 \text{ term and no } \Omega_4 \text{ term.} \end{array}\right\} \tag{141}$$

Taking the exterior derivative of (138) yields

$$\mathrm{d}\Omega_1 \;=\; \mathrm{d}\Omega_4 \,+\, a\,\mathrm{d}\dot{\Omega}_3 \,-\, \dot{\Omega}_3 \wedge \mathrm{d}a\ ,$$

and taking the time-derivative of the first equation in (140) yields

$$\mathrm{d}\dot{\Omega}_3 \;=\; \Omega_3 \wedge \dot{\Gamma}_{3,3} \,+\, \dot{\Omega}_3 \wedge \Gamma_{3,3} \,+\, \Omega_4 \wedge \dot{\Gamma}_{3,4} \,+\, \dot{\Omega}_4 \wedge \Gamma_{3,4}$$

and finally, the two above equations yield, since $\dot{\Omega}_3 = \frac{\Omega_1 - \Omega_4}{a}$,

$$\begin{array}{rcl} \mathrm{d}\Omega_1 &=& \Omega_1 \wedge \left(\Gamma_{3,3} \,-\, \dfrac{\mathrm{d}a}{a}\right) \,+\, \Omega_3 \wedge \left(\Gamma_{4,3} \,+\, a\,\dot{\Gamma}_{3,3}\right) \\ && \quad + \,\Omega_4 \wedge \left(\Gamma_{4,4} \,-\, \Gamma_{3,3} \,+\, a\,\dot{\Gamma}_{3,4} \,+\, \dfrac{\mathrm{d}a}{a}\right) \,+\, a\,\dot{\Omega}_4 \wedge \Gamma_{3,4}\ . \end{array} \tag{142}$$

On the other hand, since $(\Omega_1,\Omega_2)=(X_1,X_2)^\perp$, the pfaffian system defined by (Ω_1,Ω_2) can be defined with the help of the variable x (i.e. the four coordinates of x) only, and therefore (see the end of the Appendix), its Cartan characteristic system is at most $\mathrm{Span}\{\mathrm{d}x\}$, i.e. at most $\{\Omega_1,\Omega_2,\Omega_3,\dot\Omega_2\}$, which implies that, for some functions k_1 and k_2,

$$\left.\begin{array}{rcl}\mathrm{d}\Omega_1 & \equiv & k_1\,\Omega_3\wedge\dot\Omega_2\\ \mathrm{d}\Omega_2 & \equiv & k_2\,\Omega_3\wedge\dot\Omega_2\end{array}\right\}\text{modulo }\{\Omega_1,\Omega_2\}\ . \tag{143}$$

The first equation above implies, from (142) and (141), and using the fact that the 1-forms in (139) are a basis for all 1-forms, that $\Gamma_{4,3}+a\dot\Gamma_{3,3}$ is a linear combination of Ω_1, Ω_2, Ω_3, $\dot\Omega_2$ and Ω_4, $\Gamma_{4,4}-\Gamma_{3,3}+a\dot\Gamma_{3,4}+\frac{\mathrm{d}a}{a}$ is a linear combination of Ω_1, Ω_2, Ω_3 and Ω_4, and $\Gamma_{3,4}$ is a linear combination of Ω_1, Ω_2 and $\dot\Omega_4$, with the coefficient of Ω_4 in $\Gamma_{4,3}+a\dot\Gamma_{3,3}$ equal to the coefficient of Ω_3 in $\Gamma_{4,4}-\Gamma_{3,3}+a\dot\Gamma_{3,4}+\frac{\mathrm{d}a}{a}$:

$$\Gamma_{4,3}+a\,\dot\Gamma_{3,3} = c_1\Omega_1+c_2\Omega_2+c_3\Omega_3+c_4\dot\Omega_2+d_3\Omega_4 \tag{144}$$

$$\Gamma_{4,4}-\Gamma_{3,3}+a\,\dot\Gamma_{3,4}+\frac{\mathrm{d}a}{a} = d_1\Omega_1+d_2\Omega_2+d_3\Omega_3+d_4\Omega_4 \tag{145}$$

$$\Gamma_{3,4} = e_1\Omega_1+e_2\Omega_2+e_3\dot\Omega_4 \tag{146}$$

and finally, (142) yields

$$\left.\begin{array}{l}\mathrm{d}\Omega_1 = \Omega_1\wedge\Delta_1+\Omega_2\wedge\Delta_2+c_4\,\Omega_3\wedge\dot\Omega_2\\ \text{with}\left\{\begin{array}{rcl}\Delta_1 & = & \Gamma_{3,3}-\frac{\mathrm{d}a}{a}-c_1\Omega_3-d_1\Omega_4-ae_1\dot\Omega_4\\ \Delta_2 & = & -c_2\Omega_3-d_2\Omega_4-ae_2\dot\Omega_4\end{array}\right.\end{array}\right\} \tag{147}$$

Now, from (137),

$$\mathrm{d}\Omega_2 = \mathrm{d}\Omega_3+b\,\mathrm{d}\dot\Omega_1+\mathrm{d}b\wedge\dot\Omega_1\ ,$$

which allows, getting $\mathrm{d}\Omega_3$ from (140) and $\mathrm{d}\dot\Omega_1$ from (147)'s time-derivative, and using the fact that $\dot\Omega_1=\frac{\Omega_2-\Omega_3}{b}$ and $\dot\Omega_3=\frac{\Omega_1-\Omega_4}{a}$, to compute $\mathrm{d}\Omega_2$ and, forgetting the exterior products starting with Ω_1, Ω_2 or Ω_3, to obtain

$$\mathrm{d}\Omega_2 \equiv b\,(\frac{c_4}{a}-d_2)\,\dot\Omega_2\wedge\Omega_4-a\,b\,e_2\dot\Omega_2\wedge\dot\Omega_4+e_3\,\Omega_4\wedge\dot\Omega_4\ \text{modulo }\{\Omega_1,\Omega_2,\Omega_3\} \tag{148}$$

which, since the second identity in (143) implies $\mathrm{d}\Omega_2\equiv 0$ modulo $\{\Omega_1,\Omega_2,\Omega_3\}$, yields

$$c_4 = a\,d_2 \quad\text{and}\quad e_2 = e_3 = 0\ . \tag{149}$$

We get (130) from (147) with $e_2=0$ after substituting Ω_4 for $\Omega_1-a\dot\Omega_3$. The same substitution in (138)-(146) with $e_2=e_3=0$ yields (131). ∎

Lemma 7.8 *Around a point where the rank assumptions (49)-(50)-(51)-(52)-(53) hold,*

1. *there cannot exist functions a, b, h_1, h_2 and two invertible 2×2 matrices of degree zero J_1 and J_2, all defined on a neighborhood of the considered point, such that*
$$J_1 \begin{pmatrix} 1 & 0 \\ -a\frac{\mathrm{d}}{\mathrm{dt}} - b\frac{\mathrm{d}}{\mathrm{dt}}^2 & 1 \end{pmatrix} J_2 \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \mathrm{d}h_1 \\ \mathrm{d}h_2 \end{pmatrix} \tag{150}$$
2. *there cannot exist functions α, a, b, h_1, h_2 and an invertible 2×2 matrix of degree zero J_1, all defined on a neighborhood of the considered point, such that*
$$J_1 \begin{pmatrix} 1 & -a\frac{\mathrm{d}}{\mathrm{dt}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b\frac{\mathrm{d}}{\mathrm{dt}} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \mathrm{d}h_1 \\ \mathrm{d}h_2 \end{pmatrix} \tag{151}$$

Proof of lemma 7.8 :
Point 1 : First, let us notice that b cannot be identically zero around the considered point, because this would imply x-dynamic linearizability, which, from theorem 3.1, contradicts (52). Define Ω_1 and Ω_2 by $(\Omega_1, \Omega_2)^T = J_2(\omega_1, \omega_2)^T$; then (150) implies that the pfaffian system $\{\Omega_1, \Omega_2 - a\dot{\Omega}_1 - b\ddot{\Omega}_1\}$ is completely integrable, which implies

$$\mathrm{d}\Omega_1 = \Omega_1 \wedge \Gamma_1 + (\Omega_2 - a\dot{\Omega}_1 - b\ddot{\Omega}_1) \wedge \Gamma_2,$$

for some 1-forms Γ_1 and Γ_2. On the other hand, because $\{\Omega_1, \Omega_2\}$ span the annihilator of $\{X_1, X_2\}$, the characteristic system of this pfaffian system is included in $\mathrm{Span}\{\mathrm{d}x\}$ (see the end of the Appendix), and hence one must have $\mathrm{d}\Omega_1 \equiv k\eta_1 \wedge \eta_2$ modulo $\{\Omega_1, \Omega_2\}$ with k a function and η_1 and η_2 two form in $\mathrm{Span}\{\mathrm{d}x\}$. This implies, since b does not vanish and $\ddot{\Omega}_1$ is not in $\mathrm{Span}\{\mathrm{d}x\}$, that, in the above relation, Γ_2 is a linear combination of Ω_1, Ω_2 and $a\dot{\Omega}_1 + b\ddot{\Omega}_1$, which in turn implies, for a certain function k,

$$\mathrm{d}\Omega_1 \equiv k\,\Omega_2 \wedge (a\dot{\Omega}_1 + b\ddot{\Omega}_1) \text{ modulo } \Omega_1 .$$

This implies, on one hand that Ω_1 is in the derived system of the pfaffian system $\{\Omega_1, \Omega_2\}$, and therefore, from (55)-(56), that Ω_1 is collinear to ω_1, but ten the above relation contradicts (55) because $a\dot{\Omega}_1 + b\ddot{\Omega}_1$ is not a linear combination of ω_1, ω_2 and $\dot{\omega}_2$.

Point 2 : Suppose that (150) holds. From, lemma 7.7, the identities (130)-(131) must hold locally with

$$\begin{aligned} \Omega_1 &= \omega_1 \\ \Omega_3 &= \omega_2 + \alpha\,\omega_1 - b\,\dot\omega_1 \end{aligned}$$

and in particular, this would imply that

$$\mathrm{d}\omega_1 \equiv 0 \ \text{ modulo } \{\omega_1, \omega_2 - b\dot\omega_1, \dot\omega_2 + (\alpha - \dot b)\dot\omega_1 - b\ddot\omega_1\}$$

which is impossible, because, from (55),
$$\mathrm{d}\omega_1 \wedge \omega_1 \wedge (\omega_2 - b\dot\omega_1) \wedge \Big(\dot\omega_2 + (\alpha - \dot b)\dot\omega_1 - b\ddot\omega_1\Big) = b^2\delta^2_{1,2}\,\omega_2 \wedge \dot\omega_2 \wedge \omega_1 \wedge \dot\omega_1 \wedge \ddot\omega_1 .$$ ∎

Lemma 7.9 *If, for some functions λ_1, λ_2 and λ_3, one has*

$$\begin{aligned} \mathrm{d}\omega_2 &\equiv 0 \quad \textit{modulo } \{\omega_2, \Omega, \dot\Omega\} \\ \textit{with } \Omega &= \lambda_1\omega_1 + \lambda_2\omega_2 + \lambda_3\dot\omega_2 , \end{aligned} \tag{152}$$

then λ_1 and λ_3 are related to the functions appearing in (56) by :

$$\lambda_3 \left(2\gamma\,\lambda_1 + \delta^1_{2,1}\,\lambda_3\right) = 0 . \tag{153}$$

Proof of lemma 7.9 : The expression for Ω in (152) implies

$$\begin{aligned} \lambda_1\omega_1 &= \Omega\lambda_2\omega_2 - \lambda_3\dot\omega_2 , \\ \lambda_1\dot\omega_1 &= -\dot\lambda_1\omega_1 + \dot\Omega - \dot\lambda_2\omega_2 - (\lambda_3 + \lambda_2)\dot\omega_2 - \lambda_3\ddot\omega_2 . \end{aligned} \tag{154}$$

Using the above relations in (56), one obtains that $\lambda_1^2\mathrm{d}\omega_2$ is equal to $\lambda_3\,(\delta^1_{2,1}\lambda_3 + 2\gamma\lambda_1)\,\dot\omega_2 \wedge \dot\omega_2$ modulo $\{\omega_2, \Omega, \dot\Omega\}$. This proves the lemma. ∎

Lemma 7.10 *Let f and g be some smooth functions from an open subset $O \subset I\!R^4$ to $I\!R$. The following two assertions are equivalent :*

- $\frac{\partial g}{\partial z_4}$ *does not vanish on O and f and g are solutions of the following equations on O :*

$$2\,\frac{\partial g}{\partial z_4}\,\frac{\partial^3 g}{\partial z_4^3} - 3\left(\frac{\partial^2 g}{\partial z_4^2}\right)^2 = 0 , \tag{155}$$

$$2\,\frac{\partial g}{\partial z_4}\,\frac{\partial^3 f}{\partial z_4^3} - 3\,\frac{\partial^2 g}{\partial z_4^2}\,\frac{\partial^2 f}{\partial z_4^2} = 0 . \tag{156}$$

- *There exists* a_0*,* a_1*,* a_2*,* b_0*,* b_1*,* c_0 *and* c_1*, some smooth functions of* z_1, z_2, z_3 *defined on* O *(i.e. on its projection on* $\mathbb{R}^3$*) such that*

$$c_0(z_1,z_2,z_3)+z_4c_1(z_1,z_2,z_3) \quad \text{and} \quad \begin{vmatrix} b_0 & b_1 \\ c_0 & c_1 \end{vmatrix}(z_1,z_2,z_3) \neq 0$$

do not vanish on O *and* f *and* g *are given by (64) on* O*.*

Proof of lemma 7.10 : By simple substitution, it is clear that the forms of f and g given in (64) satisfy equations (155)-(156), let us prove the converse. Since $\frac{\partial g}{\partial z_4} \neq 0$, one may define $h = 1/\frac{\partial g}{\partial z_4}$. Equation (155) then yields

$$\left(\frac{\partial h}{\partial z_4}\right)^2 - 2\,h\,\frac{\partial^2 h}{\partial z_4^2} = 0\,,$$

whose nonvanishing solutions are exactly the squares, and opposite of squares of nonzero polynomials in z_4 of degree at most 1, with coefficients function of z_1, z_2 and z_3; if the degree is 0, g is affine in z_4, if it is 1, g is homographic in z_4, still with coefficients function of z_1, z_2 and z_3, this yields the form for g given in (64). Substituting g for its expression given by (64) in equation (156) yields $(c_0+c_1z_4)\frac{\partial^3 f}{\partial z_4^3}+3c_1\frac{\partial^3 f}{\partial z_4^2} = 0$, which states that $(c_0+c_1z_4)f$ is a polynomial of degree at most 2 in z_4 and therefore implies that f is of the form given in (64). ∎

We now proceed with the
Proof of lemma 7.6 : It is obvious that point 1 of lemma 7.5 implies (x,u)-dynamic linearizability because (see proposition 2.3) the matrix applied to (ω_1,ω_2) in (128) is obviously invertible, and (h_1,h_2) is therefore a pair of linearizing outputs, which depend only on x and u because $\mathrm{d}h_1$ and $\mathrm{d}h_2$ are linear combinations of ω_1, ω_2, $\dot\omega_1$, $\dot\omega_2$, $\ddot\omega_1$, $\ddot\omega_2$, which are all linear combinations of $\mathrm{d}x$ and $\mathrm{d}u$.

Conversely, suppose that there exists a pair of linearizing outputs (h_1,h_2) depending only on x and u. From proposition 2.3, there exists $P(\frac{\mathrm{d}}{\mathrm{dt}}) \in \mathcal{A}(U)$, with U a neighborhood $\overline{\mathcal{X}}$, such that

$$\left.\begin{array}{l} P(\frac{\mathrm{d}}{\mathrm{dt}}) \text{ is invertible in } \mathcal{A}(U) \\ \deg P \ \leq\ 2 \quad \text{on } U \\ P(\frac{\mathrm{d}}{\mathrm{dt}})\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = \begin{pmatrix}\mathrm{d}h_1\\ \mathrm{d}h_2\end{pmatrix} \end{array}\right\} \tag{157}$$

Since γ does not vanish at $\overline{\mathcal{X}}$ and the rank assumptions (49)-(50)-(51)-(52)-(53) hold at $\overline{\mathcal{X}}$, we may suppose, by possibly restricting U, that

$$\left.\begin{array}{l} \gamma \text{ does not vanish on } U \quad , \\ \text{(49)-(50)-(51)-(52)-(53) hold on } U \quad . \\ \deg P \; = \; 2 \quad \text{on an open dense subset of } U \quad . \end{array}\right\} \tag{158}$$

The last statement is implied by the second one because if $\deg P$ is strictly less than 2 on an open set, then the system is x-dynamic linearizable and this contradicts (52) from theorem 3.1.

Then, from proposition 2.4, there is an open dense subset U_0 of U such that, for all $\mathcal{Y} \in U_0$, the matrix $P(\frac{\mathrm{d}}{\mathrm{dt}})$ may be decomposed according to one of the four forms (23)-(24)-(25). From lemma 7.8 three of these four forms are forbidden, because conditions (49)-(50)-(51)-(52)-(53) hold at point $\mathcal{Y}$. Hence, around each point $\mathcal{Y} \in U_0$, there exists functions α, λ, a, b, and a matrix J_1, defined on a neighborhood of $\mathcal{Y}$ such that (128) is true on a neighborhood of $\mathcal{Y}$. By restricting possibly the open sense set U_0, we may suppose that b does not vanish on U_0 (b cannot vanish on an open set, because then P would have degree at most 1 on this open set, and therefore the linearizing outputs would depend on x only, and this would, from theorem 3.1, contradict (52). Then the conditions of point 1 of lemma 7.5 are satisfied on U_0 and hence, from lemma 7.5, one may deduce that, for all $\mathcal{Y} \in U_0$, there is a neighborhood of this point such that

- $\delta^1_{2,1}$ does not vanish on this neighborhood,
- there is a unique function $\alpha_{\mathcal{Y}}$ defined on this neighborhood such that $\mathrm{d}\left(\omega_1 + \alpha_{\mathcal{Y}}\omega_2 - \frac{2\,\gamma}{\delta^1_{2,1}}\dot{\omega}_2\right) \;\wedge\; \left(\omega_1 + \alpha_{\mathcal{Y}}\omega_2 - \frac{2\,\gamma}{\delta^1_{2,1}}\dot{\omega}_2\right) \;=\; 0$
- there are a scalar function $a_{\mathcal{Y}}$ and an invertible matrix $J_{1,\mathcal{Y}}$, both smooth functions defined on this neighborhood, so that the following equation holds on this neighborhood :

$$\begin{pmatrix} \mathrm{d}h_1 \\ \mathrm{d}h_2 \end{pmatrix} = J_{1,\mathcal{Y}} \begin{pmatrix} 1 & -a_{\mathcal{Y}}\frac{\mathrm{d}}{\mathrm{dt}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{2\gamma}{\delta^1_{2,1}}\frac{\mathrm{d}}{\mathrm{dt}} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & \alpha_{\mathcal{Y}} \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \tag{159}$$

The last point is obtained by substituting the functions λ and b by the value they must have from (132).

The second point implies in particular (by making the wedge product of both sides by ω_2 and multiplying by $(\delta^1_{2,1})^2$ that

$$\mathrm{d}\left(\delta^1_{2,1}\omega_1 - 2\gamma\dot{\omega}_2\right)\wedge\left(\delta^1_{2,1}\omega_1 - 2\gamma\dot{\omega}_2\right)\wedge\omega_2 + \alpha_{\mathcal{Y}}\delta^1_{2,1}\,\mathrm{d}\omega_2\wedge\left(\delta^1_{2,1}\omega_1 - 2\gamma\dot{\omega}_2\right)\wedge\omega_2 = 0\,. \tag{160}$$

but on the other hand, the differential form of degree 4 $\mathrm{d}\omega_2 \wedge \left(\delta^1_{2,1}\omega_1 - 2\gamma\dot{\omega}_2\right) \wedge \omega_2$ is, from (56), given by

$$\mathrm{d}\omega_2 \wedge \omega_2 \wedge (\delta^1_{2,1}\omega_1 - 2\gamma\dot{\omega}_2) = \omega_1 \wedge (\frac{1}{2}\delta^1_{2,1}\dot{\omega}_1 - \gamma\ddot{\omega}_2) \wedge \omega_2 \wedge (-2\gamma\dot{\omega}_2)\,,$$

and therefore does not vanish on U. Existence of $\alpha_{\mathcal{Y}}$ satisfying (160) may be translated in some determinants made with the coefficients of the two differential forms of degree 4 being zero, but if these determinants are zero on an open dense subset U_0, they are zero all over U, and therefore, since $\mathrm{d}\omega_2 \wedge \left(\delta^1_{2,1}\omega_1 - 2\gamma\dot{\omega}_2\right) \wedge \omega_2$ does not vanish, there is a function ν, uniquely defined all over U, such that

$$\mathrm{d}\left(\delta^1_{2,1}\omega_1 - 2\gamma\dot{\omega}_2\right) \wedge \left(\delta^1_{2,1}\omega_1 - 2\gamma\dot{\omega}_2\right) \wedge \omega_2 + nu\,\mathrm{d}\omega_2 \wedge \left(\delta^1_{2,1}\omega_1 - 2\gamma\dot{\omega}_2\right) \wedge \omega_2 = 0\,. \tag{161}$$

Of course, since on the neighborhood of each point $\mathcal{Y}$, the function $\alpha_{\mathcal{Y}}$ is uniquely defined, it must coincide with $\frac{\nu}{\delta^1_{2,1}}$ where it is defined.

Then, let us define the form ω_3 by

$$\omega_3 = \delta^1_{2,1}\omega_1 + \nu\omega_2 - 2\gamma\dot{\omega}_2\,; \tag{162}$$

equation (159) reads

$$\begin{pmatrix} \mathrm{d}h_1 \\ \mathrm{d}h_2 \end{pmatrix} = J_{1,\mathcal{Y}} \begin{pmatrix} \omega_2 - a\left(\frac{1}{\delta^1_{2,1}}\dot{\omega}_3 - \frac{\dot{\delta}^1_{2,1}}{(\delta^1_{2,1})^2}\omega_3\right) \\ \frac{1}{\delta^1_{2,1}}\omega_3 \end{pmatrix}$$

and therefore, $\mathrm{d}h_1$ and $\mathrm{d}h_2$ are linear combinations of ω_2, ω_3 and $\dot{\omega}_3$ on a neighborhood of each point $\mathcal{Y} \in U_0$. This implies that the rank of $\{\mathrm{d}h_1, \mathrm{d}h_2, \omega_2, \omega_3, \dot{\omega}_3\}$ is at most 3 on the open dense U_0, it is therefore also at most 3 on all U. Since the rank of $\{\omega_2, \omega_3, \dot{\omega}_3\}$ is three all over U (because, thanks to (50)-(51) holding all over U, γ does not vanish on U), there are six functions $\mu_{i,j}$, (uniquely) defined all over U, such that

$$\mathrm{d}h_i = \mu_{i,1}\omega_2 + \mu_{i,2}\omega_3 + \mu_{i,3}\dot{\omega}_3$$

or in other words

$$\begin{pmatrix} \mathrm{d}h_1 \\ \mathrm{d}h_2 \end{pmatrix} = \begin{pmatrix} \mu_{1,1} & \mu_{1,2}+\mu_{1,3}\frac{\mathrm{d}}{\mathrm{dt}} \\ \mu_{2,1} & \mu_{2,2}+\mu_{2,3}\frac{\mathrm{d}}{\mathrm{dt}} \end{pmatrix} \begin{pmatrix} \omega_2 \\ \omega_3 \end{pmatrix} \tag{163}$$

which implies, from (162),

$$\begin{pmatrix} \mathrm{d}h_1 \\ \mathrm{d}h_2 \end{pmatrix} = \begin{pmatrix} \mu_{1,1} & \mu_{1,2}+\mu_{1,3}\frac{\mathrm{d}}{\mathrm{dt}} \\ \mu_{2,1} & \mu_{2,2}+\mu_{2,3}\frac{\mathrm{d}}{\mathrm{dt}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \delta^1_{2,1} & \nu - 2\gamma\frac{\mathrm{d}}{\mathrm{dt}} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

which implies, from (157), and because ω_1, ω_2 and all their time-derivatives are linearly independent, that

$$P(\frac{\mathrm{d}}{\mathrm{dt}}) = \begin{pmatrix} \mu_{1,1} & \mu_{1,2}+\mu_{1,3}\frac{\mathrm{d}}{\mathrm{dt}} \\ \mu_{2,1} & \mu_{2,2}+\mu_{2,3}\frac{\mathrm{d}}{\mathrm{dt}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \delta^1_{2,1} & \nu - 2\gamma\frac{\mathrm{d}}{\mathrm{dt}} \end{pmatrix}$$

This implies that $\delta^1_{2,1}$ must not vanish on U because $P(\frac{\mathrm{d}}{\mathrm{dt}})$ cannot be invertible in the neighborhood of the zeroes of $\delta^1_{2,1}$.

Since $\delta^1_{2,1}$ does not vanish on U, the function $\alpha = \frac{\nu}{\delta^1_{2,1}}$ is defined all over U, and since (59) is satisfied on U_0 which is dense in U, it is satisfied all over U. We have proved that (x,u)-linearizability implies item 3 of lemma 7.5.

We have proved above —see for example (163), where $\omega_3 = \delta^1_{2,1}\Omega_3$— that the arbitrary pair of linearizing output that we supposed to exists is of the form described in the second paragraph of theorem 4.1, and it is clear that a pair of function meeting these conditions is a pair of linearizing outputs. This ends the proof of lemma 7.6 ∎

Proof of lemma 7.5 :

1⇔2 : This is an obvious consequence of lemma 7.7.

3⇒2 : Let b be defined by (185) :

$$b = \frac{2\,\gamma}{\delta^1_{2,1}} ,$$

and α be the one from relation (59). Define Ω_1 and Ω_3 as in (128), with $\lambda = 0$. relation (59) implies $\mathrm{d}\Omega\bar{3} \equiv 0$ modulo Ω_3, so it implies a fortiori (131). Now (130) is equivalent here to

$$\mathrm{d}\omega_2 \equiv 0 \quad \text{modulo} \; \{\, \omega_2 \,,\, \omega_1 - b\dot{\omega}_2 \,,\, \dot{\omega}_1 + (\alpha - \dot{b})\dot{\omega}_2 - b\ddot{\omega}_2 \,\}$$

but a simple computation from (56) show that this is true when $b = \frac{2\gamma}{\delta^1_{2,1}}$.

<u>4⇒3</u> : From proposition 4.5, if point 4 is true, then some other coordinates may be found where the system has the simpler form (67). We shall compute in these coordinates with the following choice

$$\begin{array}{rcl} \omega_1 & = & \mathrm{d}\zeta_2 \;-\; \zeta_3 \mathrm{d}\zeta_1 \\ \omega_2 & = & \mathrm{d}\zeta_3 \;-\; (q_0 + \zeta_4 q_1)\,\mathrm{d}\zeta_1 \;-\; (p_1 + w_1 q_1)\,\omega_1 \end{array} \tag{164}$$

On one hand, one has

$$\mathrm{d}\omega_2 \;\equiv\; 0 \;\text{ modulo } \{\omega_2, \mathrm{d}\zeta_1, \mathrm{d}w_1\} \tag{165}$$

by computing the exterior derivative of ω_2 given by (164) and replacing $\mathrm{d}\zeta_1$ and $\mathrm{d}w_1$ with zero and $\mathrm{d}\zeta_3$ with $(p_1 + w_1 q_1)\,\mathrm{d}\zeta_2$.

On the other hand, from (53), $\{\omega_1, \omega_2, \dot\omega_2\}$ is a basis of $\{\mathrm{d}\zeta_1, \mathrm{d}\zeta_2, \mathrm{d}\zeta_3\}$ and hence one has

$$\mathrm{d}\zeta_1 \;=\; \lambda_1\omega_1 \;+\; \lambda_2\omega_2 \;+\; \lambda_3\dot\omega_2$$

for some functions λ_1, λ_2 and λ_3. Applying lemma 7.9 for $\Omega = \mathrm{d}\zeta_1$, and noticing that λ_3 cannot vanish because $\omega_1 \wedge \omega_2 \wedge \mathrm{d}\zeta_1$ does not vanish from (164) yields, from (165) :

$$2\gamma\,\lambda_1 \;+\; \delta^1_{2,1}\,\lambda_3 \;=\; 0\ .$$

The above two relation imply, since by assumption $\delta^1_{2,1}$ does not vanish, that $\mathrm{d}\zeta_1$ is a linear combination of ω_2 and $\omega_1 - \frac{2\gamma}{\delta^1_{2,1}}\dot\omega_2$, and this clearly implies point 3.

<u>2⇒4</u> : This is the long and difficult part of the proof. It is all done using the symbolic computation system Maple, with the package "liesymm" to manipulate differential forms, in the coordinates of the normal for (61).

We are now working in coordinates, with system (61) for some f and g. We make the following choice for ω_1 and ω_2 :

$$\begin{array}{rcl} \omega_1 & = & \mathrm{d}z_2 \;-\; z_3\,\mathrm{d}z_1 \\ \omega_2 & = & \mathrm{d}z_3 \;-\; g\,\mathrm{d}z_1 \;-\; (\frac{\partial f}{\partial z_4} + v_1\frac{\partial g}{\partial z_4})\,\omega_1 \quad . \end{array} \tag{166}$$

The idea of the proof is quite straightforward : We suppose that there exists some α, λ and b satisfying (130)-(131), we write these equations explicitly in terms of α, λ and b, and we eliminate α, λ and b to obtain the conditions on f and g are as described in point 4.

Step 1 *With the choice (166) for ω_1 and ω_2, we have the following decomposition of $\mathrm{d}\omega_1$ and $\mathrm{d}\omega_2$, more precise than (55) and (56) in proposition 4.1 :*

$$\mathrm{d}\omega_1 \;=\; \omega_1 \wedge \left(\delta^0_{1,1}\omega_2 + \delta^2_{1,1}\dot\omega_2\right) \;+\; \delta^2_{1,2}\,\omega_2 \wedge \dot\omega_2\;, \tag{167}$$

$$\begin{aligned}\mathrm{d}\omega_2 \;=\; \omega_1 \wedge \left(\delta^0_{2,1}\omega_2 + \delta^1_{2,1}\dot\omega_1 + \delta^2_{2,1}\dot\omega_2 - \gamma\ddot\omega_2\right) \;&+\; \omega_2 \wedge \left(\delta^1_{2,2}\dot\omega_1 + \delta^2_{2,2}\dot\omega_2\right)\\ &+\; \gamma\,\dot\omega_1 \wedge \dot\omega_2\end{aligned} \tag{168}$$

for some functions $\delta^k_{i,j}$ and γ that may be computed explicitly using f, g and some of their partial derivatives.

Indeed, (55) reads

$$\mathrm{d}\omega_1 \;=\; \omega_1 \wedge \Gamma_1 \;+\; \delta^2_{1,2}\omega_2 \wedge \dot\omega_2\;,$$

for some form Γ_1, but $\mathrm{d}\omega_1 = \mathrm{d}z_1 \wedge \mathrm{d}z_3$ and $\{\omega_1, \omega_2, \dot\omega_2\}$ is a basis of $\{\mathrm{d}z_1, \mathrm{d}z_2, \mathrm{d}z_3\}$ —because it is the characteristic system of $\{\omega_1\}$ from the above equation— so that Γ_1 must be a linear combination of ω_1, ω_2 and $\dot\omega_2$. This implies (167).

Also, (56) reads

$$\mathrm{d}\omega_2 \;=\; \omega_2 \wedge \Gamma_2 \;+\; \omega_1 \wedge \left(\delta^0_{2,1}\omega_2 + \delta^1_{2,1}\dot\omega_1 + \delta^2_{2,1}\dot\omega_2 - \gamma\ddot\omega_2\right) \;+\; \gamma\dot\omega_1 \wedge \dot\omega_2\;,$$

for a certain form Γ_2, but

$$\mathrm{d}\omega_2 \;=\; \mathrm{d}z_1 \wedge \mathrm{d}g \;-\; (\frac{\partial f}{\partial z_4} + v_1\frac{\partial g}{\partial z_4})\,\mathrm{d}z_1 \wedge \mathrm{d}z_3 \;-\; \mathrm{d}(\frac{\partial f}{\partial z_4} + v_1\frac{\partial g}{\partial z_4}) \wedge \omega_1$$

and hence $\mathrm{d}\omega_2$ is, modulo $\{\omega_1\}$, a linear combination of $\mathrm{d}z_1$, $\mathrm{d}z_2$, $\mathrm{d}z_3$ and $\mathrm{d}z_4$, i.e. of ω_1, ω_2, $\dot\omega_1$ and $\dot\omega_2$; this implies that Γ_2 must be a linear combination of ω_1, ω_2, $\dot\omega_1$ and $\dot\omega_2$, and therefore (168).

Step 2 *If α, λ and b satisfy (130)-(131), then*

$$\begin{array}{l}\lambda \textit{ may depend on } z_1,\, z_2,\, z_3,\, z_4,\, v_2 - v_1 f(z_1,z_2,z_3,z_4) \textit{ only,}\\ \alpha \textit{ and } b \textit{ may depend on } z_1,\, z_2,\, z_3,\, z_4,\, v_1,\, v_2,\, \dot v_1,\, \dot v_2 \textit{ only.}\end{array} \tag{169}$$

Relations (135) and (137) imply :

$$\Omega_1 \;=\; \omega_2 \;+\; \lambda\,\omega_1 \tag{170}$$

$$\begin{aligned}
\Omega_2 &= \omega_1 + \alpha\Omega_1 = (1+\alpha\lambda)\,\omega_1 + \alpha\,\omega_2 && (171)\\
\Omega_3 &= \omega_1 + \alpha\Omega_1 - b\dot{\Omega}_1 = \alpha\,(\omega_2+\lambda\omega_1) - b\left(\dot{\omega}_2 + \lambda\dot{\omega}_1 + (\dot{\lambda}-\frac{1}{b})\,\omega_1\right) && (172)\\
\dot{\Omega}_3 &= \left(\dot{\lambda}(\alpha-\dot{b}) - b\ddot{\lambda}\right)\omega_1 + \dot{\alpha}\,(\omega_2+\lambda\omega_1)\\
&\quad + (1-2b\dot{\lambda})\,\dot{\omega}_1 + (\alpha-\dot{b})\,(\dot{\omega}_2+\lambda\dot{\omega}_1) - b\,(\ddot{\omega}_2+\lambda\ddot{\omega}_1) && (173)
\end{aligned}$$

Taking the exterior derivative of (172) and (170) yields

$$\begin{aligned}
\mathrm{d}\Omega_3 &= \mathrm{d}\omega_1 + \alpha\mathrm{d}\Omega_1 - b\mathrm{d}\dot{\Omega}_1 + \mathrm{d}\alpha\wedge\Omega_1 - \mathrm{d}b\wedge\dot{\Omega}_1 && (174)\\
\text{with}\quad \mathrm{d}\Omega_1 &= \mathrm{d}\omega_2 + \lambda\,\mathrm{d}\omega_1 + \mathrm{d}\lambda\wedge\omega_1 && (175)\\
\mathrm{d}\dot{\Omega}_1 &= \mathrm{d}\dot{\omega}_2 + \dot{\lambda}\,\mathrm{d}\omega_1 + \lambda\,\mathrm{d}\dot{\omega}_1 + \mathrm{d}\dot{\lambda}\wedge\omega_1 + \mathrm{d}\lambda\wedge\dot{\omega}_1 && (176)
\end{aligned}$$

Relation (174) implies :

$$\begin{aligned}
\mathrm{d}\Omega_3 &= (1+\alpha\lambda-b\dot{\lambda})\mathrm{d}\omega_1 + \alpha\mathrm{d}\omega_2 + \alpha\mathrm{d}\lambda\wedge\omega_1 - b\,(\mathrm{d}\dot{\omega}_2 + \lambda\mathrm{d}\dot{\omega}_1 + \mathrm{d}\dot{\lambda}\wedge\omega_1 + \mathrm{d}\lambda\wedge\dot{\omega}_1)\\
&\quad + \mathrm{d}\alpha\wedge(\omega_2+\lambda\omega_1) + \mathrm{d}b\wedge\frac{\Omega_3-\omega_1-\alpha(\omega_2+\lambda\omega_1)}{b}
\end{aligned}$$

Taking the time-derivative of both sides in (167) and (168), we have

$$\left.\begin{aligned}
\mathrm{d}\dot{\omega}_1 &= \omega_1\wedge\left(\dot{\delta}^0_{1,1}\omega_2 + (\delta^0_{1,1}+\dot{\delta}^2_{1,1})\dot{\omega}_2 + \delta^2_{1,1}\ddot{\omega}_2\right)\\
&\quad + \omega_2\wedge\left(-\delta^0_{1,1}\dot{\omega}_1 + \dot{\delta}^2_{1,2}\dot{\omega}_2 + \delta^2_{1,2}\ddot{\omega}_2\right)\\
&\quad + \delta^2_{1,1}\,\dot{\omega}_1\wedge\dot{\omega}_2\\
\mathrm{d}\dot{\omega}_2 &= \omega_1\wedge\left(\dot{\delta}^0_{2,1}\omega_2 + \dot{\delta}^1_{2,1}\dot{\omega}_1 + (\delta^0_{2,1}+\dot{\delta}^2_{2,1})\dot{\omega}_2 + \delta^1_{2,1}\ddot{\omega}_1 + (\delta^2_{2,1}-\dot{\gamma})\ddot{\omega}_2 - \gamma\omega_2^{(3)}\right)\\
&\quad + \omega_2\wedge\left((\dot{\delta}^1_{2,2}-\delta^0_{2,1})\dot{\omega}_1 + \dot{\delta}^2_{2,2}\dot{\omega}_2 + \delta^1_{2,2}\ddot{\omega}_1 + \delta^2_{2,2}\ddot{\omega}_2\right)\\
&\quad + \dot{\omega}_2\wedge\left((-\delta^2_{2,1}+\delta^1_{2,2}-\dot{\gamma})\dot{\omega}_1 - \gamma\ddot{\omega}_1\right)
\end{aligned}\right\}\qquad(177)$$

Equation (131) implies in particular that $\mathrm{d}\Omega_3 \equiv 0$ modulo $\{\omega_1, \Omega_1, \Omega_3\}$, i.e. —see (172)— modulo $\{\omega_1, \omega_2, \dot{\omega}_2 + \lambda\dot{\omega}_1\}$. Equations (167), (168) and (177) imply

$$\left.\begin{aligned}
\mathrm{d}\omega_1 &\equiv 0 & \mathrm{d}\dot{\omega}_1 &\equiv 0\\
\mathrm{d}\omega_2 &\equiv 0 & \mathrm{d}\dot{\omega}_2 &\equiv \lambda\gamma\,\dot{\omega}_1\wedge\ddot{\omega}_1
\end{aligned}\right\}\ \text{modulo}\ \{\omega_1, \omega_2, \dot{\omega}_2+\lambda\dot{\omega}_1\}\,.$$

Then, from (174), (175), (176),

$$\mathrm{d}\Omega_3 \equiv -b\,(\lambda\gamma\,\dot{\omega}_1\wedge\ddot{\omega}_1 + \mathrm{d}\lambda\wedge\dot{\omega}_1)\ \text{modulo}\ \{\omega_1, \omega_2, \dot{\omega}_2+\lambda\dot{\omega}_1\}\,,$$

which in turn implies

$$\mathrm{d}\lambda + \lambda\gamma\ddot{\omega}_1 \equiv 0 \text{ modulo } \{\omega_1, \omega_2, \dot{\omega}_1, \dot{\omega}_2\}.$$

Since $\{\mathrm{d}z_1, \mathrm{d}z_2, \mathrm{d}z_3, \mathrm{d}z_4\}$ is another basis for $\{\omega_1, \omega_2, \dot{\omega}_1, \dot{\omega}_2\}$ and, from (166) and (61),

$$\begin{array}{rcl} \dot{\omega}_1 & = & \mathrm{d}z_4 + v_1\mathrm{d}z_3 - (f + v_1 g)\,\mathrm{d}z_1 , \\ \ddot{\omega}_1 & = & \mathrm{d}\,(v_2 - f v_1) + v_1\mathrm{d}f + \dot{v}_1\mathrm{d}z_3 + \left(\dot{f} + v_1\dot{g} + \dot{v}_1 g\right)\mathrm{d}z_1 , \end{array} \tag{178}$$

that $\mathrm{d}\lambda$ is a linear combination of $\mathrm{d}\,(v_2 - f v_1)$, $\mathrm{d}z_1$, $\mathrm{d}z_2$, $\mathrm{d}z_3$ and $\mathrm{d}z_4$; this proves the statement on λ in (169).

Replacing $\dot{\omega}_1$ and $\ddot{\omega}_1$ with zero and $\dot{\omega}_2$ with $(\frac{1+\alpha\lambda}{b} - \dot{\lambda})\omega_1 + \frac{\alpha}{b}\omega_2$ in the expression of $\Omega_1 \wedge \dot{\Omega}_3$ obtained from (170) and (173) obviously yields only some terms in $\omega_1 \wedge \omega_2$, $\omega_1 \wedge \ddot{\omega}_2$ and $\omega_2 \wedge \ddot{\omega}_2$, hence

$$\Omega_1 \wedge \dot{\Omega}_3 \equiv 0 \text{ modulo } \{\Omega_3, \dot{\omega}_1, \ddot{\omega}_1, \omega_1 \wedge \omega_2, \omega_1 \wedge \ddot{\omega}_2, \omega_2 \wedge \ddot{\omega}_2\} .$$

Therefore, Equation (131) implies in particular that $\mathrm{d}\Omega_3 \equiv 0$ modulo $\{\Omega_3, \dot{\omega}_1, \ddot{\omega}_1, \omega_1 \wedge \omega_2, \omega_1 \wedge \ddot{\omega}_2, \omega_2 \wedge \ddot{\omega}_2\}$, i.e. modulo $\{b\dot{\omega}_2 - \alpha(\omega_2 + \lambda\omega_1) + (b\dot{\lambda} - 1)\omega_1, \dot{\omega}_1, \ddot{\omega}_1, \omega_1 \wedge \omega_2, \omega_1 \wedge \ddot{\omega}_2, \omega_2 \wedge \ddot{\omega}_2\}$. From (167), (168) and (177), we have :

$$\begin{array}{rclcrcl} \mathrm{d}\omega_1 & \equiv & 0 & \qquad & \mathrm{d}\dot{\omega}_1 & \equiv & 0 \\ \mathrm{d}\omega_2 & \equiv & 0 & \qquad & \mathrm{d}\dot{\omega}_2 & \equiv & -\gamma\,\omega_1 \wedge \omega_2^{(3)} \end{array}$$

modulo $\{b\dot{\omega}_2 - \alpha(\omega_2 + \lambda\omega_1) + (b\dot{\lambda} - 1)\omega_1, \dot{\omega}_1, \ddot{\omega}_1, \omega_1 \wedge \omega_2, \omega_1 \wedge \ddot{\omega}_2, \omega_2 \wedge \ddot{\omega}_2\}$. Hence, from (174), (175), (176)

$$\begin{array}{rcl} \mathrm{d}\Omega_3 & \equiv & \omega_1 \wedge \left(b\gamma\,\omega_2^{(3)} - \alpha\mathrm{d}\lambda + b\,d\mathrm{d}\dot{\lambda} + \dfrac{\mathrm{d}b}{b}\right) \\ & & + (\omega_2 + \lambda\omega_1) \wedge \left(\dfrac{\alpha}{b}\mathrm{d}b - \mathrm{d}\alpha\right) \end{array}$$

This implies in particular that

$$\left.\begin{array}{l} b\gamma\,\omega_2^{(3)} - \alpha\mathrm{d}\lambda + b\,d\mathrm{d}\dot{\lambda} + \dfrac{\mathrm{d}b}{b} \equiv 0 \\ \dfrac{\alpha}{b}\mathrm{d}b - \mathrm{d}\alpha \equiv 0 \end{array}\right\} \text{ modulo } \{\omega_1, \omega_2, \dot{\omega}_1, \dot{\omega}_2, \ddot{\omega}_1, \ddot{\omega}_2\}. \tag{179}$$

We have already shown above that $\mathrm{d}\lambda$ is a linear combination of $\omega_1, \omega_2, \dot{\omega}_1, \dot{\omega}_2, \ddot{\omega}_1$; hence $\mathrm{d}\dot{\lambda}$ is a linear combination of $\omega_1, \omega_2, \dot{\omega}_1, \dot{\omega}_2, \ddot{\omega}_1, \ddot{\omega}_2, \omega_1^{(3)}$. This and the above equations imply that $\mathrm{d}b$ and $\mathrm{d}\alpha$ are linear combinations of ω_1, ω_2, $\dot\omega_1$, $\dot\omega_2$, $\ddot\omega_1$, $\ddot\omega_2$, $\omega_1^{(3)}$, $\omega_2^{(3)}$. This yields the second statement in (169) for $\{\omega_1,\omega_2,\dot\omega_1,\dot\omega_2,\ddot\omega_1,\ddot\omega_2,\omega_1^{(3)},\omega_2^{(3)}\}$ is another basis for $\{\mathrm{d}z_1,\mathrm{d}z_2,\mathrm{d}z_3,\mathrm{d}z_4,\mathrm{d}v_1,\mathrm{d}v_2,\mathrm{d}\dot v_1,\mathrm{d}\dot v_2\}$.

Note that the second relation in (179) actually implies that $\alpha\mathrm{d}b - b\mathrm{d}\alpha$ is a linear combination of $\omega_1,\omega_2,\dot\omega_1,\dot\omega_2,\ddot\omega_1,\ddot\omega_2$, i.e. that α/b depends on z_1,z_2,z_3,z_4,v_1,v_2 only (and not on $\dot v_1,\dot v_2$).

Step 3 *If α, λ and b satisfy (130)-(131) with b non vanishing, then λ must be identically zero.*

The core of this point is a rather heavy computation conducted in Maple. Let us explain some notations.

First of all, we need to work with a finite number of variable only. The only variables that will ever be needed during the computations are

$$(z_1\,,\ z_2\,,\ z_3\,,\ z_4\,,\ v_1\,,\ v_2\,,\ \dot v_1\,,\ \dot v_2\,,\ \ddot v_1\,,\ \ddot v_2\,)$$

because, the only operation that makes some new variables appear is taking the "time-derivative" of some objects. This occurs only when computing Ω_1, Ω_3 and $\dot\Omega_3$ according to (170), (172) and (173), but from (166), it is clear that the forms Ω_1, Ω_3 and $\dot\Omega_3$ may be expressed with the help of the above variables. This may be checked in the course of the computation : we set the time-derivative of $\ddot v_1$ and $\ddot v_2$ to "$ERROR_1$" and "$ERROR_2$" and we may check that we never have to apply the time-differentiation ("DOT") to $\ddot v_1$ and $\ddot v_2$ by checking that the variables "$ERROR_1$" and "$ERROR_2$" never appear in the expressions we compute.

To take advantage of the fact that λ depends only on z_1, z_2, z_3, z_4 and $v_2 - f v_1$, we make a change of coordinates, defining w_2 by

$$w_2 \quad = \quad v_2 \; - \; f(z_1, z_2.z_3, z_4)\, v_2\ . \tag{180}$$

We work then in the coordinates

$$(z_1\,,\ z_2\,,\ z_3\,,\ z_4\,,\ v_1\,,\ w_2\,,\ \dot v_1\,,\ \dot v_2\,,\ \ddot v_1\,,\ \ddot v_2\,) \tag{181}$$

rather than the above, and the system is given by the derivative along its dynamics, i.e. along :

$$
\begin{array}{rcl}
\frac{\mathrm{d}}{\mathrm{dt}} z_1 & = & v_1 \\
\frac{\mathrm{d}}{\mathrm{dt}} z_2 & = & z_4 \;+\; z_3\, v_1 \\
\frac{\mathrm{d}}{\mathrm{dt}} z_3 & = & f(z_1,z_2,z_3,z_4) \;+\; g(z_1,z_2,z_3,z_4)\, v_1 \\
\frac{\mathrm{d}}{\mathrm{dt}} z_4 & = & w_2 \;+\; f(z_1,z_2,z_3,z_4)\, v_1 \\
\frac{\mathrm{d}}{\mathrm{dt}} v_1 & = & \dot v_1 \\
\frac{\mathrm{d}}{\mathrm{dt}} w_2 & = & \dot v_2 \;-\; f(z_1,z_2,z_3,z_4)\, \dot v_1 \\
 & & \;-\; u_1 \left(\frac{\partial f}{\partial z_1} v_1 \;+\; \frac{\partial f}{\partial z_2}(z_4 + z_3 v_1) \;+\; \frac{\partial f}{\partial z_3}(f + v_1 g) \;+\; \frac{\partial f}{\partial z_4}(w_2 + v_1 f) \right) \\
\frac{\mathrm{d}}{\mathrm{dt}} \dot v_1 & = & \ddot v_1 \\
\frac{\mathrm{d}}{\mathrm{dt}} \dot v_2 & = & \ddot v_2 \\
\frac{\mathrm{d}}{\mathrm{dt}} \ddot v_1 & = & ERROR_1 \\
\frac{\mathrm{d}}{\mathrm{dt}} \ddot v_2 & = & ERROR_2
\end{array}
\tag{182}
$$

Since b does not vanish, we may define some new functions β and ρ from b and α as follows :

$$
b \;=\; \frac{1}{\beta(z_1,z_2,z_3,z_4,v_1,w_2,\dot v_1,\dot v_2)} \tag{183}
$$

$$
\alpha \;=\; \frac{\rho(z_1,z_2,z_3,z_4,v_1,w_2,\dot v_1,\dot v_2)}{\beta(z_1,z_2,z_3,z_4,v_1,w_2,\dot v_1,\dot v_2)} \tag{184}
$$

Note that by assumption b —and therefore β— does not vanish.

If λ is not locally identically zero, then there are points arbitrarily close to the point under consideration where it does not vanish, and hence there are points where neither λ nor b —and hence β— vanish *and* the relations (130)-(131) hold.

In the following Maple session, we suppose we are at such a point, we write the equations for (130)-(131) in terms of the functions λ, β and ρ, supposing that we may divide by λ and by ρ and get a contradiction (namely that λ must be zero).

Note : The symbol

&^

in the Maple session stands for the exterior product (or wedge product).

This reproduces a session run with Maple V Release 3

```
> with(liesymm):  setup(z1,z2,z3,z4,v1,w2,v1dot,v2dot,v1dotdot,v2dotdot):
> FF:=[(z1,z2,z3,z4,v1,w2,v1dot,v2dot,v1dotdot,v2dotdot)
>        -> v1,
>    (z1,z2,z3,z4,v1,w2,v1dot,v2dot,v1dotdot,v2dotdot)
>        -> z4 + z3*v1 ,
>    (z1,z2,z3,z4,v1,w2,v1dot,v2dot,v1dotdot,v2dotdot)
>        -> f(z1,z2,z3,z4) + v1*g(z1,z2,z3,z4) ,
>    (z1,z2,z3,z4,v1,w2,v1dot,v2dot,v1dotdot,v2dotdot)
>        -> w2 + v1 * f(z1,z2,z3,z4) ,
>    (z1,z2,z3,z4,v1,w2,v1dot,v2dot,v1dotdot,v2dotdot)
>        -> v1dot ,
>    (z1,z2,z3,z4,v1,w2,v1dot,v2dot,v1dotdot,v2dotdot)
>        -> v2dot
>       - v1dot * f(z1,z2,z3,z4)
>       - v1 * diff(f(z1,z2,z3,z4),z1)* v1
>       - v1 * diff(f(z1,z2,z3,z4),z2)* ( z4 + z3*v1 )
>     - v1 * diff(f(z1,z2,z3,z4),z3)* ( f(z1,z2,z3,z4) + v1*g(z1,z2,z3,z4) )
>       - v1 * diff(f(z1,z2,z3,z4),z4)* ( w2 + v1 * f(z1,z2,z3,z4) ) ,
>    (z1,z2,z3,z4,v1,w2,v1dot,v2dot,v1dotdot,v2dotdot)
>        -> v1dotdot ,
>    (z1,z2,z3,z4,v1,w2,v1dot,v2dot,v1dotdot,v2dotdot)
>        -> v2dotdot ,
>    (z1,z2,z3,z4,v1,w2,v1dot,v2dot,v1dotdot,v2dotdot)
>        -> ERROR1 ,
>    (z1,z2,z3,z4,v1,w2,v1dot,v2dot,v1dotdot,v2dotdot)
>        -> ERROR2 ] :
> DOT := proc(forme) value(Lie(forme,FF)) end :
```

Here, we have loaded the package "liesymm" that we will use to manipulate differential forms, declared that the coordinates are $(z_1, z_2, z_3, z_4, v_1, w_2, \dot{v}_1, \dot{v}_2, \ddot{v}_1, \ddot{v}_2)$, defined the system, and finally the procedure "DOT", which is the time-derivative along the dynamics of the system, i.e. the Lie derivative along the vector field F from equation (9), this vector field is truncated here, but as explained above it will not be applied to objects that involve other variables than x_1, x_2, x_3, x_4, u_1, u_2, $\dot{u}_1$ and $\dot{u}_2$. This Lie derivative on functions as well as on forms of any degree.

Let us just check that DOT is really what we think, by applying it to the base variable functions :

```
> DOT(z1);DOT(z2);DOT(z3);DOT(z4);DOT(v1);normal(DOT(w2+ v1 * f(z1,z2,z3,z4)));
> DOT(v1dot);DOT(v2dot);DOT(v1dotdot);DOT(v2dotdot);
```

$$v1$$

$$z4 + z3\ v1$$

$$\mathrm{f}(\,z1\,,z2\,,z3\,,z4\,) + v1\ \mathrm{g}(\,z1\,,z2\,,z3\,,z4\,)$$

$$w2 + v1\ \mathrm{f}(\,z1\,,z2\,,z3\,,z4\,)$$

$$v1dot$$

$$v2dot$$

$$v1dotdot$$

$$v2dotdot$$

$$ERROR1$$

$$ERROR2$$

Note that the "ERROR" signs will never appear in the calculations since we will compute the time-derivative only of functions which do not depend on $\ddot{v}_1$ and $\ddot{v}_2$, see the explanations before (182).

We now define the forms ω_1 and ω_2, and compute $\dot{\omega}_1$, $\dot{\omega}_2$ and $\ddot{\omega}_2$ (denoted "omega1d", "omega2d" and "omega2dd") :

```
> omega1 := d(z2) - z3 * d(z1) ;
```

$$\omega 1 := \mathrm{d}(\,z2\,) - z3\ \mathrm{d}(\,z1\,)$$

```
> omega2 := wcollect(value(
>  d(z3) - g(z1,z2,z3,z4) *d(z1)
>  - (diff(f(z1,z2,z3,z4),z4) + v1 * diff(g(z1,z2,z3,z4),z4))  * omega1
> )) ;
```

$$\begin{aligned}\omega 2 := \Bigg(&- \mathrm{g}(\,z1\,,z2\,,z3\,,z4\,)\\ &+ \left(\left(\frac{\partial}{\partial z4}\,\mathrm{f}(\,z1\,,z2\,,z3\,,z4\,)\right) + v1\ \left(\frac{\partial}{\partial z4}\,\mathrm{g}(\,z1\,,z2\,,z3\,,z4\,)\right)\right)\ z3\Bigg)\ \mathrm{d}(\,z1\,)\\ &+ \left(-\left(\frac{\partial}{\partial z4}\,\mathrm{f}(\,z1\,,z2\,,z3\,,z4\,)\right) - v1\ \left(\frac{\partial}{\partial z4}\,\mathrm{g}(\,z1\,,z2\,,z3\,,z4\,)\right)\right)\ \mathrm{d}(\,z2\,)\\ &+ \mathrm{d}(\,z3\,)\end{aligned}$$

```
> omega1d := DOT(omega1):  omega2d := DOT(omega2):  omega2dd :=
```

```
> DOT(omega2d):
```

From proposition 2.2, the rank condition (53) implies that the form of degree 3 $\omega_1 \wedge \omega_2 \wedge \dot{\omega}_2$ does not vanish. We compute it, see that it is of the form $D_1 \mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_3$ where the quantity D_1 is the one from (63). We shall use a lot the fact that this D_1 does not vanish.

```
> D1 := getcoeff(factor( omega1&^omega2&^omega2d ));
```

$$
\begin{aligned}
D1 := & \left(\frac{\partial}{\partial z1} \mathrm{f}(z1, z2, z3, z4)\right) + z3 \left(\frac{\partial}{\partial z2} \mathrm{f}(z1, z2, z3, z4)\right) \\
& + \mathrm{g}(z1, z2, z3, z4) \left(\frac{\partial}{\partial z3} \mathrm{f}(z1, z2, z3, z4)\right) \\
& - \left(\frac{\partial}{\partial z4} \mathrm{g}(z1, z2, z3, z4)\right) w2 \\
& - \left(\frac{\partial}{\partial z3} \mathrm{g}(z1, z2, z3, z4)\right) \mathrm{f}(z1, z2, z3, z4) \\
& + \left(\frac{\partial}{\partial z4} \mathrm{f}(z1, z2, z3, z4)\right) \mathrm{f}(z1, z2, z3, z4) \\
& - \left(\frac{\partial}{\partial z2} \mathrm{g}(z1, z2, z3, z4)\right) z4
\end{aligned}
$$

This implements (183)-(184) and computes Ω_1, Ω_3 and $\dot{\Omega}_3$ according to (129), Ω_2 being the intermediary form defined in (135)-(137).

Note that the form Ω_3 that we use is not exactly the one in (129), it is divided by b ; this does not affect the relations (130)(131).

```
> b := 1/ beta(z1,z2,z3,z4,v1,w2,v1dot,v2dot);
> alpha := b * rho(z1,z2,z3,z4,v1,w2,v1dot,v2dot);
```

$$b := \frac{1}{\beta(z1, z2, z3, z4, v1, w2, v1dot, v2dot)}$$

$$\alpha := \frac{\rho(z1, z2, z3, z4, v1, w2, v1dot, v2dot)}{\beta(z1, z2, z3, z4, v1, w2, v1dot, v2dot)}$$

```
> Omega1 := map(normal, wcollect(
>      lambda(z1,z2,z3,z4,w2) * omega1 +  omega2
>      )):
> Omega2 := map(normal, wcollect( omega1 + alpha * Omega1)):
> Omega1d := map(normal, wcollect(  DOT(Omega1) )):
```

```
> Omega3 := map(factor,wcollect( (1/b)*Omega2 - Omega1d)):
> Omega3d := map(factor , wcollect(  DOT(Omega3) )):
```

We shall first compute $d\Omega_1$ modulo $\{\Omega_1, \Omega_3, \dot{\Omega}_3\}$. In order to compute modulo $\{\Omega_1, \Omega_3, \dot{\Omega}_3\}$, we simply substitute dz_3, dz_4 and dw_2 with the linear combinations —respectively called valdz3, valdz4 and valdw2 below— of dz_1, dz_2, dv_1 which is equal to each of them modulo $\{\Omega_1, \Omega_3, \dot{\Omega}_3\}$.

```
> map(getform,[op(Omega1)]);
> coeff( Omega1 , d(z3));
```

$$[\mathrm{d}(\,z1\,), \mathrm{d}(\,z2\,), \mathrm{d}(\,z3\,)]$$

$$1$$

```
> valdz3 := map(factor, wcollect(solve( Omega1=0, d(z3) ) )):
> map(getform,[op(valdz3)]);
```

$$[\mathrm{d}(\,z1\,), \mathrm{d}(\,z2\,)]$$

```
> map(getform,[op(Omega3)]);
> coeff( Omega3 , d(z4));
```

$$[\mathrm{d}(\,z1\,), \mathrm{d}(\,z2\,), \mathrm{d}(\,z3\,), \mathrm{d}(\,z4\,)]$$

$$-\lambda(\,z1\,, z2\,, z3\,, z4\,, w2\,)$$

```
> valdz4 := map(factor,wcollect(simplify(
>   subs( d(z3)=valdz3 ,
>   solve( Omega3=0, d(z4) )
> )))):
> map(getform,[op(valdz4)]);
```

$$[\mathrm{d}(\,z1\,), \mathrm{d}(\,z2\,)]$$

```
> map(getform,[op(Omega3d)]);
> coeff( Omega3d , d(w2));
```

$$[\mathrm{d}(\,z1\,), \mathrm{d}(\,z2\,), \mathrm{d}(\,z3\,), \mathrm{d}(\,z4\,), \mathrm{d}(\,v1\,), \mathrm{d}(\,w2\,)]$$

$$-\lambda(\,z1\,, z2\,, z3\,, z4\,, w2\,)$$

```
> valdw2 :=map(factor, wcollect(simplify(
>   subs( {d(z3)=valdz3,d(z4)=valdz4} ,
>   solve( Omega3d=0, d(w2) )
> )))):
> map(getform,[op(valdw2)]);
```

$$[\mathrm{d}(\,z1\,), \mathrm{d}(\,z2\,), \mathrm{d}(\,v1\,)]$$

```
> dOmega1 := map(factor , wcollect(value(  d(Omega1)  )) ):
> map(getform, [op(dOmega1)]);
```

$$[\mathrm{d}(\,z1\,)\,\&\hat{}\;\mathrm{d}(\,z4\,), \mathrm{d}(\,z1\,)\,\&\hat{}\;\mathrm{d}(\,w2\,), \mathrm{d}(\,z2\,)\,\&\hat{}\;\mathrm{d}(\,w2\,), \mathrm{d}(\,z1\,)\,\&\hat{}\;\mathrm{d}(\,v1\,),$$
$$\mathrm{d}(\,z2\,)\,\&\hat{}\;\mathrm{d}(\,z4\,), \mathrm{d}(\,z2\,)\,\&\hat{}\;\mathrm{d}(\,v1\,), \mathrm{d}(\,z1\,)\,\&\hat{}\;\mathrm{d}(\,z2\,),$$
$$\mathrm{d}(\,z1\,)\,\&\hat{}\;\mathrm{d}(\,z3\,), \mathrm{d}(\,z2\,)\,\&\hat{}\;\mathrm{d}(\,z3\,)]$$

```
> dOmega1mod := map( factor , wcollect ( simplify (
>   subs( {d(z3)=valdz3,d(z4)=valdz4,d(w2)=valdw2} ,
> dOmega1 ) ))):
> map(getform, [op(dOmega1mod)]);
```

$$[\,\mathrm{d}(\,z1\,)\,\&\hat{}\;\mathrm{d}(\,v1\,), \mathrm{d}(\,z2\,)\,\&\hat{}\;\mathrm{d}(\,v1\,), \mathrm{d}(\,z1\,)\,\&\hat{}\;\mathrm{d}(\,z2\,)\,]$$

In other terms, $\mathrm{d}\Omega_1 \equiv C_1 \mathrm{d}z_1 \wedge \mathrm{d}z_2 + C_2 \mathrm{d}z_2 \wedge \mathrm{d}v_1 + C_3 \mathrm{d}z_1 \wedge \mathrm{d}v_1$ modulo $\{\Omega_1, \Omega_3, \dot{\Omega}_3\}$. Hence the functions C_1, C_2 and C_3 must be identically zero

---

Let us first examine the coefficient of $\mathrm{d}z_2 \wedge \mathrm{d}v_1$. It turns out that $C_2 = 0$ allows one to express $\frac{\partial\lambda}{\partial w_2}$ as a function of f, g, λ. It is the expression "LALA" below.

```
> collect( coeff(dOmega1mod , &^(d(z2),d(v1))) ,
>        diff(lambda(z1,z2,z3,z4,w2),w2) );
```

$$\begin{aligned}
&-\Bigg(-\left(\frac{\partial}{\partial z4}\,\mathrm{f}(\,z1\,,z2\,,z3\,,z4\,)\right)\mathrm{f}(\,z1\,,z2\,,z3\,,z4\,) - z3\,\left(\frac{\partial}{\partial z2}\,\mathrm{f}(\,z1\,,z2\,,z3\,,z4\,)\right)\\
&\quad - \left(\frac{\partial}{\partial z1}\,\mathrm{f}(\,z1\,,z2\,,z3\,,z4\,)\right) + \left(\frac{\partial}{\partial z4}\,\mathrm{g}(\,z1\,,z2\,,z3\,,z4\,)\right) w2\\
&\quad + \left(\frac{\partial}{\partial z3}\,\mathrm{g}(\,z1\,,z2\,,z3\,,z4\,)\right)\mathrm{f}(\,z1\,,z2\,,z3\,,z4\,)\\
&\quad + \left(\frac{\partial}{\partial z2}\,\mathrm{g}(\,z1\,,z2\,,z3\,,z4\,)\right) z4\\
&\quad - \mathrm{g}(\,z1\,,z2\,,z3\,,z4\,)\,\left(\frac{\partial}{\partial z3}\,\mathrm{f}(\,z1\,,z2\,,z3\,,z4\,)\right)\Bigg)\\
&\quad \left(\frac{\partial}{\partial w2}\,\lambda(\,z1\,,z2\,,z3\,,z4\,,w2\,)\right) \Big/ (\,\lambda(\,z1\,,z2\,,z3\,,z4\,,w2\,)\,)\\
&\quad + \left(\frac{\partial}{\partial z4}\,\mathrm{g}(\,z1\,,z2\,,z3\,,z4\,)\right)
\end{aligned}$$

```
> LALA := solve( numer(coeff(dOmega1mod , &^(d(z2),d(v1)))) = 0 ,
>                               diff(lambda(z1,z2,z3,z4,w2),w2) );
```

$$
\begin{aligned}
LALA := & -\left(\frac{\partial}{\partial z4}\,\mathrm{g}(\,z1\,, z2\,, z3\,, z4\,)\right)\,\lambda(\,z1\,, z2\,, z3\,, z4\,, w2\,)\Bigg/\Bigg(\\
& \left(\frac{\partial}{\partial z1}\,\mathrm{f}(\,z1\,, z2\,, z3\,, z4\,)\right) + z3\,\left(\frac{\partial}{\partial z2}\,\mathrm{f}(\,z1\,, z2\,, z3\,, z4\,)\right) \\
& + \mathrm{g}(\,z1\,, z2\,, z3\,, z4\,)\,\left(\frac{\partial}{\partial z3}\,\mathrm{f}(\,z1\,, z2\,, z3\,, z4\,)\right) \\
& - \left(\frac{\partial}{\partial z4}\,\mathrm{g}(\,z1\,, z2\,, z3\,, z4\,)\right)\,w2 \\
& - \left(\frac{\partial}{\partial z3}\,\mathrm{g}(\,z1\,, z2\,, z3\,, z4\,)\right)\,\mathrm{f}(\,z1\,, z2\,, z3\,, z4\,) \\
& + \left(\frac{\partial}{\partial z4}\,\mathrm{f}(\,z1\,, z2\,, z3\,, z4\,)\right)\,\mathrm{f}(\,z1\,, z2\,, z3\,, z4\,) \\
& - \left(\frac{\partial}{\partial z2}\,\mathrm{g}(\,z1\,, z2\,, z3\,, z4\,)\right)\,z4\Bigg)
\end{aligned}
$$

The general solution of the linear PDE $\frac{\partial\lambda}{\partial w_2} = LALA$ is LALALAsymb below, where the function "*cc*" has to be equal to "coco" also given below, and λ_0 is a free function of four variables.

```
> LALALAsymb := lambda0(z1,z2,z3,z4) * (w2 + cc(z1,z2,z3,z4) );
```

$$LALALAsymb := \lambda0(\,z1\,, z2\,, z3\,, z4\,)\,(\,w2 + \mathrm{cc}(\,z1\,, z2\,, z3\,, z4\,)\,)$$

```
> coco := factor( ( lambda(z1,z2,z3,z4,w2) / LALA ) - w2 );
```

$$
\begin{aligned}
coco := & \Bigg(\left(\left(\frac{\partial}{\partial z2}\,\mathrm{g}(\,z1\,, z2\,, z3\,, z4\,)\right)\,z4 - z3\,\left(\frac{\partial}{\partial z2}\,\mathrm{f}(\,z1\,, z2\,, z3\,, z4\,)\right)\right. \\
& - \left(\frac{\partial}{\partial z4}\,\mathrm{f}(\,z1\,, z2\,, z3\,, z4\,)\right)\,\mathrm{f}(\,z1\,, z2\,, z3\,, z4\,) - \left(\frac{\partial}{\partial z1}\,\mathrm{f}(\,z1\,, z2\,, z3\,, z4\,)\right) \\
& - \mathrm{g}(\,z1\,, z2\,, z3\,, z4\,)\,\left(\frac{\partial}{\partial z3}\,\mathrm{f}(\,z1\,, z2\,, z3\,, z4\,)\right) \\
& \left. + \left(\frac{\partial}{\partial z3}\,\mathrm{g}(\,z1\,, z2\,, z3\,, z4\,)\right)\,\mathrm{f}(\,z1\,, z2\,, z3\,, z4\,)\right)\Bigg/\Bigg(\\
& \frac{\partial}{\partial z4}\,\mathrm{g}(\,z1\,, z2\,, z3\,, z4\,)\Bigg)
\end{aligned}
$$

In the sequel, we shall substitute λ with the expression LALALAsymb rather than with the expression

```
> normal( lambda0(z1,z2,z3,z4) * (w2 + coco) );
```

$$\lambda 0(\mathit{z1}, \mathit{z2}, \mathit{z3}, \mathit{z4}) \Big(- \Big(\frac{\partial}{\partial \mathit{z4}} \mathrm{f}(\mathit{z1}, \mathit{z2}, \mathit{z3}, \mathit{z4}) \Big) \mathrm{f}(\mathit{z1}, \mathit{z2}, \mathit{z3}, \mathit{z4})$$
$$- \mathit{z3} \Big(\frac{\partial}{\partial \mathit{z2}} \mathrm{f}(\mathit{z1}, \mathit{z2}, \mathit{z3}, \mathit{z4}) \Big) - \Big(\frac{\partial}{\partial \mathit{z1}} \mathrm{f}(\mathit{z1}, \mathit{z2}, \mathit{z3}, \mathit{z4}) \Big)$$
$$+ \Big(\frac{\partial}{\partial \mathit{z4}} \mathrm{g}(\mathit{z1}, \mathit{z2}, \mathit{z3}, \mathit{z4}) \Big) \mathit{w2}$$
$$+ \Big(\frac{\partial}{\partial \mathit{z3}} \mathrm{g}(\mathit{z1}, \mathit{z2}, \mathit{z3}, \mathit{z4}) \Big) \mathrm{f}(\mathit{z1}, \mathit{z2}, \mathit{z3}, \mathit{z4})$$
$$+ \Big(\frac{\partial}{\partial \mathit{z2}} \mathrm{g}(\mathit{z1}, \mathit{z2}, \mathit{z3}, \mathit{z4}) \Big) \mathit{z4}$$
$$- \mathrm{g}(\mathit{z1}, \mathit{z2}, \mathit{z3}, \mathit{z4}) \Big(\frac{\partial}{\partial \mathit{z3}} \mathrm{f}(\mathit{z1}, \mathit{z2}, \mathit{z3}, \mathit{z4}) \Big) \Big) \Big/ \Big($$
$$\frac{\partial}{\partial \mathit{z4}} \mathrm{g}(\mathit{z1}, \mathit{z2}, \mathit{z3}, \mathit{z4}) \Big)$$

where cc(z1,z2,z3,z4) is replaced by its value, because it makes the expressions shorter (without this trick, 100MegaBytes were not enough to run the Maple session).

Actually, we will rather substitute $\frac{\partial f}{\partial z_1}$ with its expression below, as a function of other partial derivatives and of the function cc than doing the contrary :

```
> valdfdz1 := factor( solve( coco = cc(z1,z2,z3,z4) ,
>                            diff( f(z1,z2,z3,z4) , z1 ) ));
```

$$\mathit{valdfdz1} := \Big(\frac{\partial}{\partial \mathit{z2}} \mathrm{g}(\mathit{z1}, \mathit{z2}, \mathit{z3}, \mathit{z4}) \Big) \mathit{z4} - \mathit{z3} \Big(\frac{\partial}{\partial \mathit{z2}} \mathrm{f}(\mathit{z1}, \mathit{z2}, \mathit{z3}, \mathit{z4}) \Big)$$
$$- \Big(\frac{\partial}{\partial \mathit{z4}} \mathrm{f}(\mathit{z1}, \mathit{z2}, \mathit{z3}, \mathit{z4}) \Big) \mathrm{f}(\mathit{z1}, \mathit{z2}, \mathit{z3}, \mathit{z4})$$
$$- \mathrm{g}(\mathit{z1}, \mathit{z2}, \mathit{z3}, \mathit{z4}) \Big(\frac{\partial}{\partial \mathit{z3}} \mathrm{f}(\mathit{z1}, \mathit{z2}, \mathit{z3}, \mathit{z4}) \Big)$$
$$+ \Big(\frac{\partial}{\partial \mathit{z3}} \mathrm{g}(\mathit{z1}, \mathit{z2}, \mathit{z3}, \mathit{z4}) \Big) \mathrm{f}(\mathit{z1}, \mathit{z2}, \mathit{z3}, \mathit{z4})$$
$$- \mathrm{cc}(\mathit{z1}, \mathit{z2}, \mathit{z3}, \mathit{z4}) \Big(\frac{\partial}{\partial \mathit{z4}} \mathrm{g}(\mathit{z1}, \mathit{z2}, \mathit{z3}, \mathit{z4}) \Big)$$

The routine “subslambda” replaces λ with its expression LALALAsymb in an expression.
The routine “subsdfdz1” replaces $\frac{\partial f}{\partial z_1}$ with its expression subsdfdz1 in an expression, it is slightly more complicated to take care of substitutions in higher order partial derivatives of f

```
> subslambda := proc ( expr )
>     simplify(subs( lambda(z1,z2,z3,z4,w2) = LALALAsymb , expr  ))
>                end :
> vald2fdz2dz1 := diff( valdfdz1 , z2 ):
> vald2fdz3dz1 := diff( valdfdz1 , z3 ):
> vald2fdz4dz1 := diff( valdfdz1 , z4 ):
> vald2fdz1dz1 := factor(simplify(subs(
>       {diff(f(z1,z2,z3,z4),z1)=valdfdz1,
>        diff(diff(f(z1,z2,z3,z4),z2),z1)=vald2fdz2dz1,
>        diff(diff(f(z1,z2,z3,z4),z3),z1)=vald2fdz3dz1,
>        diff(diff(f(z1,z2,z3,z4),z4),z1)=vald2fdz4dz1 } ,
>                                      diff( valdfdz1 , z1 ) ))):
> subsdfdz := proc(expr) simplify(subs(
>       {diff(f(z1,z2,z3,z4),z1)=valdfdz1,
>        diff(diff(f(z1,z2,z3,z4),z1),z1)=vald2fdz1dz1,
>        diff(diff(f(z1,z2,z3,z4),z2),z1)=vald2fdz2dz1,
>        diff(diff(f(z1,z2,z3,z4),z3),z1)=vald2fdz3dz1,
>        diff(diff(f(z1,z2,z3,z4),z4),z1)=vald2fdz4dz1 } , expr )) end:
```

Let us see the coefficient of $\mathrm{d}z_1 \wedge \mathrm{d}z_2$ now. It turns out that it is affine with respect to the function β with the coefficient below in front of β. Hence, $C_1 = 0$ may be solved explicitly for β, the expression for β is called "valbeta".

```
> simplify(subs( diff(lambda(z1,z2,z3,z4,w2),w2)=LALA ,
>  coeff(
>       collect( coeff(dOmega1mod , &^(d(z1),d(z2)) ) ,
>                beta(z1,z2,z3,z4,v1,w2,v1dot,v2dot) )
>              ,     beta(z1,z2,z3,z4,v1,w2,v1dot,v2dot) )  ));
```

$$2\,\frac{\frac{\partial}{\partial z4}\,\mathrm{g}(\,z1\,,z2\,,z3\,,z4\,)}{\lambda(\,z1\,,z2\,,z3\,,z4\,,w2\,)}$$

```
> valbeta := factor(solve(  coeff(dOmega1mod , &^(d(z1),d(z2))) = 0 ,
>                     beta(z1,z2,z3,z4,v1,w2,v1dot,v2dot) )):
```

Let us replace λ by its value, and $\frac{\partial f}{\partial z_1}$ by the expression subsdfdz1 in valbeta and call the new expression valbetaS :

```
> valbetaS := map(factor,collect( subsdfdz(subslambda( valbeta )),v1)) :
> degree(valbetaS,v1);
```

$$3$$

The expression valbetaS is polynomial of degree 3 with respect to v_1.

From (130), we got an expression for $\frac{\partial \lambda}{\partial w_2}$ (LALA, from which we derived an expression for λ summed up in the substitution routines "subslambda" and "subsdfdz1") and an expression for β (valbetaS). Let us check that it is all we can get, i.e. these substitution make equation (130) trivially true :

```
> subsdfdz(  subslambda(
>    simplify(subs( beta(z1,z2,z3,z4,v1,w2,v1dot,v2dot)=valbetaS  ,
>             dOmega1mod )) ) );
```

$$0$$

We now turn to (131), i.e. to $\mathrm{d}\Omega_3$.

```
> dOmega3 := factor ( value(d(Omega3)) ) :
```

The routine "modOm3" below computes the expression of a form modulo $Omega_3$ by substituting $\mathrm{d}z_4$ with "valdz4", the linear combination of $\mathrm{d}z_1$, $\mathrm{d}z_2$ and $\mathrm{d}z_3$ which is equivalent to $\mathrm{d}z_4$ modulo Ω_3 :

```
> map( getform, [op(wcollect(Omega3))]) ; coeff( Omega3, d(z4));
```

$$[\mathrm{d}(\mathit{z1}), \mathrm{d}(\mathit{z2}), \mathrm{d}(\mathit{z3}), \mathrm{d}(\mathit{z4})]$$

$$-\lambda(\mathit{z1}, \mathit{z2}, \mathit{z3}, \mathit{z4}, \mathit{w2})$$

```
> valdz4 := map(factor,wcollect(  solve( Omega3=0, d(z4) )   )):
> map(getform,[op(valdz4)]);
> modOm3 := proc(forme)
>             simplify(subs( d(z4)=valdz4 , forme )) end :
```

$$[\mathrm{d}(\mathit{z1}), \mathrm{d}(\mathit{z2}), \mathrm{d}(\mathit{z3})]$$

Let us now compute $\mathrm{d}\Omega_3$ modulo Ω_3. In the expression "dOmega3mod" below, not only have we performed the above substitution, but we have also removed all the terms containing $\mathrm{d}\dot{v}_1$ or $\mathrm{d}\dot{v}_2$, which will be useless...

```
> dOmega3mod := map(factor, wcollect(
>     simplify( subs( diff(diff(lambda(z1,z2,z3,z4,w2),w2),w2)=0,
>        modOm3(
>        simplify(subs([d(v1dot)=0,d(v2dot)=0],  dOmega3  )) )) ) )):
```

Let us check that the forms that appear in dOmega3 and dOmega3mod are these we expect :

```
> map( getform, [op(wcollect(dOmega3))]) ;
> map(getform,[op(dOmega3mod)]) ;
```

$$[\mathrm{d}(\mathit{z1})\,\&^\wedge\,\mathrm{d}(\mathit{z4}), \mathrm{d}(\mathit{z1})\,\&^\wedge\,\mathrm{d}(\mathit{w2}), \mathrm{d}(\mathit{z2})\,\&^\wedge\,\mathrm{d}(\mathit{w2}), \mathrm{d}(\mathit{z1})\,\&^\wedge\,\mathrm{d}(\mathit{v1}),$$

$$\mathrm{d}(\mathit{z2}) \mathbin{\&^\wedge} \mathrm{d}(\mathit{z4}), \mathrm{d}(\mathit{z2}) \mathbin{\&^\wedge} \mathrm{d}(\mathit{v1dot}), \mathrm{d}(\mathit{z2}) \mathbin{\&^\wedge} \mathrm{d}(\mathit{v2dot}),$$
$$\mathrm{d}(\mathit{z2}) \mathbin{\&^\wedge} \mathrm{d}(\mathit{v1}), \mathrm{d}(\mathit{z4}) \mathbin{\&^\wedge} \mathrm{d}(\mathit{w2}), \mathrm{d}(\mathit{z1}) \mathbin{\&^\wedge} \mathrm{d}(\mathit{z2}),$$
$$\mathrm{d}(\mathit{z2}) \mathbin{\&^\wedge} \mathrm{d}(\mathit{z3}), \mathrm{d}(\mathit{z1}) \mathbin{\&^\wedge} \mathrm{d}(\mathit{z3}), \mathrm{d}(\mathit{z3}) \mathbin{\&^\wedge} \mathrm{d}(\mathit{z4}),$$
$$\mathrm{d}(\mathit{z3}) \mathbin{\&^\wedge} \mathrm{d}(\mathit{v1}), \mathrm{d}(\mathit{z3}) \mathbin{\&^\wedge} \mathrm{d}(\mathit{w2}), \mathrm{d}(\mathit{z3}) \mathbin{\&^\wedge} \mathrm{d}(\mathit{v1dot}),$$
$$\mathrm{d}(\mathit{z3}) \mathbin{\&^\wedge} \mathrm{d}(\mathit{v2dot}), \mathrm{d}(\mathit{z1}) \mathbin{\&^\wedge} \mathrm{d}(\mathit{v1dot}), \mathrm{d}(\mathit{z1}) \mathbin{\&^\wedge} \mathrm{d}(\mathit{v2dot})]$$

$$[\mathrm{d}(\mathit{z1}) \mathbin{\&^\wedge} \mathrm{d}(\mathit{w2}), \mathrm{d}(\mathit{z2}) \mathbin{\&^\wedge} \mathrm{d}(\mathit{w2}), \mathrm{d}(\mathit{z1}) \mathbin{\&^\wedge} \mathrm{d}(\mathit{v1}), \mathrm{d}(\mathit{z2}) \mathbin{\&^\wedge} \mathrm{d}(\mathit{v1}),$$
$$\mathrm{d}(\mathit{z1}) \mathbin{\&^\wedge} \mathrm{d}(\mathit{z2}), \mathrm{d}(\mathit{z2}) \mathbin{\&^\wedge} \mathrm{d}(\mathit{z3}), \mathrm{d}(\mathit{z1}) \mathbin{\&^\wedge} \mathrm{d}(\mathit{z3}),$$
$$\mathrm{d}(\mathit{z3}) \mathbin{\&^\wedge} \mathrm{d}(\mathit{v1}), \mathrm{d}(\mathit{z3}) \mathbin{\&^\wedge} \mathrm{d}(\mathit{w2})]$$

We now compute $\Omega_1 \wedge \dot{\Omega}_3$, modulo Ω_3, it is the expression called “Omega33dmod”.

```
> map( getform, [op(Omega3d)]);
```

$$[\mathrm{d}(\mathit{z1}), \mathrm{d}(\mathit{z2}), \mathrm{d}(\mathit{z3}), \mathrm{d}(\mathit{z4}), \mathrm{d}(\mathit{v1}), \mathrm{d}(\mathit{w2})]$$

```
> Omega3dmod := map(factor,wcollect(simplify(
>         subs(    diff(diff(lambda(z1,z2,z3,z4,w2),w2),w2)=0,
>         modOm3( Omega3d )  )))):
> map( getform, [op(Omega3dmod)]) ;
```

$$[\mathrm{d}(\mathit{z1}), \mathrm{d}(\mathit{z2}), \mathrm{d}(\mathit{z3}), \mathrm{d}(\mathit{v1}), \mathrm{d}(\mathit{w2})]$$

```
> Omega33dmod := map( factor , wcollect (  Omega1 &^ Omega3dmod )):
```

The coefficient of $\mathrm{d}z_3$ is rather simple in “dOmega3mod” and “Omega33dmod” :

```
> cc1 := - coeff( Omega33dmod , &^(d(z3),d(w2)) ) ;
> cc2 := coeff( dOmega3mod , &^(d(z3),d(w2)) ) ;
```

$$\mathit{cc1} := \lambda(\mathit{z1}, \mathit{z2}, \mathit{z3}, \mathit{z4}, \mathit{w2})$$

$$\begin{aligned}\mathit{cc2} := \Big(&\%1\, \mathit{v1}\, \left(\frac{\partial}{\partial \mathit{z4}} \mathrm{f}(\mathit{z1}, \mathit{z2}, \mathit{z3}, \mathit{z4})\right)\\
&+ \%1\, \rho(\mathit{z1}, \mathit{z2}, \mathit{z3}, \mathit{z4}, \mathit{v1}, \mathit{w2}, \mathit{v1dot}, \mathit{v2dot})\\
&- \%1\, \left(\frac{\partial}{\partial \mathit{z3}} \mathrm{f}(\mathit{z1}, \mathit{z2}, \mathit{z3}, \mathit{z4})\right) - \%1\, \mathit{v1}\, \left(\frac{\partial}{\partial \mathit{z3}} \mathrm{g}(\mathit{z1}, \mathit{z2}, \mathit{z3}, \mathit{z4})\right)\\
&+ \%1\, \mathit{v1}^2 \left(\frac{\partial}{\partial \mathit{z4}} \mathrm{g}(\mathit{z1}, \mathit{z2}, \mathit{z3}, \mathit{z4})\right) -\end{aligned}$$

$$\left(\frac{\partial}{\partial w2}\rho(z1, z2, z3, z4, v1, w2, v1dot, v2dot)\right)\lambda(z1, z2, z3, z4, w2)$$
$$\Big)\Big/(\lambda(z1, z2, z3, z4, w2))$$
$$\%1 := \frac{\partial}{\partial w2}\lambda(z1, z2, z3, z4, w2)$$

```
> factor(  cc1*coeff(dOmega3mod,&^(d(z3),d(w2)))
>        + cc2*coeff(Omega33dmod,&^(d(z3),d(w2))) ) ;
```

$$0$$

Equation (131) implies $cc1dOmega3unmod - cc2Omega33dmod = 0$.

It turns out that the coefficient of $\mathrm{d}z_2 \wedge \mathrm{d}w_2$ allows one to solve for ρ :

```
> valrho := factor( solve(
>   simplify( cc1*coeff(dOmega3mod,&^(d(z2),d(w2)))
>          + cc2*coeff(Omega33dmod,&^(d(z2),d(w2))) ) = 0  ,
>                           rho(z1,z2,z3,z4,v1,w2,v1dot,v2dot))):
> valrhoS := map(factor,collect(
>    subsdfdz(subslambda(
>    simplify( subs( beta(z1,z2,z3,z4,v1,w2,v1dot,v2dot)=valbetaS ,
>                                          valrho )) ) ), v1)):
```

valrhoS is an expression of valrho where β and λ are substituted for the values computed above. It is shorter than valrho :

```
> [nops(expand(numer(valrho))) , nops(expand(numer(valrhoS)))];
```

$$[56, 23]$$

Let us now compute the coefficient of $\mathrm{d}z_2 \wedge \mathrm{d}w_2$, and call it EE :

```
> EE  := factor(
>           cc1*coeff(dOmega3mod,&^(d(z2),d(z3)))
>         + cc2*coeff(Omega33dmod,&^(d(z2),d(z3)))   ):
```

It turns out that it is a rather large expression :

```
> nops(numer(EE));  denom(EE);
```

$$7510$$

$$\lambda(z1, z2, z3, z4, w2)^2$$

This expression is large enough that if we simply substitute β and rho with valrhoS and valbetaS, it takes more that 100MBytes to compute the result.

To round this problem, we shall take advantage of the fact that valbetaS, valrhoS and EE are polynomial with respect to v_1, and use the expressions "rho77" and "beta77" which are polynomials of the right degree in the indeterminate v_1, with coefficients some generic functions of $(z1, z2, z3, z4, w2)$... Of course, when these coefficients are substituted for the right functions, "rho77" and "beta77" have the same values as "valrhoS" and "valbetaS"

```
> degree(valbetaS,v1);  degree(valrhoS,v1);
```

$$3$$

$$2$$

```
> BB3 := coeff( valbetaS , v1 , 3 );
> BB2 := coeff( valbetaS , v1 , 2 ):
> BB1 := coeff( valbetaS , v1 , 1 ):
> BB0 := coeff( valbetaS , v1 , 0 ):
```

$$\mathit{BB3} := -\left(\frac{\partial}{\partial \mathit{z4}}\,\mathrm{g}(\,\mathit{z1}\,,\mathit{z2}\,,\mathit{z3}\,,\mathit{z4}\,)\right)^2$$

```
> RR2 := coeff( valrhoS , v1, 2 );
> RR1 := coeff( valrhoS , v1, 1 );
> RR0 := coeff( valrhoS , v1, 0 ):
```

$$\mathit{RR2} := -\left(\frac{\partial}{\partial \mathit{z4}}\,\mathrm{g}(\,\mathit{z1}\,,\mathit{z2}\,,\mathit{z3}\,,\mathit{z4}\,)\right)$$

$$\mathit{RR1} := 2\,\lambda 0(\,\mathit{z1}\,,\mathit{z2}\,,\mathit{z3}\,,\mathit{z4}\,)\,\mathrm{cc}(\,\mathit{z1}\,,\mathit{z2}\,,\mathit{z3}\,,\mathit{z4}\,) + 2\,\lambda 0(\,\mathit{z1}\,,\mathit{z2}\,,\mathit{z3}\,,\mathit{z4}\,)\,\mathit{w2} \\ - \left(\frac{\partial}{\partial \mathit{z4}}\,\mathrm{f}(\,\mathit{z1}\,,\mathit{z2}\,,\mathit{z3}\,,\mathit{z4}\,)\right) + \left(\frac{\partial}{\partial \mathit{z3}}\,\mathrm{g}(\,\mathit{z1}\,,\mathit{z2}\,,\mathit{z3}\,,\mathit{z4}\,)\right)$$

```
> rho77 := RR2 * v1^2  +  RR1 * v1  +  R0(z1,z2,z3,z4,w2) :
> beta77 := BB3 * v1^3  +  B2(z1,z2,z3,z4,w2) * v1^2
>            +  B1(z1,z2,z3,z4,w2) * v1 + B0(z1,z2,z3,z4,w2) :
```

Let us check that when performing the correct substitutions, beta77=valbetaS and rho77=valrhoS

```
> factor(subs( R0(z1,z2,z3,z4,w2)=RR0   ,  valrhoS - rho77) );
> factor(subs( [ B0(z1,z2,z3,z4,w2)=BB0,B1(z1,z2,z3,z4,w2)=BB1,
>                     B2(z1,z2,z3,z4,w2)=BB2 ] ,  valbetaS - beta77) );
```

$$0$$

$$0$$

Now we substitute in the expression EE :

```
> EE77 := collect(simplify(subs(
>           [beta(z1,z2,z3,z4,v1,w2,v1dot,v2dot)=beta77,
```

```
>             rho(z1,z2,z3,z4,v1,w2,v1dot,v2dot)=rho77] ,   EE )),v1):
> degree( EE77 , v1);
```

$$5$$

The leading coefficient turns out to be zero when substituting λ :

```
> factor( subslambda( coeff( EE77 , v1 , 5 ) ));
```

$$0$$

And the coefficient of degree 4 is

```
> factor( subsdfdz(subslambda( simplify(
>                 subs( [R0(z1,z2,z3,z4,w2)=RR0 , B2(z1,z2,z3,z4,w2)=BB2 ] ,
>                      coeff( EE77 , v1 , 4 )        ))   ) ) );
```

$$\frac{1}{4}\left(\frac{\partial}{\partial z4}\,\mathrm{g}(\,z1\,,z2\,,z3\,,z4\,)\right)(\,w2+\mathrm{cc}(\,z1\,,z2\,,z3\,,z4\,)\,)^2\,\lambda 0(\,z1\,,z2\,,z3\,,z4\,)^3$$

It has to be zero, hence λ is identically zero.

Step 4 *If α, λ and b satisfy (130) with λ identically zero, then $\delta^1_{2,1}$ cannot vanish and b must be given by*

$$b \;=\; \frac{2\,\gamma}{\delta^1_{2,1}}\;. \tag{185}$$

This may easily be proved without the help of the program Maple.

Since $\lambda = 0$, we have $\Omega_1 = \omega_2$, $\Omega_3 = \omega_1 \;-\; b\,\dot{\omega}_2 \;+\; \alpha\,\omega_2$, but from lemma 7.7, $\mathrm{d}\Omega_1$ satisfies (130), i.e.

$$\mathrm{d}\omega_2 \;\equiv\; 0 \quad \text{modulo}\;\{\,\omega_2\;,\;\Omega_3\;,\;\dot{\Omega}_3\,\}$$

From lemma 7.9 with $\Omega = \Omega_3$, $\lambda_1 = 1$, $\lambda_2 = \alpha$ and $\lambda_3 = -b$, the above relation implies that $b\,(b\delta^1_{2,1} - 2\gamma)$ is identically zero on U, but we assume here that b does not vanish, hence $b\delta^1_{2,1} - 2\gamma$ must be identically zero, and therefore $\delta^1_{2,1}$ does not vanish (because γ does not vanish), and b is given by (185).

Step 5 *If α, λ and b satisfy (130)-(131) with λ identically zero and b is given by $b = \dfrac{2\gamma}{\delta^1_{2,1}}$, there is a unique possible value for α, and f and g must be of the form (64)-(65)*

This is done in the following Maple session which is the continuation of the previous one.

Session run with Maple V Release 3, continued

All the previous definitions remain valid, but we assign λ to be identically zero :

```
> lambda := proc(z1,z2,z3,z4,w2) 0 end :
```

First we need to compute b —i.e. β— i.e. implement step 4. To this end, we compute $d\Omega_1 \wedge \Omega_1 \wedge \Omega_3 \wedge \dot{\Omega}_3$.

```
> dOmega1ext := factor( dOmega1 &^ Omega1 &^ Omega3 &^ Omega3d ):
> getform(dOmega1ext);
> nops(getcoeff(dOmega1ext));
```

$$\&^\wedge(\mathrm{d}(\,z1\,), \mathrm{d}(\,z2\,), \mathrm{d}(\,z3\,), \mathrm{d}(\,z4\,), \mathrm{d}(\,v1\,))$$

$$2$$

This is a monomial form of degree 5, whose coefficient must therefore be identically zero. We divide by D_1 which does not vanish :

```
> EEE1 := factor(   getcoeff(dOmega1ext)  /  (-D1)   ):
```

This expression EEE1 is affine with respect to the function β, and its coefficient does not vanish :

```
> coeff( EEE1 , beta(z1,z2,z3,z4,v1,w2,v1dot,v2dot) );
```

$$-2\left(\frac{\partial}{\partial z4}\,\mathrm{g}(\,z1\,,z2\,,z3\,,z4\,)\right)$$

One may therefore get β from the equation EEE1=0 . Below, "valbeta" is the value of β as the solution of EEE1=0. For convenience, we call D_2 the constant term in EEE1, so that β will be equal to $D_2/(2\frac{\partial g}{\partial z_4}$:

```
> D2 := factor(coeff( EEE1 , beta(z1,z2,z3,z4,v1,w2,v1dot,v2dot) , 0 )):
> valbeta := solve( EEE1=0, beta(z1,z2,z3,z4,v1,w2,v1dot,v2dot) ):
```

Let us check that $\beta = D_2/(2\frac{\partial g}{\partial z_4})$:

```
> normal( valbeta - D2 / (2*diff(g(z1,z2,z3,z4),z4)) );
```

$$0$$

Here we shall check that the only possible value for b —i.e. $1/valbeta$ with valbeta computed above— is given by $\frac{2\gamma}{\delta^1_{2,1}}$. Let us first compute γ and $\delta^1_{2,1}$ and then check the equality. To compute γ and $\delta^1_{2,1}$, we use the fact that, from (56),

$$\begin{aligned} \mathrm{d}\omega_2 \wedge \omega_2 \wedge \dot\omega_2 \wedge \ddot\omega_2 &= \delta^1_{2,1}\, \omega_1 \wedge \dot\omega_1 \wedge \omega_2 \wedge \dot\omega_2 \wedge \ddot\omega_2 \\ \mathrm{d}\omega_2 \wedge \omega_2 \wedge \omega_1 &= \gamma\, \omega_1 \wedge \dot\omega_1 \wedge \omega_2 \wedge \dot\omega_2 \end{aligned}$$

```
> domega2 := wcollect(value(d(omega2))):
> omega2dd := wcollect( DOT ( omega2d)):
> Form0 := factor(omega1&^omega1d&^omega2&^omega2d&^omega2dd):
> Form1 := factor(domega2&^omega2&^omega2d&^omega2dd):
> getform(Form0);getform(Form1);
> delta211 := factor( getcoeff(Form1)/getcoeff(Form0) ):
```

$$\&^\wedge(\mathrm{d}(\mathit{z1}), \mathrm{d}(\mathit{z2}), \mathrm{d}(\mathit{z3}), \mathrm{d}(\mathit{z4}), \mathrm{d}(\mathit{v1}))$$

$$\&^\wedge(\mathrm{d}(\mathit{z1}), \mathrm{d}(\mathit{z2}), \mathrm{d}(\mathit{z3}), \mathrm{d}(\mathit{z4}), \mathrm{d}(\mathit{v1}))$$

```
> Form2 := factor(omega1&^omega1d&^omega2&^omega2d):
> Form3 := factor(domega2&^omega2&^omega1):
> getform(Form2);getform(Form3);
> Gamma := factor( getcoeff(Form3)/getcoeff(Form2) ):
```

$$\&^\wedge(\mathrm{d}(\mathit{z1}), \mathrm{d}(\mathit{z2}), \mathrm{d}(\mathit{z3}), \mathrm{d}(\mathit{z4}))$$

$$\&^\wedge(\mathrm{d}(\mathit{z1}), \mathrm{d}(\mathit{z2}), \mathrm{d}(\mathit{z3}), \mathrm{d}(\mathit{z4}))$$

$\delta^1_{2,1}$ is equal to $-D_2/D_1$ and γ to $-\frac{\partial g}{\partial z_4}/D_1$

```
> factor( delta211 * D1 / D2);
```

$$-1$$

```
> factor(  Gamma * D1 ) ;
```

$$-\left(\frac{\partial}{\partial \mathit{z4}}\, \mathrm{g}(\mathit{z1}, \mathit{z2}, \mathit{z3}, \mathit{z4})\right)$$

Hence $\frac{2\gamma}{\delta^1_{2,1}} = 2\frac{\partial g}{\partial z_4}/D_2$, which is equal to $1/valbeta$.

To compute ρ, we shall compute $\mathrm{d}\Omega_3$ modulo $\{\Omega_1, \Omega_3\}$, which is a fortiori zero from (131).

In order to compute modulo $\{\Omega_1, \Omega_3\}$, we simply substitute $\mathrm{d}z_3$, $\mathrm{d}z_2$ with "valdz3" and "valdz2", the monomial forms in $\mathrm{d}z_1$ which is equal to each of them modulo $\{\Omega_1, \Omega_3\}$.

```
> coeff(Omega1,d(z3));
```

$$1$$

```
> vvaldz3 := solve( Omega1=0 , d(z3) );
```

$$\mathit{vvaldz3} := \mathrm{d}(\mathit{z1})\,\mathrm{g}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4}) - \mathrm{d}(\mathit{z1})\,\mathit{z3}\,\left(\frac{\partial}{\partial \mathit{z4}}\mathrm{f}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4})\right)$$
$$-\,\mathrm{d}(\mathit{z1})\,\mathit{z3}\,\mathit{v1}\,\left(\frac{\partial}{\partial \mathit{z4}}\mathrm{g}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4})\right)$$
$$+\,\mathrm{d}(\mathit{z2})\,\left(\frac{\partial}{\partial \mathit{z4}}\mathrm{f}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4})\right) + \mathrm{d}(\mathit{z2})\,\mathit{v1}\,\left(\frac{\partial}{\partial \mathit{z4}}\mathrm{g}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4})\right)$$

```
> Omega3mod := wcollect( simplify(
> subs( { d(z3)=vvaldz3 , beta(z1,z2,z3,z4,v1,w2,v1dot,v2dot)=valbeta}
>       , Omega3 )
> * diff(g(z1,z2,z3,z4),z4) / D1  ));
```

$$\mathit{Omega3mod} := \left(\frac{1}{2}\,\mathit{z3}\,\mathit{v1}\,\left(\frac{\partial^2}{\partial \mathit{z4}^2}\mathrm{g}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4})\right)\right.$$
$$\left.-\left(\frac{\partial}{\partial \mathit{z4}}\mathrm{g}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4})\right) + \frac{1}{2}\,\mathit{z3}\,\left(\frac{\partial^2}{\partial \mathit{z4}^2}\mathrm{f}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4})\right)\right)\mathrm{d}(\mathit{z1}) +$$
$$\left(-\frac{1}{2}\,\mathit{v1}\,\left(\frac{\partial^2}{\partial \mathit{z4}^2}\mathrm{g}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4})\right) - \frac{1}{2}\left(\frac{\partial^2}{\partial \mathit{z4}^2}\mathrm{f}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4})\right)\right)\mathrm{d}(\mathit{z2})$$

```
> valdz2 := factor(  solve( Omega3mod=0 , d(z2)  ));
```

$$\mathit{valdz2} := \left(\mathit{z3}\,\mathit{v1}\,\left(\frac{\partial^2}{\partial \mathit{z4}^2}\mathrm{g}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4})\right) - 2\,\left(\frac{\partial}{\partial \mathit{z4}}\mathrm{g}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4})\right)\right.$$
$$\left.+\,\mathit{z3}\,\left(\frac{\partial^2}{\partial \mathit{z4}^2}\mathrm{f}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4})\right)\right)\mathrm{d}(\mathit{z1})\Big/\Bigg($$
$$\left(\frac{\partial^2}{\partial \mathit{z4}^2}\mathrm{f}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4})\right) + \mathit{v1}\,\left(\frac{\partial^2}{\partial \mathit{z4}^2}\mathrm{g}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4})\right)\Bigg)$$

```
> valdz3 := factor( subs( d(z2)=valdz2 , vvaldz3 ));
```

$$\mathit{valdz3} := \mathrm{d}(\mathit{z1})\,\left(\mathit{v1}\,\left(\frac{\partial^2}{\partial \mathit{z4}^2}\mathrm{g}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4})\right)\mathrm{g}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4})\right.$$
$$+\left(\frac{\partial^2}{\partial \mathit{z4}^2}\mathrm{f}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4})\right)\mathrm{g}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4})$$
$$\left.-\,2\,\left(\frac{\partial}{\partial \mathit{z4}}\mathrm{f}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4})\right)\%1 - 2\,\mathit{v1}\,\%1^2\right)\Big/\Bigg($$

$$\left(\frac{\partial^2}{\partial z4^2}\,\mathrm{f}(\,z1\,,z2\,,z3\,,z4\,)\right)+v1\,\left(\frac{\partial^2}{\partial z4^2}\,\mathrm{g}(\,z1\,,z2\,,z3\,,z4\,)\right)\Bigg)$$

$$\%1 := \frac{\partial}{\partial z4}\,\mathrm{g}(\,z1\,,z2\,,z3\,,z4\,)$$

```
> dOmega3mod13 := map(factor,wcollect( simplify(
> subs( {d(z2)=valdz2,d(z3)=valdz3,
>        beta(z1,z2,z3,z4,v1,w2,v1dot,v2dot)=valbeta} ,
> dOmega3 )))):
> valrho := solve( coeff(dOmega3mod13,d(z1)&^d(v1))=0 ,
>                              rho(z1,z2,z3,z4,v1,w2,v1dot,v2dot) );
```

$$\begin{aligned}
valrho := \frac{1}{2}\Bigg(&-2\,\left(\frac{\partial}{\partial z4}\,\mathrm{f}(\,z1\,,z2\,,z3\,,z4\,)\right)\,\%1^2\,v1\\
&+\left(\frac{\partial^2}{\partial z4^2}\,\mathrm{g}(\,z1\,,z2\,,z3\,,z4\,)\right)\,\left(\frac{\partial}{\partial z3}\,\mathrm{g}(\,z1\,,z2\,,z3\,,z4\,)\right)\,\mathrm{f}(\,z1\,,z2\,,z3\,,z4\,)\\
&-\left(\frac{\partial^2}{\partial z4^2}\,\mathrm{g}(\,z1\,,z2\,,z3\,,z4\,)\right)\,\mathrm{g}(\,z1\,,z2\,,z3\,,z4\,)\,\left(\frac{\partial}{\partial z3}\,\mathrm{f}(\,z1\,,z2\,,z3\,,z4\,)\right)\\
&+2\,v1\,\%1^2\,\left(\frac{\partial}{\partial z3}\,\mathrm{g}(\,z1\,,z2\,,z3\,,z4\,)\right)\\
&+\%1\,\left(\frac{\partial^2}{\partial z4^2}\,\mathrm{g}(\,z1\,,z2\,,z3\,,z4\,)\right)\,w2\\
&-\left(\frac{\partial^2}{\partial z4^2}\,\mathrm{g}(\,z1\,,z2\,,z3\,,z4\,)\right)\,\left(\frac{\partial}{\partial z4}\,\mathrm{f}(\,z1\,,z2\,,z3\,,z4\,)\right)\,\mathrm{f}(\,z1\,,z2\,,z3\,,z4\,)\\
&-\left(\frac{\partial^2}{\partial z4^2}\,\mathrm{g}(\,z1\,,z2\,,z3\,,z4\,)\right)\,\left(\frac{\partial}{\partial z1}\,\mathrm{f}(\,z1\,,z2\,,z3\,,z4\,)\right)\\
&+2\,\%1^2\,\left(\frac{\partial}{\partial z3}\,\mathrm{f}(\,z1\,,z2\,,z3\,,z4\,)\right)\\
&-z3\,\left(\frac{\partial^2}{\partial z4^2}\,\mathrm{g}(\,z1\,,z2\,,z3\,,z4\,)\right)\,\left(\frac{\partial}{\partial z2}\,\mathrm{f}(\,z1\,,z2\,,z3\,,z4\,)\right)\\
&+\left(\frac{\partial^2}{\partial z4^2}\,\mathrm{g}(\,z1\,,z2\,,z3\,,z4\,)\right)\,\left(\frac{\partial}{\partial z2}\,\mathrm{g}(\,z1\,,z2\,,z3\,,z4\,)\right)\,z4\,-2\,v1^2\,\%1^3\Bigg)\\
&\Big/\%1^2
\end{aligned}$$

$$\%1 := \frac{\partial}{\partial z4}\,\mathrm{g}(\,z1\,,z2\,,z3\,,z4\,)$$

Let us substitute the value of ρ in $\mathrm{d}\Omega_3$ modulo $\{\Omega_1, \Omega_3\}$.

```
> factor(simplify(subs( rho(z1,z2,z3,z4,v1,w2,v1dot,v2dot) = valrho ,
> (2/D1) * dOmega3mod13 )));
```

$$-(\mathrm{d}(\mathit{z1}) \mathbin{\&\hat{}} \mathrm{d}(\mathit{z4}))\Big(2\,\%1\left(\frac{\partial^3}{\partial \mathit{z4}^3}\mathrm{f}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4})\right)$$
$$+\,2\,\%1\,\mathit{v1}\left(\frac{\partial^3}{\partial \mathit{z4}^3}\mathrm{g}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4})\right)$$
$$-\,3\left(\frac{\partial^2}{\partial \mathit{z4}^2}\mathrm{g}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4})\right)\left(\frac{\partial^2}{\partial \mathit{z4}^2}\mathrm{f}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4})\right)$$
$$-\,3\,\mathit{v1}\left(\frac{\partial^2}{\partial \mathit{z4}^2}\mathrm{g}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4})\right)^2\Big)\Big/\Big(\%1$$
$$\left(\left(\frac{\partial^2}{\partial \mathit{z4}^2}\mathrm{f}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4})\right)+\mathit{v1}\left(\frac{\partial^2}{\partial \mathit{z4}^2}\mathrm{g}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4})\right)\right)\Big)$$
$$\%1 := \frac{\partial}{\partial \mathit{z4}}\mathrm{g}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4})$$

```
> EE := collect(   numer(getcoeff( " ))   ,v1);
```

$$\mathit{EE} := \left(-2\,\%1\left(\frac{\partial^3}{\partial \mathit{z4}^3}\mathrm{g}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4})\right)+3\left(\frac{\partial^2}{\partial \mathit{z4}^2}\mathrm{g}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4})\right)^2\right)\mathit{v1}$$
$$-\,2\,\%1\left(\frac{\partial^3}{\partial \mathit{z4}^3}\mathrm{f}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4})\right)$$
$$+\,3\left(\frac{\partial^2}{\partial \mathit{z4}^2}\mathrm{g}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4})\right)\left(\frac{\partial^2}{\partial \mathit{z4}^2}\mathrm{f}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4})\right)$$
$$\%1 := \frac{\partial}{\partial \mathit{z4}}\mathrm{g}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4})$$

Both the coefficient of v_1 and the constant coefficient must be zero, this gives exactly the PDEs (155)-(156) :

```
> PDE1 := -coeff(EE,v1); PDE2 := -coeff(EE,v1,0);
```

$$\mathit{PDE1} := 2\left(\frac{\partial}{\partial \mathit{z4}}\mathrm{g}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4})\right)\left(\frac{\partial^3}{\partial \mathit{z4}^3}\mathrm{g}(\mathit{z1},\mathit{z2},\mathit{z3},\mathit{z4})\right)$$

$$- 3 \left(\frac{\partial^2}{\partial z4^2} \mathrm{g}(z1, z2, z3, z4) \right)^2$$

$$PDE2 := 2 \left(\frac{\partial}{\partial z4} \mathrm{g}(z1, z2, z3, z4) \right) \left(\frac{\partial^3}{\partial z4^3} \mathrm{f}(z1, z2, z3, z4) \right) - 3 \left(\frac{\partial^2}{\partial z4^2} \mathrm{g}(z1, z2, z3, z4) \right) \left(\frac{\partial^2}{\partial z4^2} \mathrm{f}(z1, z2, z3, z4) \right)$$

Hence from lemma 7.10 f and g must be given by $f = valf$ and $g = valg$ with :

```
> valf :=  ( a0(z1,z2,z3) + a1(z1,z2,z3) * z4 + a2(z1,z2,z3) * z4^2  )
>          /  ( c0(z1,z2,z3) + c1(z1,z2,z3) * z4 ) ;
> valg := ( b0(z1,z2,z3) + b1(z1,z2,z3) * z4 )
>          /  ( c0(z1,z2,z3) + c1(z1,z2,z3) * z4 ) ;
```

$$valf := \frac{\mathrm{a0}(z1, z2, z3) + \mathrm{a1}(z1, z2, z3)\, z4 + \mathrm{a2}(z1, z2, z3)\, z4^2}{\mathrm{c0}(z1, z2, z3) + \mathrm{c1}(z1, z2, z3)\, z4}$$

$$valg := \frac{\mathrm{b0}(z1, z2, z3) + \mathrm{b1}(z1, z2, z3)\, z4}{\mathrm{c0}(z1, z2, z3) + \mathrm{c1}(z1, z2, z3)\, z4}$$

We now compute $\mathrm{d}\Omega_3 \wedge \Omega_3$ and $\Omega_1 \wedge \dot{\Omega}_3 \wedge \Omega_3$, but for this we first assign ρ and β to be equal to the values computed above :

```
> rho := proc(z1,z2,z3,z4,v1,w2,v1dot,v2dot) valrho end :
> beta  := proc(z1,z2,z3,z4,v1,w2,v1dot,v2dot) valbeta end :
>
> dOmega3ext3 := map(factor,wcollect(
>         dOmega3&^Omega3 * diff(g(z1,z2,z3,z4),z4)^2 / D1 ^2 )):
> Omega3ext1ext3dot3 := map(factor,wcollect(Omega3&^Omega1 &^Omega3d )):
> map(getform,[op(Omega3ext1ext3dot3)]);
```

$$[\&^\wedge(\mathrm{d}(z1), \mathrm{d}(z2), \mathrm{d}(z3)), \&^\wedge(\mathrm{d}(z1), \mathrm{d}(z3), \mathrm{d}(v1)), \&^\wedge(\mathrm{d}(z1), \mathrm{d}(z2), \mathrm{d}(v1)), \&^\wedge(\mathrm{d}(z2), \mathrm{d}(z3), \mathrm{d}(v1)), \&^\wedge(\mathrm{d}(z1), \mathrm{d}(z3), \mathrm{d}(z4)), \&^\wedge(\mathrm{d}(z1), \mathrm{d}(z2), \mathrm{d}(z4)), \&^\wedge(\mathrm{d}(z2), \mathrm{d}(z3), \mathrm{d}(z4))]$$

```
> factor(
>    ( coeff( Omega3ext1ext3dot3 , &^(d(z1),d(z3),d(v1)) )
>      + z3 * coeff( Omega3ext1ext3dot3 , &^(d(z2),d(z3),d(v1)) )
>    )  / D1^2 );
```

$$1$$

Hence the terms in $\mathrm{d}z_1 \wedge \mathrm{d}z_3 \wedge \mathrm{d}v_1$ and $\mathrm{d}z_2 \wedge \mathrm{d}z_3 \wedge \mathrm{d}v_1$ in the expression of $\Omega_1 \wedge \dot{\Omega}_3 \wedge \Omega_3$

cannot both vanish. But when f and g are given by valf and valg, $\mathrm{d}\Omega 3 \wedge \Omega 3$ is given by

```
> factor(simplify(
> subs( {f(z1,z2,z3,z4)=valf,g(z1,z2,z3,z4)=valg}, dOmega3ext3 )
> ));
```

$$\begin{aligned}
&\&\hat{}(\mathrm{d}(z1), \mathrm{d}(z2), \mathrm{d}(z3))\left(\left(\frac{\partial}{\partial z2}\mathrm{b1}(z1,z2,z3)\right)\mathrm{c1}(z1,z2,z3)\right.\\
&\quad - \mathrm{c1}(z1,z2,z3)\, z3\left(\frac{\partial}{\partial z2}\mathrm{a2}(z1,z2,z3)\right)\\
&\quad - \left(\frac{\partial}{\partial z1}\mathrm{a2}(z1,z2,z3)\right)\mathrm{c1}(z1,z2,z3)\\
&\quad + \left(\frac{\partial}{\partial z3}\mathrm{b1}(z1,z2,z3)\right)\mathrm{a2}(z1,z2,z3)\\
&\quad - \mathrm{b1}(z1,z2,z3)\left(\frac{\partial}{\partial z3}\mathrm{a2}(z1,z2,z3)\right)\\
&\quad + \mathrm{a2}(z1,z2,z3)\left(\frac{\partial}{\partial z1}\mathrm{c1}(z1,z2,z3)\right)\\
&\quad - \mathrm{b1}(z1,z2,z3)\left(\frac{\partial}{\partial z2}\mathrm{c1}(z1,z2,z3)\right) - \mathrm{a2}(z1,z2,z3)^2\\
&\quad \left. + z3\left(\frac{\partial}{\partial z2}\mathrm{c1}(z1,z2,z3)\right)\mathrm{a2}(z1,z2,z3)\right) \Big/\\
&\quad (\mathrm{c0}(z1,z2,z3) + \mathrm{c1}(z1,z2,z3)\, z4)^2
\end{aligned}$$

Hence this form can be a multiple of $\Omega_1 \wedge \dot{\Omega}_3 \wedge \Omega_3$ only if it is identically zero. This implies that the following expression must be identically zero :

```
> EDP3 := numer(getcoeff(  "  ));
```

$$\begin{aligned}
EDP3 := &\left(\frac{\partial}{\partial z2}\mathrm{b1}(z1,z2,z3)\right)\mathrm{c1}(z1,z2,z3)\\
&- \mathrm{c1}(z1,z2,z3)\, z3\left(\frac{\partial}{\partial z2}\mathrm{a2}(z1,z2,z3)\right)\\
&- \left(\frac{\partial}{\partial z1}\mathrm{a2}(z1,z2,z3)\right)\mathrm{c1}(z1,z2,z3)\\
&+ \left(\frac{\partial}{\partial z3}\mathrm{b1}(z1,z2,z3)\right)\mathrm{a2}(z1,z2,z3)\\
&- \mathrm{b1}(z1,z2,z3)\left(\frac{\partial}{\partial z3}\mathrm{a2}(z1,z2,z3)\right)
\end{aligned}$$

$$+ \mathrm{a2}(z1, z2, z3) \left(\frac{\partial}{\partial z1} \mathrm{c1}(z1, z2, z3)\right)$$
$$- \mathrm{b1}(z1, z2, z3) \left(\frac{\partial}{\partial z2} \mathrm{c1}(z1, z2, z3)\right) - \mathrm{a2}(z1, z2, z3)^2$$
$$+ z3 \left(\frac{\partial}{\partial z2} \mathrm{c1}(z1, z2, z3)\right) \mathrm{a2}(z1, z2, z3)$$

This is exactly the PDE (65) as shown below :

```
> Gamma := ( b1(z1,z2,z3) - z3 * a2(z1,z2,z3) ) * d(z1)
>      + a2(z1,z2,z3) * d(z2)  - c1(z1,z2,z3) * d(z3);
```

$$\Gamma := (\mathrm{b1}(z1, z2, z3) - z3\,\mathrm{a2}(z1, z2, z3))\,\mathrm{d}(z1) + \mathrm{a2}(z1, z2, z3)\,\mathrm{d}(z2) - \mathrm{c1}(z1, z2, z3)\,\mathrm{d}(z3)$$

```
> factor(value( d(Gamma) &^ Gamma ));
```

$$\&\wedge(\mathrm{d}(z1), \mathrm{d}(z2), \mathrm{d}(z3)) \left(\left(\frac{\partial}{\partial z2} \mathrm{b1}(z1, z2, z3)\right) \mathrm{c1}(z1, z2, z3)\right.$$
$$- \mathrm{c1}(z1, z2, z3)\, z3 \left(\frac{\partial}{\partial z2} \mathrm{a2}(z1, z2, z3)\right)$$
$$- \left(\frac{\partial}{\partial z1} \mathrm{a2}(z1, z2, z3)\right) \mathrm{c1}(z1, z2, z3)$$
$$+ \left(\frac{\partial}{\partial z3} \mathrm{b1}(z1, z2, z3)\right) \mathrm{a2}(z1, z2, z3)$$
$$- \mathrm{b1}(z1, z2, z3) \left(\frac{\partial}{\partial z3} \mathrm{a2}(z1, z2, z3)\right)$$
$$+ \mathrm{a2}(z1, z2, z3) \left(\frac{\partial}{\partial z1} \mathrm{c1}(z1, z2, z3)\right)$$
$$- \mathrm{b1}(z1, z2, z3) \left(\frac{\partial}{\partial z2} \mathrm{c1}(z1, z2, z3)\right) - \mathrm{a2}(z1, z2, z3)^2$$
$$\left.+ z3 \left(\frac{\partial}{\partial z2} \mathrm{c1}(z1, z2, z3)\right) \mathrm{a2}(z1, z2, z3)\right)$$

```
> factor(  EDP3 - getcoeff(  "  )  );
```

$$0$$

8 Conclusion

The present paper provides, for the 4-dimensional affine system (5), some new necessary and sufficient conditions for a restricted form of dynamic feedback linearization. These conditions are completely explicit. They also allow one to treat 3-dimensional non-affine systems.

This paper is not a general answer to dynamic feedback linearizability of 4-dimensional systems with 2 inputs for the following reasons :
- One restriction comes from the regularity assumptions. The example presented in section 5 shows that they are not necessary. A thorough treatment of singularities, or at least a clear identification of the real singularities of dynamic feedback linearization is therefore not achieved even in our "simple" case.
- We also restrict our attention to "endogenous feedback". See [19] for a discussion of the link between general dynamic feedback linearizability and endogenous dynamic feedback linearizability.
- We further restrict the class of dynamic linearization by requiring that the linearizing output depend on x and u only. The natural follow-up to this work is to decide whether systems which are not (x, u)-dynamic feedback linearizable are simply not dynamic feedback linearizable (at least endogenously), or if some are $(x, u, \dot{u})$-dynamic linearizable for example...

These three issues are still open to our knowledge, and they are of utmost interest both in the general case and to complete the picture in this four-dimensional case.

Let us finally make a remark on the method of the proofs. In a sense, the present results amount to giving conditions for some nonlinear partial differential equations to have some solutions (see section 2.6). Since the PDEs are high order —see (26)-(27)— one might think that some sophisticated tools for checking intergrability, like Spencer cohomolgy, should be involved. It turns out however that the proofs are all elementary, and never make use of more sophisticated tools than Frobenius theorem. Actually, when using the infinitesimal Brunovský form and writing the equations for the coefficients of decomposition in elementary transformations of the invertible transformation "$P(\frac{\mathrm{d}}{\mathrm{dt}})$"' instead of writing directly the equations for the linearizing outputs, as in the proof of theorem 4.1 or the "alternative" proof of case 6 in theorem 3.1, we use Frobenius theorem to write the equations in a convenient way (like the equation (129)-(130)-(131) for theorem 4.1), but then the arguments used to give conditions for existence of solutions to these equations are in a sense even not first order like Frobenius theorem, but "zeroth order", i.e. the solutions (α, λ and b in the case (129)-(130)-(131) may be explicitely computed (expression involving functions in the equations of the system) from part of the equations, and the compatibility conditions are obtained by substituting these expressions in the remaining equations. It is of course tempting to ask whether in general when using the infinitesimal Brunovskỳ form to test for existence of some linearizing outputs depending on a pre-defined number of time-derivatives of the inputs, this feature always appears —the equations for the coefficients of the invertible transformation contain enough non-differential equations to obtain them solving non-differential equations— or if this is particular to the small dimensions considered here.

Acknowledgments

The author wishes to acknowledge Claude Moog and Eduardo Aranda from Laboratoire d'Automatique de Nantes, France; the present work started essentially as an application of previous common work with them.

He also wishes to thank Pierre Rouchon, from Ecole des Mines, Paris, for communicating the preprint [25] and for discussions on the necessary condition for non-affine systems used in section 6.

Appendix : Some facts on pfaffian systems

In this section, we recall some very basic definitions on pfaffian systems, and some precise facts we are going to use. For a thorough treatment, see e.g. [28] or [6].

A **pfaffian system** I of rank r around a point can be defined as a module (over smooth functions) of differential 1-forms which is generated by r 1-forms which are pointwisely linearly independent around this point, or also as an ideal of differential forms (of arbitrary degrees, with the exterior product as "multiplication"), which has the peculiarity of being generated by such 1-forms. It is defined by giving r independent 1-forms and r 1-forms which generate the same module define the same pfaffian system.

A **congruence** like $\Omega_1 \equiv \Omega_2$ modulo $\{\eta_1, \eta_2, \ldots\}$ where the Ω_i's are 2-forms and the η_j's are 1-forms (we only need this) means modulo the ideal generated by

$\{\eta_1, \eta_2, \ldots\}$, i.e. it means that there exists some forms α_j such that $\Omega_1 - \Omega_2 = \eta_1 \wedge \alpha_1 + \eta_2 \wedge \alpha_2 + \ldots$; it is equivalent to $(\Omega_1 - \Omega_2) \wedge \eta_1 \wedge \eta_2 \wedge \ldots = 0$

A pfaffian system also defines an "orthogonal distribution", spanned by the vector fields which annihilate these 1-forms.

We will only be interested in the case $m = 1$ or $m = 2$, and we therefore speak of the pfaffian system $I = \{\omega\}$ or $I = \{\omega_1, \omega_2\}$.

It is **completely integrable** if it is, locally, generated by 1 (resp 2) exact 1-forms, or equivalently if $\mathrm{d}\omega \equiv 0$ modulo $\{\omega\}$ (resp. $\mathrm{d}\omega_i \equiv 0$ modulo $\{\omega_1, \omega_2\}$ for $i = 1, 2$); this is Frobenius theorem, and this is equivalent to the orthogonal distribution being closed under Lie brackets.

Derived System

For a given pfaffian system I, consider the module made of the forms of degree 1 which are in I, and whose exterior derivative is in I; at points where it has constant rank, this module defines a pfaffian system called the **derived system** $I^{(1)}$ of I. A pfaffian system is equal to its derived system if and only if it is integrable. In the case of a pfaffian system of rank 1, either it is integrable or its derived system is zero; in the case of a pfaffian system of rank 2, when the system $\{\omega_1, \omega_2\}$ is not integrable, one may always write

$$\mathrm{d}\omega_i \quad \equiv \quad \Gamma_i \quad \text{modulo } \{\omega_1, \omega_2\}$$

where the two-form Γ_1 (or Γ_2) is not in the ideal generated by $\{\omega_1, \omega_2\}$; then either Γ_2 may be written $\lambda\Gamma_1$ modulo $\{\omega_1, \omega_2\}$ for a certain function λ, and the derived system is $\{\omega_2 - \lambda\omega_1\}$ or it is not the case and the derived system is zero. The orthogonal distribution to the derived system of a given pfaffian system is spanned by the orthogonal distribution to this system plus all the Lie brackets between two vector fields in this distribution :

$$(I^{(1)})^{\perp} \quad = \quad I^{\perp} \, + \, [\, I^{\perp} \, , \, I^{\perp} \,] \ \ .$$

Cartan Characteristic System

The Cartan characteristic system $\mathcal{C}(I)$ of a given pfaffian system I is somehow the smallest integrable system generated by some exact forms $\mathrm{d}\psi_i$ such that the original pfaffian system is generated by 1-forms which can be expressed with the help only of the ψ_i's and the $\mathrm{d}\psi_i$'s; its dimension is the minimal number of variables needed to describe the pfaffian system. Its orthogonal distribution is given by :

$$\mathcal{C}(I)^{\perp} \;=\; \{\, X \in I^{\perp} \,/\, [X, I^{\perp}] \subset I^{\perp} \,\}\;, \tag{186}$$

The Cartan characteristic of any pfaffian system is integrable if it has constant rank. A pfaffian system is equal to its Cartan characteristic system if and only if it is integrable.

For a non-integrable system of rank 1 $\{\omega\}$, it is always possible, where the rank of the characteristic system is constant, to find $2p$ independent 1-forms η_i such that the rank of $\{\omega, \eta_1, \ldots, \eta_{2p}\}$ is $2p+1$ and

$$\mathrm{d}\omega \;\equiv\; \eta_1 \wedge \eta_2 \;+\; \eta_3 \wedge \eta_4 \;+\; \ldots \;+\; \eta_{2p-1} \wedge \eta_{2p} \quad \text{modulo } \{\omega\} \tag{187}$$

and the characteristic system is then $\{\omega, \eta_1, \ldots, \eta_{2p}\}$ (and this is automatically completely integrable).

For a non-integrable system of rank 2 $\{\omega_1, \omega_2\}$, all we need is the following : if it is possible to express this pfaffian system with 4 variables $\chi_1, \chi_2, \chi_3, \chi_4$ (i.e. there exists a basis of this pfaffian system made of two 1-forms which are linear combinations of $\mathrm{d}\chi_1, \mathrm{d}\chi_2, \mathrm{d}\chi_3, \mathrm{d}\chi_4$ with coefficients functions of $\chi_1, \chi_2, \chi_3, \chi_4$ only, ω_1 and ω_2 do not have to be of this form, but they span the same module as two forms of this form), then its characteristic system is $\{\mathrm{d}\chi_1, \mathrm{d}\chi_2, \mathrm{d}\chi_3, \mathrm{d}\chi_4\}$, and for any forms η_1 and η_2 such that $\{\omega_1, \omega_2, \eta_1, \eta_2\}$ spans the same module as $\{\mathrm{d}\chi_1, \mathrm{d}\chi_2, \mathrm{d}\chi_3, \mathrm{d}\chi_4\}$, we have

$$\mathrm{d}\omega_k \;=\; \omega_1 \wedge \Gamma_{k,1} \;+\; \omega_2 \wedge \Gamma_{k,2} \;+\; \lambda_k\, \eta_1 \wedge \eta_2 \tag{188}$$

for some 1-forms $\Gamma_{k,j}$ and some functions λ_k.

References

[1] E. Aranda, C. H. Moog, J.-B. Pomet, "A linear algebraic framework for dynamic feedback linearization", IEEE Trans. on Autom. Control, **40** (1995), No1, pp. 127–132.

[2] E. Aranda, C. H. Moog, J.-B. Pomet "Infinitesimal Brunovský form for nonlinear systems with applications to Dynamic Linearization", in B. Jakuczyk, W. Respondek, T. Rzezuchowski Eds., *Geometry in Nonlinear Control and Differential Inclusions*, Banach Center Publications **32** (1995) pp. 19–33, Warszawa.

[3] B. Bonnard : "Feedback Equivalence for Nonlinear systems and the Time-optimal Control problem". *SIAM J. on Control and Optimization*, vol. 29 (1991), pp.1300-1321.

[4] P. Brunovský, "A classification of linear controllable systems", *Kybernetica*, **6** (1970), pp. 176-188.

[5] R. W. Brockett, "Feedback invariants for nonlinear systems", in Proceedings of *IFAC World Congress*, Helsinki, 1978.

[6] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmitt, P. A. Griffiths, *Exterior Differential Systems*, Mathematical Sciences Research Institute Publications **18**, Springer-Verlag, 1991.

[7] B. Charlet, J. Lévine, R. Marino, "Sufficient conditions for dynamic feedback linearization", SIAM J. Contr. Opt., **29**, pp. 38–57, 1991

[8] M. Fliess, J. Lévine, P. Martin, P. Rouchon, "On differentially Flat Nonlinear Systems", *Proc. 2nd. IFAC NOLCOS Symposium*, Bordeaux, M Fliess, Ed., pp. 408–412, 1992.

[9] M. Fliess, J. Lévine, P. Martin, P. Rouchon, "Sur les systèmes non linéaires différentiellement plats", *C. R. Acad. Sc. Paris*, **315-I**, pp. 619–624, 1992.

[10] M. Fliess, J. Lévine, P. Martin, P. Rouchon, "Flatness and defect of non-linear systems : introductory theory and examples", *Int. J. Control*, **61**, No. 6, pp. 1327–1361, 1995.

[11] M. Fliess, J. Lévine, P. Martin, P. Rouchon, "Linéarisation par bouclage dynamique et transformations de Lie-Bäcklund", *C. R. Acad. Sc. Paris*, to appear, 1993.

[12] M. Fliess, J. Lévine, P. Martin, P. Rouchon, "Towards a new differential geometric setting in nonlinear control", presented at *International Geometrical Colloquium*, Moscow, may 1993.

[13] L.R. Hunt, R. Su, G. Meyer, *Design for multi-input nonlinear systems*, in Differential Geometric Control Theory, R. Brockett, R. Millmann, H. Sussmann Eds., pp. 268–298, Birkhäuser 1983.

[14] A. Isidori, C. H. Moog, A. De Luca, "A Sufficient Condition for full Linearization via Dynamic State Feedback", Proc. 25th. IEEE Conf. Decision Contr., pp. 203–208, 1986.

[15] B. Jakubczyk, "Equivalence and Invariants of Nonlinear Control Systems". In *Nonlinear Controllability and Optimal Control*, H.J. Sussmann Editor, Marcel Dekker, New-York, 1990.

[16] B. Jakubczyk, "Dynamic Feedback Equivalence of Nonlinear Control Systems". Preprint, 1993.

[17] B. Jacubczyk, W. Respondek, "On linearization of control systems", Bull. Acad. Pol. Sci., Ser. Sci. Math., **18**, pp. 517–522, 1980.

[18] J. Lévine, R. Marino, "On Dynamic Feedback Linearization on $\mathbf{R}^4$", Proc. 29th. IEEE Conf. Decision Contr., pp. 2088–2090, 1990.

[19] P. Martin, *Contribition à l'étude des systèmes différentiellement plats*, Thèse de doctorat, Ecole des Mines de Paris, 1992.

[20] P. Martin, Geometric sufficient conditions for flatness, *IEEE Conf. on Dec. and Cont.*, dec. 1993, also appears in the appendix of [19].

[21] P. Martin, A criterion for flatness with structure $\{1,\ldots,1,2\}$, preprint, 1992, also appears in the appendix of [19].

[22] P. Martin, P. Rouchon Systems without drift and flatness, *Int. Symp. on Math. Theory on Networks and Syst.*, Regensburg, August 1993.

[23] J.-B. Pomet, "A Differential Geometric Setting for Dynamic Equivalence and Dynamic Linearization", in B. Jakuczyk, W. Responek, T. Rzezuchowski Eds., *Geometry in Nonlinear Control and Differential Inclusions*, Banach Center Publications **32** (1995) pp. 319–339, Warszawa.

[24] J.-B. Pomet, C. H. Moog, E. Aranda, *A non-exact Brunovský form and dynamic feedback linearization*, Proc. 31st. IEEE Conf. Decision Cont., pp. 2012–2017, 1992.

[25] P. Rouchon, *Necessary condition and genericity of dynamic feedback linearization, J. of Math. Systems, Estim, & Cont.*, to appear, 1993.

[26] W. Sluis, "A necessary condition for dynamic feedback linearization", *Systems & Control Letters*, **21** (1993), pp. 277–283.

[27] W.F. Shadwick : "Absolute Equivalence and Dynamic Feedback Linearization". *Syst. Contr. Lett.*, **15** (1990), pp. 35–39.

[28] S. Sternberg, *Lectures on Differential Geometry*, Second edition, Chelsea Publishing Collinear., New-York, 1983 .

[29] H. J. Sussmann, V. Jurdjevic "Controllability of nonlinear systems", *J. of Diff. Equations*, **12**, pp. 95–116, 1972 .

[30] W. A. Wolovich, *Linear Multivariable Systems*, Springer-Verlag, Applied mathematical sciences, vol. 11, 1974.

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INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
ISSN 0249-6399