A Note on Convex Combinations of Stable Polynomials

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Abstract. We show an equivalence between conditions for linear systems to be stabilizable by stable controllers, and conditions for convex combinations of polynomials to be stable.

Une remarque concernant les combinaisons convexes de polynômes stables

Résumé: Nous démontrons une équivalence entre une condition pour qu'un système linéaire soit stabilisable par un controlleur stable et une condition pour que toutes les combinaisons convexes de deux polynômes soient stables.

A note on convex combinations of polynomials

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Key Words - stability, stabilization, strong stabilization.

Abstract

We show an equivalence between conditions for linear systems to be stabilizable by stable controllers, and conditions for convex combinations of polynomials to be stable.

In this note we point to a surprising equivalence between conditions for linear systems to be stabilizable by stable controllers, and conditions for convex combinations of polynomials to be stable.

For the proof of our result we shall need the following extension of the classical zero exclusion principle (see e.g. Theorem 7.3.3 in [3]). Let Q be a pathwise connected subset of $\mathbb{R}^m$ and suppose that the family of polynomials $\mathcal{P} := \{p(\cdot, q) : q \in Q\}$ has invariant degree and has continuous coefficient functions $a_i(q)$. Then the members of $\mathcal{P}$ all have the same number of zeros in the right half plane iff $p(j\omega, q) \neq 0$ for all $q \in Q$ and $\omega \in \mathbb{R}$.

Theorem 1 *Suppose p_0, p_1 are two stable polynomials and assume that $p_0(0)p_1(0) > 0$. Let $p_0(s) = p_{00}(-s^2) + sp_{01}(-s^2)$, $p_1(s) = p_{10}(-s^2) + sp_{11}(-s^2)$ and define $n(s) = s(p_{01}(s)p_{10}(s) - p_{00}(s)p_{11}(s))$, $d(s) = p_{00}(s)p_{10}(s) + sp_{01}(s)p_{11}(s)$. Then the following are equivalent*

1. *$\lambda p_0(s) + (1-\lambda)p_1(s)$ are stable for all $0 \leq \lambda \leq 1$,*
2. *$p_0(s)p_1(-s) + \mu$ have equally many zeros in the right half plane when $\mu > 0$,*
3. *the system $n(s)/d(s)$ is stabilizable by a stable controller.*

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Proof.

($1 \Leftrightarrow 2$) By the extended version of the zero exclusion principle given above and by using the fact that p_0 and p_1 are stable we deduce that the polynomials $\lambda p_0(s) + (1-\lambda)p_1(s)$ are stable for all $0 \leq \lambda \leq 1$ iff

$$\lambda p_0(j\omega) + (1-\lambda)p_1(j\omega) \neq 0 \qquad 0 < \lambda < 1,\ \omega \in \mathbb{R}.$$

(The family $\lambda p_0(s) + (1-\lambda)p_1(s)$ has invariant degree for $0 < \lambda < 1$ since p_0, p_1 are stable and $p_0(0)p_1(0) > 0$.) This last condition can equivalently be expressed by requiring that

$$p_0(j\omega) + \mu p_1(j\omega) \neq 0 \qquad 0 < \mu,\ \omega \in \mathbb{R}.$$

Multiplying both sides by $\overline{p_1(j\omega)} = p_1(-j\omega)$ we arrive at the equivalent condition

$$p_0(j\omega)p_1(-j\omega) + \mu \neq 0 \qquad 0 < \mu,\ w \in \mathbb{R}.$$

A second application of the extended zero exclusion principle leads us to the conclusion.

($2 \Leftrightarrow 3$) From the extended zero exclusion principle we infer an equivalence between the condition that $p_0(s)p_1(-s) + \mu$ have equally many zeros in the right half plane when $\mu > 0$ and

$$p_0(j\omega)p_1(-j\omega) + \mu \neq 0 \qquad 0 < \mu,\ \omega \in \mathbb{R}.$$

The decompositions $p_0(s) = p_{00}(-s^2) + sp_{01}(-s^2)$ and $p_1(s) = p_{10}(-s^2) + sp_{11}(-s^2)$ lead to $p_0(j\omega) = p_{00}(\omega^2) + j\omega p_{01}(\omega^2)$ and $p_1(-j\omega) = p_{10}(\omega^2) - j\omega p_{11}(\omega^2)$. Hence, an equivalent condition is given by

$$\begin{gathered}(p_{00}(\omega^2)p_{10}(\omega^2) + \omega^2 p_{01}(\omega^2)p_{11}(\omega^2) + \mu) \\ +j\omega(p_{01}(\omega^2)p_{10}(\omega^2) - p_{00}(\omega^2)p_{11}(\omega^2)) \neq 0 \qquad 0 < \mu,\ \omega \in \mathbb{R}.\end{gathered}$$

By looking at the real and imaginary parts of this expression we deduce the equivalent condition

$$p_{00}(\omega^2)p_{10}(\omega^2) + \omega^2 p_{01}(\omega^2)p_{11}(\omega^2) \geq 0$$

whenever

$$\omega(p_{01}(\omega^2)p_{10}(\omega^2) - p_{00}(\omega^2)p_{11}(\omega^2)) = 0, \qquad w \in \mathbb{R}.$$

In other words, the polynomial $n(s) = p_{00}(s)p_{10}(s) + sp_{01}(s)p_{11}(s)$ must take positive values whenever $d(s) = s(p_{01}(s)p_{10}(s) - p_{00}(s)p_{11}(s))$ is equal to 0 on the positive real axis. By the assumption $n(0)d(0) > 0$ this condition is satisfied if and only if $d(s)$ has an even number of zeros between each pair of positive real zeros of $n(s)$. By the parity interlacing condition (see [5]) this is exactly equivalent to the requirement that $n(s)/d(s)$ is stabilizable by a stable controller. $\square$

Example. Let $p_0(s) = 10s^3 + s^2 + 6s + 0.57$ and $p_1(s) = 10s^3 + 2s^2 + 8s + 1.57$. It is shown in [3] that although both polynomials are stable, not all convex combinations of p_0 and p_1 are stable. We verify this result by direct application of Theorem 1. From $p_0(s) = -(-s^2) + 0.57 + s(-10(-s^2) + 6)$ and $p_1(s) = (-2(-s^2) + 1.57 + s(-10(-s^2) + 8)$ we construct $n(s) = 10s^3 - 14s^2 + 4.86s$ and $d(s) = 100s^3 - 138s^2 + 45.29s + 0.8949$. The polynomial $d(s)$ has a single zero (at 0.5991) between the pair $(0, 0.6368)$ of zeros of $n(s)$ and the system n/d is therefore not stabilizable by a stable controller. The convex combinations of p_0 and p_1 are thus not all stable. This can be verified by checking that $2/3p_0 + 1/3p_1$ is unstable.

Remarks. The existence of a stable stabilizing controller for a system can be checked by the techniques described in [1] and [2]. These conditions can thus be used as alternatives to the conditions given in [4] for testing the stability of convex combinations of polynomials.

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