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Francis Collino

INRIA

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET AUTOMATIQUE

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Abstract: Two inovative techniques for the simulation of the effect of a lossy dielectric half-space are derived and analysed. The first is an adapted impedance surface condition imposed at the interface to replace the dielectric medium. This condition, when coupled with Maxwell's equations yields a stable system which can be written in a variational form. It allows us to take into acccount both the polarization and the angle of incidence of the out-going waves. The second technique is a generalization of the Bérenger perfectly matched layers: the dielectric is now replaced by a short length layer in which the waves are governed by a modification of Maxwell's equations. For this model, the two reflection coefficients are the same as for the true dielectric while the damping inside the layer is enforced.

Key-words: Absorbing Boundary Condition, Bérenger Perfectly Matched Layer, Maxwell equations

(Résumé : tsvp)

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Unité de recherche INRIA Rocquencourt
Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
Téléphone : (33 1) 39 63 55 11 – Télécopie : (33 1) 39 63 53 30

Conditions aux limites et technique de couches pour la simulation d' ondes electromagnétiques sur un milieu conducteur

Résumé : Deux nouvelles méthodes pour la simulation des effets d'un demi-plan diélectrique sur les ondes électromagnétiques sont proposés et analysés. La première consiste en une condition d'impédance de surface imposée à l' interface pour remplacer le milieu diélectrique. Cette condition, couplée avec les équations de Maxwell fournissent un système stable qui peut être ècrit sous forme variationnelle. Cette condition permet la prise en compte de la polarisation et de l'angle d'incidence des ondes incidentes sur le conducteur. Une deuxième technique est une géneralisation des couches parfaitement adaptées de Bérenger: le dielectrique est alors remplacé par une couche de faible épaisseur dans laquelle les ondes sont régies par une modification des équations de Maxwell. Pour ce modèle, les coefficients de réflection sont identiques à ceux du vrai diélectrique tandis que l'amortissement dans la couche est artificiellement renforcé.

Mots-clé : Condition aux limites absorbante, Couches de Bérenger, équations de Maxwell

1 Introduction

Finite-difference and finite-element time-domain techniques have been successfully applied to a large number of scattering or interaction problems in electromagnetism. While these methods provide great accuracy, they can be quite expensive, especially for large sizes problems occuring in realistic situations. For an electromagnetic problem set in unbounded domains, the reduction of the computational domain is essential to making the use of such a technique feasible. For the case in which the surrounding media is a homogeneous free space, two types of solutions have been proposed for reducing the problem to a finite domain : absorbing boundary condition (ABC) and absorbing layer techniques. ABC technique consist in replacing the surrounding media by some appropriate boundary condition imposed at the artificial boundary. For Maxwell's equation second order ABC were first proposed in the early eighties to replace the first order Silver-Muller condition, [13], [7], (the order of an ABC being related to its accuracy). These conditions were generalized to higher order in [17], [6] and to lossy media in [5]. The absorbing layer technique is an alternative to the ABCs. The idea is to surround the domain of interest by a layer in which the waves are governed by some appropriate damped wave equation. Recently, this technique has received a revival of interest because of Bérenger's work on perfectly matched layer (PML), [3]. The PML model differs from the classical layer models by their astonishing property of generating absolutely no reflection at the interface between the free space and the layer. Numerical experiments have demonstrated the potentiality of the method, [8].

In this paper, we are concerned with the more complex situation of a medium composed of a free half-space covered by a lossy dielectric medium, the domain of interest being a part of the free half-space. Our notation is the following :

The lossy dielectric mediun is characterized by ϵ_s, μ_s and σ_s, its electric permittivity, magnetic permeability and conductivity. The propagation of the electromagnetic waves is governed by the 3D Maxwell system of equations

$$\left\{\begin{array}{l} \epsilon\dfrac{\partial\vec{E}}{\partial t}+\sigma\vec{E}-\mathrm{curl}\vec{H}=0 \\ \mu\dfrac{\partial\vec{H}}{\partial t}+\mathrm{curl}\vec{E}=0 \end{array}\right. \tag{1}$$

where

$$(\epsilon,\mu,\sigma)(x,y,z)=\left[\begin{array}{lll} (\epsilon_0,\mu_0,0) & \text{if } z<0 & \text{in the free half-space} \\ (\epsilon_s,\mu_s,\sigma_s) & \text{if } z>0 & \text{in the lossy medium} \end{array}\right. \tag{2}$$

At time $t=0$, the supports of the initial conditions are supposed to be strictly contained in the half space $z<0$.

The problem is how to circumvent the computation of the electromagnetic-field inside the dielectric. Drawing a parallel with the previous situation, we have two possible techniques for treating our problem. The first one consists in replacing the lossy dielectric with some adapted surface impedance conditions (ASIC). These boundary conditions, imposed at the interface, are constructed in such a way as to correctly replace the effect of the dielectric on the out-going wave. The second technique is to extend Bérenger's PML to some perfectly adapted layer model (PAL) : the idea is to design a model more absorbing than the real dielectric media but restoring exactly the same reflection coefficients at the interface between the free-media and the layer.

- The adapted impedance surface conditions (AISC):

  The restricted domain is:

  $$\Omega^-=\{(x,y,z),\ z<0\} \tag{3}$$

  and the problem is completed with a boundary condition on $\Gamma=\partial\Omega^-=\{z=0\}$

  This technique have been already addressed by several authors with some limitations on the constitutive parameters. For examples, Beggs and Yee, [2], assume that the lossy dielectric is a good conductor,

  $$\frac{\sigma_s}{\omega\epsilon_s}\gg 1,$$

where ω varies over the frequency spectrum of the incident wave, whereas Maloney, [12], uses the weaker condition

$$\left|1 - i\frac{\sigma_s}{\omega\epsilon_s}\right| \gg \sin(\theta_i),$$

where θ_i is the maximum angle of incidence with respect to the normal of the interface present in the incident wave. Both authors propose to replace the exact impedance boundary condition by some approximate impedance surface condition under the above mentionned assumptions. The approximated impedance operator amounts to a time-convolution operator that can be approximated by some exponential approximants. In [9], this technique was generalized to the case in which the incidence of the in-coming wave is fixed.

In this paper, we propose a new impedance boundary condition able to deal both with polarizations and with no limitation on the angles of incidence. This condition is written as follows:

$$\begin{array}{l} \left(\epsilon_s\mu_s\dfrac{\partial^2}{\partial t^2} + \sigma_s\mu_s\dfrac{\partial}{\partial t} + \vec{\mathrm{curl}}_\Gamma(\mathrm{curl}_\Gamma)\right)\ \vec{E}_t - \ \epsilon_s\mu_s Z_a\dfrac{\partial^2}{\partial t^2}\ \vec{H}_\tau = 0 \\ \vec{E}_t = \hat{z}\wedge(\vec{E}\wedge\hat{z}), \quad \vec{H}_\tau = \vec{H}\wedge\hat{z} \end{array} \tag{4}$$

where Z_a is a differential operator which operates on functions of the variables (x, y, t).

In section 2.1, we show that an exact impedance boundary condition can be written as in (4) but with Z_a replaced by some scalar integro-differential operator Z The approximation, constructed in section 2.2, consists in simply replacing Z by a family of appropriate local operators (see [15] for a similar idea for the wave equation). In section 2.3, we prove that the coupling between the Maxwell's equation and the boundary condition (4) is well-posed when the velocity of the lossy dielectric is less than the velocity in the free media. Finally, in section 2.4, a variational formulation is proposed for our approximate conditions. This formulation allows us to use this condition with either finite-difference or finite-element methods.

- An extension of Berenger's Perfectly Matched Layers : the PAL

  The idea consists in substituting for the real dielectric an artificial model, that provides the same reflection coefficient as does the real dielectric while enforcing the attenuation inside the layer. Let us remark that this is the case for Bérenger's PMLs when the dielectric is a trivial dielectric, i.e. a free space, and consequently the reflection coefficients are zero. Our purpose is then to generalize the PML's to the case of a non-trivial dielectric.

  The model that we shall design in section (3.1) is the following. The restricted domain is

$$\Omega^\delta = \{(x,y,z),\ z < +\delta\} \tag{5}$$

  and the Maxwell system of equations is replaced by a system of 12 equations with 12 unknowns

$$\left\{\begin{array}{l}
\epsilon\dfrac{\partial E_z}{\partial t} + \sigma E_z = \dfrac{\partial(H_{yz}+H_{yx})}{\partial x} - \dfrac{\partial(H_{xy}+H_{xz})}{\partial y} \\
\epsilon\dfrac{\partial E_{yz}}{\partial t} + \sigma E_{yz} = \epsilon\dfrac{\partial \Psi_y}{\partial t}, \qquad \epsilon\dfrac{\partial E_{yx}}{\partial t} + \sigma E_{yx} = -\dfrac{\partial H_z}{\partial x} \\
\epsilon\dfrac{\partial \Psi_y}{\partial t} + \sigma_z \Psi_y = \dfrac{\partial\left(H_{xy}+H_{xz}\right)}{\partial z} \\
\epsilon\dfrac{\partial E_{xy}}{\partial t} + \sigma E_{xy} = \dfrac{\partial H_z}{\partial y}, \qquad \epsilon\dfrac{\partial E_{xz}}{\partial t} + \sigma E_{xz} = \epsilon\dfrac{\partial \Psi_x}{\partial t} \\
\epsilon\dfrac{\partial \Psi_x}{\partial t} + \sigma_z \Psi_x = -\dfrac{\partial\left(H_{yz}+H_{yx}\right)}{\partial z} \\
\mu\dfrac{\partial H_z}{\partial t} = \dfrac{\partial(E_{xy}+E_{xz})}{\partial y} - \dfrac{\partial(E_{yx}+E_{yz})}{\partial x} \\
\mu\dfrac{\partial H_{yx}}{\partial t} = \dfrac{\partial E_z}{\partial x}, \qquad \mu\dfrac{\partial H_{yz}}{\partial t} + \sigma_z^\star H_{yz} = -\dfrac{\partial(E_{xy}+E_{xz})}{\partial z} \\
\mu\dfrac{\partial H_{xz}}{\partial t} + \sigma_z^\star H_{xz} = \dfrac{\partial\left(E_{yz}+E_{yx}\right)}{\partial z} \qquad \mu\dfrac{\partial H_{xy}}{\partial t} = -\dfrac{\partial E_z}{\partial y}
\end{array}\right. \tag{6}$$

where

$$(\epsilon, \mu, \sigma, \sigma_z, \sigma_z^\star) = \left[\begin{array}{ll} (\epsilon_0, \quad \mu_0, \quad 0, \quad 0, \quad\quad 0) & \text{if } z < 0 \\ \\ (\epsilon_s, \quad \mu_s, \quad \sigma_s, \quad \epsilon_s \sigma^\star, \quad \mu_s \sigma^\star) & \text{if } 0 < z < \delta \end{array} \right. \tag{7}$$

and $\sigma^\star$, a positive function of z, can be interpreted as a damping factor. On the boundary $\partial\Omega^\delta$, we complete the system with a Dirichlet Boundary Condition

$$E_{yz} + E_{yx} = E_{xz} + E_{xy} = 0, \text{ on } z = \delta \tag{8}$$

Remark that in the free space (i.e. for $z < 0$) the field

$$\left\{ \begin{array}{l} (E_x = E_{zx} + E_{zy}, \quad E_y = E_{yx} + E_{yz}, \quad E_z), \\ \\ (H_x = H_{zx} + H_{zy}, \quad H_y = H_{yx} + H_{yz}, \quad H_z) \end{array} \right.$$

satisfies Maxwell's equations (1). The equations are only changed in the dielectric medium. Moreover, in subsection 3.2, we will show that the reflection of an incident wave on the interface is exactly the same for this model as for the model (1) when the length of the layer is infinite. The only difference arises from the reflection due to the Dirichlet Boundary Condition (8). By choosing $\sigma^\star$ sufficiently large, we can make the damping strong enough to render this reflection very weak.

2 Derivation and Analysis of the Impedance Boundary Condition

2.1 Derivation of an exact impedance boundary condition

We are looking for a relationship satisfied at the boundary $z = 0$ relating the continuous quantities (E_x, E_y) and (H_x, H_y). The procedure is classical and is based on the use of the Fourier-Laplace transformation

$$\hat{v}\,(k_x, k_y, z, \omega) = \int\int\int v(x, y, z, t) \exp^{-i(\omega t - k_x x - k_y y)} \, dx dy dt. \tag{9}$$

Denoting by $\left(\tilde{E}_x^0, \tilde{E}_y^0, \tilde{E}_z^0\right)$ and $\left(\tilde{H}_x^0, \tilde{H}_y^0, \tilde{H}_z^0\right)$ the Fourier transforms with respect to the space variables (x, y) of the initial conditions at time $t = 0$, we have

$$\left\{\begin{array}{ll} \epsilon\left(i\omega\hat{E}_x - \tilde{E}_x^0\right) + \sigma\hat{E}_x = -ik_y\hat{H}_z - \dfrac{\partial\hat{H}_y}{\partial z}, & \mu\left(i\omega\hat{H}_x - \tilde{H}_x^0\right) = +\dfrac{\partial\hat{E}_y}{\partial z} + ik_y\hat{E}_z \\ \epsilon\left(i\omega\hat{E}_y - \tilde{E}_y^0\right) + \sigma\hat{E}_y = +\dfrac{\partial\hat{H}_x}{\partial z} + ik_x\hat{H}_z, & \mu\left(i\omega\hat{H}_y - \tilde{H}_y^0\right) = -ik_x\hat{E}_z - \dfrac{\partial\hat{E}_x}{\partial z} \\ \epsilon\left(i\omega\hat{E}_z - \tilde{E}_z^0\right) + \sigma\hat{E}_z = -ik_x\hat{H}_y + ik_y\hat{H}_x, & \mu\left(i\omega\hat{H}_z - \tilde{H}_z^0\right) = -ik_y\hat{E}_x + ik_x\hat{E}_y. \end{array}\right. \tag{10}$$

By hypothesis, the supports of the initial conditions are zero for $z > 0$. Hence we obtain in $z > 0$

$$\left\{\begin{array}{ll} \epsilon_s i\omega\hat{E}_x + \sigma_s\hat{E}_x = -\dfrac{\partial\hat{H}_y}{\partial z} - ik_y\hat{H}_z, & \mu_s i\omega\hat{H}_x = ik_y\hat{E}_z + \dfrac{\partial\hat{E}_y}{\partial z} \\ \epsilon_s i\omega\hat{E}_y + \sigma_s\hat{E}_y = ik_x\hat{H}_z + \dfrac{\partial\hat{H}_x}{\partial z}, & \mu_s i\omega\hat{H}_y = -\dfrac{\partial\hat{E}_x}{\partial z} - ik_x\hat{E}_z \\ \epsilon_s i\omega\hat{E}_z + \sigma_s\hat{E}_z = ik_y\hat{H}_x - ik_x\hat{H}_y, & \mu_s i\omega\hat{H}_z = ik_x\hat{E}_y - ik_y\hat{E}_x. \end{array}\right. \tag{11}$$

Solving this set of ordinary differential equations, we obtain

$$\begin{aligned} \hat{E} &= (A_{te}^{out}\omega\mu_s\vec{r} - A_{tm}^{out}\vec{g}_{out}^s)\exp^{-ik_z^s z} + (A_{te}^{in}\omega\mu_s\vec{r} - A_{tm}^{in}\vec{g}_{in}^s)\exp^{+ik_z^s z} \\ \hat{H} &= (A_{te}^{out}\vec{g}_{out}^s + A_{tm}^{out}(\omega\epsilon_s - i\sigma_s)\vec{r})\exp^{-ik_z^s z} + (A_{te}^{in}\vec{g}_{in}^s + A_{tm}^{in}(\omega\epsilon_s - i\sigma_s)\vec{r})\exp^{+ik_z^s z} \end{aligned} \tag{12}$$

where we have defined

$$\vec{r} = \begin{bmatrix} k_y \\ -k_x \\ 0 \end{bmatrix} \quad \vec{g}_{out}^s = \begin{bmatrix} k_x k_z^s \\ k_y k_z^s \\ -k_x^2 - k_y^2 \end{bmatrix} \quad \vec{g}_{in}^s = \begin{bmatrix} -k_x k_z^s \\ -k_y k_z^s \\ -k_x^2 - k_y^2 \end{bmatrix} \tag{13}$$

and

$$\left\{\begin{array}{l} k_z^s = \left(\epsilon_s\mu_s\omega^2 - i\omega\sigma_s\mu_s - (k_x^2 + k_y^2)\right)^{\frac{1}{2}} \\ \Im m(k_z^s) < 0 \text{ for } \Im m(\omega) < 0. \end{array}\right. \tag{14}$$

Our choice of the determination of the square root is made in order to separate in-going and out-going waves. It can be shown that

$$k_z^s = \omega\sqrt{\epsilon_s\mu_s}\left[1 - \frac{i\omega\sigma_s\mu_s + k_x^2 + k_y^2}{\epsilon_s\mu_s\omega^2}\right]^{\frac{1}{2}} \quad \forall\omega,\ \Im m(\omega) < 0, \tag{15}$$

where the square root is defined via the principal determination of the logarithm.

Now, we have the energy estimate

$$\begin{aligned} &\text{if } E(t) = \int_{R^3} \left(\epsilon|\vec{E}|^2 + \mu|\vec{H}|^2\right) dxdydz, \\ &\frac{dE(t)}{dt} = -\int_{R^3_+} \sigma_s \left|\frac{\partial\vec{E}}{\partial t}\right|^2 dxdydz \quad \Rightarrow E(t) \le E(0) \end{aligned} \tag{16}$$

which shows that each component of the electro-magnetic field is bounded in the set $\mathbb{L}^\infty\left(0,\infty;\mathbb{L}^2(dxdydz)\right)$ and thus that its Fourier transform belongs to $\mathbb{V}$, the set of $\mathbb{L}^2(dk_xdk_ydz)$-valued functions, analytic on $P^- = \{\omega \in \mathbb{C}\,;\ \Im m\omega < 0\}$ and bounded on every subset of P^- which is bounded away from the real axis. An element of $\mathbb{V}$ can not increase exponentially in z and so

$$A_{te}^{in} = A_{tm}^{in} = 0. \tag{17}$$

Thus, the tangential components of the field are given by

$$\begin{aligned} \hat{E}_x &= +\omega\mu_s k_y A_{te}^{out} - k_x k_z^s A_{tm}^{out} \\ \hat{E}_y &= -\omega\mu_s k_x A_{te}^{out} - k_y k_z^s A_{tm}^{out} \\ \hat{H}_x &= k_x k_z^s A_{te}^{out} + k_y(\omega\epsilon_s - i\sigma_s)A_{tm}^{out} \\ \hat{H}_y &= k_y k_z^s A_{te}^{out} - k_x(\omega\epsilon_s - i\sigma_s)A_{tm}^{out}. \end{aligned}$$

In particular, after a little algebra, we get the following relationship between the tangential magnetic and electric fields in the ground

$$
\left\{
\begin{array}{l}
((i\omega)^2\mu_s\epsilon_s + i\omega\sigma_s\mu_s)\begin{bmatrix}\hat{E}_x\\ \hat{E}_y\end{bmatrix} + \begin{bmatrix} -(ik_y)^2 & ik_x\, ik_y \\ ik_x\, ik_y & -(ik_x)^2\end{bmatrix}\begin{bmatrix}\hat{E}_x\\ \hat{E}_y\end{bmatrix} + \\
\qquad +\omega k_z^s\mu_s\begin{bmatrix}\hat{H}_y\\ -\hat{H}_x\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}.
\end{array}
\right. \tag{18}
$$

Passing to the limit, $z \to 0^+$, taking into account the continuity of the tangential components of the electromagnetic field at the interface, and performing the inverse Fourier transform on the equation obtained, we get the following relationship at the interface

$$
\left\{
\begin{array}{l}
\left((\epsilon_s\mu_s\dfrac{\partial^2}{\partial t^2} + \sigma_s\mu_s\dfrac{\partial}{\partial t}\right)\begin{bmatrix}E_x\\ E_y\end{bmatrix} + \begin{bmatrix} -\partial^2_{yy} & \partial^2_{xy} \\ \partial^2_{xy} & -\partial^2_{xx}\end{bmatrix}\begin{bmatrix}E_x\\ E_y\end{bmatrix} - \\
\qquad -\epsilon_s\mu_s Z\cdot\dfrac{\partial^2}{\partial t^2}\begin{bmatrix}H_y\\ -H_x\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}, \qquad \text{on } z = 0^- \text{ and } z = 0^+,
\end{array}
\right. \tag{19}
$$

where Z is the scalar integro-differential operator defined by the commutative diagram

$$
\begin{array}{llll}
Z: & \varphi(x,y,t) & \longrightarrow & Z\cdot\varphi \\
 & F\downarrow & & \uparrow F^{-1} \\
\hat{Z}: & \hat{\varphi}(k_x,k_y,\omega) & \longrightarrow & \hat{Z}\hat{\varphi} = \dfrac{k_z^s}{\omega\epsilon_s}\hat{\varphi}.
\end{array} \tag{20}
$$

Another expression for $\hat{Z}$ is given by

$$
\hat{Z}(k_x,k_y,\omega) = \sqrt{\frac{\mu_s}{\epsilon_s}}\left[1 - \frac{i\omega\sigma_s\mu_s + k_x^2 + k_y^2}{\epsilon_s\mu_s\omega^2}\right]^{\frac{1}{2}}. \tag{21}
$$

We note the presence of a square root : Z is not a differential operator.

2.2 Construction of approximate impedance boundary conditions

Because of the presence of the square root in (21), the exact impedance operator Z is non-local both in space and time. The idea of the approximate impedance operators is to replace Z by a differential operator, more tractable from a numerical point of view. These approximations are constructed from high frequency approximation of the symbol $\hat{Z}$ with rational fractions. More precisely, we use

$$[1-z]^{\frac{1}{2}} \approx \hat{Z}_a = 1 - \beta_0 z - \sum_{l=1}^{L} \beta_l \frac{z}{1-\alpha_l z}, \quad z = \frac{i\omega\sigma_s\mu_s + k_x^2 + k_y^2}{\epsilon_s\mu_s\omega^2}, \tag{22}$$

where $\{\beta_0, (\alpha_l, \beta_l), 1 \leq l \leq L\}$ is a set of positive real coefficients. A possible choice is based on Padé expansions, ([1])

$$\begin{cases} \beta_0 = 0, \quad \beta_l = \dfrac{2}{2L+1}\sin^2\left(\dfrac{l\pi}{2L+1}\right), \quad \alpha_l = \cos^2\left(\dfrac{l\pi}{2L+1}\right) & \text{even approximant} \\ \beta_0 = \dfrac{1}{2L}, \quad \beta_l = \dfrac{2}{2L}\sin^2\left(\dfrac{l\pi}{2L}\right), \quad \alpha_l = \cos^2\left(\dfrac{l\pi}{2L}\right), & \text{odd approximant} \end{cases} \tag{23}$$

for which we have

$$[1-z]^{\frac{1}{2}} = 1 - \beta_0 z - \sum_{l=1}^{L} \beta_l \frac{z}{1-\alpha_l z} + O\left(|z|^{2L+\eta}\right), \quad \eta = \begin{cases} 2, \text{ even case} \\ 1, \text{ odd case} \end{cases}. \tag{24}$$

The exact impedance boundary condition is now replaced by

$$\begin{cases} ((i\omega)^2\epsilon_s\mu_s + i\omega\sigma_s\mu_s)\begin{bmatrix} \hat{E}_x \\ \hat{E}_y \end{bmatrix} + \begin{bmatrix} -(ik_y)^2 & ik_x\, ik_y \\ ik_x\, ik_y & -(ik_x)^2 \end{bmatrix}\begin{bmatrix} \hat{E}_x \\ \hat{E}_y \end{bmatrix} - \\ \qquad -(i\omega)^2\epsilon_s\mu_s\hat{Z}_a\begin{bmatrix} \hat{H}_y \\ -\hat{H}_x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{cases} \tag{25}$$

In order to return to a space-time formulation, we use auxiliary functions, see [11], [4]. We define ($\alpha_0 \equiv 0$)

$$\left[\begin{array}{c} +\hat{\phi}^{(z)}_{l,y} \\ \\ -\hat{\phi}^{(z)}_{l,x} \end{array}\right] = \frac{i\omega\sigma_s\mu_s + k_x^2 + k_y^2}{\omega^2\epsilon_s\mu_s - \alpha_l(i\omega\sigma_s\mu_s + k_x^2 + k_y^2)} \cdot \left[\begin{array}{c} +\hat{H}_y \\ \\ -\hat{H}_x \end{array}\right], \quad l = 0, ..., L \tag{26}$$

so we can rewrite condition (25)

$$\left\{\begin{array}{l} ((i\omega)^2\epsilon_s + i\omega\sigma_s)\left[\begin{array}{c} \hat{E}_x \\ \\ \hat{E}_y \end{array}\right] + \left[\begin{array}{cc} -(ik_y)^2 & ik_x\, ik_y \\ \\ ik_x\, ik_y & -(ik_x)^2 \end{array}\right]\left[\begin{array}{c} \hat{E}_x \\ \\ \hat{E}_y \end{array}\right] - \\ \\ \quad -(i\omega)^2\epsilon_s\mu_s\sqrt{\dfrac{\mu_s}{\epsilon_s}}\left[\begin{array}{c} \hat{H}_y \\ \\ -\hat{H}_x \end{array}\right] + \displaystyle\sum_{l=0}^{L}\beta_l(i\omega)^2\epsilon_s\mu_s\sqrt{\dfrac{\mu_s}{\epsilon_s}}\left[\begin{array}{c} +\hat{\phi}^{(z)}_{l,y} \\ \\ -\hat{\phi}^{(z)}_{l,x} \end{array}\right] = \left[\begin{array}{c} 0 \\ \\ 0 \end{array}\right], \\ \\ \omega^2\epsilon_s\mu_s\hat{\phi}^{(z)}_{l,a} - (i\omega\sigma_s\mu_s + k_x^2 + k_y^2)\left(\alpha_l\hat{\phi}^{(z)}_{l,a} + \hat{H}_a\right) = 0, \quad l = 0, ..., L, \quad a = x, y \end{array}\right. \tag{27}$$

It is now easy to perform the inverse Fourier transform. We obtain

$$\left\{\begin{array}{l} \left(\left(\epsilon_s\mu_s\dfrac{\partial^2}{\partial t^2} + \sigma_s\mu_s\dfrac{\partial}{\partial t}\right)\right)\left[\begin{array}{c} E_x \\ \\ E_y \end{array}\right] + \left[\begin{array}{cc} -\partial^2_{yy} & \partial^2_{xy} \\ \\ \partial^2_{xy} & -\partial^2_{xx} \end{array}\right]\left[\begin{array}{c} E_x \\ \\ E_y \end{array}\right] - \\ \\ \quad -\epsilon_s\mu_s\sqrt{\dfrac{\mu_s}{\epsilon_s}}\dfrac{\partial^2}{\partial t^2}\left[\begin{array}{c} +\hat{H}_y \\ \\ -\hat{H}_x \end{array}\right] + \displaystyle\sum_{l=1}^{L}\beta_l\epsilon_s\mu_s\sqrt{\dfrac{\mu_s}{\epsilon_s}}\dfrac{\partial^2}{\partial t^2}\left[\begin{array}{c} +\phi^{(z)}_{l,y} \\ \\ -\phi^{(z)}_{l,x} \end{array}\right] = \left[\begin{array}{c} 0 \\ \\ 0 \end{array}\right], \end{array}\right. \tag{28}$$

and

$$\left\{\begin{array}{l} \epsilon_s\mu_s\dfrac{\partial^2\phi^{(z)}_{l,a}}{\partial t^2} + \left(\sigma\mu_s\dfrac{\partial\psi^{(z)}_{l,a}}{\partial t} - \dfrac{\partial^2\psi^{(z)}_{l,a}}{\partial x^2} - \dfrac{\partial^2\psi^{(z)}_{l,a}}{\partial y^2}\right) = 0, \quad \psi^{(z)}_{l,a} = \alpha_l\phi^{(z)}_{l,a} + H_a, \\ \qquad\qquad\qquad \text{for } l = 1, \ldots, L, \quad a = x, y. \end{array}\right. \tag{29}$$

2.3 Stability analysis

In this section, we address the problem of the stability of problem (P) for smooth initial data.

$$(\mathrm{P}) \quad \left\{ \begin{array}{l} \bullet \text{ Maxwell's equations in } z < 0 \\ \bullet \text{ Boundary conditions (28)-(29) at } z = 0 \\ \bullet \text{ Initial conditions } (\vec{E}^0, \vec{H}^0) \text{ at time } t = 0 \\ \bullet \text{ Zero initial conditions for } \vec{\phi}_l^{(z)} \text{ and } \partial_t \vec{\phi}_l^{(z)} \text{ at time } t = 0 \end{array} \right.$$

We establish the following result

Theorem 2.1 *Let suppose that the initial conditions satisfy*

$$\vec{H}_0, \vec{E}_0 \in \mathcal{C}^\infty_{comp}\left(\mathbf{R}^3_-)\right)^3. \tag{30}$$

$$(\Leftrightarrow distance(K, z=0) \neq 0, \quad K = Support\vec{E}_0 \cup Support\vec{H}_0) \tag{31}$$

then problem (P) admits a unique solution

$$\left\{ \begin{array}{l} \vec{E}, \vec{H} \; \in \; \mathcal{C}^\infty\left(0, T, \mathcal{C}^\infty\left(\mathbf{R}^3_-)\right)^3\right) \\ \vec{\phi}_l^{(z)} \; \in \; \mathcal{C}^\infty\left(0, T, \mathcal{C}^\infty\left(\mathbf{R}^2)\right)^2\right) \end{array} \right. \tag{32}$$

as soon as the following conditions are satisfied

$$0 \leq \alpha_l < 1, \;\; 0 < \beta_l < 1, \;\; \beta_0 + \sum_{l=1}^{L} \frac{\beta_l}{1 - \alpha_l} < 1 \tag{33}$$

$$\epsilon_s \mu_s > \epsilon_0 \mu_0, \;\; \sigma_s \geq 0. \tag{34}$$

In this case, the estimations

$$\begin{array}{l} \dfrac{1}{T}\displaystyle\int_0^T \sqrt{\mathcal{E}(t)} dt \;\leq\; 2C_\star\left(1 + 2\varepsilon\,\frac{\sigma_s T}{\epsilon_s}\right)\sqrt{\mathcal{E}_0} \\ \mathcal{E} = \displaystyle\int_{-\infty}^{0} dz \int_{R^2} dx dy\, \epsilon_0 \left|\vec{E}(x,y,z,t)\right|^2 + \mu_0 \left|\vec{H}(x,y,z,t)\right|^2 \end{array} \tag{35}$$

where

$$\begin{array}{l} C_\star = 1, \quad if\, \sigma_s = 0, \quad C_\star \geq 1, \quad if\, \sigma_s \neq 0, \\ \varepsilon = 1 \; if\, \beta_0 = 0, \quad \varepsilon = 0 \; if\, \beta_0 \neq 0 \end{array} \tag{36}$$

may be established.

Remark 1 Condition (33) is the same condition required for the stability of the solutions of the wave equation coupled with an absorbing boundary condition involving the same approximation of the square root (cf. [18]).

Remark 2 In practice, condition (34) is not restrictive since the velocity in the ground is always smaller then the velocity in the air.

Proof of the theorem The proof is carried out in several steps and is closely related to the Kreiss analysis (see [18]) as it equally uses the Fourier-Laplace transform and is based on the properties of analyticity of a particular function in a complex half-plane. But, it differs in that the presence of the conducting term (σ_s) causes the loss of homogeneity between spatial variables and the time variable, a crucial property in the Kreiss techniques.

The Fourier Laplace solution of the problem The solution is sought as the sum of an incident field and a reflected field.

$$\vec{E} = \vec{E}_{inc} + \vec{E}_{ref} \tag{37}$$

The incident field is the restriction to $\mathbb{R}^3_-$ of the unique solution of Maxwell's equations posed in the whole plane with $(\vec{E}^0, \vec{H}^0)$ as initial data (or their prolongation to be more precise).

The reflected field satisfies Maxwell's equations in $\mathbb{R}^3_-$ with zero initial conditions and the boundary condition (28)-(29) with an additional right hand side, function of the incident field.

The use of the Fourier-Laplace transform (9) allows us to eliminate the auxiliary functions and to determine both fields from a set of differential equations in the variable z. We solve these equations using a process similar to the above (see (10)-(17)). We obtain, for the incident field and for z located above the support of the initial data,

$$\hat{E}_{inc} = \left(A^{out}_{te}\omega\mu_0\vec{r} - A^{out}_{tm}\vec{g}_{out}\right)\exp^{-ik_z z} \qquad \hat{H}_{inc} = \left(A^{out}_{te}\vec{g}_{out} + A^{out}_{tm}\omega\epsilon_0\vec{r}\right)\exp^{-ik_z z}, \tag{38}$$

where $\vec{r}$, $\vec{g}_{in}$, $\vec{g}_{out}$, are defined in (13) with k^s_z replaced by k_z,

$$k_z = \omega\sqrt{\epsilon_0\mu_0}\left[1 - \frac{k_x^2 + k_y^2}{\epsilon_0\mu_0\omega^2}\right]^{\frac{1}{2}}. \tag{39}$$

In the same way, for every $z \leq 0$, the reflected field is given by

$$\hat{E}_{ref} \;=\; \left(B_{te}^{in}\omega\mu_0\vec{r} - B_{tm}^{in}\vec{g}_{in}\right)\, \exp^{+ik_z z}, \quad \hat{H}_{ref} \;=\; \left(B_{te}^{in}\vec{g}_{in} + B_{tm}^{in}\omega\epsilon_0\vec{r}\right)\, \exp^{+ik_z z}. \tag{40}$$

The coefficients A_{te}^{out} and A_{tm}^{out} are linked to the values of the initial data and are supposed to be known. The coefficients B_{te}^{in} and B_{tm}^{in} for the reflected field are determined by means of the Fourier Laplace transformed boundary condition at $z = 0$, namely

$$\left\{ \begin{array}{l} \left(((i\omega)^2\mu_s\epsilon_s + i\omega\sigma_s\mu_s) + \begin{bmatrix} -(ik_y)^2 & ik_x\, ik_y \\ ik_x\, ik_y & -(ik_x)^2 \end{bmatrix}\right) e_t - (i\omega)^2\hat{Z}_a\epsilon_s\mu_s h_\tau = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \\ \text{with } \; e_t = \left(\hat{z}\wedge((\hat{E}_{inc} + \hat{E}_{ref})\wedge\hat{z}\right)_{z=0}, \quad h_\tau = \left((\hat{H}_{inc} + \hat{H}_{ref})\wedge\hat{z}\right)_{z=0}. \end{array} \right. \tag{41}$$

Defining

$$\vec{r}_t = \begin{bmatrix} k_y \\ -k_x \end{bmatrix} = \hat{z}\wedge(\vec{r}\wedge\hat{z}), \quad k_z\vec{g}_t = \begin{bmatrix} k_z k_x \\ k_z k_y \end{bmatrix} = \hat{z}\wedge(\vec{g}_{out}\wedge\hat{z}) = -\hat{z}\wedge(\vec{g}_{in}\wedge\hat{z}), \tag{42}$$

we have

$$\vec{r}\wedge\hat{z} = -\vec{g}_t, \quad \vec{g}_{out}\wedge\hat{z} = k_z\vec{r}_t, \quad \vec{g}_{in}\wedge\hat{z} = -k_z\vec{r}_t, \tag{43}$$

and so

$$\left\{ \begin{array}{l} e_t = \omega\mu_0\,(A_{te}^{out} + B_{te}^{in})\,\vec{r}_t - (A_{tm}^{out} - B_{tm}^{in})\,k_z\vec{g}_t \\ \\ h_\tau = (A_{te}^{out} - B_{te}^{in})\,k_z\vec{r}_t - \omega\epsilon_0\,(A_{tm}^{out} + B_{tm}^{in})\,\vec{g}_t. \end{array} \right. \tag{44}$$

By plugging (44) into (41) and using the remarkable identities

$$\begin{bmatrix} -(ik_y)^2 & ik_x\, ik_y \\ ik_x\, ik_y & -(ik_x)^2 \end{bmatrix}\vec{g}_t = 0, \quad \begin{bmatrix} -(ik_y)^2 & ik_x\, ik_y \\ ik_x\, ik_y & -(ik_x)^2 \end{bmatrix}\vec{r}_t = (k_x^2 + k_y^2)\vec{r}_t, \tag{45}$$

the transverse electric and transverse magnetic coefficients can be separated into two distinct equations

$$\left\{\begin{array}{l} (-\omega^2\epsilon_s\mu_s + i\omega\mu_s\sigma_s + k_x^2 + k_y^2)\omega\mu_0\,(A_{te}^{out} + B_{te}^{in}) = -\omega^2\epsilon_s\mu_s\hat{Z}_a k_z\,(A_{te}^{out} - B_{te}^{in}) \\ \\ (-\omega^2\epsilon_s\mu_s + i\omega\mu_s\sigma_s)k_z\,(A_{tm}^{out} - B_{tm}^{in}) = -\omega^2\epsilon_s\mu_s\hat{Z}_a\omega\epsilon_0\,(A_{tm}^{out} + B_{tm}^{in})\,, \end{array}\right. \tag{46}$$

and, finally

$$B_{te}^{in} = R_{te}\,A_{te}^{out}, \quad B_{tm}^{in} = R_{tm}\,A_{tm}^{out}, \tag{47}$$

with

$$\left\{\begin{array}{l} R_{te} = \dfrac{-\mu_0(\omega^2\epsilon_s\mu_s - i\omega\mu_s\sigma_s - k_x^2 - k_y^2) + \hat{Z}_a k_z\epsilon_s\mu_s\omega}{\mu_0(\omega^2\epsilon_s\mu_s - i\omega\mu_s\sigma_s - k_x^2 - k_y^2) + \hat{Z}_a k_z\epsilon_s\mu_s\omega} \\ R_{tm} = \dfrac{k_z(\epsilon_s - i\frac{\sigma_s}{\omega}) - \hat{Z}_a\epsilon_0\epsilon_s\omega}{k_z(\epsilon_s - i\frac{\sigma_s}{\omega}) + \hat{Z}_a\epsilon_0\epsilon_s\omega}. \end{array}\right. \tag{48}$$

At this stage, we have obtained an expression for the Fourier Laplace transform of the solution (see (37) via (38)-(40) with B_{te}^{in} and B_{te}^{out} given by (47)-(48)). This expression involves two functions "reflection coefficients". The next step is to establish a uniform estimation for both functions. This will be done by using the hypothesis concerning the approximate operator Z_a.

Properties of the reflection coefficients

The main properties of the reflexion coefficient are given in the following lemma :

Lemma 2.1 *If α_l and β_l verify the conditions of the theorem 2.1, then the reflection coefficients defined in (48) are analytical in the half space $\Im m\ \omega\ < 0$ and satisfy*

$$\left\{\begin{array}{l} \exists C_\star \geq 1,\ C_\star = 1 \ if\, \sigma_s = 0/ \quad \forall k_x, k_y \in \mathbb{R}^2, \quad \forall\omega \in \mathbb{C}, \Im m\omega < 0, \\ \\ \left|\hat{R}_{te}(k_x, k_y, \omega)\right| \leq 1, \quad \left|\hat{R}_{tm}(k_x, k_y, \omega)\right| \leq C_\star \left(1 + \varepsilon\,\dfrac{\sigma_s}{\epsilon_s\eta}\right) \end{array}\right. \tag{49}$$

where $\varepsilon = 0$ if $\beta_0 \neq 0$ and 1 otherwise.

Proof: the proof is purely technical and can be found in appendix A. However, we note that it amounts to showing that there are no non-trivial TE or TM incoming waves of type (40) satisfying the boundary condition with zero second term. This is nothing other than the Kreiss criterion for stability in the sense of Kreiss for hyperbolic boundary value problems.

Remark : the calculation of the reflection coefficients gives us some information about the accuracy of the boundary condition. Indeed, the true reflection coefficients, i.e. those corresponding to the true lossy dielectric media, merely (48) with Z_a replaced by Z. We thus have

$$\begin{aligned} &|R_{te} - R_{te}^{true}| + |R_{tm} - R_{tm}^{true}| \leq C^{ste}\,|Z - Z_a| = C^{ste} O(u^{2L+1}) \\ &\text{with } \; u = \frac{i\sigma_s}{\omega\epsilon_s} + \frac{k_x^2 + k_y^2}{\epsilon_s\mu_s\omega^2}. \end{aligned} \tag{50}$$

This inequality shows that our method can be of higher-order : by taking L large enough we can deal with the situation when $|u|$ is close to 1.

To Obtain the estimation

Once the estimation (49) is obtained, it is easy to deduce the estimation (35).

We use Plancherel's identity

$$\forall \varphi(x,y,t) \in \mathbb{L}^2\left(\mathbb{R}^2_{x,y} \times \mathbb{R}^+_t\right),$$

$$\int_0^\infty e^{-2\eta t}\left(\int_{-\infty}^\infty \varphi^2(x,y,t)\,dx dy\right)dt = \int_{-\infty}^\infty \frac{d\omega}{2\pi}\int_{R^2} \frac{dk_x dk_y}{4\pi^2}\,|\hat\varphi(k_x,k_y,\omega - i\eta)|^2$$

to evaluate

$$\int_0^\infty e^{-2\eta t}\mathcal{E}(t)dt = \int_0^\infty e^{-2\eta t}\left(\epsilon_0\mathcal{E}_E(t) + \mu_0\mathcal{E}_H(t)\right)dt$$

with

$$\mathcal{E}_E(t) = \int_{-\infty}^0 dz \int_{R^2} dx dy\,\left|\vec{E}(x,y,z,t)\right|^2$$

and a similar expression for $\mathcal{E}_H$.

We get

$$\int_0^\infty e^{-2\eta t}\mathcal{E}_E(t)dt = \int_{-\infty}^0 dz \int_{-\infty}^\infty \frac{d\omega}{2\pi}\int_{R^2}\frac{dk_x dk_y}{4\pi^2}\,|\hat{E}(k_x,k_y,z,\omega - i\eta)|^2.$$

Now we have,

$$|\hat{E}|^2 = \left|\hat{E}_{inc} + \hat{E}_{ref}\right|^2 \leq 2\left|\hat{E}_{inc}\right|^2 + 2\left|\hat{E}_{ref}\right|^2$$

with

$$\begin{aligned}
\left|\hat{E}_{ref}\right|^2 &= \left|R_{te}A_{te}^{out}\omega\mu_0\vec{r} - R_{tm}A_{tm}^{out}\vec{g}_+\right|^2 \exp^{-2\Im m(k_z)z} \\
&= \left(\left|R_{te}A_{te}^{out}\omega\mu_0\vec{r}\right|^2 + \left|R_{tm}A_{tm}^{out}\vec{g}_+\right|^2\right)\exp^{-2\Im m(k_z)z} && (\text{since } \vec{g}_+\cdot\vec{r}=0) \\
&\leq \max(|R_{te}|^2,|R_{tm}|^2)\left(\left|A_{te}^{out}\omega\mu_0\vec{r}\right|^2 + \left|A_{tm}^{out}\vec{g}_-\right|^2\right)\exp^{-2\Im m(k_z)z} && (\text{since } |\vec{g}_+|=|\vec{g}_-|) \\
&\leq \max(|R_{te}|^2,|R_{tm}|^2)\left|A_{te}^{out}\omega\mu_0\vec{r} + A_{tm}^{out}\vec{g}_-\right|^2 \exp^{-2\Im m(k_z)z} && (\text{since } \vec{g}_-\cdot\vec{r}=0) \\
&= C_\star^2\left(1+\varepsilon\frac{\sigma_s}{\epsilon_s|\omega|}\right)^2\left|\hat{E}_{inc}\right|^2(k_x,k_y,-z,\omega-i\eta) && (\text{see (38) and (49)}).
\end{aligned}$$

Hence

$$\begin{aligned}
\int_0^\infty e^{-2\eta t}\mathcal{E}_E(t)dt &\leq C(\eta)\int_{-\infty}^0 dz\int_{-\infty}^\infty\frac{d\omega}{2\pi}\int_{R^2}\frac{dk_xdk_y}{4\pi^2}\left(|\hat{E}_{inc}(.,.,z,.)|^2 + |\hat{E}_{inc}(.,.,-z,.)|^2\right) \\
&= C(\eta)\int_{-\infty}^{+\infty}dz\int_{-\infty}^\infty\frac{d\omega}{2\pi}\int_{R^2}\frac{dk_xdk_y}{4\pi^2}\,|\hat{E}_{inc}(k_x,k_y,z,\omega-i\eta)|^2 \\
&= C(\eta)\int_0^\infty e^{-2\eta t}\int_{R^3}dxdydz\,|\vec{E}_{inc}|^2(x,y,z,t),
\end{aligned}$$

with ($C_\star \geq 1$)

$$C(\eta) = 2C_\star^2\left(1+\varepsilon\,\frac{\sigma_s}{\epsilon_s\eta}\right)^2.$$

Performing the same calculations on the magnetic field, we obtain

$$\int_0^\infty e^{-2\eta t}\mathcal{E}(t)dt \leq C(\eta)\int_0^\infty e^{-2\eta t}\mathcal{E}_{inc}(t)dt,$$

$$\text{with}\quad \mathcal{E}_{inc}(t) = \int_{R^3}dxdydz\,\left(\epsilon_0|\vec{E}_{inc}|^2 + \mu_0|\vec{H}_{inc}|^2\right).$$

A simple a priori estimate gives the conservation of this energy for the incident field

$$\mathcal{E}_{inc}(t) = C^{ste} = \mathcal{E}_{inc}(0) = \mathcal{E}_0,$$

and so

$$\int_0^\infty e^{-2\eta t}\mathcal{E}(t)dt \le C(\eta)\mathcal{E}_0 \int_0^\infty e^{-2\eta t}dt = \frac{C(\eta)\mathcal{E}_0}{2\eta}.$$

It is now easy to deduce an estimation in finite time :

$$\begin{aligned}\frac{1}{T}\int_0^T \sqrt{\mathcal{E}(t)}dt \;&\le\; \frac{1}{T}\int_0^\infty e^{\eta t}1_{[0,T]}(t)\; e^{-\eta t}\sqrt{\mathcal{E}(t)}dt \\ &\le\; \frac{1}{T}\left(\int_0^T e^{2\eta t}dt\right)^{1/2}\left(\int_0^\infty e^{-2\eta t}\mathcal{E}(t)dt\right)^{1/2} \\ &\le\; \frac{1}{T}\left(\frac{e^{2\eta T}-1}{2\eta}\right)^{1/2}\sqrt{\frac{C(\eta)}{\eta}}\;\sqrt{\mathcal{E}_0}.\end{aligned}$$

Thus,

$$\frac{1}{T}\int_0^T \sqrt{\mathcal{E}(t)}dt \;\le\; \gamma C_\star\left(1+\varepsilon\, x_\star\frac{\sigma_s T}{\epsilon_s}\right)\sqrt{\mathcal{E}_0},$$

with

$$x_\star = \mathrm{Argmin}\gamma(x) \approx 1.59,\; \gamma = \gamma(x_\star) \equiv 1.24,\; \gamma(x) = \left(\frac{e^x-1}{x^2}\right)^{1/2}.$$

This is the sought estimation.

Regularity of the solution

To obtain the regularity of the solution described in the theorem, we apply the same previously used technique to the solution of the problem differentiated once in time. Indeed, this derivative is a solution to exactly the same problem studied above. We remark that the finite propagation velocity of the solutions of Maxwell's equations together with the assumption of compact support of the initial data makes the solution at time t identically equal to the incident field for small t. We need no compatibility relations between the initial data and the boundary conditions.

We get

$$\int_0^T \|\mathrm{curl}\vec{E}\| + \|\mathrm{curl}\vec{H}\|\, dt = \int_0^T \|\frac{d\vec{E}}{dt}\| + \|\frac{d\vec{H}}{dt}\|\, dt \leq\ C(T)\left(\|\mathrm{curl}\vec{E}_0\| + \|\mathrm{curl}\vec{H}_0\|\right).$$

This inequality with the previous estimation gives the L^1 regularity in time of both fields, their curl, and their time derivative.

Of course, we can reiterate the process, differentiating the equations in time as many time as necessary and finally obtaining the $\mathcal{C}^\infty$ regularity for the electro-magnetic field.

Next, the regularity for the auxiliary functions stems from the usual theory for the solution of the 2D, possibly damped, wave equation with infinitely smooth second term.

2.4 Variational formulation

We now turn to the derivation of a variational formulation for Maxwell's equations in the free space $\{z < 0\}$ associated with the boundary condition (28), (29).

Let us recall that the natural setting for Maxwell's equation is in the $H(\mathrm{curl})$ functional framework :

$$E \in H(\mathrm{curl}) = \left\{E \in L^2(R^3_-)^3, \mathrm{curl}E \in L^2(R^3_-)^3\right\}. \tag{51}$$

In this space, the trace mapping, ($\Gamma = \{z = 0\} \equiv R^2_{xy}$)

$$E \in H(\mathrm{curl}) \quad \hookrightarrow \quad n \wedge (E \wedge n)_{/\Gamma} = \gamma_t E \tag{52}$$

is well defined in an appropriate functional framework ($\gamma_t E \in H^{-\frac{1}{2}}(\mathrm{curl}, \Gamma)$, see [10] for more details). In the following we shall look for the electric field in the space

$$V = \{E \in H(\mathrm{curl}, \Omega)), \gamma_\tau E \in V_\Gamma\},$$

where

$$V_\Gamma = H(\mathrm{curl}_\Gamma, \Gamma) = \left\{u \in L^2(\Gamma)^3, u \cdot n = 0, \mathrm{curl}_\Gamma u \in L^2(\Gamma)^3\right\}.$$

We shall also need to define another space of tangent vector fields for some auxiliary unknowns ($\vec{\nabla}_\Gamma$ here denotes the tangential gradient)

$$H^1_\Gamma(\Gamma) = \left\{ u \in L^2(\Gamma)^3, u \cdot n = 0, \vec{\nabla}_\Gamma u \in L^2(\Gamma)^3 \right\}.$$

Let us introduce the following additional unknowns on the boundary,

$$\left\{ \begin{array}{lcl} r & = & \dfrac{1}{\mu_0}(\mathrm{curl} E \wedge n) = -\partial_t (H \wedge n) = \; {}^t(\partial_t H_y, -\partial H_x) \\ \Phi_l & = & \dfrac{\partial}{\partial t} \begin{bmatrix} +\phi^{(z)}_{l,y} \\ \\ -\phi^{(z)}_{l,x} \end{bmatrix} \\ D_l & = & \mathrm{div}_\Gamma \dfrac{1}{\mu_s}(\alpha_l \Phi_l + r) = \dfrac{1}{\mu_s}\, \partial_t \left(\partial_x \left(\alpha_l \phi^{(z)}_{l,y} + H_y \right) - \partial_y \left(\alpha_l \phi^{(z)}_{l,x} + H_x \right) \right). \end{array} \right. \tag{53}$$

We claim that this set of unknowns allows us to obtain a variational formulation.

From Maxwell's equation and the definition of r, we first easily obtain

$$\forall F \in V, \quad \frac{d^2}{dt^2} \int_{R^3_-} \epsilon_0 E \cdot F \; dx + \int_{R^3_-} \frac{1}{\mu_0} \mathrm{curl} E \cdot \mathrm{curl} F \; dx - \int_\Gamma r \cdot \gamma_\tau F \; d\sigma = 0. \tag{54}$$

Using the identity

$$\int_\Gamma \vec{\mathrm{curl}}_\Gamma \left(\mathrm{curl}_\Gamma \gamma_t E \right) \cdot s d\sigma = \int_\Gamma (\mathrm{curl}_\Gamma \gamma_t E) \cdot (\mathrm{curl}_\Gamma s) d\sigma, \tag{55}$$

with (28) provides a second variational identity

$$\left\{ \begin{array}{c} \forall s \in V_\Gamma, \quad \dfrac{d}{dt} \left(\displaystyle\int_\Gamma \sqrt{\epsilon_s \mu_s}\, r \cdot s \; d\sigma - \sum_{l=1}^{L} \beta_l \int_\Gamma \sqrt{\epsilon_s \mu_s}\, \Phi_l \cdot s \; d\sigma \right) + \\ + \dfrac{d^2}{dt^2} \displaystyle\int_\Gamma \epsilon_s (\gamma_t E) \cdot s \; d\sigma + \dfrac{d}{dt} \int_\Gamma \sigma_s (\gamma_t E) \cdot s \; d\sigma + \int_\Gamma \dfrac{1}{\mu_s} (\mathrm{curl}_\Gamma \gamma_t E) \cdot (\mathrm{curl}_\Gamma s) \; d\sigma = 0. \end{array} \right. \tag{56}$$

There remains to write the L 2-dimensional vectorial wave equations in a variational setting. The problem is that for $s \in V_\Gamma$, $\nabla_\Gamma s$ is not necessarily square integrable, that is $\mathrm{curl}_\Gamma s$ is defined in L^2 but $\mathrm{div}_\Gamma s$ is not. This is the reason for which we have defined the auxiliary unknowns D_l. Using

$$\vec{\Delta}_\Gamma = \vec{\nabla}_\Gamma \mathrm{div}_\Gamma - \vec{\mathrm{curl}}_\Gamma \mathrm{curl}_\Gamma$$

we obtain

$$\left\{ \begin{array}{l} \forall\, s \in V_\Gamma,\ \ \forall l = 1,..L \qquad \dfrac{d^2}{dt^2}\displaystyle\int_\Gamma \epsilon_s \Phi_l \cdot s\ d\sigma + \dfrac{d}{dt}\displaystyle\int_\Gamma \sigma_s \Phi_l \cdot s\ d\sigma + \\ \quad \displaystyle\int_\Gamma \frac{1}{\mu_s} \mathrm{curl}_\Gamma(\alpha_l \Phi_l + r) \cdot \mathrm{curl}_\Gamma s\ d\sigma = \int_\Gamma \vec{\nabla}_\Gamma D_l \cdot s\ d\sigma, \end{array} \right. \tag{57}$$

and

$$\forall\, p \in H^1_\Gamma,\ \ \forall l = 1,..L \qquad \int_\Gamma D_l \cdot p d\sigma = \int_\Gamma \frac{1}{\mu_s}(\alpha_l \Phi_l + r) \cdot \vec{\nabla}_\Gamma p d\sigma. \tag{58}$$

The problem to be solved can be viewed as a system with $2L+2$ unknowns which are

- E : a 3D vector field defined in Ω, belonging to V
- r : a tangent field defined on Γ, in $V_\Gamma = H(\mathrm{curl}_\Gamma, \Gamma)$
- Φ_l : L tangent fields defined on Γ, belonging to V_Γ
- D_l : L scalar fields defined on Γ, belonging to H^1_Γ.

Then problem (54), (56), (57) and (58) can be discretized in space in a conforming way using

- 3D edge elements for E, see [14]
- 2D edge elements for r and Φ_l
- 2D conforming, piecewise linear, elements for D_l.

3 Extension of the perfectly matched layers model to the case of a lossy medium

3.1 Derivation of the model

The idea is inspired by the work of Bérenger on absorbing layers for the electromagnetic waves in a free media. It consists in adding some additional conductivity inside the lossy medium in such a way that the waves are more damped than in reality whereas the reflection coefficient at the interface remains unchanged.

To design our model, we start again from the expression for the solution of Maxwell's equations in the Fourier-Laplace domain (see equations (11), (12), (17))

$$\begin{aligned} \hat{E} &= (A_{te}^{out}\omega\mu_s\vec{r} - A_{tm}^{out}\vec{g}_{out}^{s}) \exp^{-ik_z^s z} \\ \hat{H} &= (A_{te}^{out}\vec{g}_{out}^{s} + A_{tm}^{out}(\omega\epsilon_s - i\sigma_s)\vec{r}) \exp^{-ik_z^s z} . \end{aligned} \tag{59}$$

Generalizing the idea of [16], let us define the change of variable

$$z^\star(z) = z + \frac{1}{i\omega}\int_0^z \sigma^\star(\xi)d\xi, \tag{60}$$

where $\sigma^\star$ is a positive function. Let us remark that (59) can be extended analytically to the complex plane $z \in C$ so we can define

$$\hat{E}^\star(z) = \hat{E}(z^\star), \quad \hat{H}^\star(z) = \hat{H}(z^\star). \tag{61}$$

As claimed by the following lemma, the new electro-magnetic field is more damped than the original one.

Lemma 3.1 *$(\hat{E}^\star,\ \hat{H}^\star)$ satisfy*

$$\left|\hat{E}^\star(z)\right| < \left|\hat{E}(z)\right|, \quad \left|\hat{H}^\star(z)\right| < \left|\hat{H}(z)\right|. \tag{62}$$

Proof : it is equivalent to prove that $0 \geq \Im m(k_z^s(\tilde{z} - z)) = -\int_0^z \sigma^\star(\xi)d\xi\ \Re e\frac{k_z^s}{\omega}$, and this is clearly true as $\frac{k_z^s}{\omega}$ is defined as the square root with positive real part of some complex number (see (15)).

Let us look for the equations satisfied by the new electro-magnetic field, At first, we remark that the differential equations (11) remain valid for $z \in C$ and, in particular, at point $z^\star$.

Using the chain rule

$$\partial_z(f(z^\star)) = \frac{\partial z^\star}{\partial z} \cdot \partial_z f(z = z^\star),$$

we obtain

$$\partial_z f(z = z^\star) = \left(\frac{\partial z^\star}{\partial z}\right)^{-1} \partial_z(f(z^\star)) = \frac{i\omega}{i\omega + \sigma^\star(z)}\, \partial_z(f^\star(z)), \tag{63}$$

and so

$$\left\{\begin{array}{ll} \epsilon_s i\omega \hat{E}_x^\star + \sigma_s \hat{E}_x^\star = -\dfrac{i\omega}{i\omega + \sigma^\star(z)}\dfrac{\partial \hat{H}_y^\star}{\partial z} - ik_y \hat{H}_z^\star, & \mu_s i\omega \hat{H}_x^\star = ik_y \hat{E}_z^\star + \dfrac{i\omega}{i\omega + \sigma^\star(z)}\dfrac{\partial \hat{E}_y^\star}{\partial z} \\ \epsilon_s \omega \hat{E}_y^\star + \sigma_s \hat{E}_y^\star = ik_x \hat{H}_z^\star + \dfrac{i\omega}{i\omega + \sigma^\star(z)}\dfrac{\partial \hat{H}_x^\star}{\partial z}, & \mu_s i\omega \hat{H}_y^\star = -\dfrac{i\omega}{i\omega + \sigma^\star(z)}\dfrac{\partial \hat{E}_x^\star}{\partial z} - ik_x \hat{E}_z^\star \\ \epsilon_s i\omega \hat{E}_z^\star + \sigma_s \hat{E}_z^\star = ik_y \hat{H}_x^\star - ik_x \hat{H}_y^\star, & \mu_s i\omega \hat{H}_z^\star = ik_x \hat{E}_y^\star - ik_y \hat{E}_x^\star. \end{array}\right. \tag{64}$$

The problem now is to return to the (x, y, z, t) domain. At first, we easily obtain

$$\left\{\begin{array}{l} \epsilon_s \dfrac{\partial E_z^\star}{\partial t} + \sigma_s E_z^\star = \dfrac{\partial H_y^\star}{\partial x} - \dfrac{\partial H_x^\star}{\partial y} \\ \mu_s \dfrac{\partial H_z^\star}{\partial t} = \dfrac{\partial E_x^\star}{\partial y} - \dfrac{\partial E_y^\star}{\partial x}. \end{array}\right. \tag{65}$$

These two equations remain unchanged with respect to the usual Maxwell's equations.

The equations relative to $\hat{H}_x^\star$ and $\hat{H}_y^\star$ are treated with Bérenger's tricky method. We first split the four tangential components of the electro-magnetic field according to

$$\hat{H}_x^\star = \hat{H}_{xy}^\star + \hat{H}_{xz}^\star, \qquad \hat{H}_y^\star = \hat{H}_{yx}^\star + \hat{H}_{yz}^\star, \tag{66}$$

For the x components, $\hat{H}^\star_{xy}$ and $\hat{H}^\star_{xz}$ are defined as

$$\left\{\begin{array}{l} \mu_s i\omega \hat{H}^\star_{xy} = ik_y \hat{E}^\star_z \\ \\ \mu_s(i\omega+\sigma^\star)\hat{H}^\star_{xz} = \dfrac{\partial \hat{E}^\star_y}{\partial z}, \end{array}\right. \tag{67}$$

so we get

$$\left\{\begin{array}{l} \mu_s \dfrac{\partial H^\star_{xy}}{\partial t} = -\dfrac{\partial \hat{E}^\star_z}{\partial y} \\ \mu_s \dfrac{\partial H^\star_{xz}}{\partial t} + \mu_s \sigma^\star H^\star_{xz} = \dfrac{\partial E^\star_y}{\partial z}. \end{array}\right. \tag{68}$$

In the same way, we obtain

$$\left\{\begin{array}{l} \mu_s \dfrac{\partial H^\star_{yx}}{\partial t} = \dfrac{\partial \hat{E}^\star_z}{\partial x} \\ \mu_s \dfrac{\partial H^\star_{yz}}{\partial t} + \mu_s \sigma^\star H^\star_{yz} = -\dfrac{\partial E^\star_x}{\partial z}. \end{array}\right. \tag{69}$$

The transformation for the equations relative to $\hat{E}^\star_x$ and $\hat{E}^\star_y$ are more delicate. We have

$$\left\{\begin{array}{l} \epsilon_s i\omega \hat{E}^\star_{xy} + \sigma_s \hat{E}^\star_{xy} = -ik_y \hat{H}^\star_z \\ \\ (i\omega+\sigma^\star)(\epsilon_s i\omega+\sigma_s)\hat{E}^\star_{xz} = -i\omega \dfrac{\partial \hat{H}^\star_y}{\partial z} \Leftrightarrow \left\{\begin{array}{l} (i\omega+\sigma^\star)\hat{\Psi}_x = -\dfrac{\partial \hat{H}^\star_y}{\partial z} \\ (\epsilon_s i\omega+\sigma_s)\hat{E}_{xz} = i\omega\hat{\Psi}_x \end{array}\right. , \end{array}\right. \tag{70}$$

and we get

$$\left\{\begin{array}{l} \epsilon_s \dfrac{\partial E^\star_{xy}}{\partial t} + \sigma_s E^\star_{xy} = \dfrac{\partial H^\star_z}{\partial y} \\ \epsilon_s \dfrac{\partial \Psi_x}{\partial t} + \sigma^\star \epsilon_s \Psi_x = -\dfrac{\partial H^\star_y}{\partial z} \\ \epsilon_s \dfrac{\partial E_{xz}}{\partial t} + \sigma_s E_{xz} = \epsilon_s \dfrac{\partial \Psi_x}{\partial t}. \end{array}\right. \tag{71}$$

Similarly, we obtain

$$\begin{cases} \epsilon_s \dfrac{\partial E^\star_{yx}}{\partial t} + \sigma_s E^\star_{yx} = -\dfrac{\partial H^\star_z}{\partial x} \\ \epsilon_s \dfrac{\partial \Psi_y}{\partial t} + \epsilon_s \sigma^\star \Psi_y = \dfrac{\partial H^\star_x}{\partial z} \\ \epsilon_s \dfrac{\partial E_{yz}}{\partial t} + \sigma_s E_{yz} = \epsilon_s \dfrac{\partial \Psi_y}{\partial t}. \end{cases} \tag{72}$$

Collecting the different equations, we obtain the system (6).

3.2 Analysis of the reflection coefficients

In practical situations, one uses layers of finite width. If δ denotes this width, we choose to close our system of equations by the Dirichlet boundary condition

$$E^\star_x(z=\delta) = E^\star_y(z=\delta) = 0.$$

Once again, the use of the Fourier Laplace transform allows us to obtain an expression for the solution.

In the free medium, we write

$$\vec{E} = \vec{E}_{inc} + \vec{E}_{ref}, \quad z \leq 0. \tag{73}$$

The incident field is the restriction to $\mathbb{R}^3_-$ of the unique solution of Maxwell's equations posed in the whole plane with $(\vec{E}^0, \vec{H}^0)$ as initial data. The reflected field satisfies Maxwell's equations in $\mathbb{R}^3_-$ with zero initial conditions. For z located above the support of the initial data, we get

$$\hat{E}^\star_{inc} = \left(A^{out}_{te} \omega \mu_0 \vec{r} - A^{out}_{tm} \vec{g}_{out}\right) \exp^{-ik_z z}, \quad \hat{H}^\star_{inc} = \left(A^{out}_{te} \vec{g}_{out} + A^{out}_{tm} \omega \epsilon_0 \vec{r}\right) \exp^{-ik_z z}. \tag{74}$$

In the same way, for every $z \leq 0$, the reflected field is given by

$$\begin{aligned} \hat{E}^\star_{ref} &= \left(R^\star_{te} A^{out}_{te} \omega \mu_0 \vec{r} - R^\star_{tm} A^{out}_{tm} \vec{g}_{in}\right) \exp^{+ik_z z} \\ \hat{H}^\star_{ref} &= \left(R^\star_{te} A^{out}_{te} \vec{g}_{in} + R^\star_{tm} A^{out}_{tm} \omega \epsilon_0 \vec{r}\right) \exp^{+ik_z z}. \end{aligned} \tag{75}$$

The two reflection coefficients are determined by the transmission conditions.

Inside the layer, $0 < z < \delta$, the solutions of our problem are obtained by considering the general solution of problem (11) (i.e. the usual Maxwell's equations in a conducting medium written in the (k_x, k_y, z, ω) domain) but taken at a point $(k_x, k_y, z^\star, \omega)$. For $0 < z < \delta$,

$$\begin{aligned} \hat{E}^\star &= (B_{te}^{out}\omega\mu_s\vec{r} - B_{tm}^{out}\vec{g}_{out}^s)\exp^{-ik_z^s z^\star} + (B_{te}^{in}\omega\mu_s\vec{r} - B_{tm}^{in}\vec{g}_{in}^s)\exp^{+ik_z^s z^\star} \\ \hat{H}^\star &= (B_{te}^{out}\vec{g}_{out}^s + B_{tm}^{out}(\omega\epsilon_s - i\sigma_s)\vec{r})\exp^{-ik_z^s z^\star} + (B_{te}^{in}\vec{g}_{in}^s + B_{tm}^{in}(\omega\epsilon_s - i\sigma_s)\vec{r})\exp^{+ik_z^s z^\star} \end{aligned} \quad (76)$$

The solutions such that $E_a^\star(z=\delta) = 0$ and $E_a^\star(z=0) = E_{inc,a}^\star(z=0) + E_{ref,a}^\star(z=0)$, $a = x, y$, are

$$\begin{aligned} \hat{E}^\star &= \left(A_{te}^{out}(1+R_{te}^\star)\frac{\mu_0}{\mu_s}\omega\mu_s\vec{r} - A_{tm}^{out}(1-R_{tm}^\star)\frac{k_z}{k_z^s}\frac{\vec{g}_{out}^s - \vec{g}_{in}^s}{2}\right) f(z) \\ &\quad - \left(A_{tm}^{out}(1-R_{tm}^\star)\frac{k_z}{k_z^s}\frac{\vec{g}_{out}^s + \vec{g}_{in}^s}{2}\right) g(z) \\ \hat{H}^\star &= \left(A_{te}^{out}(1+R_{te}^\star)\frac{\mu_0}{\mu_s}\frac{\vec{g}_{out}^s - \vec{g}_{in}^s}{2} + A_{tm}^{out}(1-R_{tm}^\star)\frac{k_z}{k_z^s}(\omega\epsilon_s - i\sigma_s)\vec{r}\right) g(z) \\ &\quad + \left(A_{te}^{out}(1+R_{te}^\star)\frac{\mu_0}{\mu_s}\frac{\vec{g}_{out}^s + \vec{g}_{in}^s}{2}\right) f(z), \end{aligned} \quad (77)$$

where we have defined,

$$\begin{aligned} f(z) &= \frac{e^{-ik_z^s(z^\star-\delta^\star)} - e^{ik_z^s(z^\star-\delta^\star)}}{e^{ik_z^s\delta^\star} + e^{-ik_z^s\delta^\star}} \\ g(z) &= \frac{e^{-ik_z^s(z^\star-\delta^\star)} + e^{ik_z^s(z^\star-\delta^\star)}}{e^{ik_z^s\delta^\star} + e^{-ik_z^s\delta^\star}}, \end{aligned} \quad (78)$$

(remark that $f(0) = 1$ and $f(\delta) = 0$).

$R_{te}^\star$ and $R_{tm}^\star$ are now defined to ensure the continuity of the two tangential components of the magnetic field (i.e. $H_a^\star(z=0) = H_{inc,a}^\star(z=0) + H_{ref,a}^\star(z=0)$, $a = x, y$).

We obtain,

$$\begin{aligned} \frac{1+R^\star_{te}}{1-R^\star_{te}} &= \frac{\mu_s}{\mu_0}\,\frac{k_z}{k_z^s}\,g_0 \\ \frac{1+R^\star_{tm}}{1-R^\star_{tm}} &= \frac{\epsilon_s+\sigma_s/(i\omega)}{\epsilon_0}\,\frac{k_z}{k_z^s}\,g_0 \end{aligned} \tag{79}$$

with

$$g_0 = g(0) = \frac{1+e^{-2ik_z^s\delta^\star}}{1-e^{-2ik_z^s\delta^\star}} \tag{80}$$

We note that if δ tends to infinity, g_0 tends toward 1 and

$$\begin{aligned} \frac{1+R^\star_{te}}{1-R^\star_{te}} &\longrightarrow \frac{\mu_s}{\mu_0}\,\frac{k_z}{k_z^s} = \frac{1+R_{te}}{1-R_{te}} \\ \frac{1+R^\star_{tm}}{1-R^\star_{tm}} &\longrightarrow \frac{\epsilon_s+\sigma_s/(i\omega)}{\epsilon_0}\,\frac{k_z}{k_z^s} = \frac{1+R_{tm}}{1-R_{tm}}, \end{aligned} \tag{81}$$

where R_{te} and R_{tm} are nothing other than the reflection coefficients for the original problem set in the whole space (i.e. the problem of a free half-space above a lossy media). In other words, we find again that the introduction of an additional and artificial conductivity does not change the solution in the domain of interest ($z \leq 0$); our model is perfectly adapted.

Now, let us return to the case of layer of finite width. If $(R, R^\star)$ denotes $(R_{te}, R^\star_{te})$ or $(R_{tm}, R^\star_{tm})$, the equation

$$\frac{1+R^\star}{1-R^\star} = \frac{1+R}{1-R}\,\frac{1+a}{1-a}, \quad a = e^{-2ik_z^s\delta^\star} \tag{82}$$

can be rewritten

$$R^\star = \frac{R+a}{1+Ra}, \tag{83}$$

and so

$$|R-R^\star| = |a|\,\frac{|1-R^2|}{|1+Ra|}, \quad |a| = \exp\left(\frac{2\Re e(k_z^s)}{\omega}\int_0^\delta \sigma^\star(\xi)d\xi\right)\,\exp\left(2\Im m(k_z^s)\delta\right). \tag{84}$$

This relation proves that both reflection coefficients with the additional conductivity $\sigma^\star$ tend to the value of the reflection coefficient for the lossy dielectric half-space problem as $|a|$ tends to 0, i.e. as $\int_0^\delta \sigma^\star(\xi)d\xi$ tends to infinity.

We have not yet succeed in obtaining estimations as in the previous problem. Indeed, even in the case of the Bérenger's layer model (i.e. $\sigma_s = 0$, $\epsilon_s = \epsilon_0$, $\mu_s = \epsilon_0$), no result on the well-posedness of the problem has been obtained yet, at least to our knowledge. Due to the importance of this model in the Electro-magnetic domain, we think that this work would be of great importance.

4 Conclusion

We have derived and analyzed two inovative methods for the simulation of the effect of a lossy dielectric half-space. The first one is an adapted impedance surface condition (AISC) and the second one a thin absorbing layer. We have demonstrated the stability of the AISC method and given it a variational formulation. It is then easy to use this formulation to construct a finite-element method (edge elements). The absorbing layer is a generalization of Bérenger PMLs to lossy dielectric media. This method is appropriate for finite-difference-time-domain method. We expect that the good numerical results obtained by Bérenger will also hold for our generalization. Numerical experiments should confirm these expectations and should also allow us to compare the potential of the two techniques. This will be the objet of a forthcoming paper.

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A Appendix: proof of lemma 1

Let

$$G(\omega) = \left[1 - \frac{k_x^2 + k_y^2}{\epsilon_0 \mu_0 \omega^2}\right]^{\frac{1}{2}}, \quad \Im m(\omega G(\omega)) < 0 \text{ for } \Im m(\omega) < 0,$$

and

$$u = u(\omega) = \frac{k_x^2 + k_y^2}{\epsilon_s \mu_s \omega^2} + \frac{i\sigma_s}{\epsilon_s \omega}.$$

We define

$$\hat{R}: \quad \omega \in P^- = \{\omega \in \mathbb{C};\ \Im m\omega < 0\} \quad \longrightarrow \quad \hat{R}(\omega) = \frac{A-B}{A+B},$$

with

$$\left\{ \begin{array}{l} A(\omega) = \sqrt{\dfrac{\mu_0}{\epsilon_0}} G(\omega) \left(1 - \dfrac{i\sigma_s}{\omega\epsilon_s}\right) \\ B(\omega) = \sqrt{\dfrac{\mu_s}{\epsilon_s}} \left(1 - \beta_0 u - \displaystyle\sum_{l=1}^{L} \beta_l \dfrac{u}{1-\alpha_l u}\right) \end{array} \right. , \quad \text{for TM modes,}$$

and

$$\left\{ \begin{array}{l} A(\omega) = \sqrt{\dfrac{\mu_s}{\epsilon_s}}\ G(\omega) \\ B(\omega) = \sqrt{\dfrac{\mu_0}{\epsilon_0}} \dfrac{1-u}{1 - \beta_0 u - \displaystyle\sum_{l=1}^{L} \beta_l \dfrac{u}{1-\alpha_l u}} \end{array} \right. , \text{for TE modes.}$$

It is straightforward to verify that $\hat{R} = \hat{R}_{te}$ or $\hat{R} = \hat{R}_{tm}$ as given by (48).

We assume

$$\beta_0 \geq 0, \ \ \beta_l > 0, \ \ 0 < \alpha_l < 1, \ \ \beta_0 + \sum_{l=1}^{L} \frac{\beta_l}{1-\alpha_l} < 1, \tag{85}$$

and

$$\left\{ \begin{array}{l} \mu_s,\ \epsilon_s,\ \mu_0,\ \epsilon_0 > 0, \quad k_x^2 + k_y^2 \in \ \mathbf{R}^+ \\ \sigma_s \geq 0, \ \ \epsilon_s \mu_s \geq \epsilon_0 \mu_0. \end{array} \right. \tag{86}$$

We claim that

- if $\sigma_s = 0$:
$$\sup_{\omega \in P^-} |\hat{R}(\omega)| \leq 1 \quad \text{(TE and TM mode)} \tag{87}$$
- $\sigma_s \geq 0$ and $\beta_0 \neq 0$:
$$\begin{array}{ll} \displaystyle\sup_{\omega \in P^-} |\hat{R}(\omega)| \leq 1 & \text{(TE mode)} \\ \displaystyle\sup_{\omega \in P^-} |\hat{R}(\omega)| \leq C & \text{(TM mode)} \end{array} \tag{88}$$

- $\sigma_s \geq 0$ and $\beta_0 \geq 0$:

$$\begin{aligned} &\sup_{\omega \in P^-} |\hat{R}(\omega)| \leq 1 && \text{(TE mode)} \\ &\sup_{\omega \in P^-} |\hat{R}(\omega)| \leq C\left(1 + \frac{\sigma_s}{|\epsilon_s \omega|}\right) && \text{(TM mode)}, \end{aligned} \tag{89}$$

where C is a constant greater than one, independent of k_x and k_y.

The proof is carried out in three steps. First, we prove that $\hat{R}$ is an analytic function on the half-plane P^- when α_l and β_l satisfy (85). Then we demonstrate that the appropriate inequality, (87), (88) or (89), is satisfied both at infinity and on the real axis as soon as conditions (85) and (86) are fulfilled. Finally, we conclude using the maximum principle

$$\hat{R} \text{ analytic in } P^- \quad \Rightarrow \quad \sup_{\omega \in P^-} |\hat{R}(\omega)| = \sup_{\omega \in \partial P^-} |\hat{R}(\omega)| \leq C.$$

Analyticity in P^-:

It is clear that $\hat{R}$ is analytic throughout the complex plane except at the points of the branch cut of $G(\omega)$ and at the possible zeros of $A + B$. As the branch cut of G, let $]-|k|^2/\epsilon_0\mu_0, |k|^2/\epsilon_0\mu_0[$, is located on the real axis, the only thing we have to prove is that $A + B$ has no zero in P^-. Toward this end , we establish

$$\Im m(\omega A) < 0 \text{ and } \Im m(\omega B) < 0 \text{ for } \Im m\omega < 0, \tag{90}$$

so that the denominator of $\hat{R}$ can not vanish on P^-.

We note that for $\omega \in P^-$,

$$\Im m(\omega u) = \frac{k_x^2 + k_y^2}{\epsilon_s \mu_s |\omega|^2} \Im m(\overline{\omega}) + \frac{\sigma_s}{\epsilon_s} > 0,$$

and

$$\Im m(\omega G(\omega)) < 0, \text{ and } \Re e(G(\omega)) > 0.$$

Hence

- TM mode

$$
\begin{aligned}
\Im m(\omega B) &= \sqrt{\frac{\mu_s}{\epsilon_s}}\left(1+\sum_{l=1}^{L}\frac{\beta_l\alpha_l|u|^2}{|1-\alpha_l u|^2}\right)\Im m(\omega) \\
&\quad -\sqrt{\frac{\mu_s}{\epsilon_s}}\left(\beta_0+\sum_{l=1}^{L}\frac{\beta_l}{|1-\alpha_l u|^2}\right)\Im m(\omega u) \ < 0 \\
\Im m(\omega A) &= \sqrt{\frac{\mu_0}{\epsilon_0}}\left(\Im m(\omega G(\omega)) - \frac{\sigma_s}{\epsilon_s}\Re e(G(\omega))\right) < 0
\end{aligned}
$$

- TE mode

$$
\Im m(\omega A) = \sqrt{\frac{\mu_s}{\epsilon_s}}\Im m(\omega G(\omega)) < 0.
$$

The sign of $\Im m(\omega B)$ requires a little more attention. We have

$$
\begin{aligned}
\sqrt{\frac{\mu_0}{\epsilon_0}}\,\frac{1}{\omega B} &= \frac{1}{\omega}\left(\frac{1-\beta_0 u}{1-u} - \sum_{l=1}^{L}\frac{\beta_l}{1-\alpha_l u}\,\frac{1}{1-u}\right) \\
&= \beta_0\frac{\overline{\omega}}{|\omega|^2} + \left(1-\beta_0-\sum_{l=1}^{L}\frac{\beta_l}{1-\alpha_l}\right)\left(\frac{1}{|\omega|^2}\frac{\overline{\omega}}{|1-u|^2} - \frac{|u|^2}{|1-u|^2}\frac{1}{\omega u}\right) + \\
&= \frac{1}{\omega}\left(\beta_0 + \left(1-\beta_0-\sum_{l=1}^{L}\frac{\beta_l}{1-\alpha_l}\right)\frac{1}{1-u} + \sum_{l=1}^{L}\frac{\beta_l}{1-\alpha_l}\,\frac{1}{1-\alpha_l u}\right) \\
&= \beta_0\frac{\overline{\omega}}{|\omega|^2} + \left(1-\beta_0-\sum_{l=1}^{L}\frac{\beta_l}{1-\alpha_l}\right)\left(\frac{1}{|\omega|^2}\frac{\overline{\omega}}{|1-u|^2} - \frac{|u|^2}{|1-u|^2}\frac{1}{\omega u}\right) + \\
&\quad + \sum_{l=1}^{L}\frac{\beta_l}{1-\alpha_l}\left(\frac{1}{|\omega|^2}\frac{\overline{\omega}}{|1-\alpha_l u|^2} - \frac{\alpha_l|u|^2}{|1-\alpha_l u|^2}\frac{1}{\omega u}\right),
\end{aligned}
\tag{91}
$$

and we conclude using (85) together with the following implications

$$
\Im m\,\overline{\omega} > 0 \text{ and } \Im m\frac{1}{\omega u} < 0 \ \Rightarrow\ \Im m\frac{1}{\omega B} > 0 \ \Rightarrow\ \Im m\,\omega B < 0.
$$

Maximum on ∂P^-:

At infinity we have

$$
\lim_{\omega\in P^-\uparrow\infty}|\hat{R}(\omega)| = \left|\frac{\sqrt{\frac{\epsilon_0}{\mu_0}}-\sqrt{\frac{\epsilon_s}{\mu_s}}}{\sqrt{\frac{\epsilon_0}{\mu_0}}+\sqrt{\frac{\epsilon_s}{\mu_s}}}\right| \le 1.
$$

On the real axis, we define

$$\hat{R}(\omega_r) = \lim_{\omega_i \uparrow 0^-} \hat{R}(\omega_r + i\omega_i), \quad A(\omega_r) = \lim_{\omega_i \uparrow 0^-} A(\omega_r + i\omega_i), \quad B(\omega_r) = \lim_{\omega_i \uparrow 0^-} B(\omega_r + i\omega_i), \tag{92}$$

and we treat separately the case of the different modes.

- **TE modes**
  For TE modes, we want to prove that the modulus of $\hat{R}$ is less than or equal to 1. We have the lemma,

  Lemma A.1 *If* $\hat{R} = \dfrac{A-B}{A+B}$, *A or* $B \neq 0$, *we have*

  $$\left\{ \begin{array}{lll} (a): & \Re e(A)\cdot\Re e(B) & \geq 0 \\ (b): & \Im m(A)\cdot\Im m(B) & \geq 0 \end{array} \right. \quad \Rightarrow \quad |\hat{R}| \leq 1. \tag{93}$$

  (Proof : $|A-B|^2 \leq |A+B|^2 \iff 2\left(\Re e(A)\Re e(B) + \Im m(A)\Im m(B)\right) \geq 0.$*)*

  and the property, see (90),

  $$\forall \omega_i < 0,\ \left(\Im m(\omega A)\cdot\Im m(\omega B)\right)(\omega = \omega_r + i\omega_i) > 0 \ \Rightarrow\ \omega_r^2\left(\Im m(A)\cdot\Im m(B)\right)(\omega_r) \geq 0.$$

  It is thus sufficient to prove that (93-(a)) holds for ω_r real.

  We consider the two following cases

  1. $\dfrac{k_x^2 + k_y^2}{\omega_r^2} < \epsilon_0\mu_0$:
     As $A(\omega_r) = \sqrt{\frac{\mu_s}{\epsilon_s}}\left[1 - \frac{|k|^2}{\epsilon_0\mu_0\omega^2}\right]^{\frac{1}{2}}$ is real and positive it is sufficient to prove that the real part of B (or equivalently the real part of $\sqrt{\frac{\mu_0}{\epsilon_0}}\frac{1}{B}$) is positive. However, if

     $$f(u) = \Re e\sqrt{\frac{\mu_0}{\epsilon_0}}\frac{1}{B}(u) \quad \text{for} \quad u = u_r + iu_i,$$

     with $0 \leq u_r = \dfrac{\epsilon_0\mu_0}{\epsilon_s\mu_s}\,\dfrac{|k|^2}{\omega_r^2\epsilon_0\mu_0} \leq \dfrac{\epsilon_0\mu_0}{\epsilon_s\mu_s} \leq 1$ and $u_i = \dfrac{\sigma_s}{\epsilon_s\omega_r} \in \mathbf{R}$, we have, cf. (91),

$$f(u_r,u_i)=\beta_0+\sum_{l=0}^{L}\tilde\beta_l\frac{1-\alpha_l u_r}{(1-\alpha_l u_r)^2+u_i^2\alpha_l^2}$$
$$\tilde\beta_l=\frac{\beta_l}{1-\alpha_l},\ \tilde\beta_0=1-\beta_0-\sum_{l=1}^{L}\frac{\beta_l}{1-\alpha_l},\ \alpha_0=1.$$

Thus, if (85) holds, f is really positive when holds (note that $(1-\alpha_l u_r)$ is positive since indeed $u_r\le|u_r|\le 1\le 1/\alpha_l$ and that $\tilde\beta_0>0$ by assumption).

2. $\dfrac{k_x^2+k_y^2}{\omega_r^2}\ge\epsilon_0\mu_0$:
   We have
   $$A(\omega_r)=-i\ \mathrm{sign}(\omega_r)\sqrt{\frac{\mu_s}{\epsilon_s}}\sqrt{\frac{|k|^2}{\epsilon_0\mu_0\omega_r^2}-1}.$$
   In this case $\Re e(A)$ is 0 and (93-(a)) is satisfied.

- TM modes
  One more time we distinguish two cases:

  1. $\dfrac{k_x^2+k_y^2}{\omega_r^2}\le\epsilon_0\mu_0$: In this case, we use lemma (A.1) again and show that $\hat R$ is less than one. As before, (93-b) stems from (90) and (92) and we only have to prove (93-a).
     On the one hand, we have
     $$A(\omega_r)=\sqrt{\frac{\mu_0}{\epsilon_0}}\left(1-\frac{|k|^2}{\epsilon_0\mu_0\omega_r^2}\right)^{\frac{1}{2}}\left(1-i\frac{\sigma_s}{\omega_r\epsilon_s}\right),$$
     and $\Re e(A)$ is positive.
     On the other hand, if we define
     $$f_r(u)=\Re e\sqrt{\frac{\epsilon_s}{\mu_s}}B(u)\quad\text{for}\ \ u=u_r+iu_i,$$
     $$\text{with}\ \ 0\le u_r=\frac{\epsilon_0\mu_0}{\epsilon_s\mu_s}\,\frac{|k|^2}{\omega_r^2\epsilon_0\mu_0}\le\frac{\epsilon_0\mu_0}{\epsilon_s\mu_s}\le 1\ \text{and}\ u_i=\frac{\sigma_s}{\epsilon_s\omega_r}\in\mathbf{R},$$

we get

$$f_r(u_r, u_i) = 1 - \beta_0 u_r - \sum_{l=1}^{L} \beta_l \frac{\alpha_l(u_r^2 + u_i^2) - u_r}{(1-\alpha_l u_r)^2 + u_i^2 \alpha_l^2}.$$

Now f_r appears as a sum of homographic functions in the variable u_i^2 which increase from $g(u_r) = 1 - \beta_0 u_r - \sum \beta_l \frac{u_r}{(1-\alpha_l u_r)}$ to $1 - \beta_0 u_r \sum \frac{\beta_l}{\alpha_l}$ as u_i^2 increases from 0 to infinity. The function $u_r \rightarrow g(u_r)$ is a decreasing continuous function of u_r, $u_r \in]0,1[$ so its minimum value is obtained at the point $u_r = 1$ with value $1 - \beta_0 - \sum \frac{\beta_l}{1-\alpha_l}$. The positiveness of this term ensures the positiveness of f_r.

Finally $\Re e(A)$ and $\Re e(B)$ are both positive and (93-(a)) holds.

2. $\dfrac{k_x^2 + k_y^2}{\omega_r^2} < \epsilon_0 \mu_0$:
   - $\sigma_s = 0$

     In this case, A is purely imaginary, B is real and $|\hat{R}|$ is identically equal to 1.
   - $\sigma_s > 0$, $\beta_0 \neq 0$

     This case require a little more analysis. We begin with a change in variables. Defining

$$\chi = \frac{\epsilon_s \mu_s}{\epsilon_0 \mu_0}, \quad \eta = \sqrt{\frac{\epsilon_0 \mu_s}{\epsilon_s \mu_0}}, \quad x = \frac{k^2}{\epsilon_s \mu_s \omega^2}, \quad y = \frac{\sigma_s}{\epsilon_s |\omega|}, \tag{94}$$

we get the following expression for $|\hat{R}|$

$$\left\{ \begin{array}{l} |\hat{R}|^2(x,y) = \dfrac{(a_r + \eta b_r)^2 + (a_i - \eta b_i)^2}{(a_r - \eta b_r)^2 + (a_i + \eta b_i)^2} \\ a_r = y\sqrt{\chi x - 1}, \quad a_i = \sqrt{\chi x - 1} \\ b_r = 1 - \beta_0 x + \displaystyle\sum_{l=1}^{L} \beta_l \frac{\alpha_l(x^2+y^2) - x}{(1-\alpha_l x)^2 + \alpha_l^2 y^2}, \\ b_i = \beta_0 y + \displaystyle\sum_{l=1}^{L} \beta_l \frac{y}{(1-\alpha_l x)^2 + \alpha_l^2 y^2}, \end{array} \right. \tag{95}$$

Now, $|\hat{R}|^2$ is a positive continuous function on the set $\Omega = \{x \geq \chi^{-1},\ y \geq 0\}$. We intend to prove that it is bounded at infinity.

(a) *when x is large enough, b_r is negative and $|\hat{R}|^2$ is less than 1*

We have $b_r = 1 + \sum \beta_l f_l(x, y^2) - \beta_0 x$ with f_l a homographic function in the variable y^2. Its derivative has the sign of $-\alpha_l x + 1$ and f_l is a decreasing function in y^2 when $x > \max 1/\alpha_l$. If $x > \max 2/\alpha_l$, we have

$$b_r < 2b_\infty - \beta_0 x \quad \left(b_\infty = 1 + \sum \frac{\beta_l}{\alpha_l} \right).$$

Let $x_\infty = \max\,(2\alpha_l^{-1}, b_\infty/\beta_0)$. Then $b_r < 0$ for $x > x_\infty$. As a_r, a_i, b_i are all positive, it is easy to verify that $|\hat{R}|$ is less than 1.

(b) *when $x \in [\chi^{-1}(1+\epsilon), x_\infty]$, $|\hat{R}|^2$ is bounded as $y \to \infty$*

If $x \leq x_\infty$, b_r is uniformly bounded :

$$|b_r| < 1 + \beta_0 |x_\infty| + \sum \beta_l \frac{C^{ste} + \alpha_l y^2}{\alpha_l y^2} < M \text{ when } y > 1.$$

Let α, $0 < \alpha < 1$, we always have $\left|\frac{(a_i - \eta b_i)}{(a_i + \eta b_i)}\right| < 1$ and $|\hat{R}| < 1 + \alpha$ occurs when $\left|\frac{(a_r + \eta b_r)}{(a_r - \eta b_r)}\right| < 1 + \alpha$. But,

$$\left|\frac{(a_r + \eta b_r)}{(a_r - \eta b_r)}\right| < \frac{a_r + \eta |b_r|}{|a_r - \eta |b_r||} < 1 + \alpha,$$

as soon as $a_r > M\eta\tau_\alpha$, $\tau_\alpha = \max(1, (2-\alpha)/\alpha)$. Thus, $|\hat{R}|$ is less than $1 + \alpha$ when $y > \max\,(1, \frac{M\eta\tau_\alpha}{\sqrt{\epsilon}})$ and $x \in [\chi^{-1}(1+\epsilon), x_\infty]$.

(as a matter of fact, $|\hat{R}| \to 1$ in this case).

(c) *when $x \to \chi^{-1}$ and $y \to +\infty$, R tends toward 1*

When $x \to \chi^{-1}$ and $y \to +\infty$, we get $b_r \to b_\chi = b_\infty - \beta_0 \chi^{-1}$, $b_i \sim y\beta_0$ and $a_i \to 0$. So

$$|\hat{R}|^2 \sim \frac{(u + 2b_\chi \eta)^2 + y^2 \beta_0^2}{u^2 + y^2 \beta_0^2}, \quad u = y\sqrt{\chi x - 1} - 2b_\chi \eta,$$

and the result follows from the fact that $|\hat{R}|$ tends toward 1 uniformly with respect to u when y tends toward infinity.

The collection of these three results implies that $|\hat{R}|$ is bounded at infinity hence bounded on Ω.

- $\sigma_s \neq 0$, $\beta_0 = 0$

  In this case, R is no longer bounded. Indeed, $a_i, b_i \to 0$, $b_r \to b_\infty$, when $x \to \chi^{-1}$ and $y \to \infty$ but $a_r = y\sqrt{\chi x - 1}$ has no defined limit, in particular, if a_r tends toward ηb_∞, $|\hat{R}| \sim \frac{2\eta b_\infty}{|a_r - \eta b_\infty|}$ which is unbounded.

  However, we have the following bound :

  if $u = (a_r - \eta b_r)(a_i + \eta b_i)^{-1}$, $v = (2\eta b_r)(a_i + \eta b_i)^{-1}$,

$$\begin{aligned} |\hat{R}|^2 &\leq \left((u+v)^2 + \tfrac{(a_i-\eta b_i)^2}{(a_i+\eta b_i)^2}\right)\cdot(u^2+1)^{-1} &&\leq \left((u+v)^2+1\right)\cdot\left(u^2+1\right)^{-1} \\ &\leq 1 + 2v\tfrac{u}{u^2+1} + v^2\tfrac{1}{u^2+1} &&\leq (1+2v+v^2) = \left(1+\tfrac{\eta|b_r|}{a_i+\eta b_i}\right)^2. \end{aligned}$$

  Now, we have $|b_r| \leq \max\left(b_\infty, 1 + \sum \dfrac{\beta_l}{\alpha_l}\dfrac{|\alpha_l x|}{|\alpha_l x - 1|}\right) < 2b_\infty$ as soon as $\dfrac{|\alpha_l x|}{|\alpha_l x - 1|} < 2$ or $x > x_\infty = \max_l(2\alpha_l^{-1})$.

  Therefore, if $|x| > x_\infty$ then $\dfrac{\eta|b_r|}{a_i + \eta b_i} < \dfrac{2\eta b_\infty}{\sqrt{\chi x_\infty - 1}} = C^{ste}$.

  On the other hand, if $|x| < x_\infty$,

  $|b_r| < 1 + \sum \dfrac{\beta_l}{\alpha_l}(1 + \dfrac{|\alpha_l x - 1|}{(\alpha_l x - 1)^2 + \alpha_l^2 y^2}) < 1 + \sum \dfrac{\beta_l}{\alpha_l}(1 + \dfrac{1}{2\alpha_l y}) < 2b_\infty$ as soon as $y > x_\infty$ in which case $\dfrac{\eta b_r}{\eta b_i + a_r} < \dfrac{b_r}{b_i} < \dfrac{2b_\infty y}{\sum \beta_l \alpha_l^{-2}}$.

  Then, if $x^2 + y^2 \geq 2x_\infty^2$ we get $|\hat{R}|(x,y) \leq C^{ste}(1+y)$, as inequality that we can extend to the whole set Ω by continuity of $|\hat{R}|$ in the compact set $x \geq \chi^{-1}$, $y \geq 0$, $x^2 + y^2 \leq 2x_\infty^2$.

  To get the estimation (89), there remains to consider the function,

$$R'(\omega) = \hat{R}(\omega)\left(1 - \frac{i\sigma_s}{\epsilon_s \omega}\right)^{-1}. \qquad (96)$$

  R' is still analytical in the half space P^-, bounded both at infinity and, according to what has been shown, over the real axis. The maximum principle allows us to conclude.

Unité de recherche Inria Lorraine, Technopôle de Nancy-Brabois, Campus scientifique,
615 rue du Jardin Botanique, BP 101, 54600 Villers Lès Nancy
Unité de recherche Inria Rennes, Irisa, Campus universitaire de Beaulieu, 35042 Rennes Cedex
Unité de recherche Inria Rhône-Alpes, 46 avenue Félix Viallet, 38031 Grenoble Cedex 1
Unité de recherche Inria Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, 78153 Le Chesnay Cedex
Unité de recherche Inria Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 Sophia-Antipolis Cedex

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