Upper and Lower Bounds on Overflow Probabilities for a Multiplexer with Multiclass Markovian Sources

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Abstract: In this paper, we consider a multiplexer with constant output rate and infinite buffer capacity fed by independent Markovian fluid on-off sources. We do not assume that the model is symmetrical: there is an arbitrary number K of different traffic classes, and for each class k, an arbitrary number N_k of sources of class k. We derive lower and upper bounds for the stationary distribution of the backlog X of the form

$$B\exp(-\theta^\star x) \;\leq\; P\{X > x\} \;\leq\; C\exp(-\theta^\star x).$$

When $K = 2$ or $K = 1$, we numerically compare our bounds to the exact distribution of X. as well as to other previously known results. Through various examples, we discuss the behavior of $P\{X > x\}$ and the tightness of the bounds.

Key-words: Exponential bound; Effective bandwidth; Markov fluid input; Statistical multiplexing; Large deviations; Tail distribution.

(Résumé : tsvp)

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Bornes Inférieures et Supérieures pour le multiplexage de sources markoviennes fluides

Résumé : Nous étudions un multiplexeur avec taux de service constant dont le flux d'entrée est la superposition de sources markoviennes de type on-off. Le modèle est en temps continu: chaque source alterne entre un état d'émission à taux constant (on) et un état de silence (off), le temps de séjour dans chacun de ces états étant une variable exponentielle. Nous ne supposons pas que le modèle soit symétrique: il existe un nombre arbitraire K de classes de trafic différentes, et un nombre arbitraire N_k de sources de classe k.

Nous montrons qu'il existe des bornes inférieures et supérieures sur la distribution de la charge stationnaire X du multiplexeur ayant la forme

$$B \exp(-\theta^\star x) \;\leq\; P\{X > x\} \;\leq\; C \exp(-\theta^\star x).$$

Lorsque $K = 2$ ou $K = 1$, nous comparons ces bornes avec la distribution exacte de X calculée numériquement. Certains résultats précédemment connus sont retrouvés comme des cas particuliers et sont généralisés. Au travers de plusieurs exemples, nous analysons le comportement de $P\{X > x\}$ et des bornes.

Mots-clé : Bornes exponentielles; bande passante équivalente; sources markoviennes fluides; multiplexage statistique; grandes déviations.

1 Introduction

In a network node, a large number of incoming data streams are multiplexed and share the common buffer space and bandwidth. The advantage of statistical multiplexing is to allow an inferior output bandwidth than what would be required if the input streams were all emitting at their maximum rate. The price to pay for this saving is the risk of overflow and congestion, which may harm network users in two different ways. Some of the traffic is time sensitive and suffers mainly from queueing delays building up when congestion occurs, another part of the traffic can tolerate some delay but will not accept a single lost cell in the transmitted data. Estimating the delay and cell loss probability is thus an important part of network control. Two main problems arise: the first difficulty is to define mathematical models which render as closely as possible the principal characteristics of real traffic; the second difficulty is to analyze these models and derive accurate bounds or estimates. We are here mainly concerned with addressing the second difficulty.

The model considered here, where the input data stream is represented as the superposition of a given number of Markovian fluid on-off sources, has been the subject of numerous studies in the past few years. Even for such a simple queueing model, bounds or approximations of the backlog distribution are not easily obtained. Nevertheless, some significant progress has been made by using a few different techniques. The recent theory of effective bandwidth provides some insights on the asymptotical behavior of the backlog distribution; papers of interest dealing with effective bandwidth include Chang [3], Elwalid and Mitra [6], Kesidis, Walrand, and Chang [12]. The use of large deviation theory also leads to asymptotical results, examples can be found in Weiss [16], Hui [10], and Hsu and Walrand [9]. Duffield, in [5], and Buffet and Duffield, in [2], introduced some martingale techniques to treat the problem.

So far, most of the papers on this subject either considered only symmetrical models, where all sources are of the same type, or presented only asymptotical results, for large buffer size or large number of sources. The originality of our work is to find upper and lower bounds which hold for any number of sources, any number of different source classes, and any buffer size.

Our results are obtained by following the approach presented by Liu, Nain, and Towsley in [14], which extends the work of Kingman in [13]. The results in

[14] are for discrete time processes defined by a recursive equation, while the model which we consider here is in continuous time. For this reason, the bounds in the continuous time model are not immediate to obtain: we first define a discretized model, where the time line is split into segments of length δ; we find bounds for the discretized model from [14]; and when the discretization parameter δ tends to 0, we show that these bounds tend to finite positive values, which are the bounds for the continuous time model.

From the precedent remark, it would seem easier to analyse directly a problem with a discrete time formulation and avoid the limiting scheme necessary to study the continuous time model. But we find that the bounds in the discretized model are complicated and not easily computable, while their limits in the continuous time model have much simpler expressions. This is the reason why we choose to formulate the problem in continuous time.

The paper is organized as follows. In the next section, we define the model and show how to make use of the bounds proposed in [14]: we define the discretized model and give the expression of the discrete time bounds. In section 3, we take the limit and find the continuous time bounds. In section 4, we present numerical results and discuss the validity of our bounds. We study the particular case of symmetrical sources in section 5.

2 Approximation by a Discrete-Time Model

2.1 Problem Formulation

A multiplexer with constant service rate c and infinite buffer capacity is fed by independent Markovian on-off streams. There are K classes of traffic ($K \geq 1$) and N_k streams of class k. A source of class k emits data at a constant rate r_k when in state on and is idle when in state off. The time spent by the source in the off or on state before changing state is exponentially distributed with mean value $1/\lambda_k$ and $1/\mu_k$ respectively, $\lambda_k/(\lambda_k+\mu_k)$ is then the stationary probability to be in the on state. Let $\hat{r} = \sum_{1\leq k\leq K} N_k r_k$ be the maximum instantaneous input rate, and $\bar{r} = \sum_{1\leq k\leq K} N_k r_k \lambda_k/(\lambda_k + \mu_k)$ be the mean input rate. We assume that $\bar{r} < c < \hat{r}$.

Let $Y = (Y_t)_{t\in\mathbb{R}}$ be the Markov process describing the state of the sources, and $S = \{0,1\}^{\sum N_k}$ its finite state space. If the process Y is in state $s \in S$

at time t, with $s = (s_k^i; k = 1, \ldots, K; i = 1, \ldots, N_k)$, $s_k^i = 0$ (resp. 1) if the ith source of class k is in state off (resp. on), and we call s_k the number of class k sources which are active at time t, the corresponding instantaneous input rate being then $r(s) = \sum_{1 \le k \le K} s_k r_k$. We assume that the process Y is stationary. For $t \ge 0$, let W_t be the excess workload in the interval $(-t, 0]$, i.e. the difference between the amount of work arrived in that interval and the total available service, and let X be the backlog at time $t = 0$. We have

$$W_t = \int_{-t}^{0} (r(Y_u) - c) du \tag{1}$$

$$X = \sup_{t \ge 0} W_t. \tag{2}$$

Our objective is to find bounds on the probability $P\{X > x\}$. We first show that the continuous time model considered here is the limit of a discretized model defined in the following way: we change the original model only by sampling the input rate $r(Y_t)$ every δ units of time (with $\delta > 0$), and by making it constant in the interval $[n\delta, (n+1)\delta)$, for all integer n. In the discretized model, the amount of work arrived in the time interval $[n\delta, (n+1)\delta)$ is $\delta r(Y_{n\delta})$ and the offered service δc. Thus, if $X^\delta(n)$ denotes the backlog at time $n\delta$, the dynamics of the new system are described by the equation $X^\delta(n+1) = [X^\delta(n) + \delta(r(Y_{n\delta}) - c)]^+$, where $(Y_{n\delta})_{n \in \mathbb{Z}}$ is a stationary Markov chain on S. Let $(Y_t^\delta)_{t \in \mathbb{R}}$ be the process defined by $Y_t^\delta = Y_{n\delta}$, with $n \in \mathbb{Z}$ and $n\delta \le t < (n+1)\delta$, and let X^δ be the backlog at time $n = 0$, then, as in (1) and (2):

$$W_t^\delta = \int_{-t}^{0} (r(Y_u^\delta) - c) du \tag{3}$$

$$X^\delta = \sup_{t \ge 0} W_t^\delta. \tag{4}$$

For such a discrete-time model with a Markovian environment, Liu, Nain, and Towsley [14] extended the results found by Kingman in [13] and derived exponential bounds of the following form:

$$B_\delta \exp(-\theta_\delta^\star x) \le P\{X^\delta > x\} \le C_\delta \exp(-\theta_\delta^\star x), \quad x \ge 0. \tag{5}$$

In section 3, we show that B_δ, C_δ, and $\theta_\delta^\star$ tend to finite positive values B, C, and $\theta^\star$ as δ tends to 0; we also show in Appendix A that X^δ tends almost surely to X, so that for all $x \geq 0$, $P\{X^\delta > x\}$ tends to $P\{X > x\}$ as δ tends to 0, which finally yields for the continuous time model:

$$B \exp(-\theta^\star x) \leq P\{X > x\} \leq C \exp(-\theta^\star x), \quad x \geq 0 \tag{6}$$

We now show how to calculate the decay rate $\theta^\star_\delta$ and the coefficients B_δ and C_δ.

2.2 Discrete-Time Coefficients

Between the instants $n\delta$ and $(n+1)\delta$, a source of class k switches from off to on with probability p_k and from on to off with probability q_k ($0 < p_k < 1$ and $0 < q_k < 1$), then, the transition matrix of the state of this source is

$$P_k = \begin{pmatrix} 1-p_k & p_k \\ q_k & 1-q_k \end{pmatrix}.$$

The transition matrix $P = [p_\delta(t,s)]_{(t,s)\in S\times S}$ of the Markov chain $(Y_{n\delta})_{n\in\mathbb{Z}}$ is given by the Kronecker product (see Graham [8]) of the matrices P_k: $P = \bigotimes_{1\leq k\leq K} P_k^{\otimes N_k}$. We further define $\Psi_k(\theta) = \mathrm{Diag}(1, \exp(\delta\theta r_k))$ and $J_k(\theta) = P_k^T \Psi_k(\theta)$. By the Perron-Frobenius theorem, we know that the spectral radius $\tau_k(\theta)$ of the positive matrix $J_k(\theta)$ is its unique largest (real) eigenvalue and that there is a unique corresponding positive eigenvector $z_k(\theta)$ with L^1 norm $|z_k(\theta)|$ equal to 1. Finally, let $\Psi(\theta) = \mathrm{Diag}(\exp(\delta\theta r(s)), s \in S)$, $J(\theta) = P^T\Psi(\theta)$, let $\tau(\theta)$ be the spectral radius of $J(\theta)$ and $z(\theta)$ be the corresponding eigenvector with $|z(\theta)| = 1$. The following properties hold (see [8]):

$$\begin{aligned}
\Psi(\theta) &= \bigotimes_{1\leq k\leq K} \Psi_k(\theta)^{\otimes N_k} \\
J(\theta) &= \bigotimes_{1\leq k\leq K} J_k(\theta)^{\otimes N_k} \\
z(\theta) &= \bigotimes_{1\leq k\leq K} z_k(\theta)^{\otimes N_k} && (7) \\
\tau(\theta) &= \prod_{1\leq k\leq K} \tau_k(\theta)^{N_k}. && (8)
\end{aligned}$$

From [14] (Proposition 2.3), the decay rate $\theta^\star_\delta > 0$ is now given implicitly as the unique positive solution of the equation:

$$\tau(\theta^\star_\delta) = \exp(\delta\theta^\star_\delta c) \tag{9}$$

We denote by $z_s(\theta^\star_\delta)$, for all $s \in S$ the components of the vector $z(\theta^\star_\delta)$, and by $\pi = (\pi_s)_{s\in S}$ the stationary distribution of the Markov chain $(Y_{n\delta})_{n\in\mathbb{Z}}$ (note that π does not depend on δ). Then, the coefficients B_δ and C_δ are (from [14]):

$$B_\delta = \inf_{\substack{x>0\\ s\in S}} F(s,x) \tag{10}$$

$$C_\delta = \sup_{\substack{x>0\\ s\in S}} F(s,x) \tag{11}$$

where

$$F(s,x) = \frac{\sum_{t\in S(x)} \, p_\delta(t,s)\,\pi_t}{\sum_{t\in S(x)} \, p_\delta(t,s)\,z_t(\theta^\star_\delta)\exp(\theta^\star_\delta(\delta(r(t)-c)-x))}$$

and $S(x) = \{t \in S \,/\, x \leq \delta(r(t)-c)\}$. We now simplify the above formulas to get looser bounds B'_δ and C'_δ: in the denominator of $F(s,x)$,

$$1 \leq \exp(\theta^\star_\delta(\delta(r(t)-c)-x)) \leq \exp(\theta^\star_\delta \delta \hat{r})$$

and thus

$$\exp(-\theta^\star_\delta\delta\hat{r})\frac{\sum_{t\in S(x)} \, p_\delta(t,s)\,\pi_t}{\sum_{t\in S(x)} \, p_\delta(t,s)\,z_t(\theta^\star_\delta)} \leq F(s,x) \leq \frac{\sum_{t\in S(x)} \, p_\delta(t,s)\,\pi_t}{\sum_{t\in S(x)} \, p_\delta(t,s)\,z_t(\theta^\star_\delta)}.$$

From the definition of the set $S(x)$, the numerator and denominator in the above fractions are constant functions of x on every interval $(a,b]$, where a and b are two consecutive values of $\delta(r(h)-c)$ with $h \in S$. Thus, the extrema of the fraction over $x > 0$ are equal to the extrema over the values of x of the form $\delta(r(h)-c)$, where h is in the set $T = \{s \in S \,/\, r(s) > c\}$. Define

$$B'_\delta = \min_{\substack{h\in T\\ s\in S}} \frac{\sum_{r(t)\geq r(h)} \, p_\delta(t,s)\,\pi_t}{\sum_{r(t)\geq r(h)} \, p_\delta(t,s)\,z_t(\theta^\star_\delta)}\exp(-\theta^\star_\delta\delta\hat{r}) \tag{12}$$

$$C'_\delta = \max_{\substack{h\in T\\ s\in S}} \frac{\sum_{r(t)\geq r(h)} \, p_\delta(t,s)\,\pi_t}{\sum_{r(t)\geq r(h)} \, p_\delta(t,s)\,z_t(\theta^\star_\delta)}, \tag{13}$$

then $B'_\delta \leq B_\delta \leq C_\delta \leq C'_\delta$. It is easier to deal with B'_δ and C'_δ than with B_δ and C_δ because the extremum is over a finite set of variables. Note that the ratio between the two lower or upper bounds is no more than $\exp(\theta^\star_\delta\delta\hat{r})$, so that they have the same limit as δ tends to 0, and thus the simplification does not affect the tightness of the bounds in the continuous time model.

3 Upper and Lower Bounds

3.1 Exponential Decay Rate $\theta^\star$

We now show that the decay rate $\theta^\star_\delta$ defined in equation (9) for the discrete time model has a finite positive limit when δ goes to 0. The first step is to find a Taylor expansion of $\tau_k(\theta)$. As the larger root of the 2nd degree characteristic polynomial of $J_k(\theta)$, $\tau_k(\theta)$ has a simple explicit expression, and if we let δ tend to 0 in this expression, by using the fact that $p_k = \delta\lambda_k + o(\delta)$ and $q_k = \delta\mu_k + o(\delta)$, we find:

$$\tau_k(\theta) = 1 + \delta G_k(\theta) + o(\delta)$$

where we have defined

$$G_k(\theta) = \frac{1}{2}\left(\sqrt{(\lambda_k + \mu_k + r_k\theta)^2 - 4\mu_k r_k\theta} - (\lambda_k + \mu_k - r_k\theta)\right).$$

Let $G(\theta) = \sum_k N_k G_k(\theta)$, then from (8) we have

$$\tau(\theta) = 1 + \delta G(\theta) + o(\delta)$$

and thus

$$\lim_{\delta\to 0} \frac{1}{\delta}\log\tau(\theta) = G(\theta)$$

Note that $G(\theta)/\theta$ is nothing else than the effective bandwidth of the input process (see e.g. [12] and [6]). We also know from [14] (Proposition 2.3) that, for all $\theta > 0$,

$$\theta < \theta^\star_\delta \iff \frac{1}{\delta}\log\tau(\theta) < \theta c \qquad (14)$$

$$\theta > \theta^\star_\delta \iff \frac{1}{\delta}\log\tau(\theta) > \theta c \qquad (15)$$

The equation $G(\theta) = \theta c$ has a unique positive solution which we call $\theta^\star$. The existence and uniqueness of $\theta^\star$ come from the following properties of G, which are easily established: $G(0) = 0$, G is differentiable and strictly convex

$(G'' > 0)$, $G'(\theta)$ increases from $\bar{r}$ for $\theta = 0$ to $\hat{r}$ when θ goes to $+\infty$. From the above, we also have the following characterization of $\theta^\star$: for all $\theta > 0$

$$\theta < \theta^\star \iff G(\theta) < \theta c \tag{16}$$

$$\theta > \theta^\star \iff G(\theta) > \theta c \tag{17}$$

Now choose θ such that $0 < \theta < \theta^\star$, then $\lim_{\delta \to 0} \frac{1}{\delta} \log \tau(\theta) = G(\theta) < \theta c$. Thus, there is a $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$, $\frac{1}{\delta} \log \tau(\theta) < \theta c$, and then from (14), $\theta < \theta_\delta^\star$. This shows that $\liminf_{\delta \to 0} \theta_\delta^\star \geq \theta$, and because the inequality is true for any choice of θ in $(0, \theta^\star)$, we have $\liminf_{\delta \to 0} \theta_\delta^\star \geq \theta^\star$. In the same way, we prove that $\limsup_{\delta \to 0} \theta_\delta^\star \leq \theta^\star$, and we have finally

$$\lim_{\delta \to 0} \theta_\delta^\star = \theta^\star.$$

Unless $K = 1$, in which case $\theta^\star$ is given by a very simple formula (see section 5), there is no explicit expression for the decay rate $\theta^\star$, and the equation $G(\theta) = \theta c$ has to be solved numerically. Nevertheless, the analytical study of this equation yields asymptotic results in light or heavy traffic conditions. We consider $\theta^\star$ as a function of the parameter c varying between $\bar{r}$ and $\hat{r}$, then,

$$\text{for} \quad c \searrow \bar{r}: \quad \theta^\star = a(c - \bar{r}) + o(c - \bar{r}) \tag{18}$$

$$\text{for} \quad c \nearrow \hat{r}: \quad \theta^\star = \mu/(\hat{r} - c) - b + o(1), \tag{19}$$

where

$$\begin{aligned} a &= \left(\sum_k N_k r_k^2 \frac{\lambda_k \mu_k}{(\lambda_k + \mu_k)^3} \right)^{-1} \\ \mu &= \sum_k N_k \mu_k \\ b &= \frac{1}{\mu} \sum_k N_k \frac{\lambda_k \mu_k}{r_k}. \end{aligned}$$

In sections 3.2 and 3.3, we suppose that $\theta^\star$ is known and calculate the limits of B'_δ and C'_δ.

3.2 Lower Bound Coefficient

We show in this section that the coefficient B'_δ tends to a positive number B as δ goes to 0. The first step is to find a limit to the following fraction as δ tends to 0:

$$A_\delta(h,s) \;=\; \frac{\sum_{r(t)\geq r(h)} \; p_\delta(t,s)\, \pi_t}{\sum_{r(t)\geq r(h)} \; p_\delta(t,s)\, z_t(\theta^\star_\delta)} \tag{20}$$

Let $z_{k,0}(\theta^\star_\delta)$ and $z_{k,1}(\theta^\star_\delta)$ be the components of the eigenvector $z_k(\theta^\star_\delta)$, and t_k be the number of active sources of class k in state t, then from (7),

$$z_t(\theta^\star_\delta) = \prod_{1\leq k\leq K} z_{k,0}(\theta^\star_\delta)^{N_k - t_k}\, z_{k,1}(\theta^\star_\delta)^{t_k}.$$

We can obtain the limit of $z_t(\theta^\star_\delta)$ by calculating $z_{k,0}(\theta^\star_\delta)$ and $z_{k,1}(\theta^\star_\delta)$ explicitly and by letting δ tend to 0 in the above formula. We omit the details and give only the final formula in Lemma 3.1 below. We define for all $k = 1, \ldots, K$

$$u_{k,0} \;=\; \frac{1}{2(\lambda_k+\mu_k)}\left(\sqrt{(\lambda_k+\mu_k+r_k\theta^\star)^2 - 4\mu_k r_k\theta^\star} + (\lambda_k+\mu_k+r_k\theta^\star)\right) \tag{21}$$

$$u_{k,1} \;=\; \frac{1}{2(\lambda_k+\mu_k)}\left(\sqrt{(\lambda_k+\mu_k+r_k\theta^\star)^2 - 4\mu_k r_k\theta^\star} + (\lambda_k+\mu_k-r_k\theta^\star)\right) \tag{22}$$

It is easily seen that $0 < u_{k,1} < 1 < u_{k,0}$. For any state $t \in S$, we also define

$$u(t) = \prod_{1\leq k\leq K} u_{k,0}^{N_k - t_k}\, u_{k,1}^{t_k},$$

then the following result holds:

Lemma 3.1 *For all state t in S,*

$$\lim_{\delta\to 0} z_t(\theta^\star_\delta) = \pi_t / u(t)$$

The next lemma gives a Taylor expansion of $p_\delta(t,s)$. For any pair of states s and t in S, we denote by $|t-s|$ the number of sources whose state differs in s and t: $|t-s| = \sum_{k,i} |t^i_k - s^i_k|$.

Lemma 3.2 *For all s, t in S, there is a positive number $b(t,s)$ independent of δ such that*

$$p_\delta(t,s) = b(t,s)\delta^{|t-s|} + o(\delta^{|t-s|}).$$

Lemma 3.2 is established by writing the explicit expression of $p_\delta(t,s)$ in terms of p_k and q_k and by using the properties $p_k = \delta\lambda_k + o(\delta)$ and $q_k = \delta\mu_k + o(\delta)$. From the above two lemmas, we now derive that for each pair h, s with $h \in T$ and $s \in S$, $A_\delta(h,s)$ has a positive limit $A(h,s)$ when δ tends to 0, where $A(h,s)$ is defined as follows, by considering two cases:

1. $r(s) \geq r(h)$. In this case, in the numerator and denominator of $A_\delta(h,s)$, all terms of the sum tend to zero with δ except the term corresponding to $t = s$, so that we have simply

$$A(h,s) = u(s)$$

2. $r(s) < r(h)$. In this case, let i be the minimum of $|t-s|$ over all states t such that $r(t) \geq r(h)$. We now introduce the set $S(h,s) = \{t \in S \,/\, r(t) \geq r(h) \text{ and } |t-s| = i\}$, then,

$$A(h,s) = \frac{\sum_{t \in S(h,s)} b(t,s)\,\pi_t}{\sum_{t \in S(h,s)} b(t,s)\,\pi_t / u(t)}.$$

Recall that $B'_\delta = \exp(-\theta^\star_\delta \delta \hat{r}) \times \min\{A_\delta(h,s) \,/\, h \in T, s \in S\}$. As δ tends to 0, $A_\delta(h,s)$ tends to $A(h,s)$ for all pairs (h,s) with h in T and s in S, thus the minimum of $A_\delta(h,s)$ tends to the minimum of $A(h,s)$, because there is a finite number of pairs (h,s). The limit of $\exp(-\theta^\star_\delta \delta \hat{r})$ is 1 and so, as δ tends to 0, B'_δ tends to a positive number B defined by:

$$B = \inf\{A(h,s) \,/\, h \in T, s \in S\}.$$

The minimum of $A(h,s)$ is found without any difficulty: let $\hat{s}$ be the state where all the sources are in the state on ($s_k = N_k$ for all k), then, from $u_{k,1} < 1 < u_{k,0}$, we have $u(t) \geq u(\hat{s})$ for all state t, and thus $A(h,s) \geq u(\hat{s})$, which entails $B \geq u(\hat{s})$. On the other hand, $\hat{s}$ is in the set T ($r(\hat{s}) = \hat{r} > c$) and thus $B \leq A(\hat{s},\hat{s}) = u(\hat{s})$. Finally:

$$B \;=\; \prod_{1 \leq k \leq K} u_{k,1}^{N_k}. \tag{23}$$

3.3 Upper Bound Coefficient

By using the same line of arguments as in the previous section, one can show that the coefficient C'_δ tends to a finite positive number C as δ tends to 0, with

$$C = \sup\{A(h,s) \,/\, h \in T, s \in S\}.$$

Let m be a state in T such that $u(m)$ is maximum. Then, $A(h,s) \leq u(m)$ for all $h \in T$ and $s \in S$, and thus $C \leq u(m)$, but we also have $C \geq A(m,m) = u(m)$, hence $C = u(m)$. Calculating C is thus equivalent to finding some integers m_k satisfying the conditions:

$$0 \leq m_k \leq N_k \tag{24}$$
$$\sum_{1\leq k\leq K} m_k r_k > c \tag{25}$$
$$\prod_{1\leq k\leq K} u_{k,0}^{N_k - m_k} u_{k,1}^{m_k} \quad \text{is maximum.} \tag{26}$$

The problem of finding these numbers can be expressed as an integer linear programming problem of the knapsack type: let $m'_k = N_k - m_k$, then the above conditions can be rewritten as

$$0 \leq m'_k \leq N_k \tag{27}$$
$$\sum_{1\leq k\leq K} m'_k r_k < \hat{r} - c \tag{28}$$
$$\sum_{1\leq k\leq K} m'_k \log(u_{k,0}/u_{k,1}) \quad \text{is maximum.} \tag{29}$$

Note that for all k, $\log(u_{k,0}/u_{k,1}) > 0$. The coefficient C is then given by the formula

$$\log(C) \;\; = \;\; \log(B) + \sum_{1\leq k\leq K} m'_k \log(u_{k,0}/u_{k,1}). \tag{30}$$

In terms of complexity, the exact calculation of the coefficient C is a difficult problem: the simplest algorithm uses a Dynamic Programming approach where all possible combinations of numbers m'_k are considered, and has a complexity of $O(\prod_k N_k)$ (see Garfinkel and Nemhauser [7]). To overcome this difficulty, we consider the equivalent problem in real numbers: let D be the number defined by

$$\log(D) \quad = \quad \log(B) + \sum_{1 \leq k \leq K} x_k \log(u_{k,0}/u_{k,1}), \tag{31}$$

where the x_k are real numbers satisfying $0 \leq x_k \leq N_k$, $\sum_k x_k r_k \leq \hat{r} - c$, and such that $\sum_k x_k \log(u_{k,0}/u_{k,1})$ is maximum. Because the numbers m'_k have to satisfy stronger conditions than the x_k, we have clearly $C \leq D$, thus $D \exp(\theta^\star x)$ is also an upper bound of $P\{X > x\}$, although possibly not as tight as $C \exp(\theta^\star x)$. The gain in considering the coefficient D instead of C is that the computational complexity is much lower: assume that the classes are sorted in decreasing order of $\log(u_{k,0}/u_{k,1})/r_k$ (the sorting requires $O(K \log K)$ computations), then the numbers x_k are given by the following straightforward algorithm, of complexity $O(K)$. Initial parameters are Volume $= \hat{r} - c$, Value $= \log(B)$, $x_i = 0$ for all i, and $k = 0$.

While (Volume > 0)
  do
    $x_k = \min(N_k, \text{Volume}/r_k)$
    Volume $=$ Volume $-x_k r_k$
    Value $=$ Value $+x_k \log(u_{k,0}/u_{k,1})$
    $k = k + 1$
  done

The condition $0 < \hat{r} - c < \sum_k N_k r_k$ ensures the correct termination of the algorithm, and $\log(D)$ is given by the parameter Value at the exit of the while loop. The numerical comparisons in section 4 show that the difference between the two bounds $\log(C)$ and $\log(D)$ is relatively small, which suggests that, in most cases, the improvement of tightness from one bound to the other is not worth the extra computational cost.

We now summarize our results: we have defined a function $G(\theta)$ and have shown that the exponential decay rate $\theta^\star$ is the unique positive solution of the equation $G(\theta) = \theta c$. We have then introduced three coefficients B, C, and D depending on $\theta^\star$ and satisfying $B \leq C \leq D$. B has an explicit form (assuming $\theta^\star$ is known) and is easily calculated, $\log(C)$ is the value of an integer programming problem, and $\log(D)$ is an approximation of $\log(C)$ which is obtained through a simple and fast algorithm. These coefficients define bounds on the distribution of the backlog X in the following way:

$$\forall x \geq 0, \quad B \exp(-\theta^{\star} x) \leq P\{X > x\} \leq C \exp(-\theta^{\star} x) \leq D \exp(-\theta^{\star} x).$$

We conclude this section by making a remark which shows that, in some sense, the coefficients B and C are the best possible. We consider the case where the service rate c is such that the only state s of S with input rate $r(s)$ larger than c is the state $\hat{s}$, where all sources are on. This condition is true if we have $\hat{r} - \min_k r_k < c < \hat{r}$. In this case, it is easily seen that $C = B$, and thus, the lower and upper bounds are equal and give the exact queue length distribution: $P\{X > x\} = B \exp(-\theta^{\star} x)$. The bounds are then obviously the best possible. Although the study of this particular case is not necessarily of great practical importance, it suggests that the tightness of the coefficients B and C may not be easily improved.

4 Numerical Comparisons when $K = 2$

We now present some numerical experiments which compare our bounds to the exact value of the probability $P\{X > x\}$. For every system considered, the bounds were obtained by following the steps described in section 3: we solved numerically the equation $G(\theta) = \theta c$, calculated $\log(B)$, and derived $\log(C)$ through a Dynamic Programming algorithm. The exact backlog distribution was computed via the procedure described by Elwalid and Mitra in [6], by solving a linear system of differential equations. The difficult part in this procedure is to find all the eigenvalues and eigenvectors of a matrix of dimension $\prod_k (1 + N_k)$. This spectral analysis and the resolution of a linear system were implemented by using some functions of the library Meschach, a freeware package in C language for linear algebra (see reference manual [15]). In theory, the exact distribution can be obtained for any number of sources, but in practice, only very small systems can be studied because of the high computation time.

Figures 1, 2, and 3 show a few examples of the the models that we have considered, each has two different classes of traffic ($K = 2$) and a relatively small number of sources of each class ($N_k \leq 12$). The mean durations of off and on periods, respectively $1/\lambda_k$ and $1/\mu_k$, are given in seconds, and the rate r_k in Mb/s. In each figure, $\log_{10}(P\{X > x\})$ is plotted as a function of the queueing delay in the multiplexer, which is equal to x/c, and compared to the lower and to the two upper bounds. All three bounds are represented by straight lines with the same slope, and correspond to the three previously defined coefficients B, C, and D.

In Figure 1, the load $\bar{r}/c$ is 0.6, sources of class 1 represent voice channels, with parameters $1/\lambda_1 = 0.650$, $1/\mu_1 = 0.352$, $r_1 = 0.064$, and sources of class 2 model data streams with average on and off periods of 0.2 and 0.8 seconds and peak rate 320 Kb/s. In Figure 2, the parameters are the same but for the load which is taken equal to 0.40.

Because of the difficulty to analyze exactly large systems, it would be interesting to know how to quickly and accurately extrapolate the behavior of $P\{X > x\}$ with many sources from the study of smaller systems, where the exact distribution of X or some approximations of this distribution are more easily obtained. More precisely, let a be a positive integer, which will act as a "scaling factor", let X_a be the stationary backlog in a system with aN_k sources of class k and with a service rate equal to ac: the size of the system has been "multiplied" by a. How does $P\{X_a > ax\}$ compare to $P\{X > x\}$? We do not have any clear answer to this question; however, a very simple remark can be made about the scaling property of our bounds: we observe that the decay rate $\theta^\star$ does not depend on a (this is because the function $G(\theta)$ is linear in respect to the size of the system: $G_a(\theta) = aG(\theta)$), and if we let B_a and D_a be the coefficients of the bounds in the new system, it is also easily shown that $\log(B_a) = a\log(B)$ and $\log(D_a) = a\log(D)$. We have finally:

$$a(\log(B) - \theta^\star x) \;\leq\; \log(P\{X_a > ax\}) \;\leq\; a(\log(D) - \theta^\star x).$$

Thus we can easily compute the bounds for large systems from the bounds calculated for small systems. The above formula also reveals a difficulty: our ability to estimate $\log(P\{X > x\})$ is determined by the difference between the upper and the lower bound, namely $\log(D) - \log(B)$. This difference grows linearly with the size of the system, so that we may expect a loss of accuracy of at least one of the bounds when the number of sources is large. To address this difficulty, we now derive from the numerical results a couple of heuristic

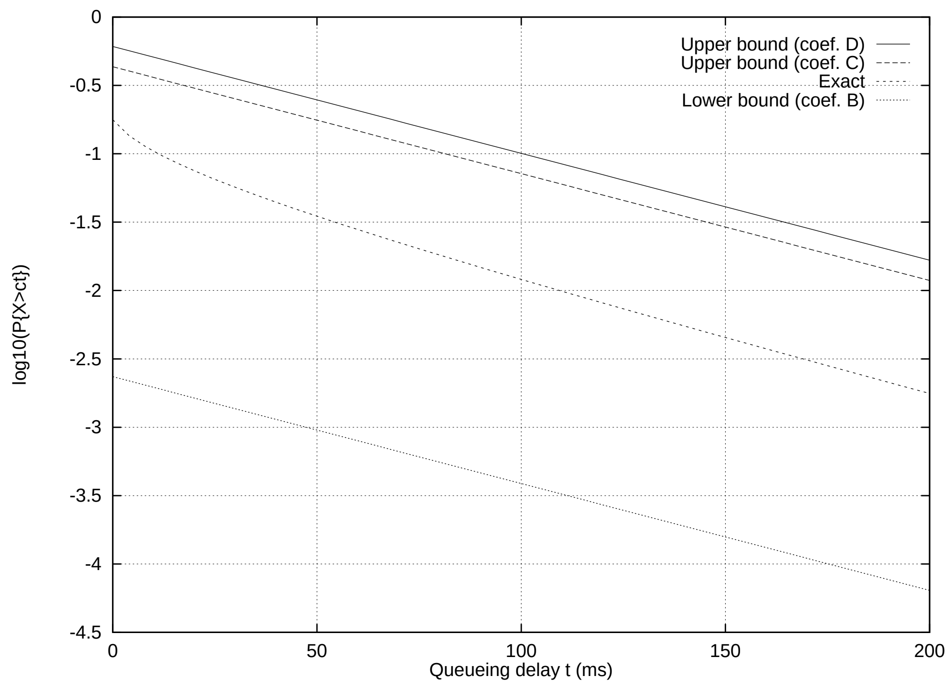

Figure 1: Load=0.6 $(N_1, 1/\lambda_1, 1/\mu_1, r_1) = (12, 0.650, 0.352, 0.064)$, $(N_2, 1/\lambda_2, 1/\mu_2, r_2) = (6, 0.8, 0.2, 0.32)$

rules which can help to decide whether the actual distribution lies closer to the upper or to the lower bound.

We first notice that in all the cases that we have considered, $\log(P\{X > x\})$ as a function of x is decreasing and appears to be convex. Thus, the difference with the upper bound (which is linear in x) is smallest for $x = 0$, and the difference with the lower bound is smallest for large x.

The second observation is that $\log(P\{X > x\})$ can be very close to the lower bound when x is not close to 0, and when the load of the system is very low, i.e. when the service rate c is of the same order as $\hat{r}$ (but still with $c < \hat{r} - \min_k r_k$). This phenomenon is illustrated in Figure 3 where we have

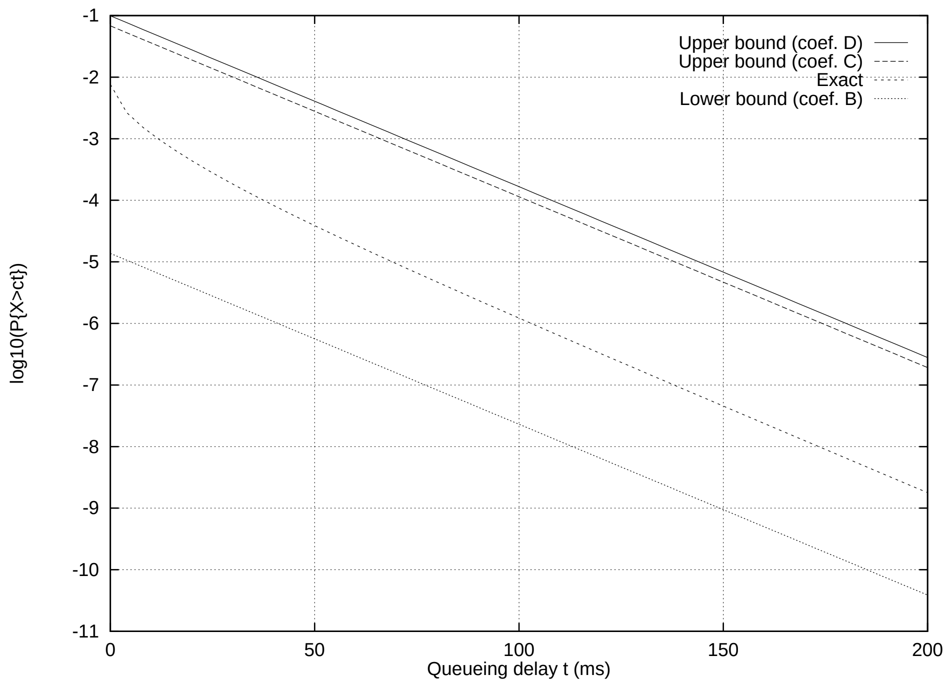

Figure 2: Load=0.4 $(N_1, 1/\lambda_1, 1/\mu_1, r_1) = (12, 0.650, 0.352, 0.064)$, $(N_2, 1/\lambda_2, 1/\mu_2, r_2) = (6, 0.8, 0.2, 0.32)$

taken $\bar{r}/c = 0.125$. However, for medium or high loads, the exact value is usually closer to the upper bound, even for larger x.

We continue this discussion in the next section, when there is only one class of traffic, and when the number of sources is larger.

5 Symmetrical Model ($K = 1$)

We consider a system with N sources of the same class, with parameters λ, μ, and r. We denote by c_1 the service rate per source, i.e. $c_1 = c/N$. In this

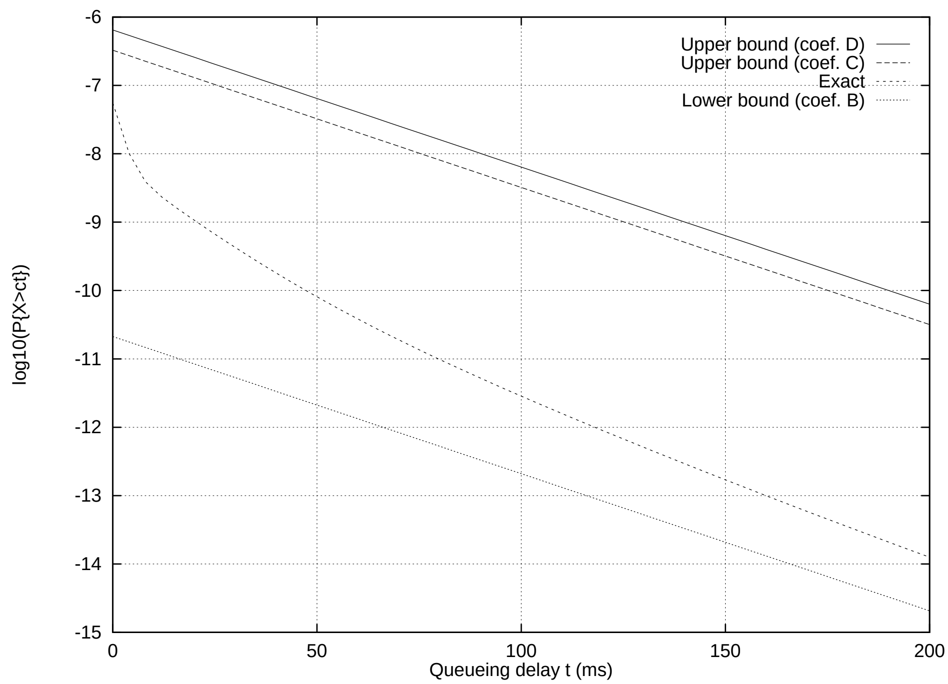

Figure 3: Load=0.125 $(N_1, 1/\lambda_1, 1/\mu_1, r_1) = (12, 9.0, 1.0, 1.0)$, $(N_2, 1/\lambda_2, 1/\mu_2, r_2) = (4, 9.0, 1.0, 2.0)$

case, the exponential decay rate $\theta^\star$ and the coefficients of our bounds have explicit expressions which we give in section 5.1. Then, we consider separately the model with or without buffer and compare numerically the upper bound with the exact distribution. When $K = 1$, the simplicity of the formulas and the ability to confront them with other known results enables us to get a more precise insight on the behavior of the bounds.

5.1 Decay rate and Bounds

When $K = 1$, the positive solution of the equation $G(\theta) = \theta c$ is :

$$\theta^\star = \frac{\mu}{r - c_1} - \frac{\lambda}{c_1}. \tag{32}$$

The above formula is not new and can be found for instance in [1] and [16]. If we report this value in (21) and (22) to calculate the numbers u_0 and u_1, we find:

$$\begin{aligned} u_0 &= \frac{\mu r}{(\lambda + \mu)(r - c_1)} \\ u_1 &= \frac{\lambda r}{(\lambda + \mu) c_1}. \end{aligned}$$

We now define $m = \lfloor c/r \rfloor + 1$, the integer m is the minimum number of active sources such that the total input rate exceeds c. Note that $m > Nc_1/r$. Then, the coefficients of the bounds are

$$\begin{aligned} B &= u_1^N \\ C &= u_0^{N-m} u_1^m \\ D &= (u_0^{1-c_1/r} u_1^{c_1/r})^N. \end{aligned}$$

We write $D = \exp(-NI(c_1))$, with $I(c_1) = -(1 - c_1/r)\log(u_0) - (c_1/r)\log(u_1)$. The formula for the coefficient D was previously known as a large deviation approximation for the overflow probability in a bufferless model with large N (see for instance Weiss [16]), our work shows that this approximation is really an upper bound. This upper bound was also obtained by Buffet and Duffield in [2] for a discrete time model: we can derive D as the limit of their formula, if we let the discrete time model which they study tend to a continuous time model, as we have done in section 2.

5.2 Buffered Model

We present here some numerical comparisons in the symmetrical system. Three different curves are plotted: the upper bound with coefficient C, the exact distribution, and an approximation for small buffers due to Hsu and Walrand (see

[9]). The exact value of $P\{X > x\}$ was computed by following the method proposed by Anick, Mitra, and Sondhi in [1], their approach is not fundamentally different from the heterogeneous case, but leads to a simpler and quicker algorithm, which makes it possible to analyze larger systems. The approximation found in [9] is of the form $A(N)\exp(-NC_2\sqrt{x})$, where $A(N)$ is an estimate of the probability that the input rate exceeds c, and the coefficient C_2 is derived for small buffer asymptotics by Weiss [16].

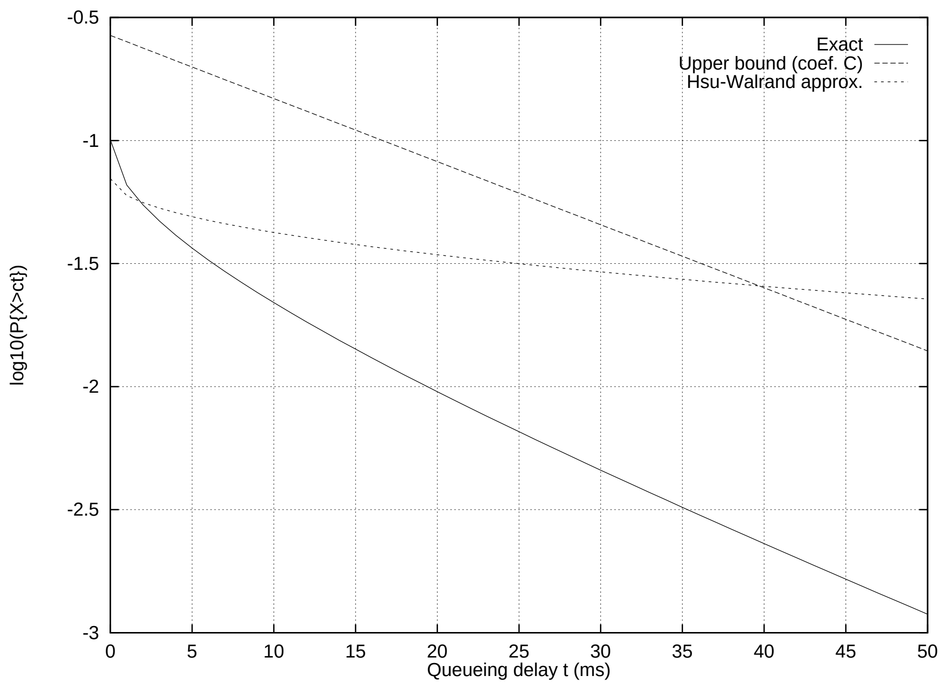

Figure 4: Load=0.82, $N = 100$

We consider a system of 100 sources modelling voice channels ($1/\lambda = 0.650$, $1/\mu = 0.352$, $r = 0.064$). In Figure 4, the service rate c is such that the load of the system is 0.82, and in Figure 5, the load is 0.66. The buffer occupation X is represented by the corresponding queueing delay in milliseconds. On both figures, we observe that our upper bound is close to the real distribution for small x ($x = c\times$delay), but, as noted also in the heterogeneous model, the gap may increase by several orders of magnitude for larger x. The small buffer approximation is also very close for small x.

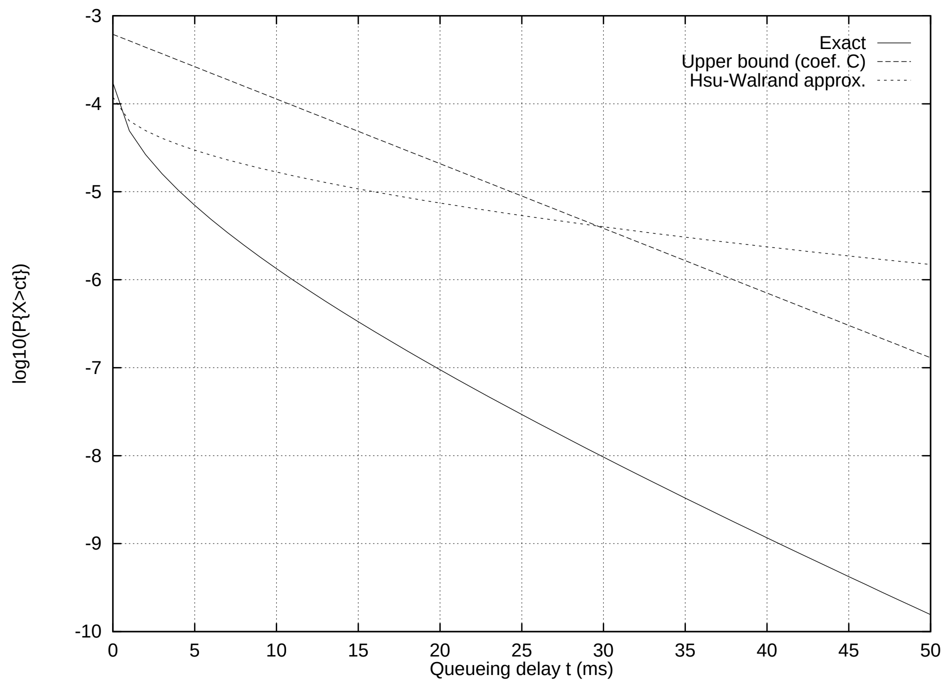

Figure 5: Load=0.66, $N = 100$

The lower bound with coefficient B is in both cases very inferior to the real probability ($\log(B)$ would be about -8 in Figure 4 for $x = 0$ and -18 in Figure 5), and was left out of the picture. This is not a surprise: as mentioned in section 4, the difference between $\log(D)$ and $\log(B)$ grows linearly with the size of the system and thus, at least one of the two bounds is expected to miss the exact value by a large margin when the number of sources increases. When $K = 1$, we have seen that $\log(D)$ is the large deviation approximation of $\log(P\{X > 0\})$

when N goes to $+\infty$, the difference between these two terms is thus $o(N)$, which is consistent with our observations: the upper bound $\log(D)$ is good when x is small, even for large N. This implies that $\log(P\{X > 0\}) - \log(B)$ has to grow linearly with N. Thus, for large systems, we can expect our lower bound to be very inferior to the exact probability, and the upper bound to be reasonably good.

5.3 Bufferless Model

We finally consider a system with no buffer. Let $\Phi(N)$ be the stationary probability that the input rate exceeds c, i.e. the probabiliy of having m or more emitting sources at one time, which is given by the formula

$$\Phi(N) = \sum_{k=m}^{N} \binom{N}{k} \left(\frac{\lambda}{\lambda+\mu}\right)^k \left(\frac{\mu}{\lambda+\mu}\right)^{N-k}.$$

We have $\Phi(N) \leq P\{X > 0\}$, because in the model with buffer, the queue length is positive whenever the input rate is larger than c, and thus $\Phi(N) \leq C$. In Figures 6 and 7, we compare numerically $\log(\Phi(N))$ and $\log(C)$ for up to 200 sources with two different sets of parameters. We also plot the approximation $\log(A(N))$ calculated by Hsu and Walrand in [9], $A(N)$ is a refinement of the large deviation formula and has the following form:

$$\log(A(N)) = -h(c_1) - \frac{1}{2}\log(N) - NI(c_1).$$

We can see on the curves that this estimate is very close to the exact value of $\log(\Phi(N))$. Recall that $\log(D) = -NI(c_1)$. If we now think of $\Phi(N)$ as an approximation of $P\{X > 0\}$, the above formula indicates that the difference between $\log(P\{X > 0\})$ and the upper bound $\log(D)$ is roughly given by $h(c_1) + \frac{1}{2}\log(N)$.

6 Conclusion

The multiplexing of Markovian on-off sources has received a lot of attention in the recent past, with most of the results concerning symmetrical systems. We

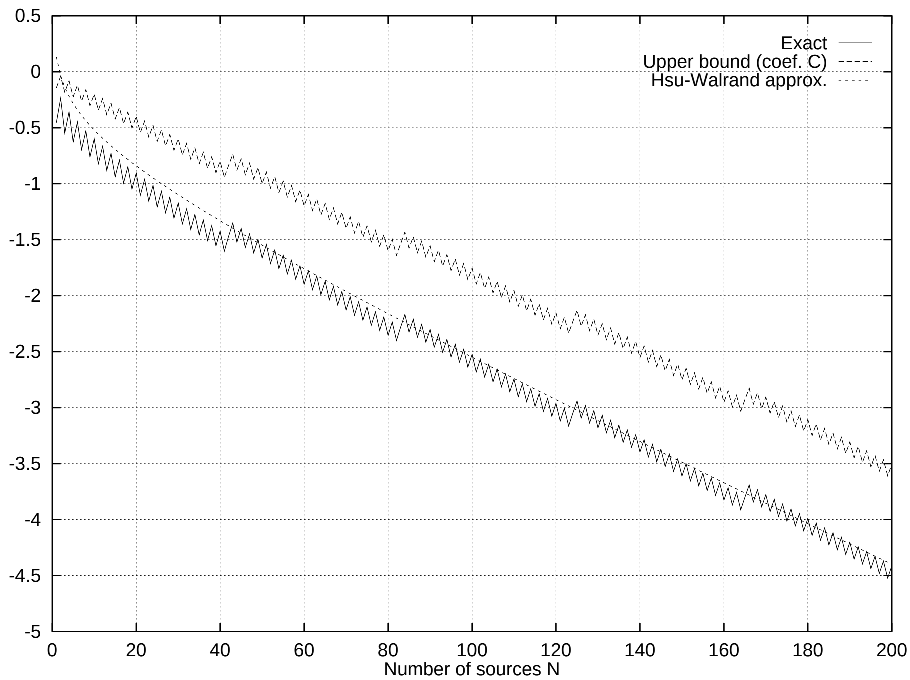

Figure 6: Plot of $\log_{10}(\Phi(N))$, with load=0.72 and $(1/\lambda, 1/\mu, r) = (0.650, 0.352, 0.064)$

have proposed exponential upper and lower bounds of the queue length distribution which are easily computed, and which hold for any number of different traffic classes. We have conducted numerical experiments to test the validity of the bounds, and have compared them with other authors' results when possible. When considering a symmetrical system, we retrieve some previously well known formulas as a special case. We have argued that in large systems, our lower bound may greatly underestimate the exact value, whereas the upper bound, as observed in the symmetrical case, is presumably reasonably close for small buffers.

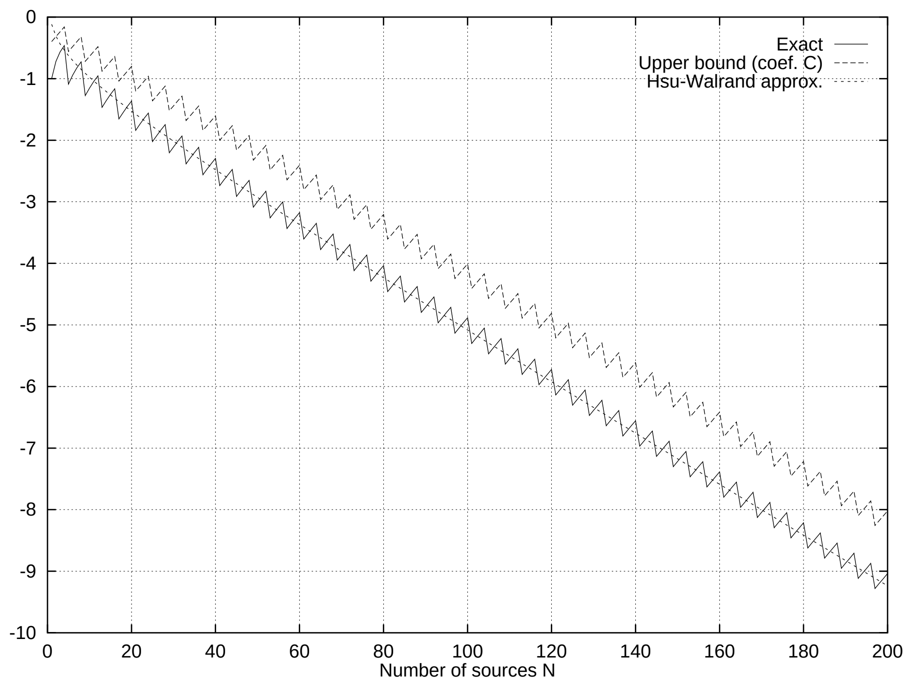

Figure 7: Plot of $\log_{10}(\Phi(N))$, with load=0.4 and $(1/\lambda, 1/\mu, r) = (45.0, 5.0, 1.0)$

A Approximation of X by X^δ

We prove here the following result, used in section 2.1:

$$\lim_{\delta \to 0} X^\delta = X \qquad \text{a. s.}$$

For all t in $\mathbb{R}$, we have $|r(Y_t^\delta) - r(Y_t)| \leq \hat{r}$, but if the process Y_t stays in the same state in some time interval $[n\delta, (n+1)\delta)$, then $Y_t^\delta = Y_t$ for all t in that interval. Furthermore, if $N(t)$ denotes the number of state transitions of the process Y in the interval $(-t, 0]$, there are no more than $N(t) + 1$ intervals of the form $[n\delta, (n+1)\delta)$ intersecting $(-t, 0]$ and containing a state transition of

Y, thus

$$\begin{aligned} |W_t^\delta - W_t| &= |\int_{-t}^{0} (r(Y_u^\delta) - r(Y_u))du| \\ &\leq \delta\hat{r}(N(t)+1), \end{aligned}$$

which yields

$$\frac{W_t^\delta}{t} \leq \frac{W_t}{t} + \delta\hat{r}\frac{(N(t)+1)}{t}.$$

It can be shown that, almost surely, W_t/t tends to $\bar{r} - c$ and $N(t)/t$ tends to a finite positive number ν as t goes to $+\infty$. We now adopt a sample path approach, and consider one given trajectory of the process Y such that the above limits exist. Because $\bar{r} - c < 0$, there is a positive number M_1 such that for any $t \geq M_1$, $W_t < 0$. Let δ_0 be a positive number such that $\bar{r} - c + \delta_0\hat{r}\nu < 0$, then, there is a number M_2 such that, for $t \geq M_2$, the term $W_t/t + \delta_0\hat{r}(N(t)+1)/t$, which tends to $\bar{r} - c + \delta_0\hat{r}\nu$, is negative. If $t \geq M_2$ and $\delta < \delta_0$, we have then

$$\begin{aligned} \frac{W_t^\delta}{t} &\leq \frac{W_t}{t} + \delta\hat{r}\frac{(N(t)+1)}{t} \\ &\leq \frac{W_t}{t} + \delta_0\hat{r}\frac{(N(t)+1)}{t} \\ &< 0. \end{aligned}$$

Let $M = \max(M_1, M_2)$, then for $t \geq M$ and for all $\delta < \delta_0$, W_t and W_t^δ are negative, which yields

$$\begin{aligned} X &= \sup_{0 \leq t \leq M} W_t \\ X^\delta &= \sup_{0 \leq t \leq M} W_t^\delta. \end{aligned}$$

Thus we can write for all $\delta < \delta_0$,

$$\begin{aligned} |X - X^\delta| &= |\sup_{0 \leq t \leq M} W_t - \sup_{0 \leq t \leq M} W_t^\delta| \\ &\leq \sup_{0 \leq t \leq M} |W_t - W_t^\delta| \\ &\leq \delta\hat{r}(N(M)+1). \end{aligned}$$

When we let δ tend to 0, $\delta\hat{r}(N(M)+1)$ tends to 0, and we have finally $\lim_{\delta\to 0} X^\delta = X$, which concludes the proof.

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