The Diamond Torus : a Cayley Graph on the 6-valent Grid

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N 2702

Novembre 1995

PROGRAMME 1

Rapport de recherche

ISSN 0249-6399

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Programme 1 — Architectures parallèles, bases de données, réseaux et systèmes distribués
Projet API

Rapport de recherche n° 2702 — Novembre 1995 — 19 pages

Abstract: We pursue our analysis of a family of Cayley graphs defined on the 6-valent grid from generators and relations and provided with a high level of symmetry. The grid $H = (V, E)$ is generated by three families of straight lines. We adopt the *non-isotropic* orientation $N \to S$, $NE \to SW$, $NW \to SE$ and define the system of generators $S = \{s_1, s_2, s_3\}$ whose elements are the three respective translations. The multiplication on S defines a group acting on the vertices of V with a basic set of relations. The "diamond torus" is the graph of a finite group generated by superimposing a cyclic relation for each direction. The diamond interconnection network has several important advantages. It has a bounded valence as a grid and the highest allowed valence for a 2D regular grid. As a Cayley graph, it allows recursive constructions and divide-and-conquer schemes for information dissemination, it is also vertex-transitive hence all routers will behave in a similar way. From construction it appears finally as a good host for embedding subvalent topologies like the usual grid. This paper is the replication of a former presentation of the *arrowhead torus* in the isotropic case.

Key-words: interconnection networks, Cayley graphs, hexavalent grid.

(Résumé : tsvp)

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Le tore en "rhombe" : un graphe de Cayley sur la grille 6-valente

Résumé : On poursuit l'analyse d'une famille de graphes de Cayley définis sur la grille 6-valente à partir de générateurs et de relations et pourvus d'un haut degré de symétrie. La grille $H = (V, E)$ est engendrée par trois familles de droites. On adopte l'orientation *anisotrope* $N \to S$, $NE \to SW$, $NW \to SE$ et on définit le système de générateurs $S = \{s_1, s_2, s_3\}$ dont les éléments sont les trois translations respectives. La multiplication sur S définit un groupe opérant sur les sommets de V selon un ensemble de relations de base. Le "tore en rhombe" est le graphe d'un groupe fini engendré en surimposant une relation cyclique sur chaque direction. Le réseau d'interconnexion en rhombe a plusieurs avantages importants. En tant que grille, il est de valence bornée, la plus haute valence possible pour une grille 2D régulière. En tant que graphe de Cayley, il est récursivement constructible et autorise des schémas de type diviser-conquérir pour la dissémination d'information, il est de plus sommet-transitif et tous les routeurs se comporteront ainsi de la même manière. Par construction, il apparaît finalement comme un hôte apte au plongement de topologies subvalentes comme la grille usuelle. Ce papier est le pendant d'une présentation antérieure du *tore en sagette* pour le cas isotrope.

Mots-clé : réseaux d'interconnexion, graphes de Cayley, grille hexavalente.

1 Introduction

We pursue our analysis of a family of Cayley graphs defined on the *6*-valent grid from generators and relations and provided with a high level of symmetry. The grid is generated by three families of straight lines with respective directions *N–S*, *SW–NE* and *SE–NW*. It had been pointed out in [4] that the *arrowhead torus* resulted from an *isotropic* configuration of the set of generators. Reversing any generator will yield a *non-isotropic*, diamond-shaped version of *6*-valent torus. Consequently, for the sake of clarity, we choose to name this peculiar graph a "*6*-valent diamond torus", or *diamond* for short. The reason is highlighted in Fig. 5.

The diamond is the graph of a discrete group generated from an abstract definition – or "presentation" – composed of a finite set of relations. We adopt the non-isotropic orientation $N \rightarrow S$, $NE \rightarrow SW$, $NW \rightarrow SE$ and define the system of generators $S = \{s_1,\, s_2,\, s_3\}$ whose elements are the three respective translations. The multiplication on S defines a group acting on the vertices of the grid according to the presentation. In order to generate a finite torus on the infinite grid we enlarge the presentation by imposing a cyclic relation for each direction. As for the arrowhead, a high symmetry is carried out by applying the same relation for any generator of S. The generation will follow the same recursive scheme : the smallest non-trivial cyclic group has order 2, so we decide to start the process in such a way that the new relation generates unidirectional cycles of length 2, then merely doubles their length at each step. The graph of the group is usually known as its *Cayley diagram.* It is a coloured digraph with one colour per generator. We require further that S be closed under inverses and define the diamond as the non-oriented version of the digraph. In the sequel, the digraph will be denoted by $\Gamma_{(n)}$ and the non-oriented diamond by $\mathcal{D}_n = (V_n, E_n)$.

The diamond is closely related to a more usual interconnection topology, the k-ary *2*-cube [2] and appears as a skewed k-ary *2*-cube with a peculiar value of k and on which a diagonal direction of links has been added. Consequently this feature provides the diamond with a good host capability for embedding subvalent grids like the usual grid.

Section 2 describes the generating process of the diamond. General conventions are first set up on the infinite *6*-valent grid. We analyse the recursive generation of $\Gamma_{(n)}$ and the cases $n = 1$ and $n = 2$ are emphasized before dealing with the general case. A definition of the non-oriented version is issued afterwards and an orthogonal representation of the diamond is also displayed. Section 3 exhibits some basic properties regarding vertices, edges and diameter. In particular, a recursive labelling scheme to define V_n as well as a recursive connecting scheme to define E_n are relevant. Section 4 concludes upon the similar properties between arrowhead and diamond and plans our future work. This paper is a word for word remake of the companion paper which gave a first presentation of the arrowhead [4]. For a relevant background about abstract groups and graphs of groups the reader should refer to it and references therein.

2 Generating the diamond

We first set up general metric conventions that define the infinite (*6*-valent) lattice in some convenient coordinate system and give the induced Cayley representation in terms of abstract definition of a group. Then, in order to generate a torus, we enlarge the previous definition by imposing a cyclic relation and show how the diamond can be recursively generated.

2.1 General conventions

Although the construction has a combinatorial nature, the diamond arises from a metrical definition of the lattice. We adopt the orientation *N-NW-SW-S-SE-NE* as shown in Fig. 1. The lattice $H = (V, E)$, composed of a set V of vertices and a set E of edges connecting pairs of vertices, is generated by three families of straight lines with respective directions *N–S*, *SW–NE* and *SE–NW*. For a reason of symmetry, we provide the Euclidean plane with a hexagonal coordinate system ("HCS") with the point 000 (named O) as the origin (see [3] for detail). Vertices are labelled with indices (h_1, h_2, h_3) in the HCS – the bar over an integer is a short notation meaning that it is negative –, so V is defined as :

$$V = \{(h_1, h_2, h_3) \in \mathbb{Z}^3 : h_1 + h_2 + h_3 = 0\} \tag{1}$$

and E is such that any vertex $\mathbf{h}$ is connected to the six neighbours $\mathbf{h} \pm\varepsilon_1$, $\mathbf{h} \pm\varepsilon_2$, $\mathbf{h} \pm\varepsilon_3$ where :

$$\Sigma = (\varepsilon_1, \varepsilon_2, \varepsilon_3) = ((1, 0, -1), (-1, 1, 0), (0, -1, 1)) \tag{2}$$

defines a *3*-fold system of generators in the HCS (Fig 1.a).

Without loss of generality we adopt an arbitrary, non-isotropic orientation for the lines, say $N \to S$, $NE \to SW$ and $NW \to SE$, or *S–SW–SE* for short. Note that there exist altogether eight possible orientations, split up into both types, say "isotropic" (such as *N–SW–SE* and *S–NE–NW* by a rotation of π) and "anisotropic" (such as *S–SW–SE* plus five equivalent configurations by $j\pi/3$ $(j = 1, ..., 5)$ rotational symmetries).

The above metric conventions lead to a Cayley representation for the lattice H in Fig. 1.b with the set of generators $\mathrm{S} = \{s_1, s_2, s_3\}$ whose elements are the three corresponding translations related to the system:

$$\Sigma' = (\varepsilon'_1, \varepsilon'_2, \varepsilon'_3) = (-\varepsilon_1, \varepsilon_2, \varepsilon_3) \tag{3}$$

The multiplication on S defines a group G acting on the vertices of V, with the set of relations

$$R_{(h)} : (s_2 s_3 = s_1) \ ; \qquad R_{(c)} : (s_2 s_3 s_2^{-1} s_3^{-1} = e) \tag{4}$$

where e is the identity. The *commutator* $R_{(c)}$ holds with any combination of elements of S because G is abelian. The relation $R_{(h)}$ holds on an elementary cycle of the *hexavalent* grid ; note that the dissymmetry results from the non-isotropic choice in (3). The abstract definition $\left(S; R_{(h)}, R_{(c)}\right)$ is called a *presentation* of G. It is clear that all elements of G will be of the form $s_1^{m_1} s_2^{m_2} s_3^{m_3}$ where the m_k take on integer values.

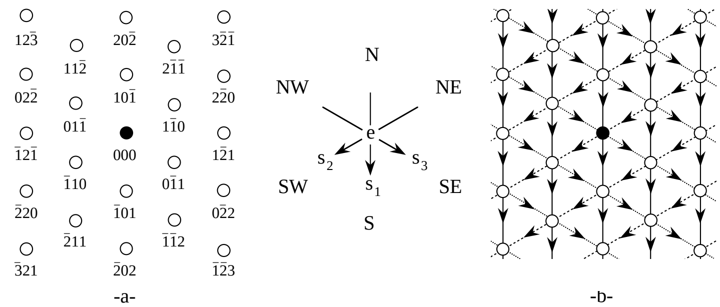


Figure 1: a/ The hexavalent grid in the hexagonal coordinate system
b/ Cayley graph of the group acting on the vertices of the grid

2.2 Generating $\mathcal{D}_n$

In order to build a torus, we choose to generate unidirectional cycles of length 2^n, so we impose now a new set of cyclic relations R_n defined by

$$R_n : (s_1^{2^n} = s_2^{2^n} = s_3^{2^n} = e) \tag{5}$$

and the presentation $\big(S; R_{(h)}, R_{(c)}, R_n\big)$.

2.2.1 Generating $\mathcal{D}_1$ and $\mathcal{D}_0$

Consider the case $n = 1$, that is : $s_1^2 = s_2^2 = s_3^2 = e$ and define the set $S_{(1)} = \{e, s_1, s_2, s_3\}$. For consistency with the following we set up $S^*_{(0)} = \{e\}$ and $S^*_{(1)} = S_{(1)}$. The presentation $(S; R_{(h)}, R_{(c)}, R_1)$ clearly defines a finite abelian group on the set $S^*_{(1)}$ displayed in the multiplication table of Fig. 2. Any binary operation is computable from the presentation. For example we have :

$$s_2 s_3 \overset{R_{(h)}}{=} s_1$$

$$s_1 s_2 \overset{R_{(h)}}{=} s_2 s_3 s_2 \overset{R_{(c)}}{=} s_2^2 s_3 \overset{R_1}{=} s_3$$

(" $\overset{\rho}{=}$" means that ρ is the relevant relation). Let G_1 be the set of elements of G that now equal e as a direct consequence : those elements are of the form $s_1^{2m_1} s_2^{2m_2} s_3^{2m_3}$. We claim that G_1 is a normal subgroup of G of index *4*. Clearly G_1 is a subset of G and whenever s and t belong to G_1, then st^{-1} belongs to G_1. So G_1 is a subgroup of G. Furthermore, since G is abelian, G_1 is normal. Since G_1 is a normal subgroup of G the function $\phi_1 : G \to G/G_1$ defined by $\phi_1(s) = sG_1$ is a homomorphism. The image of ϕ_1 is the factor group

$$G/G_1 = \{G_1, s_1 G_1, s_2 G_1, s_3 G_1\} \tag{6}$$

$S^*_{(1)}$	e	s_1	s_2	s_3
e	e	s_1	s_2	s_3
s_1	s_1	e	s_3	s_2
s_2	s_2	s_3	e	s_1
s_3	s_3	s_2	s_1	e

Figure 2: Multiplication table of $S^*_{(1)}$

and its kernel is precisely G_1. The cosets of G/G_1 have elements of the form:

$$\begin{array}{rcl} G_1 & : & (s_1^{2m_1} s_2^{2m_2} s_3^{2m_3}) \\ s_1 G_1 & : & (s_1^{2m_1+1} s_2^{2m_2} s_3^{2m_3}) \\ s_2 G_1 & : & (s_1^{2m_1} s_2^{2m_2+1} s_3^{2m_3}) \\ s_3 G_1 & : & (s_1^{2m_1} s_2^{2m_2} s_3^{2m_3+1}) \end{array}$$

and for any s', $s'' \in G$ we have $s'G_1 s''G_1 = s's''G_1$. One could check that – as a consequence of the First isomorphism theorem – the table of G/G_1 and the table of $S^*_{(1)}$ display two isomorphic groups: the group of cosets and the group of their representative. It should also be pointed out that there exist exactly two groups of order 4, namely the Klein's four-group (or dihedral group D_2) and the cyclic group C_4; the reader can refer to the relevant literature to check that the table of $S_{(1)}$ displays a group isomorphic to D_2. Figure 3 displays the *Cayley diagram* of the group $S^*_{(1)}$ – let us call it $\Gamma_{(1)}$. Orientation *S–SW–SE* of all connections are emphasized for clarity. A vertex x stands for one element of the group and there exists one arc for each entry $x \rightarrow xs$ ($x \in S^*_{(1)}, s \in S$) in the multiplication table.

Finally we define the *diamond* $\mathcal{D}_1 = (V_1 = S^*_{(1)}, E_1)$ as the non-oriented version of the digraph $\Gamma_{(1)}$.The torus $\mathcal{D}_1$ is thus a regular *2*-graph with four vertices and is readily *6*-valent. In addition we define $\mathcal{D}_0 = (V_0 = S^*_{(0)}, E_0)$ as the non-oriented version of the Cayley diagram $\Gamma_{(0)}$ of the trivial group defined by the presentation $(S; R_{(h)}, R_{(c)}, R_0) = (S; R_0)$.

2.2.2 Generating $\mathcal{D}_2$ from $\mathcal{D}_1$

Given the relation $R_2 : (s_1^4 = s_2^4 = s_3^4 = e)$ and the presentation $(S; R_{(h)}, R_{(c)}, R_2)$, let us define the set

$$S_{(2)} = \{e,\ s_1^2,\ s_2^2,\ s_3^2\}.$$

$S_{(2)}$ is a group and its multiplication table would display that $S_{(2)}$ is isomorphic to $S_{(1)}$. We should be aware that the relation R_2 replaces the relation R_1 and that R_1 is no longer

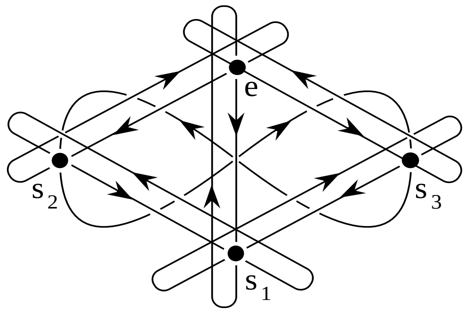


Figure 3: Graph of the group $S^*_{(1)}$

fulfilled in this presentation – note incidentally that $R_1 \Rightarrow R_2$. Any operation is computable from the presentation and it can be easily shown from (4) that:

$$s_\alpha^2\, s_\beta^2 = s_\gamma^2$$

where α, β, γ are all distinct. Let G_2 be the set of elements of G that equal e as a direct consequence of R_2, clearly:

$$G_2 = \{s \in G : s = s_1^{4m_1} s_2^{4m_2} s_3^{4m_3};\quad m_k \in \mathbb{Z}\}$$

and for the same reason as above, G_2 is a normal subgroup of G but it is also normal in G_1 hence G_1 is decomposable into the cosets:

$$\begin{array}{rcl} G_2 & : & (s_1^{2(2m_1)} s_2^{2(2m_2)} s_3^{2(2m_3)}) \\ s_1^2 G_2 & : & (s_1^{2(2m_1+1)} s_2^{2(2m_2)} s_3^{2(2m_3)}) \\ s_2^2 G_2 & : & (s_1^{2(2m_1)} s_2^{2(2m_2+1)} s_3^{2(2m_3)}) \\ s_3^2 G_2 & : & (s_1^{2(2m_1)} s_2^{2(2m_2)} s_3^{2(2m_3+1)}) \end{array}$$

and we obtain a factor group:

$$G_1/G_2 = \{G_2, s_1^2 G_2, s_2^2 G_2, s_3^2 G_2\} \tag{7}$$

isomorphic to $S_{(2)}$. The index of G_2 in G_1 is $[G_1 : G_2] = 4$, therefore the index of G_2 in G is : $[G : G_2] = [G : G_1][G_1 : G_2] = 16$. More precisely – refer to the Third isomorphism theorem – G_1/G_2 is a normal subgroup of G/G_2 and then :

$$(G/G_2)/(G_1/G_2) = \{G_1/G_2, (G_1/G_2)s_1, (G_1/G_2)s_2, (G_1/G_2)s_3\}. \tag{8}$$

Let us define $I = (0, 1, 2, 3)$ and consider the set of representatives :

$$S^*_{(2)} = S_{(2)} S^*_{(1)} = \{s_{q_1}^2\, s_{q_0} \mid q_0, q_1 \in I\}. \tag{9}$$

It should be clear from (7) and (8) – and the First isomorphism theorem – that G/G_2 is isomorphic to $S^*_{(2)}$ and we shall say that the words of $S^*_{(2)}$ have the "canonical" form for the given presentation. To carry out a full connection of the Cayley diagram $\Gamma_{(2)}$ in Fig. 4 with arcs $x \to xs$ ($x \in S^*_{(2)}, s \in S$), we detail the following relations

$$
\begin{array}{ll}
 & s_\alpha\, s_\beta \overset{R_{(h)}}{=} s_1 \\
R' : & s_\alpha\, s_1 \overset{R_{(h)},R_{(c)}}{=} s_\alpha(s_\alpha\, s_\beta) = s_\alpha^2 s_\beta \\
 & \\
 & s_\alpha^2\, s_\beta^2 \overset{R_{(c)}}{=} (s_\alpha s_\beta)^2 \overset{R_{(h)}}{=} s_1^2 \\
R'' : & s_\alpha^2\, s_1^2 \overset{R_{(c)}}{=} (s_\alpha s_1)^2 \overset{R'}{=} s_\alpha^4\, s_\beta^2 \overset{R_2}{=} s_\beta^2 \\
 & \\
 & s_\alpha^3\, s_\beta = s_\alpha^2(s_\alpha s_\beta) \overset{R_{(h)}}{=} s_\alpha^2\, s_1 \\
 & s_\alpha^3\, s_1 = s_\alpha^2(s_\alpha s_1) \overset{R'}{=} s_\alpha^2(s_\alpha^2 s_\beta) = s_\alpha^4 s_\beta \overset{R_2}{=} s_\beta \\
 & s_1^3\, s_\alpha = s_1^2(s_1 s_\alpha) \overset{R'}{=} s_1^2(s_\alpha^2 s_\beta) = (s_1^2 s_\alpha^2) s_\beta \overset{R''}{=} s_\beta^2 s_\beta \\
 & \\
 & s_1^2\, s_\alpha\, s_\beta \overset{R_{(h)}}{=} s_1^2\, s_1 \\
 & s_\alpha^2\, s_\beta\, s_1 \overset{R'}{=} s_\alpha^2(s_\beta^2 s_\alpha) = (s_\alpha^2 s_\beta^2) s_\alpha = s_1^2\, s_\alpha
\end{array}
$$

between elements in S, where $\alpha \neq \beta \neq 1$. They help to transform any product xs into the canonical form of its representative (white vertices are replications of their homologue in Fig. 4). Finally we define the diamond $\mathcal{D}_2 = (V_2, E_2)$ as the non-oriented version of the digraph $\Gamma_{(2)}$. Incidentally, we note that this graph was known by Shrikhande [5] about Latin Square designs and association schemes and mentioned by Biggs [1] because some special properties of its spectrum.

2.2.3 Generating $\mathcal{D}_n$

Theorem 1 *The presentation $\left(S; R_{(h)}, R_{(c)}, R_n\right)$ defines a group G/G_n of order 4^n isomorphic to the set of representatives:*

$$S^*_{(n)} = \{ s_{q_{n-1}}^{2^{n-1}} s_{q_{n-2}}^{2^{n-2}} \ldots s_{q_1}^2 s_{q_0} \mid q_k \in I = (0,1,2,3), (0 \le k \le n-1)\}.$$

Proof Applying the Third isomorphism theorem enables an inductive proof. Define :

$$G_k = \{ s \in G \mid R_k \Rightarrow (s = e)\} = \{ s_1^{2^k m_1} s_2^{2^k m_2} s_3^{2^k m_3};\quad m_k \in \mathbb{Z}\}$$

$$S_{(k)} = \{ e,\, s_1^{2^{k-1}},\, s_2^{2^{k-1}},\, s_3^{2^{k-1}} \}$$

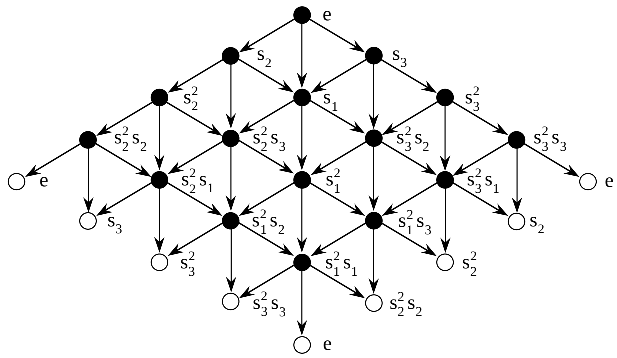


Figure 4: Graph of the group $S^*_{(2)}$

for any k $(1 \leq k \leq n)$. Clearly $G_n \subset G_{n-1} \subset G$ and both G_{n-1} and G_n are normal in G, then G_{n-1}/G_n is a normal subgroup of G/G_n and the quotient $(G/G_n)/(G_{n-1}/G_n)$ is isomorphic to G/G_{n-1}. But by induction we know that $[G : G_{n-1}] = 4^{n-1}$ and that G/G_{n-1} is isomorphic to the set

$$S^*_{(n-1)} = S_{(n-1)} \ldots S_{(2)} S_{(1)}.$$

Furthermore $S_{(n)}$ is readily a group and is the set of representatives of G_{n-1}/G_n. Hence the index of G_n in G is $[G : G_n] = [G : G_{n-1}][G_{n-1} : G_n] = 4^n$ and the set :

$$S^*_{(n)} = S_{(n)} S^*_{(n-1)} = \{s's'' \in G \mid s' \in S_{(n)}, s'' \in S^*_{(n-1)}\}$$

is the set of representatives of G/G_n. □

Let us call $\Gamma_{(n)}$ the digraph of the group $S^*_{(n)}$. To carry out a full connection of $\Gamma_{(n)}$ with arcs $x \to xs$ $(x \in S^*_{(n)}, s \in S)$ we must adopt a reduction process to turn any word xs into one – and only one – canonical form $s^{2^{n-1}}_{q'_{n-1}} s^{2^{n-2}}_{q'_{n-2}} \ldots s^2_{q'_1} s_{q'_0}$ by using the relations of the presentation as it was done for $n = 2$. A convenient way is provided by the above induction which defines a *compound scheme* of $\Gamma_{(n)}$ from four "copies" of $\Gamma_{(n-1)}$, one copy corresponding with one coset of G_{n-1}/G_n. We process as follows:

1. Remove any arc in $\Gamma_{(n-1)}$ induced by the relation R_{n-1} – since R_{n-1} is no longer fulfilled – and perform the operation for each copy.

2. Replace any missing arc in $\Gamma_{(n)}$ by a new arc induced by the relation R_n.

Figure 5: A view of $\mathcal{D}_6$ as a compound of $\mathcal{D}_1$ by $\mathcal{D}_5$

Indeed a more general compound scheme can be settled from any subgroup of G as detailed thereafter.

Corollary 2 *For any k $(1 \leq k \leq n-1)$, $\Gamma_{(n)}$ can be arranged as a compound of 4^{n-k} copies of $\Gamma_{(k)}$.*

Proof $G_n \subset G_k \subset G$ and both G_k and G_n are normal in G. Therefore G_k/G_n is normal in G/G_n and the quotient $(G/G_n)/(G_k/G_n)$ is isomorphic to $S^*_{(k)}$ whose graph is $\Gamma_{(k)}$. The number of copies of $\Gamma_{(k)}$ is $[G_k : G_n]$, that is the index of G_n in G_k. $\square$

As above the connecting scheme consists in removing any arc induced now by the relation R_k, in each of the 4^{n-k} copies of $\Gamma_{(k)}$; then applying R_n to replace any missing arc in $\Gamma_{(n)}$. Two typical values of k are relevant, namely $k = 1$, so $\Gamma_{(n)}$ will be arranged with 4^{n-1} copies of the graph $\Gamma_{(1)}$ displayed in Fig. 3 ; or $k = n-1$, as in the above case highlighted for $n = 6$ in Fig. 5 – for clarity, only the metric distribution of the vertex set is drawn.

2.3 Definition of the Diamond

Definition 3 *The diamond $\mathcal{D}_n = (V_n, E_n)$ is the non-oriented version of the digraph $\Gamma_{(n)}$, the Cayley graph of the group $\big(S; R_{(h)}, R_{(c)}, R_n\big)$. It has $N = 4^n$ vertices and $3 \cdot 4^n$ edges. The set of vertices is $V_n = S^*_{(n)}$. In addition we require that S be closed under inverses so that the set of edges can be defined from $\Gamma_{(n)}$ as $E_n = \{\{x, y\} \in V_n \times V_n \mid x^{-1}y \in S\}$.*

All definitions which will be set up in the sequel for the diamond result from the generating process of $\Gamma_{(n)}$. In particular, the compound scheme of $\Gamma_{(n)}$ will be applied on $\mathcal{D}_n$ as well. A complete view of $\mathcal{D}_0$, $\mathcal{D}_1$, $\mathcal{D}_2$ is displayed in Fig. 6 (compare with the digraphs $\Gamma_{(1)}$ and $\Gamma_{(2)}$ in Fig. 3 and Fig. 4).

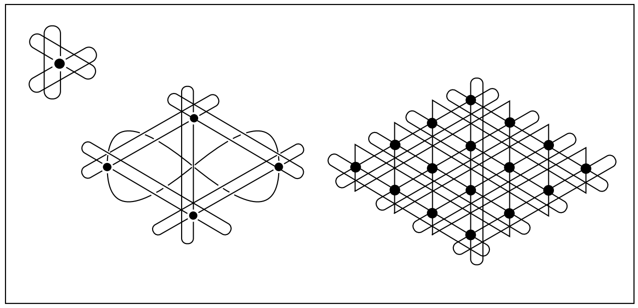


Figure 6: A view of $\mathcal{D}_0$, $\mathcal{D}_1$, $\mathcal{D}_2$

2.4 Orthogonal representation of the diamond

An orthogonal representation of the diamond is depicted in Fig. 7a for $n = 2$. In the case of infinite grids, it has been shown in [3] that both representations belong to the same topological (and therefore combinatorial) type of grid. The orthogonal representation provides a straightforward notation for labelling the vertices in the Cartesian coordinate system and there exists an easy correspondence between the hexagonal system of Fig. 1a and the usual Cartesian system (see again [3] for detail). Since both graphs are equivalent, we shall call the transformed one the *orthogonal* diamond $\mathcal{D}_n$, or still the *diamond* for short, if it is clear from context.

The orthogonal $\mathcal{D}_n$ is closely related to the k-ary 2-cube [2], with $k = 2^n$ and on which a "diagonal" direction of links has been added. Fig. 7b displays a combination of three k-ary 2-cubes embedded onto $\mathcal{D}_2$. Each one results from a restriction of the set of relations in the presentation of $\mathcal{D}_n$ and can be defined by :

$$s_\alpha s_\beta s_\alpha^{-1} s_\beta^{-1} = e \ ; \qquad s_\alpha^{2^n} = s_\beta^{2^n} = e$$

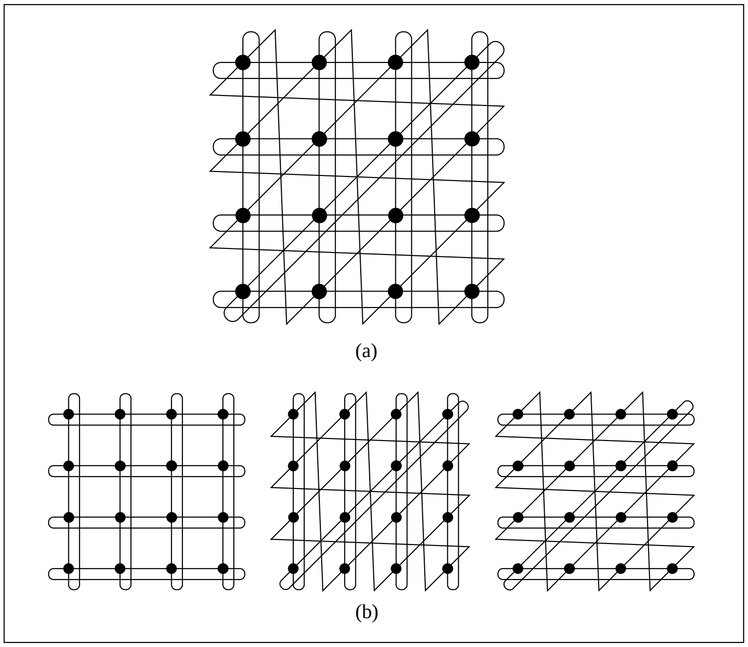
successively for $(\alpha, \beta) = (2, 3), (1, 3), (1, 2)$. This feature shows that the diamond appears as a good host for embedding subvalent grids like the usual grid.

3 Basic topological properties

3.1 Vertices

3.1.1 Vertex-transitivity

Vertex-transitivity (or vertex-symmetry) is an interesting property by which all vertices have a similar behaviour. One can easily check that, as a Cayley graph, $\mathcal{D}_n$ is vertex-transitive.



(a)

(b)

Figure 7: a/ Orthogonal representation of $\mathcal{D}_2$
b/ A combination of *4*-ary *2*-cubes embedded onto $\mathcal{D}_2$

In other words, for every pair of vertices there exists an automorphism of $\mathcal{D}_n$ that maps one into the other. For any fixed vertex $a \in V_n$ define the permutation $\pi_a : \sigma \to a\sigma$ ($\sigma \in V_n$) and check that adjacency is preserved by π_a. For any edge $\{x, y\} \in E_n$, the above definition of $\mathcal{D}_n$ implies $x^{-1}y \in S$. But $x^{-1}y = (ax)^{-1}(ay)$ hence $\{ax, ay\} \in E_n$ that is $\{\pi_a(x), \pi_a(y)\} \in E_n$.

3.1.2 A recursive labelling scheme

The compound schemes of $\mathcal{D}_n$ provide a straightforward quaternary notation with words of n digits $Q_n = q_{n-1}q_{n-2}\ldots q_1q_0$ for numbering the elements of V_n.

1. Assume first we compose $\mathcal{D}_1$ by $\mathcal{D}_{n-1}$. Let ${}^i\mathcal{D}_{n-1}$ be the root, south, southwest or southeast copy of $\mathcal{D}_{n-1}$ respectively for $i = 0, 1, 2$ or 3. To any vertex labelled $Q_{n-1} = q_{n-2}\ldots q_1q_0$ in ${}^i\mathcal{D}_{n-1}$ we assign the new word $Q_n = i\,Q_{n-1}$ in $\mathcal{D}_n$.
2. Assume now we compose $\mathcal{D}_{n-1}$ by $\mathcal{D}_1$. A vertex labelled Q_{n-1} in $\mathcal{D}_{n-1}$ is replaced by the tetrad $\{Q_n = Q_{n-1}\,i \; ; \; i \in I\}$ in $\mathcal{D}_n$.

Both schemes are consistent in the sense that the notation will be the same whatever compound alternative we choose. This combinatorial notation allows to bypass any metric notation that would have been defined on the infinite grid. An illustration of the labelling scheme is given in Fig. 8 for $\mathcal{D}_2$ and $\mathcal{D}_3$. Let us now define in V_n the subset :

$$\mathbf{4}^k\, V_n = \{x \in V_n : \; x \equiv 0 \; (mod\; 4^k)\} \tag{10}$$

for any k ($0 \le k \le n$). Clearly we have : $\mid \mathbf{4}^k\, V_n \mid = 4^{n-k}$. It may be useful to partition V_n into a hierarchy of $n+1$ members :

$$V_n = \{0\} \cup \bigcup_{p=1}^{n} \left(\mathbf{4}^{n-p}\, V_n - \mathbf{4}^{n-p+1}\, V_n\right) \tag{11}$$

and to regard any p–member ($p \ge 1$) as a “descendant” of the $(p-1)$–member.

3.1.3 Labelling vertices in the hexagonal coordinate system

It will be sometimes convenient to express V_n in the hexagonal coordinate system of Sect. 2.1. Let $\Sigma' = (\varepsilon'_1, \varepsilon'_2, \varepsilon'_3)$ be the system of generators defined in (3) and ε'_0 the zero vector. Since an arc $x \to xs$ in $\Gamma_{(n)}$ denotes a translation in the space spanned by Σ', then the coordinates $\mathbf{h}_{Q_n} = (h_1, h_2, h_3)$ of a vertex labelled with the word Q_n in the 4–ary system satisfy

$$\mathbf{h}_{q_{n-1}q_{n-2}\ldots q_1q_0} = \sum_{k=0}^{n-1} 2^k \varepsilon'_{q_k}.$$

for the *generic* representation in Fig. 5. Moreover, it would be easy to show by induction on n and using the compound scheme of $\Gamma_{(n)}$ from $\Gamma_{(n-1)}$ that V_n can be shortly defined from (1) as :

$$V_n = \{(h_1,\, h_2,\, h_3) \in V : -(2^n - 1) \le h_1 \le 0 \; ; \quad 0 \le h_2,\, h_3 \le 2^n - 1\}$$

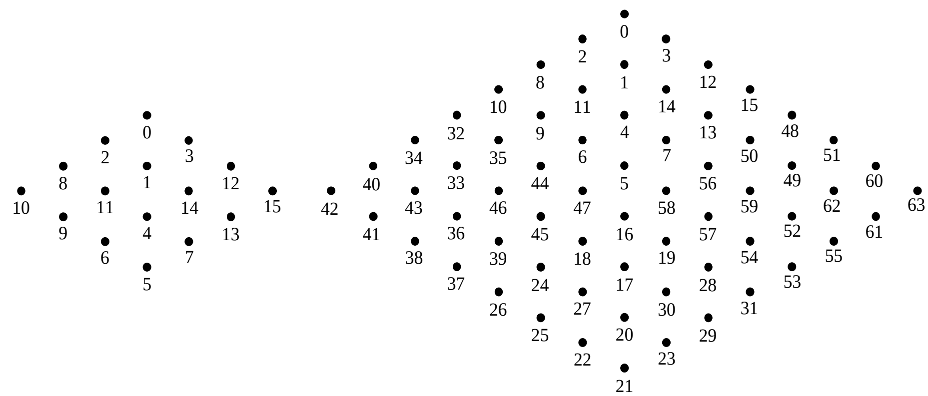


Figure 8: Labelling $\mathcal{D}_2$ and $\mathcal{D}_3$

As far as the *orthogonal* diamond is concerned, its vertex set can be shortly defined from (1) as :

$$\widehat{V}_n = \{(x, y) \in \mathbb{Z}^2 : 0 \leq x, y \leq 2^n - 1\}$$

(see also [3] for detail).

3.2 Edges

3.2.1 Edge-connectivity

The edge-connectivity λ of a graph is the minimum number of edges whose removal disconnects the graph. Knowing λ involves the number of disjoint paths joining a given pair of vertices and is helpful for finding disjoint spanning trees for information dissemination.

Theorem 4 *The diamond $\mathcal{D}_n$ has an edge-connectivity $\lambda(\mathcal{D}_n) = 6$.*

Proof We assume the non-trivial case $n > 0$. First the valence gives a upper bound of the edge-connectivity, so $\lambda(\mathcal{D}_n) \leq 6$. We must show that $\lambda(\mathcal{D}_n) \geq 6$ and find that for any pair of vertices (x, y) there exist at least six edge-disjoint paths between x and y. There are three cycles (of length 2^n) going through any vertex and induced by the three generators of S. Let us denote $c_{x,i}$ such a cycle going through x in the direction i and $c_{y,j}$ a cycle going through y in the direction j ($i,\ j = 1, 2, 3$). In general, all those cycles are edge-disjoint, $c_{x,i}$ and $c_{y,j}$ ($i \neq j$) intersect at one vertex, $z_{i,j}$ say, and there exist exactly six intersections. We analyse all the paths of the form $(x, z_{i,j}, y)$ with one change of direction. For a given $z_{i,j}$ there exist four paths depending on whether we follow the direction s_i or s_i^{-1} from x to $z_{i,j}$ and the direction s_j or s_j^{-1} from $z_{i,j}$ to y. The rule is that the chosen path should not cut through another intersecting vertex. There exists one such path $(x, z_{i,j}, y)$ for a given $z_{i,j}$ leading to six edge-disjoint paths for all $z_{i,j}$, according to this rule. There is a particular case when x

and y belong to a same unidirectional cycle. Without loss of generality, assume $c_{x,1} = c_{y,1}$. Therefore, $z_{2,1} = z_{3,1} = x$ whereas $z_{1,2} = z_{1,3} = y$. First there exist two unidirectional paths (x, y) following either the direction s_1 or the direction s_1^{-1}. On the other hand there exist two edge-disjoint paths $(x, z_{2,3}, y)$ either by following s_2 then s_3^{-1} or by following s_2^{-1} then s_3 and two edge-disjoint paths $(x, z_{3,2}, y)$ with a similar alternative, hence six edge-disjoint paths altogether. □

3.2.2 A recursive connecting scheme

A recursive connecting scheme follows from the statement below, where $\nu_k : V_k \to V_k$ defines the neighbour, in a given direction, of any u of V_k (clearly $\nu_k \cdot \nu_k^{-1}(u) = u$).

Theorem 5 $\nu_0(0) = 0\,;\quad \nu_n(4\,x) = \nu_n^{-1}(4\,\nu_{n-1}(x))\quad (\forall x \in V_{n-1}, n > 0).$

Proof Observe first that $\nu_0(0) = 0$ and moreover : $\nu_1^{-1}(4\,\nu_0(0)) = \nu_1^{-1}(0) = \nu_1(0)$. Assume now we compose $\mathcal{D}_{n-1}$ by $\mathcal{D}_1$, then recall from the definition of V_n that a vertex labelled Q_{n-1} in $\mathcal{D}_{n-1}$ is replaced by the tetrad $\{Q_n = Q_{n-1}\,i\ ;\ i \in I\}$ in $\mathcal{D}_n$. In other words, let $\psi_n : V_{n-1} \to \mathbf{4}V_n$ be the one-to-one mapping defined by $\psi_n(x) = 4x$. (Since the diamond was initially defined on the Euclidean plane, note that it amounts to saying that ψ_n acts on V_{n-1} as a dilatation with center O and ratio 2). Hence we have also: $\psi_n(\nu_{n-1}(x)) = 4\nu_{n-1}(x)$ but : $4\nu_{n-1}(x) = \nu_n^2(4x)$ rewritten in the form : $\nu_n(4\,x) = \nu_n^{-1}(4\,\nu_{n-1}(x))$. □

The compound scheme readily defines the *S–SW–SE* connection of any vertex of $\mathbf{4}V_n$, that is, a vertex labelled $4x$ is connected to the neighbours $4x+1$, $4x+2$, $4x+3$. So the previous statement will be applied to carry out the opposite *N–NE–NW* connection. To express this fact more precisely, let X be any direction in the ordered set (*S*, *SW*, *SE*), $(\delta_S, \delta_{SW}, \delta_{SE})$ $= (1, 2, 3)$ be the set of associated increments and $\overline{X}$ the opposite direction of X in such a way that ${}^U\nu_k : V_k \to V_k$ can define the neighbour in the direction U of any vertex of V_k (clearly we have: ${}^U\nu_k^{-1} = {}^{\overline{U}}\nu_k$).

Corollary 6 *For any* $x \in V_{n-1}$ $(n > 0)$:

$$ {}^X\nu_n(4x) = 4x + \delta_X $$

$$ {}^{\overline{X}}\nu_n(4x) = 4\,({}^{\overline{X}}\nu_{n-1}(x)) + \delta_X $$

Proof We examine the latter case, the other is trivial. By Theorem above :

$$ {}^{\overline{X}}\nu_n(4x) = {}^X\nu_n(4\,({}^{\overline{X}}\nu_{n-1}(x))) $$

whence the result. □

For illustration, knowing the whole configuration of $\mathcal{D}_2$, let us examine what are the N, NE and NW neighbours of vertex 40 in $\mathcal{D}_3$ (see Fig. 8) :

$$^{N}\nu_3(40) = 4\,(^{N}\nu_2(10)) + \delta_S = 4 \cdot 7 + 1 = 29$$

$$^{NE}\nu_3(40) = 4\,(^{NE}\nu_2(10)) + \delta_{SW} = 4 \cdot 8 + 2 = 34$$

$$^{NW}\nu_3(40) = 4\,(^{NW}\nu_2(10)) + \delta_{SE} = 4 \cdot 5 + 3 = 23$$

Consequently we can organize the connection from any element, $4x$ say, of $\mathbf{4}V_n$ by splitting E_n into three parts :

1. a *S–SW–SE 3*-fold connection : $4x$ is connected to the neighbours $4x+1$, $4x+2$, $4x+3$.
2. a *N–NE–NW 3*-fold reversed connection : $4x$ is therefore connected to the neighbours one can yield from the above recurrence.
3. a *6*-fold ring surrounding $4x$: for example the *SW*–neighbour of the *N*–neighbour of $4x$ is the *NW*–neighbour of $4x$ and so forth.

Hence E_n is wholly defined since 4^{n-1} *12*-fold disjoint connections are thus achieved.

3.3 Diameter

A rough estimation of the diameter can first be given through a geometric approach. Looking at Fig. 9 and assuming n large, it should be clear that the origin O has *antipodal* vertices – that is, whose distance is the diameter – within a close neighbourhood of the barycentre G' (resp. G'') of the (equilateral) triangle O–S–SW (resp. O–S–SE) (observe that vertices O, S, SW lie at distance 1 in the torus). A shortest path (or "geodesic") from the origin O to any antipodal vertex must follow only *two* directions among the six ones allowed by the three generators and their inverses : to reach for example G', first follow the southward direction until the barycentre G of O-G'-G", then follow the southwestward direction until destination. Since the southward path from O to S has length $2^n - 1$, thus the diameter is close to $2 \cdot \frac{1}{3} \cdot (2^n - 1) \approx \frac{2}{3}\sqrt{N}$. However, a thorough computation of the diameter will take another path composed of convenient subpaths following the recursive scheme below. Without loss of generality, we can choose both antipodal vertices of the example since the diamond is vertex-transitive. In the sequel, ω_k $(0 \leq k \leq n)$ stands for the origin of a subdiamond $\mathcal{D}_k$, α stands for one antipodal vertex of ω_n in $\mathcal{D}_n$ and close to G' – we do not claim that α is unique – and σ_k is a word expressed in terms of generators which defines in $\mathcal{D}_k$ a geodesic from ω_k to α.

Lemma 7 *A geodesic σ_n from ω_n to α satisfies the recurrence relation:*

$$\sigma_0 = e \qquad \sigma_1 = s_1$$

$$\sigma_n = s_1^{2^{n-2}} s_2^{2^{n-2}} \sigma_{n-2}.$$

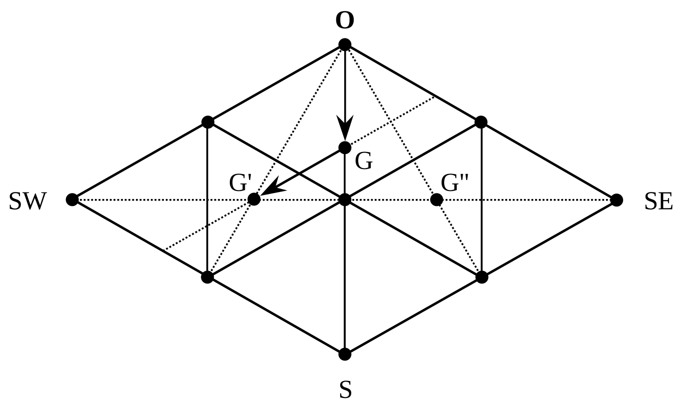


Figure 9: Origin and antipodal points

Proof The proof is obvious for $n < 2$ so we focus on the general case. We construct a path from successive subpaths $[\omega_n, \omega_{n-1}]$, $[\omega_{n-1}, \omega_{n-2}], ...,$ $[\omega_1, \omega_0]$, where $[\omega_k, \omega_{k-1}]$ denotes a unidirectional subpath of length 2^{k-1}, following a sequence of orientations alternating from southwest to southeast and from southeast to southwest, in such a way that the subdiamond $\mathcal{D}_k$ whose ω_k is the origin still includes the vertex α. In terms of generators that yields the recurrence :

$$\sigma_n = s_2^{2^{n-1}} s_3^{2^{n-2}} \sigma_{n-2}.$$

Since $s_2 s_3 = s_1$, then rearranging the terms from the presentation we obtain the irreducible relation on the geodesic :

$$\sigma_n = s_2^{2^{n-2}} s_1^{2^{n-2}} \sigma_{n-2}.$$

□

Lemma 8 *The diameter D_n of $\mathcal{D}_n$ satisfies the recurrence relation :*

$$D_0 = 0 \qquad D_1 = 1$$

$$D_n = D_{n-2} + 2^{n-1}.$$

Proof The diameter is the length of the geodesic. So, as a result of Lemma 7 we have :

$$D_0 = |\sigma_0| \qquad D_1 = |\sigma_1|$$

$$D_n = |\sigma_n| = |\sigma_{n-2}| + 2 \cdot 2^{n-2} = D_{n-2} + 2^{n-1}.$$

□

Theorem 9 *The diamond $\mathcal{A}_n$ has a diameter*

$$D_n = \frac{2}{3}\left[\sqrt{N} - 1\right] \quad \text{or} \quad D_n = \frac{2}{3}\left[\sqrt{N} + 1\right] - 1 \tag{12}$$

depending on whether n *is even or odd.*

Proof Assume n even. Then from Lemma 8:

$$D_{2k} = D_{2k-2} + 2^{2k-1} \qquad (1 \leq k \leq \frac{n}{2})$$

By summation and elimination of intermediate terms we obtain:

$$D_n = \sum_{k=0}^{\frac{n}{2}-1} 4^k = \frac{2}{3}\left[2^n - 1\right]$$

(note that the term within brackets is divisible by 3).

Assume now n odd. Then from Lemma 8:

$$D_{2k+1} = D_{2k-1} + 2^{2k} \qquad (1 \leq k \leq \frac{n-1}{2})$$

In the same way we obtain after some little arrangement:

$$D_n = \sum_{k=0}^{\frac{n-1}{2}} 4^k = \frac{2^{n+1} - 1}{3} = \frac{2}{3}\left[2^n + 1\right] - 1$$

(the term within brackets is also divisible by 3). □

We close the study of the diameter with two useful relations:

Corollary 10 $D_0 = 0$ *and for any* n > 0*:*

- $D_{n-1} + D_n = 2^n - 1$
- $D_n = \begin{cases} 2\,D_{n-1} + 1 & \textit{if } \text{n } \textit{odd} \\ 2\,D_{n-1} & \textit{if } \text{n } \textit{even} \end{cases}$

Proof Immediate from the theorem above. □

4 Conclusion

This paper is a first presentation of the "diamond torus", a new interconnection topology generated on the hexavalent grid. The diamond results from a non-isotropic configuration of the system of generators and is merely the replication of the arrowhead torus for the isotropic case. Like the arrowhead, the diamond reveals the same important properties of Cayley graphs, such as vertex-transitivity, or recursive construction schemes on vertices and edges ; both have an equal edge-connectivity and an equal diameter. Moreover, the diamond readily appears as a good host for embedding subvalent topologies like the k-ary 2-cube and, therefore, the usual grid. Although diamond and arrowhead display a rather different morphology, the fact that their respective systems of generators be closed under inverses –and therefore coincide– allows us to claim that their undirected version is isomorphic. So, our future work will consist in showing up whether there exists a natural one-to-one mapping between their vertex set. That can be useful in particular for embedding one topology onto another.

Acknowledgments The author wants again to acknowledge the French broadcasting group RUMEUR and specially Charles Delorme for their helpful assistance.

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Éditeur
INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
ISSN 0249-6399