Nonlinear Resonance in Systems with Hysteresis

Pierre-Alexandre Bliman, Alexander M. Krasnosel'Skii, Michel Sorine, Alexander A. Vladimirov

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Abstract: A general approach is suggested for the analysis of nonlinear resonance in control systems including an integral linear term and a nonlinear term with hysteresis. This approach uses a special property of nonlinear terms which is valid for many important classes of hysteresis operators. For systems with such hysteresis nonlinearities, we introduce a method to derive conditions for the existence of forced oscillations and (for systems with a parameter) of nonlinear resonance at infinity.

Key-words: hysteresis, nonlinear resonance, forced periodic oscillations, Landesman-Lazer type nonlinearities, bifurcation at infinity

(Résumé : tsvp)

The first and third authors are with INRIA Rocquencourt, `pierre-alexandre.bliman@inria.fr`, `michel.sorine@inria.fr`. The second and fourth authors are with Institute for Information Transmission Problems, 19 Bol. Karetny per., 101447 Moscow, Russia, `amk@ippi.msk.su`. The research described in this publication was made possible in part by Grant MD4000 from the International Scientific Foundation and by Grant 93-01-00884 of Russian Foundation of Fundamental Researchs. A big part of this work has been done during visit of A.M. Krasnosel'skii to INRIA Rocquencourt.

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Unité de recherche INRIA Rocquencourt
Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
Téléphone : (33 1) 39 63 55 11 – Télécopie : (33 1) 39 63 53 30

Résonance non linéaire dans des sytèmes avec hystérèsis

Résumé : Une approche générale pour l'analyse de la résonance non linéaire dans des systèmes dynamiques ayant au moins un intégrateur, en présence d'hystérèsis est proposée. Cette approche utilise une propriété de la non linéarité hystérètique valide pour de nombreuses et importantes classes d'opérateurs d'hystérèsis. Pour des systèmes avec de telles non linéarités, nous introduisons une méthode permettant d'exprimer des conditions d'existence d'oscillations périodiques et, dans le cas de systèmes avec un paramètre, de résonance non linéaire à l'infini.

Mots-clé : hystérèsis, résonance non linéaire, oscillations périodiques forcées, non linéarités de type Landesman-Lazer, bifurcation à l'infini

Contents

1 Introduction

In the early seventies, first results were presented (Landesman and Lazer [18]) on solvability of equations which are degenerate in linear at the infinity, in terms of the behaviour of bounded nonlinearities. This field of nonlinear analysis was investigated by number of researchers (for example, S. Fučik [6], S. Fučik and A. Kufner [7], S. Fučik, J. Nečas, M. Kučera [8], J. Mawhin and M. Willem [20], P. Hess [9] and many others).

The common way to deal with this problem is to consider superposition nonlinearities $x(t) \mapsto f(t, x(t))$, generated by the functions f that satisfy the Landesman-Lazer conditions

$$\lim_{x \to +\infty} f(t,x) = f^+(t), \qquad \lim_{x \to -\infty} f(t,x) = f^-(t).$$

Such nonlinearities satisfy the following main property:

$$\lim_{R \to \infty} \sup_{\|h(t)\|_{E_1} \le c} \|f[t, Re(t) + h(t)] - F(t)\|_{E_2} = 0. \tag{1}$$

This property (with appropriate choice of functions $e(t)$ and $F(t) = F(t; e)$ and spaces E_1 and E_2) allows to study degenerate equations with bounded nonlinearities; it plays a main role in the proof of various statements concerning equations with the nonlinearity $f(t, x)$ and makes it possible to calculate topological characteristics of the corresponding vector fields (the index at infinity). These characteristics define important properties of the systems considered.

The functions $e(t)$ in (1) are determined by the linear part of the problems. In applications to systems with hysteresis (to problems on forced T-periodic oscillations in control systems), equality (1) is generally considered with $e(t) = const$ and $e(t) = \sin(k\omega t + \varphi)$. Here and in the sequel k is a natural number, $\omega = 2\pi T^{-1}$ is the main frequency. For the functions $e(t)$ mentioned above, we will formulate and prove corresponding theorems. One may also consider some other functions $e(t)$ satisfying a condition of the type $mes\{t: \ e(t) = 0\} = 0$ or $mes\{t: \ \dot{e}(t) = 0\} = 0$.

For various types of hysteresis terms (for the general theory, see [16]), analogues of (1) are satisfied. The first results on forced T-periodic oscillations in systems with hysteresis nonlinearities of Landesman-Lazer type were given in [11], where hysteresis nonlinearities (*hysterons with saturation*) satisfying (1) were presented.

In the present paper we introduce an approach to resonant problems on forced T-periodic oscillations in systems whose dynamics can be described by the equation $L(p)x = M(p)G(x) + b(t)$, $p = d/dt$. We suppose that nonlinearity $G(x)$ (this nonlinearity may be of hysteresis type, or functional type, or may include delays and derivatives) satisfies some analogue of (1). Theorems are formulated on the calculation of the index at infinity of corresponding vector fields; this makes it possible to derive existence theorems and theorems on nonlinear resonance.

The main part of this paper is the description of particular classes of hysteresis nonlinearities satisfying equations of the type (1). In the next section we describe a class of hysterons with saturation. In section 3 we describe the hysteresis nonlinearity of friction type introduced by P.-A. Bliman and M. Sorine [1, 2]. In section 4 we consider the so-called *stop* nonlinearity, section 5 concerns relays and Preisach models. In section 6 we formulate results on abstract nonlinear operators with some hysteresis properties.

For systems with one of the hysteresis nonlinear terms mentioned above, it is possible to prove theorems on existence of forced periodic oscillations and on nonlinear resonance for the corresponding equations. In sections 8-9 examples are given for systems with dry friction and with the stop nonlinearity. To prove these (and others) theorems a general approach is useful, it is given in section 7. Sections 10 to 14 contain the proofs.

We use various spaces of functions defined on $[0, T]$ (namely, $L^p,\ C^k,\ W^{k,p}$) as the spaces E_1 and E_2 in (1). The particular choice is determined by the properties of the hysteresis nonlinearity under consideration.

Operator equations which appear in the problems considered, usually include two components: the equation of closure of the system and the equation of periodicity of the state of hysteresis nonlinearity (such equations are studied in [17]). In some cases it is possible to consider only the first equation; the periodicity of the state holds either automatically or with the use of some special approachs.

2 Hysteron with saturation

The theorem formulated in this section is used in the proofs of results from [11].

First, we describe briefly the nonlinearity considered, namely, the *hysteron*. Consider the graphs of two continuous functions $H_1(x)$ and $H_2(x)$ in the plane $\{x, g\}$ and suppose $H_1(x) < H_2(x) \quad (x \in I\!R)$. Let the set $\Omega = \{\{x, g\} : x \in I\!R, H_1(x) \le g \le H_2(x)\}$ be included in the union of nonintersected graphs of a family of continuous functions $g_\alpha(x)$, where α is a parameter. Every function $g_\alpha(x)$ is defined on its own finite interval $[\eta_\alpha^1, \eta_\alpha^2]$ ($\eta_\alpha^1 < \eta_\alpha^2$ for every α) and $g_\alpha(\eta_\alpha^1) = H_1(\eta_\alpha^1)$, $g_\alpha(\eta_\alpha^2) = H_2(\eta_\alpha^2)$. This means that the ends of the graphs of the functions $g_\alpha(x)$ lie on the graphs of the functions $H_1(x)$ and $H_2(x)$ (see Fig. 1).

The output $\mathcal{H}(g_0)x(t)\,(t \ge 0)$ (it is also the state of the hysteron at time t) is defined for monotonous for $t \ge t_0$ inputs as

$$\mathcal{H}(g_0)u(t) = \begin{cases} g_\alpha(u(t)), & \eta_\alpha^1 \le u(t) \le \eta_\alpha^2, \\ H_1(u(t)), & u(t) \le \eta_\alpha^1, \\ H_2(u(t)), & \eta_\alpha^2 \le u(t); \end{cases}$$

the value of α is choosen such that $g_0 = g_\alpha(u(t_0))$. For piecewise monotonous inputs the output is constructed by the semigroup identity. Piecewise monotonous functions are dense in C^0, we define our operator onto C^0 by continuity. The correctness of this procedure see in [16]. The hysteron $\mathcal{H}(g_0)x(t)$ is defined for every continuous input and for every admissible initial state $g_0 \in [H_1(x(t_0)), H_2(x(t_0))]$; it is continuous as an operator from $I\!R \times C^0$ into C^0.

Introduce the following notations:

$$u_*(t) = u_*(t,\varphi,k) = \begin{cases} \sin(k\omega t+\varphi), & k \geq 1, \\ 1, & k = 0, \end{cases}$$
$$u_R(t) = u_R(t,\varphi,h,k) = Ru_*(t,\varphi,k) + h(t),$$
$$d_1(R) = H_1(u_R(0,\varphi,h)), \quad d_2(R) = H_2(u_R(0,\varphi,h)).$$

Theorem 1 *Let the functions $H_1(x)$ and $H_2(x)$ satisfy*

$$\lim_{x\to\pm\infty} H_1(t,x) = \lim_{x\to\pm\infty} H_2(t,x) = g_\pm \tag{2}$$

for some real g_+ and g_-. Then for every integer $k \geq 0$

$$\lim_{R\to\infty} \sup_{\substack{\|h(t)\|_{L^1}\leq c,\ g_0\in[d_1(R),d_2(R)] \\ \varphi\in[0,2\pi]}} \|\mathcal{H}(g_0)u_R(t) - f(t)\|_{L^1} = 0 \tag{3}$$

where

$$f(t) = \frac{g_+ + g_-}{2} + \frac{g_+ - g_-}{2}\operatorname{sgn} u_*(t,\varphi,k).$$

The supremum with respect to g_0 in formula (3) is considered for all initial values $g_0 \in [d_1, d_2]$ which are admissible for the input $Ru_*(t,\varphi,k)+h(t)$. The proof of theorem 1 is similar to the proofs of analogous statements for functional nonlinearities (see for example, [6, 10]).

3 A dry friction model

In this section we consider a hysteresis model suggested by P.-A. Bliman and M. Sorine [1, 2] in order to describe the dry friction effects. In this model, the state $\mathbf{x}$ of the hysteresis nonlinearity is an n-dimensional vector; both the input and the output are scalars.

Let us consider a stable square $n \times n$-matrix $\tilde{A}$ and two n-dimensional vectors $\mathbf{b}$ and $\mathbf{c}$. We denote by $(\cdot,\cdot)$ the scalar product in $I\!R^n$. Vectors are denoted by bold symbols, matrices have tildas; $\tilde{I}$ is the identity matrix.

Consider an absolutely continuous scalar-valued input $u(t)$, $t \geq 0$ and an initial state $\mathbf{x}_0 \in I\!R^n$. The output $\mathcal{F}(\mathbf{x}_0)u(t)$ is defined by the following formulas. Let $\mathbf{x}(t)$ be the solution of

$$\dot{\mathbf{x}} = \tilde{A}\mathbf{x}(t)|\dot{u}(t)| + \dot{u}(t)\,\mathbf{b} \tag{4}$$

satisfying $\mathbf{x}(0) = \mathbf{x}_0$. Then

$$\mathcal{F}(\mathbf{x}_0)u(t) = (\mathbf{c}, \mathbf{x}(t)).$$

Theorem 2 *For every piecewise monotonous, piecewise-C^1 function $u \in W^{1,1}$ such that*

$$mes\{t:\ t\in(0,T),\ \dot{u}(t) = 0\} = 0$$

we have

$$\lim_{R\to\infty} \sup_{\|\mathbf{x}_0\|,\ \|h(t)\|_{W^{1,1}}\le c} \|\mathcal{F}(\mathbf{x}_0)(Ru+h) + (\mathbf{c}, \tilde{A}^{-1}\mathbf{b}) \operatorname{sgn} \dot{u}(t)\|_{L^1} = 0. \tag{5}$$

In particular we deduce the corollary

Corollary. *For $k > 0$*

$$\lim_{R\to\infty} \sup_{\|\mathbf{x}_0\|,\ \|h(t)\|_{W^{1,1}}\le c} \|\mathcal{F}(\mathbf{x}_0)(Ru_*(t,\varphi,k)+h(t)) + (\mathbf{c}, \tilde{A}^{-1}\mathbf{b}) \operatorname{sgn} \dot{u}_*(t,\varphi,k)\|_{L^1} = 0$$

for any $p < \infty$.

The proof of theorem 2 is given in section 10 as a consequence of theorem 7.

Remark. *In* [1] *a more general model of dry friction $\mathcal{F}(\mathbf{x}_0)u + D$ sign $\dot{u}$ $D \ge 0$ is considered, where the set-valued operator* sign *satisfies* sign$0 = [-1,1]$. *This model also satisfies a natural analog of* (5). *For study of forced oscillations in systems with such nonlinearities it is necessary to overcome an additional difficulty: the set-valued character of* sign.

4 The stop

A hysteron (see section 2) is called a *stop* if

$$H_1(x) \equiv -1,\quad H_2(x) \equiv 1;\quad g_\alpha = x - \alpha,\quad \alpha - 1 \le x \le \alpha + 1,\quad \alpha \in I\!R.$$

(see Fig. 2). We denote by $\mathcal{S}(g_0)$ the corresponding hysteresis operator. The state variable g of the stop belongs to the interval $[-1, 1]$.

Theorem 3 *For $k > 0$ the following equality holds:*

$$\lim_{R\to\infty} \sup_{\|h(t)\|_{W^{1,1}}\le c,\ g_0\in[-1,1]} \|\mathcal{S}(g_0)u_R(t) - \operatorname{sgn} \dot{u}_*(t)\|_{L^1} = 0. \tag{6}$$

This theorem follows from the abstract results given in section 6. Note that in (6) one cannot substitute the norm of $W^{1,1}$ by the norm of L^p or C^0.

To show this, let us consider a saw-tooth function

$$h(t) = \min\{frac(t), frac(-t)\},\quad t \in I\!R$$

where $frac(t)$ is the fractional part of t. The graph of the function is given at Fig.3. Consider the family of functions $h_b(t) = h(bt)$, $b > 1$ in $W^{1,1}$. Obviously, $\|h_b\|_{C^0} = .5$ and $\|h_b\|_{W^{1,1}} = bT$. The derivative $\dot{h}_b(t)$ exists for almost all $t \in [0,T]$ and $|\dot{h}_b(t)| = b$, so for large values of $b = b(R)$, the function $Ru_* + h_b$ increases if sgn $\dot{h}(t) > 0$ and decreases if sgn $\dot{h}(t) < 0$. Therefore, for $b = R^2$, the value $\|\mathcal{S}u_R - \operatorname{sgn} \dot{u}_*\|_{L^p}$ does not tend to 0 as $R \to \infty$ although $\|h_b\|_{C^0}$ is bounded.

5 Nonideal relay and the Preisach model

Denote by $\mathcal{R} = \mathcal{R}(\alpha, \beta)$ the nonideal relay (see Fig. 4, detailed description can be found in [3, 4, 5, 16, 21]) with the switch values α and β. The relay is defined for every continuous input and initial state $r_0 \in \{0, 1\}$ by the formula ([19])

$$\mathcal{R}(\alpha, \beta)u(t) = \begin{cases} 0 & \text{if } u(t) \le \beta; \\ 1 & \text{if } u(t) \ge \alpha; \\ 0 & \text{if } u(t) \in (\beta, \alpha) \text{ and } u(\tau) = \beta; \\ 1 & \text{if } u(t) \in (\beta, \alpha) \text{ and } u(\tau) = \alpha \end{cases}$$

where $\tau = \sup\{s : \ s \le t, \ u(s) = \beta \text{ or } u(s) = \alpha\}$ is the value of time at the last threshold attained. If this τ does not exist (i.e. $u(t) \in (\beta, \alpha)$ for all $s < t$ then $\mathcal{R}(\alpha, \beta)u(t) \equiv r_0$. Since $\mathcal{R}u(t) = 1$ for $u(t) > \alpha$ and $\mathcal{R}u(t) = 0$ for $u(t) < \beta$, we get

Theorem 4 *The following relation holds for every integer k*

$$\lim_{R\to\infty} \sup_{\|h(t)\|_{L^1}, \alpha, \beta \le c} \|\mathcal{R}u_R(t) - \frac{1}{2}(1 + \operatorname{sgn} u_*(t, \varphi, k))\|_{L^1} = 0.$$

The nonideal relay has a lot of "bad" properties which make it difficult to study equations with this nonlinearity: the state space is nonconvex, $\mathcal{R}$ does not act in C^0, $\mathcal{R}$ is discontinuous as an operator from C^r to L^p, etc.

Theorem 4 has a counterpart for the Preisach model.

Let Ω be a bounded measurable set in the plane $\{\alpha, \beta\}$. Let some weight function $\rho(\alpha, \beta)$ be defined on Ω. The output of the Preisach model is described by the formula

$$\mathcal{P}(x)u(t) = \int_\Omega \rho(\alpha, \beta)\, \mathcal{R}(\alpha, \beta)u(t)\, d\alpha\, d\beta.$$

We shall choose as the state x of the Preisach hysteresis, the set of the states of all the relays $\mathcal{R}(\alpha, \beta)$, i.e. the characteristic function of some subset of Ω (other choices are possible [3, 4, 5]).

Theorem 5 *The following equality holds*

$$\lim_{R\to\infty} \sup_{\|h(t)\|_{L^1} \le c} \|\mathcal{P}(x_0)u_R(t) - \frac{1}{2}(1 + \operatorname{sgn} u_*(t, \varphi, k)) \int_\Omega \rho(\alpha, \beta)\, d\alpha\, d\beta\|_{L^1} = 0. \quad (7)$$

The last theorem follows from theorem 4.

6 Theorems on abstract operators

In this section we formulate two abstract theorems. They demonstrate which properties of hysteresis allow to prove analogs of theorem 2 and 3.

Consider a nonlinear operator $F : \ C^0(0, T) \to C^0(0, T)$ and two numbers $F^+, \ F^-$. The domain of F can be a subset of C^0, the natural case is $D(F) = W^{1,1}$.

Definition 1 *We say that F satisfies the M-property (M stands for monotonicity) if for every $\varepsilon > 0$ there exists a $d = d(\varepsilon)$ such that for every $[a,b] \subset [0,T]$ and every function $u(t)$ monotonous on $[a,b]$, the following implication is valid:*

$$|u(b) - u(a)| > d \quad \Longrightarrow \quad \left\{ \begin{array}{lll} |Fu(b) - F^+| < \varepsilon & \textit{if} & u(t) \textit{ increases on } [a,b], \\ |Fu(b) - F^-| < \varepsilon & \textit{if} & u(t) \textit{ decreases on } [a,b]. \end{array} \right. \tag{8}$$

The stop satisfies the M-property for every initial state g_0. For the stop, we have $F^{\pm} = \pm 1$ and $d = 2$ uniformly with respect to g_0. The dry friction model from section 3 also satisfies M-property (see section 10 for the proof) uniformly with respect to initial values $\mathbf{x}_0$ from every ball.

The hysteron with saturation from section 2 as well as the relays, Preisach model and classical Landesman-Lazer functional nonlinearities do not satisfy M-property.

The M-property is close to the properties of convergence and ε_0-convergence of tranducers (see [16]).

Theorem 6 *Suppose a nonlinear operator F is uniformly bounded ($|Fu(t)| \leq B < \infty$) and satisfies the M-property. Let $e(t)$ be piecewise-monotonous, piecewise-C^1 and satisfy*

$$mes\{t : \dot{e}(t) = 0\} = 0.$$

Then, for every $c > 0$

$$\lim_{R \to \infty} \sup_{\|h(t)\|_{C^1} \leq c} \|F(Re + h) - F^{\operatorname{sgn} \dot{e}}\|_{L^1} = 0. \tag{9}$$

Here and below $F^{\operatorname{sgn} \dot{e}}$ is the function of t which is equal to F^+ if $\dot{e}(t) > 0$, and to F^- if $\dot{e}(t) < 0$.

Consider some family F_α of operators F (for example, $F_\alpha = \mathcal{S}(\alpha)$, $\alpha \in [-1, 1]$). Let all the operators from F_α satisfy M-property with common F^+, F_- and $d(\varepsilon)$. Then in (9) the supremum with respect to h can be replaced by the supremum with respect to h and α.

Theorem 6 is proved in section 11. Without any additional assumptions, the norm of C^1 in (9) can not be replaced by the norm of $W^{1,1}$. Let us give a counterexample.

Consider the following Madelung-type hysteresis nonlinearity F (see Fig. 5). If $u(t)$ is a monotone increasing input, then we move along the lines $x = u - a$ until we reach the line $x = 1$ and then $x(t) = 1$. If $u(t)$ is a monotone decreasing input, then we move along the lines $x = (1 + R^4)(u - R)$ until we reach the line $x = -1$. This nonlinearity satisfies the M-property with $d = 2$ and $F^{\pm} = \pm 1$.

Let $e(t) = t + 1$, $x_0 = -1$. Fix any large R. Now we shall construct $h(t) = h(t; R)$ with the bounded norm in $W^{1,1}$ such that $F[Re(t) + h(t)] \leq 0$. The last relation contradicts (9) since $F^{\operatorname{sgn} \dot{e}} \equiv 1$ for this $e(t)$.

Put $\dot{h}(t) = 0$ for $t \in [0, 1/R]$ until $u(t) = R(t+1)$ reaches the value $R+1$ and $x(t)$ reaches 0; then put $\dot{h}(t) = -2R$ until $x(t)$ comes back to the value -1; then put $\dot{h}(t) = 0$ until $t = 1$. The main part of the time $\dot{h}(t) = 0$ holds (this is true approximately R times $1/R$) and the function $Re(t)+h(t)$ increases. The time when $\dot{h}(t) = -2R$ tends to 0 approximately as fast as $1/R^2$. It means that the norm $\|h\|_{W^{1,1}}$ even tends to zero as $R \to \infty$!

Consequently, if we want to change the norm of h in (9), we need some additional assumptions.

Definition 2 *We will say that F satisfies the U-property (U stands for uniformity), if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every $[a,b] \subset [0,T]$, for every piecewise-C^1 input $u(\cdot)$ monotonous on $[a,b]$, and for every perturbation $h(\cdot) \in W^{1,1}$ satisfying $\|h\|_{W^{1,1}(a,b)} < \delta$, one has $\|F(u(t)+h(t)) - Fu(t)\|_{C^0(a,b)} < \varepsilon$.*

The U-property is the local uniform continuity of F as an operator from $W^{1,1}(a,b)$ to $C^0(a,b)$ at monotonous $u(t)$ (if F can be considered as an operator on functions, defined on (a,b)). The stop possesses the U-property because it satisfies more restrictive Hölder inequality (see [16]) as an operator from $W^{1,1}$ to C^0. The friction model also satisfies U-property, see [1, 2]. Both these nonlinearities satisfy the U-property uniformly with respect to any bounded initial values.

Theorem 7 *Let all the assumptions of the previous theorem be valid and F have the U-property. Then, for every $c > 0$,*

$$\lim_{R\to\infty} \sup_{\|h(t)\|_{W^{1,1}} \le c} \|F(Re+h) - F^{\operatorname{sgn}\dot{e}}\|_{L^1} = 0. \tag{10}$$

This theorem will be proved in section 12.

In theorems 6 and 7 the space L^1 can be replaced by L^p for any $p < \infty$.

7 Theorem on index at infinity

In this section we give a general construction which allows to formulate theorems on forced periodic oscillations in systems with hysteresis nonlinearities. Such ideas were used by many authors, precise formulations (for existence theorems) can be found, for example, in [6].

Consider a vector field $\Phi x = x - Ax - Fx \equiv x - Bx$ in a Banach space E. Let the operator A be linear and completely continuous and let F be nonlinear completely continuous and bounded.

Let the rotation γ of the field Φx on the sphere $S_\rho = \{\|x\| = \rho\}$ of sufficiently large radius ρ be independent from ρ. Then this value γ is called the *index at infinity* of the field Φx and is denoted by *ind* Φ. If 1 is not an eigenvalue of A, then γ equals $(-1)^r$, where r is the sum of multiplicities of all real eigenvalues of A which are greater than 1. In this case, the rotation γ is defined by the linear part $I - A$ of Φx only.

If $Ker(I-A) = E_0 \neq \{0\}$ the problem of the calculation of *ind* Φ can be reduced to the problem of the calculation of the rotation of a vector field on the unit circle in the finite-dimensional space E_0.

Let an eigenvalue 1 of A corresponds only to eigenvectors of A (adjucent vectors do not exist). Denote by E^0 a subspace invariant for A such that $E_0 \oplus E^0 = E$; denote by P and Q the corresponding projectors onto E_0 and E^0.

Theorem 8 *Let the nonlinear operator F satisfy*

$$\lim_{R\to\infty} \sup_{\|h\|\leq c} \|PF(Re+h) - P\Psi e\| = 0, \tag{11}$$

for any $e \in E_0$, $\|e\| = 1$ and any $c > 0$, where Ψe is a vector field. Let the field $P\Psi e$ be nonzero for $e \in E_0$, $\|e\| = 1$. Then the index $\operatorname{ind}\Phi$ is defined and equals $(-1)^r\gamma_0$, where r is the sum of multiplicities of all real eigenvalues of A that are greater than 1, and γ_0 is the rotation of the vector field $P\Psi e$ on the unit circle $S_1 = \{\|e\| = 1\} \subset E_0$.

The proof of this theorem is given in section 13. This theorem is close to the main result from [10]. Note that if (11) holds, then $P\Psi ae = P\Psi e$, $a > 0$.

8 Theorems on existence of forced oscillations

We study a system whose dynamics can be described by the equations

$$L(\frac{d}{dt})x = M(\frac{d}{dt})\,[\mathcal{G}(g_0)x(t) + b(t)] \tag{12}$$

or by analogous equations with a parameter

$$L(\frac{d}{dt};\lambda)x = M(\frac{d}{dt};\lambda)\,[\mathcal{G}_\lambda(g_0)x(t) + b(t;\lambda)]\,. \tag{13}$$

Here L and M are real coprime polynomials with constant coefficients

$$\begin{array}{l} L(p) = p^l + a_1p^{l-1} + \ldots + a_l, \\ M(p) = b_0p^m + b_1p^{m-1} + \ldots + b_m, \end{array} \tag{14}$$

and $l > m$. We denote by $\mathcal{G}$ one of the hysteresis nonlinearities considered above.

For the equations (13), we suppose that the coefficients of polynomials (14) depend continuously on λ and the nonlinearity is continuous with respect to the set of variables "input + initial state + parameter". We suppose also that the period T is common for all the values of λ. This last restriction can be removed by a change of the variable t.

If the polynomial $L(p)$ has no zeros of the type $\frac{2k\pi i}{T}$ for integer k and $i^2 = -1$, then (see [12, 15, 17]) equation (12) with continuous $\mathcal{G}$ has at least one T-periodic solution and the set of such solutions is *a priori* bounded.

Let the set of initial values for hysteresis nonlinearity $\mathcal{G}(g_0)$ be a finite dimensional space $\mathbb{R}^q$ (for $\mathcal{G} = \mathcal{F}$ $q = n$; for $\mathcal{G} = \mathcal{S}$ $q = 1$).

Theorem 9 *Let the polynomial $L(p)$ have a pair of zeros $\pm\frac{2s\pi i}{T}$, where s is a positive integer number, and let it have no others of the same type. Let the hysteresis nonlinearity $\mathcal{G}(g_0)$ satisfy the M-property with some limit values F^+ and F^- and the U-property uniformly for* *initial values from every ball. Let $\mathcal{G}(g_0)u(t)$ be continuous as an operator from $I\!R^q \times W^{1,1}$ to C^0. Let $F^+ \neq F^-$ and*

$$\frac{\pi}{T}\left|\int_0^T b(t)e^{\frac{2s\pi i}{T}t}dt\right| < |F^+ - F^-|.$$

Then equation (12) has at least one T-periodic solution.

This theorem is proved in section 14.

Collorary 1. *Let $\mathcal{G} = \mathcal{F}$. Let the polynomial $L(p)$ have a pair of zeros $\pm\frac{2s\pi i}{T}$, where s is a positive integer number, and let it have no others of the same type. Let*

$$\frac{\pi}{T}\left|\int_0^T b(t)e^{\frac{2s\pi i}{T}t}dt\right| < 2|(\mathbf{c}, \tilde{A}^{-1}\mathbf{b})|.$$

Then equation (12) has at least one T-periodic solution.

Collorary 2. *Let $\mathcal{G} = \mathcal{S}$. Let the polynomial $L(p)$ have a pair of zeros $\pm\frac{2s\pi i}{T}$, where s is a positive integer number, and let it have no others of the same type. Let*

$$\frac{\pi}{T}\left|\int_0^T b(t)e^{\frac{2s\pi i}{T}t}dt\right| < 2.$$

Then equation (12) has at least one T-periodic solution.

9 Nonlinear resonance (bifurcation at infinity)

In this section we introduce theorems on nonlinear resonance in control systems with hysteresis, in particular with dry friction and with the stop. First, let us recall the notion of *asymptotic bifurcation points* introduced by M.A. Krasnosel'skii in the early fifties (see [14]).

Definition 3 *Let us have an equation $x = F(x;\lambda)$ in a Banach space with a real parameter $\lambda \in \Lambda = (\lambda_1, \lambda_2)$. A value $\lambda_0 \in \Lambda$ of the parameter is called an asymptotic bifurcation point if, for every $\varepsilon > 0$, there exists a $\lambda = \lambda(\varepsilon) \in \Lambda \bigcap (\lambda - \varepsilon, \lambda + \varepsilon)$ such that the equation $x = F(x;\lambda)$ has at least one solution x_λ such that $\|x_\lambda\| > \varepsilon^{-1}$.*

M.A. Krasnosel'skii developed an important topological tool for analysis of such asymptotic bifurcation points called *the principle of changing index.* Let us give one of its possible formulations.

Let the operator $x \mapsto F(x;\lambda)$ be completely continuous for some λ and let the equation $x = F(x;\lambda)$ have no solutions outside a ball. Then the *index at infinity* (the common value of the rotation for the sufficiently large spheres) of the vector field $x - F(x;\lambda)$ is defined; denote it by $ind\,(\lambda)$. If, for every $\lambda < \lambda_0$ and λ close to λ_0, the index $ind\,(\lambda)$ is defined and does not depend on λ, we denote this common value by $ind\,(\lambda_0 - 0)$. Analogously, the value $ind\,(\lambda_0 + 0)$ is defined.

Statement 1 (The principle of changing index [14]) *Let at least two of these three numbers:*

$$ind\,(\lambda_0 - 0), \quad ind\,(\lambda_0), \quad ind\,(\lambda_0 + 0)$$

be different. Then λ_0 is an asymptotic bifurcation point for $x = F(x;\lambda)$.

The theorems formulated above make it possible to prove various statements concerning asymptotic bifurcation points for systems with hysteresis nonlinearities.

Let, again, the input $b(t;\lambda)$ in system (13) be T-periodic.

Theorem 10 *Let the coefficient $a_l(\lambda)$ of $L(p;\lambda)$ be equal to zero for $\lambda = \lambda_0$ and take both positive and negative values in every neighborhood of the point λ_0. Let the hysteresis nonlinearity $\mathcal{G}$ be continuous from $W^{1,1}$ to C^0 and let the state space of $\mathcal{G}$ be finite dimensional linear space. Then λ_0 is an asymptotic bifurcation point for T-periodic problem for* (13).

Under assumptions of this theorem, for $\lambda = \lambda_0$, the value 1 is an eigenvalue of odd multiplicity for the leading (at infinity) linear part (for equivalent operator equation, see section 14) so the theorem may be proved as a corollary of general theorems by M.A. Krasnosel'skii [14, 15].

Below we formulate theorems for equations with hysteresis nonlinearities, satisfying M- and U- properties, and examples for dry friction and stop. In these theorems, only the polynomials L and M and the input $b(t)$ depend on the parameter. These theorems can be easily modified for the case, where hysteresis also depends on the parameter.

The two following theorems concern the case, where $L(p;\lambda_0)$ has a pair of zeros $\pm\frac{2s\pi i}{T}$ for some natural s and has no other roots of the same type. These theorems concern the case of 2-dimensional degeneration of the linear part and can not be obtained by general methods.

Let hysteresis satisfy all the conditions of theorem 9: the set of initial values for hysteresis nonlinearity $\mathcal{G}(g_0)$ is a finite dimensional space $I\!R^q$, hysteresis nonlinearity $\mathcal{G}(g_0)$ satisfy M-property with some limit values F^+ and F^- and U-property uniformly for initial values from every ball. Let $\mathcal{G}(g_0)u(t)$ be continuous as an operator from $I\!R^q \times W^{1,1}$ to C^0.

Theorem 11 *Let $L(\frac{2s\pi i}{T};\lambda) \equiv 0$ on some neighborhood of the point λ_0. Let the function*

$$\varphi_s(\lambda) = \frac{\pi}{T}\left|\int_0^T b(t;\lambda)e^{\frac{2s\pi i}{T}t}\,dt\right| - |F^+ - F^-|$$

take both positive and negative values in every neighborhood of λ_0. Then λ_0 is an asymptotic bifurcation point for T-periodic problem for (13).

Corollary 1. *Let $L(\frac{2s\pi i}{T};\lambda) \equiv 0$ on some neighborhood of the point λ_0. Let the function*

$$\varphi_s^{\mathcal{F}}(\lambda) = \frac{\pi}{T}\left|\int_0^T b(t;\lambda)e^{\frac{2s\pi i}{T}t}\,dt\right| - 2|(\mathbf{c}, \tilde{A}^{-1}\mathbf{b})|$$

take both positive and negative values in every neighborhood of λ_0. *Then* λ_0 *is an asymptotic bifurcation point for* T*-periodic problem for* (13) *with* $\mathcal{G} = \mathcal{F}$.

Corollary 2. *Let* $L(\frac{2s\pi i}{T};\lambda) \equiv 0$ *on some neighborhood of the point* λ_0. *Let the function*

$$\varphi_s^{\mathcal{S}}(\lambda) = \frac{\pi}{T}|\int_0^T b(t;\lambda)e^{\frac{2s\pi i}{T}t}\,dt| - 2$$

take both positive and negative values in every neighborhood of λ_0. *Then* λ_0 *is an asymptotic bifurcation point for* T*-periodic problem for* (13) *with* $\mathcal{G} = \mathcal{S}$.

Theorem 12 *Let* $L(\frac{2s\pi i}{T};\lambda) \neq 0$ *for* $0 < |\lambda - \lambda_0| < \varepsilon$. *Let* $\varphi_s(\lambda_0) > 0$. *Then* λ_0 *is an asymptotic bifurcation point for* T*-periodic problem for* (13).

Corollary 1. *Let* $L(\frac{2s\pi i}{T};\lambda) \neq 0$ *for* $0 < |\lambda - \lambda_0| < \varepsilon$. *Let* $\varphi_s^{\mathcal{F}}(\lambda_0) > 0$. *Then* λ_0 *is an asymptotic bifurcation point for* T*-periodic problem for* (13) *with* $\mathcal{G} = \mathcal{F}$.

Corollary 2. *Let* $L(\frac{2s\pi i}{T};\lambda) \neq 0$ *for* $0 < |\lambda - \lambda_0| < \varepsilon$. *Let* $\varphi_s^{\mathcal{S}}(\lambda_0) > 0$. *Then* λ_0 *is an asymptotic bifurcation point for* T*-periodic problem for* (13) *with* $\mathcal{G} = \mathcal{S}$.

Brief proofs are given in section 14.

10 Proof of theorem 2

Theorem 2 follows from theorem 7. We have only to prove, that the operator $\mathcal{F}(\mathbf{x}_0)$ satisfy M-property with $F^{\pm} = \mp(\mathbf{c}, \tilde{A}^{-1}\mathbf{b})$. The U-property follows from Lipschitz condition for $\mathcal{F}(\mathbf{x}_0)$ as an operator from $W^{1,1}$ to C^0, proved in [1].

Note first, that all the numbers $\|\mathbf{x}_0\|$ are uniformly (with respect to u) bounded.

All the solutions of the linear differential equation (4) have the form

$$\mathbf{x}(t) = e^{\tilde{A}\int_a^t |\dot{u}(\tau)|\,d\tau}\mathbf{x}_a + \mathbf{x}^*(t),$$

where

$$\mathbf{x}^*(t) = \left(\int_a^t e^{\tilde{A}\int_\tau^t |\dot{u}(s)|\,ds}\dot{u}(\tau)\,d\tau\right)\,\mathbf{b}$$

is a solution of (4). Therefore to prove the M-property it is sufficient to prove that for any function $u(t)$ monotonous on $[a,b]$ with $|u(b) - u(a)| > d$ one has

$$\left\|\mathbf{x}^*(b) + \tilde{A}^{-1}\mathbf{b}\,\mathrm{sgn}\,\dot{u}\right\|_{L^1} < \varepsilon.$$

Let $u(t)$ increase on $[a,b]$. The case when $u(t)$ decreases is completely analogous. We have

$$\|\mathbf{x}^*(b) + \tilde{A}^{-1}\mathbf{b}\| = \|\left(\int_a^b e^{\tilde{A}(u(b)-u(\tau))}\dot{u}(\tau)\,d\tau\right)\mathbf{b} + \tilde{A}^{-1}\mathbf{b}\| = \|e^{\tilde{A}(u(b)-u(a))}\tilde{A}^{-1}\mathbf{b}\| \to 0$$

when $u(b) - u(a) \to \infty$. The theorem is proved.

11 Proof of theorem 6

The set $[0,T]$ will be splitted into subsets according to which $\int_0^T |\dots| dt$ will be decomposed. Each term will be estimated separately.

Fix $\varepsilon > 0$. Choose a $r_0 > 0$ such that $mes\,\Omega(r_0) < \frac{\varepsilon}{6B}$, where

$$\Omega(r) = \{t:\ t \in [0,T],\ |\dot e(t)| \le r\}, \quad r < r_0.$$

We have

$$\int_{\Omega(r)} |F[Re(t)+h(t)] - F^{\operatorname{sgn}\dot e(t)}|dt < \frac{\varepsilon}{3}, \quad r \le r_0. \tag{15}$$

The set $[0,T] \setminus \Omega(r)$ is the union of a finite number N of intervals $[a_j,b_j],\ j=1,\dots,N$ such that $\operatorname{sgn}\dot e(t) = const$ for $t \in [a_j,b_j]$. This number N depends only on the function e and on ε. On each $[a_j,b_j]$ the function $e(t)$ is monotonous and $|\dot e(t)| \ge r_0$.

Let $R > \frac{2c}{r_0}$, $t \in [a_j,b_j]$ for some j. Then $\operatorname{sgn}\dot e(t) = \operatorname{sgn}[R\dot e(t) + \dot h(t)]$ and $\frac{d}{dt}[Re(t) + h(t)]| \ge \frac{Rr_0}{2}$. Hence $Re(t)+h(t)$ is monotone on every $[a_j,b_j]$, and we can use the M-property of H. Let us choose a $d = d\left(\frac{\varepsilon}{3NT}\right)$ such that $|F[Re(t)+h(t)] - F^{\operatorname{sgn}\dot e(t)}| < \frac{\varepsilon}{3NT}$ if $|Re(t)+h(t) - Re(a_j) - h(a_j)| > d$. If $t \in \left(a_j + \frac{2}{Rr_0}d\left(\frac{\varepsilon}{3NT}\right), b_j\right)$, then the last inequality holds. Therefore,

$$|F[Re(t)+h(t)] - F^{\operatorname{sgn}\dot e(t)}| < \frac{\varepsilon}{3NT}, \quad t \in \Omega = \bigcup_{j=1}^{N}\left(a_j + \frac{2}{Rr_0}d\left(\frac{\varepsilon}{3NT}\right), b_j\right)$$

and

$$\int_{\Omega} |F[Re(t)+h(t)] - F^{\operatorname{sgn}\dot e(t)}|dt < \frac{\varepsilon}{3}. \tag{16}$$

Relations (15) and (16) together with

$$mes \bigcup_{j=1}^{N}\left(a_j, a_j + \frac{2}{Rr_0}d\left(\frac{\varepsilon}{3NT}\right)\right) < \frac{2N}{Rr_0}d\left(\frac{\varepsilon}{3NT}\right)$$

prove the theorem.

12 Proof of theorem 7

First, let us as in section 11 split the set $[0,T]$ into the intervals of monotonicity of the function $e(t)$, where $|\dot e(t)| \ge r$ and the set $\{t:\ |\dot e(t)| < r\}$. The last set has an arbitrarily small measure for r small enough. Let us fix some small value of r and prove the statement of the theorem for every interval $[a,b]$ of monotonicity of $\dot e(t)$.

Without loss of generality we shall suppose that $e(t)$ is an increasing function satisfying $\dot e(t) \ge r > 0$.

We have to prove that, for every $c > 0$,

$$\lim_{R\to\infty} \sup_{\|h(t)\|_{W^{1,1}}\leq c} \|F(Re+h) - F^{+}\|_{L^1} = 0. \tag{17}$$

Consider an arbitrary function $h \in W^{1,1}$. Split the interval $[a,b]$ in $N = N(R)$ equal intervals $[a + k\frac{(b-a)}{N}, a + (k+1)\frac{(b-a)}{N}]$, $k = 0, 1, \ldots, N-1$. Since

$$\int_a^b |\dot{h}(t)|dt \leq \int_0^T |\dot{h}(t)|dt \leq c,$$

then the number n of the intervals for which we have

$$\int_{a+k(b-a)/N}^{a+(k+1)(b-a)/N} |\dot{h}(t)|dt \geq \alpha(R)$$

satisfies the inequality $n\alpha(R) \leq c$ for any $\alpha(R)$. The joint measure $n\frac{(b-a)}{N}$ of these intervals is not more than $\beta(R) = \frac{c(b-a)}{N(R)\alpha(R)}$. Let us now choose $\alpha(R)$ such that $\alpha(R) \to 0$ and $\alpha(R)N(R) \to \infty$ as $R \to \infty$. Then $\beta(R) \to 0$, and the measure of the part of the integral in (17) corresponding to all these intervals tends to zero.

So we only need to consider the intervals for which

$$\int_{a+k\frac{(b-a)}{N}}^{a+(k+1)\frac{(b-a)}{N}} |\dot{h}(t)|dt \leq \alpha(R) \tag{18}$$

and estimate the values

$$\int_{a+k\frac{(b-a)}{N}}^{a+(k+1)\frac{(b-a)}{N}} |F[Re(t)+h(t)] - F^{+}|dt.$$

Since $\alpha(R) \to 0$, then, for sufficiently large R, due to the U-property, we can make the quantity

$$\sup_{t\in\Delta} |F[Re(t)+h(t)] - F[Re(t)+h(0)]|$$

arbitrary small. Consequently, we must prove that

$$\sum_k \left(\int_{a+k\frac{(b-a)}{N}}^{a+(k+1)\frac{(b-a)}{N}} |F[Re(t)+h(0)] - F^{+}| \right) \to 0. \tag{19}$$

where the summation concerns the numbers k such that (18) holds.

Choose now $N(R)$ such that $N(R)/R \to 0$ as $R \to \infty$. The function $Re(t)+h(0)$ is strictly increasing. Then, on every interval in time of order d/R, we will reach the ε neighborhood of F^{+} and, after that, $|F[Ru(t)+h(0)] - F^{+}| < \varepsilon$. The sum of these times for all the intervals is less than $d \cdot N(R)/R \to 0$. This finishes the proof.

13 Proof of theorem 8

The proof is similar to the proof of the main result from [10] (in [10] it is more detailed).

First, the equation $x = Bx$ ($Bx = Ax + Fx$)) in E is transformed by projection into the system of two equations: $Px = PBx$ and $Qx = QBx$.

From the equation $Qx = QBx$, after easy transformations, the following estimate follows: $||Qx||_E \leq const$. This estimate implies that all zero points of the field $x - Bx$ lie in the infinite cylinder $\{||Qx|| \leq const\} \subset E$. The relation $||Px|| \to \infty$, due to (11) and $P\Psi e \neq 0$, implies $||Px - PBx|| \neq 0$ for sufficiently large $||Px||$. Therefore, we get an *a priori* estimate for the fixed points x^* of operator B: $||x^*|| \leq const$. Consequently, the rotation of the field $x - Bx$ on the spheres $||x|| = \rho$ of sufficiently large radii r coincide with the rotation of this field on the boundary surface of the cylinders $\{||Qx|| \leq c,\ ||Px|| \leq \rho\}$.

Now we can use the theorem on the product of rotations ([10, 15]) for the calculation of the rotation on the surface of the cylinder. This theorem implies that the rotation of the field $x - Bx$ on the surface coincide with the rotation of the limit field $P\Psi e$ on the unit circle $||e|| = 1$ in E_0 multiplied by $(-1)^r$.

14 Proof of theorems 9 and 11, 12

We start our proofs by the transformation of T-periodic problem for equation (12) to some system of operator equations.

Let a real number α be such that $L(\frac{2k\pi i}{T}) \neq \alpha M(\frac{2k\pi i}{T})$ for every integer k. Consider the linear integral operator

$$A_\alpha x(t) = \int_0^T K_\alpha(t-\tau)x(\tau)d\tau,$$

whose kernel $K_\alpha(\tau)$ is the unit impulse response of the rational transfer function $W_\alpha(p) = M(p)/[L(p) - \alpha M(p)]$. This operator associates to every summable T-periodic function $u(t)$ a unique T-periodic solution of

$$[L(\frac{d}{dt}) - \alpha M(\frac{d}{dt})]x(t) = M(\frac{d}{dt})u(t).$$

A_α is completely continuous as an operator from L^2 to $W^{1,2}$.

Consider the system of equations

$$\left\{\ y(t) = \mathcal{G}(g_0)A_\alpha y(t) - \alpha A_\alpha y(t) + b(t), \quad g_0 = \mathcal{G}(g_0)A_\alpha y(t)\ \right|_{t=T}$$

of two unknowns: $y(t) \in L^2$ and initial state g_0. Every solution $\{y(t),\ g_0\}$ of this system generates a T-periodic solution of (12), given by $x(t) = A_\alpha y(t)$. The operator

$$B\{y(t), g_0\} = \{\mathcal{G}(g_0)A_\alpha y(t) - \alpha A_\alpha y(t) + b(t),\ \mathcal{G}(g_0)A_\alpha y(t)\ \Big|_{t=T}\}$$

in $L^2 \times I\!R^q$ satisfies all the conditions of theorem 8. The linear (at ∞) part of $B\{y(t), g_0\}$ is the operator $\{-\alpha A_\alpha y(t),\ 0\}$. If $L(\frac{2s\pi i}{T}) = \alpha M(\frac{2s\pi i}{T})$, then 1 is an eigenvalue of this linear part, with 2-dimensional eigenspace E_0 of $L^2 \times I\!R^q$ given by

$$E_0 = \left\{ \{y(t), g_0\} : \ y(t) = \rho \sin(\frac{2s\pi}{T} t + \varphi),\ g_0 = 0; \quad \rho,\ \varphi \in I\!R \right\}.$$

The rotation of the two-dimensional field $P\Psi e$ in our case can be calculated in an obvious form. Analogous calculations were made by many authors [6, 12, 20] and we do not repeat them.

To prove theorem 9 it is sufficient to show that the index at infinity of the corresponding vector field is different from zero. To prove theorems 11 and 12 it is also sufficient to calculate the corresponding indices and to use the principle of changing index. These tree theorems are consequences of the following lemma.

Lemma 1 *Let the polynomial $L(p)$ have two and exactly two zeros $\pm\frac{2s\pi i}{T}$ with a positive integer s. If*

$$\frac{\pi}{T} \left| \int_0^T b(t) e^{\frac{2s\pi i}{T} t} dt \right| < |F_+ - F_-|, \tag{20}$$

then the index at infinity of the field $I - B$ is equal to ± 1; if

$$\frac{\pi}{T} \left| \int_0^T b(t) e^{\frac{2s\pi i}{T} t} dt \right| > |F_+ - F_-|, \tag{21}$$

then it is equal to 0.

In the lemma, the mapping $P\Psi e$ is a one-to-one mapping of the unit circle onto the circle Y with center $\{\sqrt{2/T} \int_0^T b(t) e^{\frac{2s\pi i}{T} t} dt, 0\}$ and radius $\sqrt{2T}|F_+ - F_-|/\pi$. Therefore if (20) holds, then Y surrounds the origin, and the rotation is equal to ± 1. If (21) holds then Y does not surround the origin and the rotation is equal to 0.

References

[1] Bliman P.-A., Sorine M. *A system-theoretic approach of systems with hysteresis. Application to friction modelling and compensation*, Proc. of the 2nd Eur. Cont. Conf., Groningen, The Netherlands, 1993, 1844-1849

[2] Bliman P.-A., Sorine M. *Easy-to-use realistic dry friction models for automatic control*, Proc. of the 3rd Eur. Cont. Conf., Roma, Italy, 5-8 Sept. 1995, 3788-3794

[3] Brokate M. *Some BV properties of the Preisach hysteresis operator*, Bericht 33, Universität Kaiserslautern, 1988

[4] Brokate M. *On a characterization of the Preisach model for hysteresis*, Bericht 35, Universität Kaiserslautern, 1989

[5] Brokate M., Visintin A. *Properties of the Preisach model for hysteresis*, Bericht 30, Universität Kaiserslautern, 1988

[6] Fučik S. *Solvability of Nonlinear Equations and Boundary Value Problems*, Prague, 1980

[7] Fučik S., Kufner A., *Nonlinear Differential Equations*, Elsevier, Oxford, 1980

[8] Fučik S., Nečas J., Kučera M. Ranges of nonlinear asymptotically linear operators, *J. Diff. Equ.*, **17**, 1975, 375-394

[9] Hess P. On a theorem by Landesman and Lazer, *Indiana Univ. Math. J.*, **23**, 1974, 827-829

[10] Krasnosel'skii A.M. On bifurcation points of equations with Landesman-Lazer type nonlinearities, *Nonlinear Analysis. Theory, Methods & Applications*, **18**, #12, 1992, 1187-1199

[11] Krasnosel'skii A.M. On nonlinear resonance in systems with hysteresis, in *"Models of Hysteresis"*, Pitman Res. Notes in Math. Ser, **286**, Longman Sci & Tech, 1993, 71-76

[12] Krasnosel'skii A.M. *Asymptotics of Nonlinearities and Operator Equations*, Nauka, Moscow, 1992 [Russian], English translation in Operator Theory Advances and Applications vol. 76, Birkhäuser Verlag, Basel Boston Berlin, 1995

[13] Krasnosel'skii A.M. Large-amplitude oscillations in systems with saturation, *Sov. Phys. Dokl.*, **32**, #6, 1992, 419-421

[14] Krasnosel'skii M.A. *Topological Methods in the Theory of Nonlinear Integral Equations*, Pergamon Press, Oxford, 1964

[15] Krasnosel'skii M.A., Zabrejko P.P. *Geometrical Methods of Nonlinear Analysis*, Springer, Berlin, 1984

[16] Krasnosel'skii M.A., Pokrovskii A.V. *Systems with Hysteresis*, Springer, 1989

[17] Krasnosel'skii M.A., Pokrovskii A.V. Operators of forced oscillations problems in systems with hysteresis, *Sov.Math.Dokl.*, **44**, #1, 1992, 228-232

[18] Landesman E.N., Lazer A.C. Nonlinear perturbations of linear elliptic boundary problems at resonance, *J. Math. Mech*, **19**, 1970, 609-623

[19] Macki J. W., Nistri P., Zecca P. Mathematical models for hysteresis, *SIAM Review*, **35**, #1, 1993, 94-123

[20] Mawhin J., Willem M. *Critical Point Theory and Hamiltonian Systems*, Springer, Berlin, 1989

[21] Visintin A. *Mathematical Models of Hysteresis*, Topics in nonsmooth analysis, J.J. Moreau, P.D. Panagiotopulos, G. Strang eds, Birkhaüser, Basel Boston Berlin, 1988, 295-326

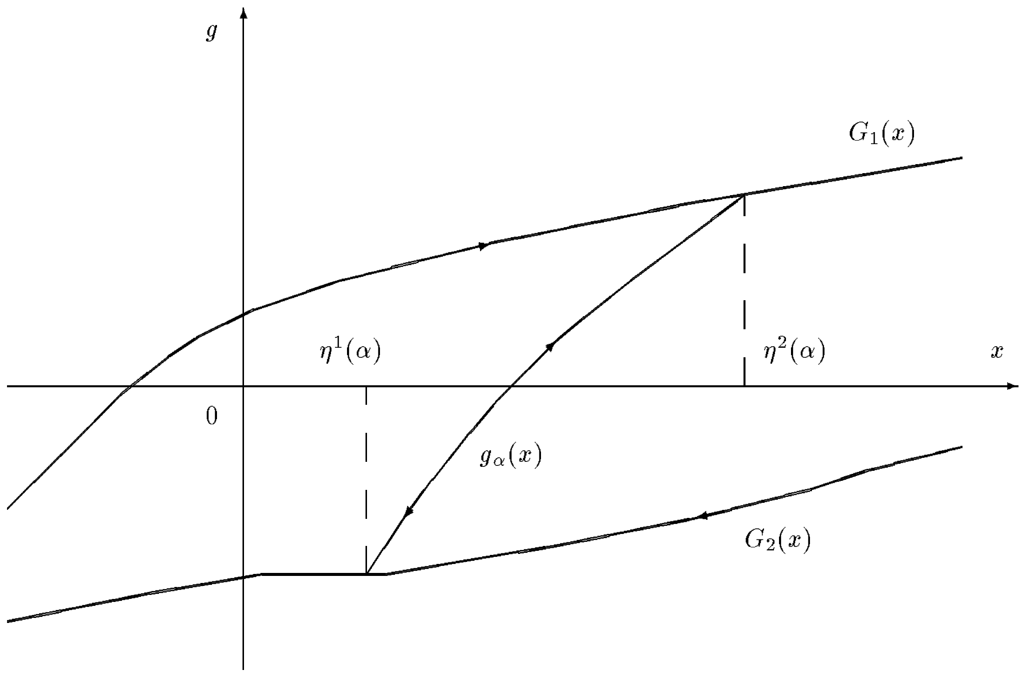


Fig. 1

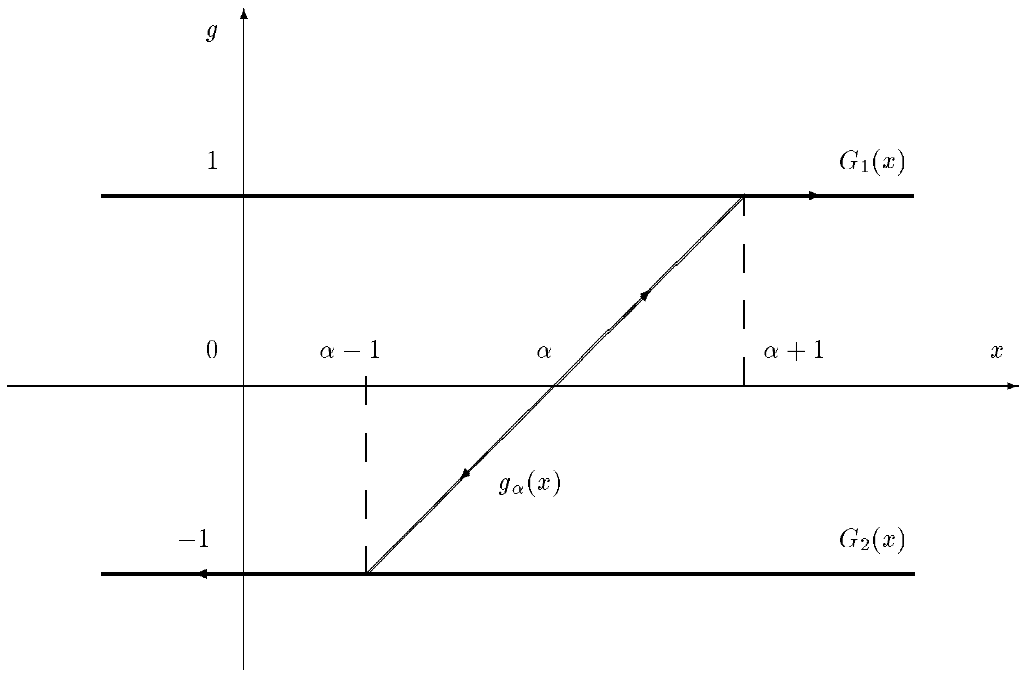
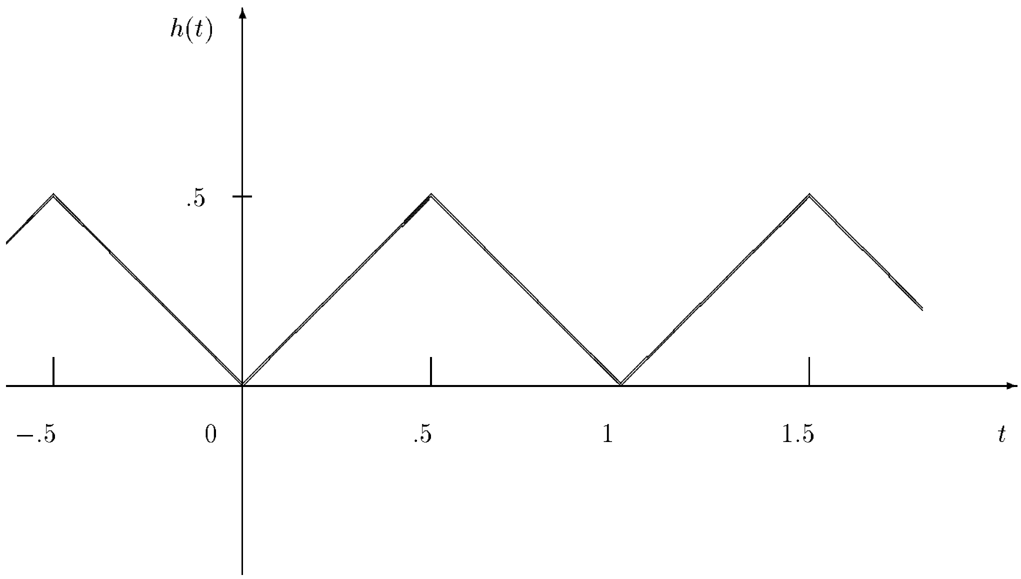


Fig. 2



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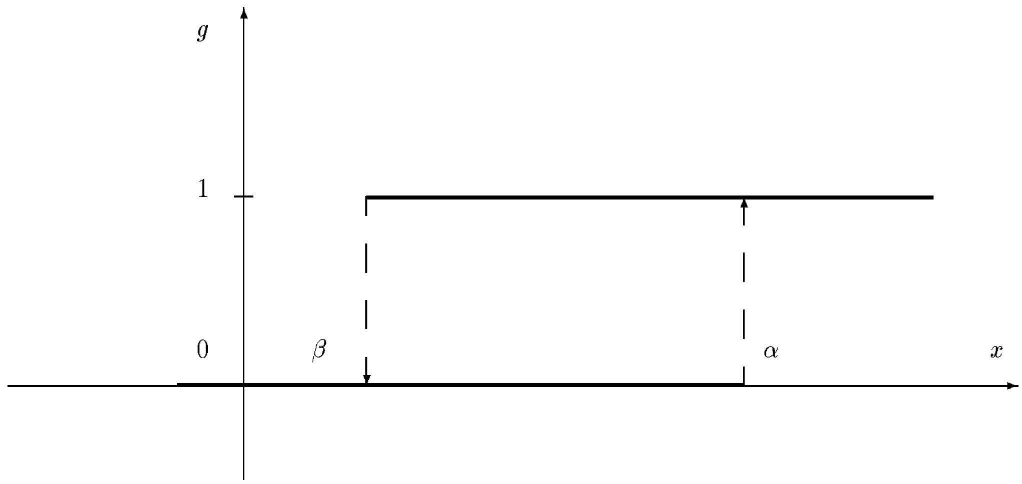

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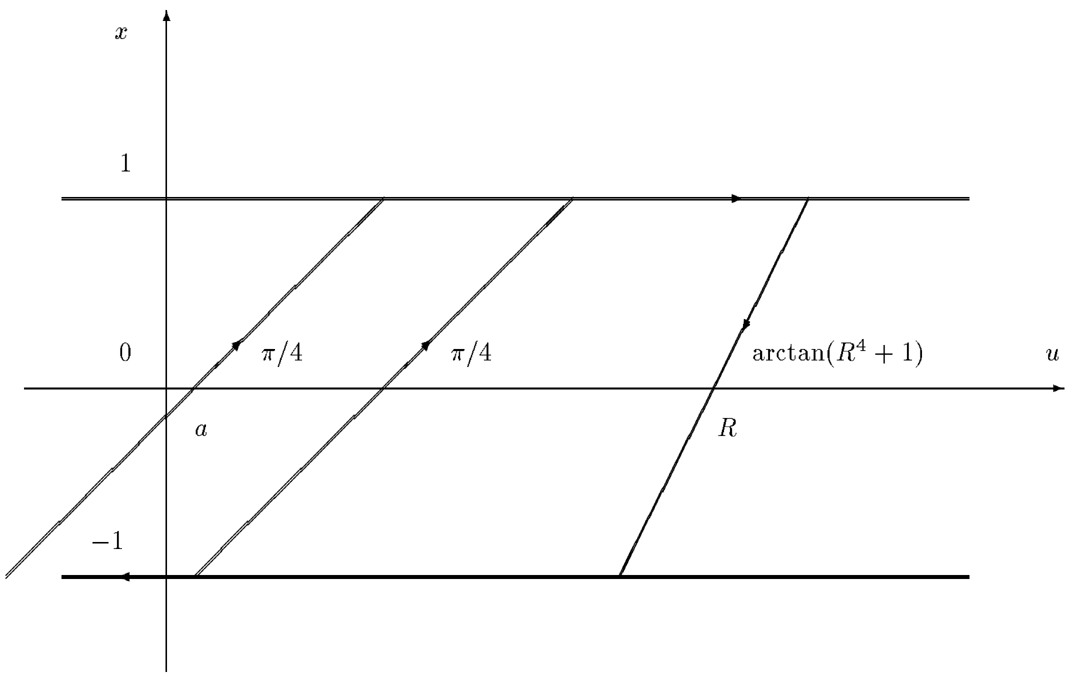

Fig. 5

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