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INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

***A Posteriori Error Estimators for Linearized  
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# A Posteriori Error Estimators for Linearized Semiconductor Equations with Mixed Finite Elements Approach in $H(\text{div}) \times L^2$

Abderrazzak EL BOUKILI\*, Manolo CASTRO-DIAZ \*\*

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**Abstract:** In the framework of Mixed Finite Element Methods, the mathematical analysis of an error indicator, which relies on the residual of a linearized Drift-Diffusion model of the transport equation for electrons in heterojunction semiconductor devices using Fermi-Dirac statistic, is presented. Just now, no numerical experiences have been carried out in order to prove the efficiency of the proposed isotropic estimator. However, from its numerical interpretation, it appears that they are coherent and well indicate the boundary layer and abrupt heterojunction problems.

**Key-words:** Automatic mesh adaptation, Mixed finite elements, Abrupt heterojunction semiconductors, Transport equations (for electrons), Drift-Diffusion model, Local error indicators, Isotropic mesh.

*(Résumé : tsvp)*

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# Estimateur d'Erreur A Posteriori pour les Équations Linéarisées des Semiconducteurs par l'Approche Éléments Finis Mixtes dans $H(\text{div}) \times L^2$

**Résumé :** Dans le cadre de la méthode des éléments finis mixtes, nous présentons une analyse mathématique des indicateurs d'erreur locale pour l'équation linéarisée de transport des particules chargées (électrons). Celle-ci est issue du modèle de Dérive-Diffusion dégénéré pour les semiconducteurs à hétérojonction. La mise en oeuvre, permettant de justifier l'efficacité de ces indicateurs isotropes, n'est pas encore faite. Cependant, leur interprétation numérique montre qu'ils sont cohérents et qu'ils mettent bien en évidence les problèmes de couches limites et de l'hétérojonction abrupte.

**Mots-clé :** Adaptation automatique du maillage, Éléments finis mixtes, Semiconducteurs, Hétérojonction abrupte, Équation de transport des électrons, Dérive-diffusion, Indicateurs d'erreur locale, Maillage isotrope.

## 1 Introduction.

In computational heterojunction semiconductor devices, as in other problems of fluid dynamics or engineering, one often encounters the difficulty that the overall accuracy of the numerical solution is disturbed by local singularities such as those near heterojunction regions, critical points or boundary layers at ohmic contacts (see [5]).

An obvious solution is to refine the discretization near the critical regions, i.e., to place more grid-points where the solution is less regular. The question then is how to identify these regions automatically and how to guarantee a good balance of the number of grid-points in the refined and un-refined regions so that the overall accuracy is optimal. Another closely related problem is to obtain reliable estimates of the accuracy of the computed numerical solution. A priori error estimates are in general not sufficient since they only yield asymptotic estimates which are not known explicitly. Moreover, they often require regularity assumptions about the solution which, for practical problems, are hardly satisfied.

The problem addressed in this paper is principally to establish an ‘a posteriori’ error estimator for linearized transport equation for electrons in semiconductor devices. Initially, classical formulation of this equation is considered, then introducing the current density  $\vec{u}$  as independent variable, we deduce a mixed formulation for the problem that is later linearized to obtain a linear mixed equations in the variables  $p$  (electron quasi-Fermi level) and  $\vec{u}$  (current density for electrons). A mixed Raviart-Thomas Finite Element method of minimal order is used to approximate the continuous problem in  $H(\text{div}) \times L^2$ . A priori estimations are also provided. Finally, two theoretical results which guarantee the existence of an isotropic ‘a posteriori’ local estimator which controls the approximation error are proved. No numerical experiences have been carried out by now. Future works will consider the extensions to anisotropic estimators as well as numerical comparison between isotropic and anisotropic estimators. .

## 2 Continuous problem.

Considering the boundary conditions, the transport equation for electrons in semiconductor devices is given by:

$$\begin{cases} -\text{div}(\alpha(x, p)\nabla p) + c(x, p) = 0 & \Omega, \\ p = g_1 & \partial\Omega_D, \\ (\alpha(x, p)\nabla p) \cdot n = 0 & \partial\Omega_N, \end{cases} \quad (1)$$

where  $\Omega \in \mathbb{R}^2$  is a regular domain with  $\partial\Omega_D \cup \partial\Omega_N = \partial\Omega$ .

## 2.1 Mixed formulation.

Let us consider

$$\vec{u} = \alpha(x, p)\nabla p,$$

then the following mixed problem can be derived from equation (1): *find  $p$  and  $\vec{u}$ , two real value functions such that*

$$\begin{cases} a(x, p)\vec{u} = \nabla p & \Omega, \\ -\operatorname{div}(\vec{u}) + c(x, p) = 0 & \Omega, \\ p = g_1 & \partial\Omega_D, \\ \vec{u} \cdot n = 0 & \partial\Omega_N, \end{cases} \quad (2)$$

with  $a(x, p) = \alpha^{-1}(x, p)$ .

## 2.2 Linearization of the continuous model.

A linear version of the problem (2) in a neighborhood of a point  $(\vec{u}^0, p^0)$  is given by the following equations:

$$\begin{cases} a(x)\vec{u} - \nabla p - \vec{b}(x)p = f(x) & \Omega, \\ -\operatorname{div}(\vec{u}) + c(x)p = g(x) & \Omega, \\ p = g_1 & \partial\Omega_D, \\ \vec{u} \cdot n = 0 & \partial\Omega_N, \end{cases} \quad (3)$$

with

$$\begin{aligned} a(x) &= a(x, p^0(x)), \\ \vec{b}(x) &= a'_p(x, p^0(x))\vec{u}^0(x), \\ f(x) &= a'_p(x, p^0(x))\vec{u}^0(x)p^0(x), \\ c(x) &= c'_p(x, p^0(x)), \\ g(x) &= c'_p(x, p^0(x))p^0(x) - c(x, p^0(x)). \end{aligned}$$

The problem (3) is going to be solved by a mixed variational method, thus, the functions  $a(\cdot)$ ,  $\vec{b}(\cdot)$ ,  $c(\cdot)$ ,  $f$ ,  $g$  and  $g_1$  and the domain  $\Omega$  are supposed to be regular enough so that the mixed variational formulation to be well posed in the function spaces  $L^2(\Omega)$  and  $H(\operatorname{div}, \Omega)$  and to guarantee the existence, uniqueness and regularity of the weak solution.

### 2.3 Mixed variational formulation of the linear model equations.

Let us define

$$\begin{aligned} X &= X(\Omega) = H(\operatorname{div}, \Omega), \\ X_0 &= X_0(\Omega) = H_{0,N}(\operatorname{div}, \Omega), \\ Y &= Y(\Omega) = L^2(\Omega), \end{aligned}$$

where

$$\begin{aligned} H(\operatorname{div}, \Omega) &= \{\vec{v} \in (L^2(\Omega))^2 : \operatorname{div}(\vec{v}) \in L^2(\Omega)\}, \\ H_{0,D}^1(\Omega) &= \{\vec{v} \in H^1(\Omega) : \vec{v}|_{\partial\Omega} = 0 \text{ over } \partial\Omega_D\}, \\ H_{0,N}(\operatorname{div}, \Omega) &= \{\vec{v} \in H(\operatorname{div}, \Omega) : \langle \vec{v} \cdot n, \vec{w} \rangle = 0 \ \forall \vec{w} \in H_{0,D}^1(\Omega)\}. \end{aligned}$$

The following mixed variational problem is considered:  
find  $(\vec{u}, p) \in X_0 \times Y$  so that:

$$\left\{ \begin{array}{l} \int_{\Omega} a(x) \vec{u} \cdot \vec{v} \, dx + \int_{\Omega} p \operatorname{div}(\vec{v}) \, dx \\ \qquad \qquad \qquad - \int_{\Omega} \vec{b}(x) p \cdot \vec{v} \, dx = \langle f, \vec{v} \rangle + \langle \vec{v} \cdot n, g_1 \rangle \quad \forall \vec{v} \in X_0, \\ \int_{\Omega} \operatorname{div}(\vec{u}) q \, dx - \int_{\Omega} c(x) p q \, dx = - \int_{\Omega} g q \, dx \quad \forall q \in Y \end{array} \right. \quad (4)$$

Now, we note

$$\begin{aligned} Z &= Z(\Omega) = X \times Y, \\ Z_0 &= Z_0(\Omega) = X_0 \times Y, \\ U &= (\vec{u}, p); \quad V = (\vec{v}, q). \end{aligned}$$

Let  $\mathcal{A}$  be defined as

$$\begin{aligned} \mathcal{A}(U, V) &= \int_{\Omega} a(x) \vec{u} \cdot \vec{v} \, dx + \int_{\Omega} \operatorname{div}(\vec{v}) p \, dx - \int_{\Omega} \vec{b}(x) p \cdot \vec{v} \, dx \\ &\quad + \int_{\Omega} \operatorname{div}(\vec{u}) q \, dx - \int_{\Omega} c(x) p q \, dx, \end{aligned}$$

by adding the two equations of problem (4). The same process is applied to define the continuous linear form

$$\langle \mathcal{F}, V \rangle = \int_{\Omega} f \vec{v} \, dx + \langle g_1, \vec{v} \cdot n \rangle - \int_{\Omega} g q \, dx.$$



Therefore, the mixed variational problem (4) is equivalent to the following problem:  
find  $U \in Z_0$  so that:

$$\mathcal{A}(U, V) = \langle \mathcal{F}, V \rangle \quad \forall V \in Z_0. \quad (5)$$

Let us consider an arbitrary  $F = (f, g) \in Z'_0$ , with  $f \in X'_0$  and  $g \in Y$ . The second equation of (4) is equivalent to

$$\int_{\Omega} c(x)pq \, dx = \int \operatorname{div}(\vec{u})q \, dx + \langle g, q \rangle. \quad (6)$$

Let us define  $C: L^2(\Omega) \longrightarrow L^2(\Omega)$  as

$$\int_{\Omega} C(p)q \, dx = \int_{\Omega} c(x)pq \, dx,$$

that is,  $\langle C(p), q \rangle = \langle c(x)p, q \rangle$ , and suppose the following hypothesis:  $\exists \gamma > 0$  so that, for all  $x \in \Omega$ ,  $c(x) \geq \gamma$ . In that case,  $C$  is invertible and

$$\|C^{-1}\| \leq \frac{1}{\gamma}.$$

Let us define  $B: H(\operatorname{div}, \Omega) \longrightarrow L^2(\Omega)$  as

$$\int_{\Omega} B(\vec{u})q \, dx = \int_{\Omega} \operatorname{div}(\vec{u})q \, dx,$$

i.e.,  $\langle B(\vec{u}), q \rangle = \langle \operatorname{div}(\vec{u}), q \rangle$ . Thus, the equation (6) can be written as

$$C(p) = B(\vec{u}) + g \Leftrightarrow p = C^{-1}(B(\vec{u}) + g). \quad (7)$$

Let us consider the following hypothesis for  $a(x)$ :  $\exists \alpha > 0$  so that for all  $x \in \Omega$ ,  $a(x) \geq \alpha$ , and define  $A: (L^2(\Omega))^2 \longrightarrow (L^2(\Omega))^2$  as

$$\int_{\Omega} A(\vec{u}) \cdot \vec{v} \, dx = \int_{\Omega} a(x)\vec{u} \cdot \vec{v} \, dx, \quad (8)$$

that is,  $\langle A(\vec{u}), \vec{v} \rangle = \langle a(x)\vec{u}, \vec{v} \rangle$ . With these notations, the first equation of the mixed formulation can be written as:

$$\langle A(\vec{u}), \vec{v} \rangle + \langle B(\vec{v}), p \rangle - \langle p, \vec{b} \cdot \vec{v} \rangle = \langle f, \vec{v} \rangle \quad (9)$$

but  $p = C^{-1}(B(\vec{u}) + g)$ , therefore

$$\langle A(\vec{u}), \vec{v} \rangle + \langle B(\vec{v}) - \vec{b} \cdot \vec{v}, C^{-1}(B(\vec{u}) + g) \rangle = \langle f, \vec{v} \rangle, \quad \forall \vec{v} \in X_0, \quad (10)$$

and this expression is equivalent to

$$\begin{aligned} \langle A(\vec{u}), \vec{v} \rangle + \langle C^{-1}B(\vec{u}), B(\vec{v}) \rangle - \langle C^{-1}B(\vec{u}), \vec{b} \cdot \vec{v} \rangle = \\ \langle f, \vec{v} \rangle - \langle C^{-1}g, B(\vec{v}) \rangle + \langle C^{-1}g, \vec{b} \cdot \vec{v} \rangle, \quad \forall \vec{v} \in X_0. \end{aligned}$$

Let us define the application

$$\vec{v} \mapsto \langle f, \vec{v} \rangle - \langle C^{-1}g, B(\vec{v}) \rangle + \langle C^{-1}g, \vec{b} \cdot \vec{v} \rangle = l(\vec{v}), \quad (11)$$

for which the following inequality holds:

$$\begin{aligned} |l(\vec{v})| &\leq \|f\|_{X'_0} \|\vec{v}\|_{X_0} + \|C^{-1}g\|_{L^2} \|B(\vec{v})\| + \|C^{-1}g\|_{L^2} \|\vec{b} \cdot \vec{v}\|_{L^2} \\ &\leq C(\|f\|_{X'_0} + \frac{1}{\gamma} \|g\|_{L^2} + \frac{1}{\gamma} \|g\|_{L^2} \|\vec{b}\|_{L^\infty}) \|\vec{v}\|_{X_0}, \end{aligned}$$

that is,  $l \in X'_0$ . Let us study the properties of the bilinear form defined as

$$a(\vec{u}, \vec{v}) = \langle A(\vec{u}), \vec{v} \rangle + \langle C^{-1}B(\vec{u}), B(\vec{v}) \rangle - \langle C^{-1}B(\vec{u}), \vec{b} \cdot \vec{v} \rangle.$$

$$\begin{aligned} |a(\vec{u}, \vec{v})| &\leq \|A(\vec{u})\|_{L^2} \|\vec{v}\|_{L^2} + \|C^{-1}B(\vec{u})\|_{L^2} \|B(\vec{v})\|_{L^2} + \|C^{-1}B(\vec{u})\|_{L^2} \|\vec{b} \cdot \vec{v}\|_{L^2} \\ &\leq \|a\|_{L^\infty} \|\vec{u}\|_{L^2} \|\vec{v}\|_{L^2} + \|C^{-1}\|_{L^\infty} \|\operatorname{div} \vec{v}\|_{L^2} \|\operatorname{div} \vec{u}\|_{L^2} \\ &\quad + \|C^{-1}\|_{L^\infty} \|\operatorname{div} \vec{u}\|_{L^2} \|\vec{b}\|_{L^\infty} \|\vec{v}\|_{L^2} \\ &\leq \left( \|a\|_{L^\infty} + \|C^{-1}\|_{L^\infty} (1 + \|\vec{b}\|_{L^\infty}) \right) \|\vec{u}\|_{H(\operatorname{div})} \|\vec{v}\|_{H(\operatorname{div})}. \end{aligned}$$

To prove the ellipticity of the bilinear form  $a(\cdot, \cdot)$  note that

$$\langle C^{-1}B(\vec{v}), B(\vec{v}) \rangle = \langle C^{-1}B(\vec{v}), C \circ C^{-1}B(\vec{v}) \rangle \geq \gamma \|C^{-1}B(\vec{v})\|_{L^2}^2$$

and

$$-\langle C^{-1}B(\vec{v}), \vec{b} \cdot \vec{v} \rangle \geq -\frac{1}{2} \left( \epsilon \|C^{-1}B(\vec{v})\|_{L^2}^2 + \frac{1}{\epsilon} \|\vec{b} \cdot \vec{v}\|_{L^2}^2 \right), \quad \epsilon > 0$$

$$a(\vec{v}, \vec{v}) \geq \alpha \|\vec{v}\|_{L^2}^2 + \gamma \|C^{-1}B(\vec{v})\|_{L^2}^2 - \frac{1}{2} \left( \epsilon \|C^{-1}B(\vec{v})\|_{L^2}^2 + \frac{1}{\epsilon} \|\vec{b} \cdot \vec{v}\|_{L^2}^2 \right).$$

Imposing  $\epsilon = \gamma$ , we obtain

$$\frac{1}{\epsilon} \|\vec{b} \cdot \vec{v}\|_{L^2}^2 = \frac{1}{\gamma} \|\vec{b} \cdot \vec{v}\|_{L^2}^2 \leq \frac{1}{\gamma} \|\vec{b}\|_{L^\infty}^2 \|\vec{v}\|_{L^2}^2,$$

Finally, the following expression can be derived,

$$a(\vec{v}, \vec{v}) \geq \left( \alpha - \frac{\|\vec{b}\|_{L^\infty}^2}{2\gamma} \right) \|\vec{v}\|_{L^2}^2 + \frac{\gamma}{2} \|C^{-1}B(\vec{v})\|_{L^2}^2, \quad (12)$$

supposing that  $\|\vec{b}\|_{L^\infty}$  is sufficiently small to verify

$$\alpha - \frac{\|\vec{b}\|_{L^\infty}^2}{2\gamma} \geq \delta \geq 0, \quad (13)$$

then,

$$\begin{aligned} a(\vec{v}, \vec{v}) &\geq \delta \|\vec{v}\|_{L^2}^2 + \frac{\gamma}{2} \|C^{-1}B(\vec{v})\|_{L^2}^2 \\ &\geq \delta \|\vec{v}\|_{L^2}^2 + \frac{\gamma}{2\|C\|_{L^\infty}^2} \|\operatorname{div} \vec{v}\|_{L^2}^2 \end{aligned}$$

Therefore, under the hypothesis:

1.  $a \in L^\infty$ ,  $a(x) \geq \alpha > 0$ ,  $\forall x \in \Omega$ ,
2.  $c \in L^\infty$ ,  $c(x) \geq \gamma > 0$ ,  $\forall x \in \Omega$ ,
3.  $\vec{b} \in L^\infty$ , with  $\|\vec{b}\|_{L^\infty}$  verifying equation (13),

the mixed problem (5) is well posed, i.e., for all  $f \in X'_0$ ,  $g \in L^2$ , the mixed problem (5) has an unique solution  $(\vec{u}, p)$  and continuous dependence of the solution with respect to data.

**Remark 2.1** *It is easy to verify that the application  $\tilde{f}$  defined by*

$$\langle \tilde{f}, \vec{v} \rangle = \int_{\Omega} \tilde{f} \cdot \vec{v} \, dx + \langle g_1, \vec{v} \cdot \vec{n} \rangle$$

*is a linear form in  $X'_0$ .*

As the mixed problem is well posed, then the bilinear form  $\mathcal{A}(U, V)$  verifies the *inf-sup* condition, i.e., there exists a constant  $\beta > 0$  so that:

$$\inf_{U, V \neq 0} \sup \frac{\mathcal{A}(U, V)}{\|U\|_Z \|V\|_Z} \geq \beta > 0, \quad (14)$$

### 3 Internal approximation by a mixed F.E.M of minimal order.

The mixed variational problem (4) is discretized using the Raviart-Thomas F.E. of minimal order. Let us consider a regular family of triangulations  $\mathcal{T}_h$  of  $\Omega$ ,  $0 < h \leq 1$ , that is, there exists a constant  $\sigma$  independent of  $h$  so that  $\frac{h_K}{\rho_K} < \sigma$ , for all triangle  $K \in \mathcal{T}_h$ , where  $h_K$  is the diameter of a triangle  $K$  and  $\rho_K$  is the diameter of the circumscript circle to  $K$ . Geometrically speaking, the previous conditions is equivalent to ‘minimal angles of triangles bounded from below’. Let us consider

$$RT_0(K) = (P_0(K))^2 + xP_0(K); \quad x \in \mathbb{R}^2,$$

$$R_0(\partial K) = \{q : q \in L^2(\partial K), q|_{F_i} \in P_0(F_i), i = 1, 2, 3\},$$

where  $F_i$ ,  $i = 1, 2, 3$  are the three edges of  $K$  and  $\dim RT_0(K) = 3$ . The degrees of freedom for a triangle  $K$  are

$$\sum_K = \{(l_i)_{i=1,3} : RT_0(K) \rightarrow \mathbb{R}\}$$

where  $l_i$  is a linear form defined by

$$l_i(\vec{v}) = \int_{\partial K} \vec{v} \cdot \vec{n} q \, ds; \quad \forall q \in R_0(\partial K); \quad \forall \vec{v} \in RT_0(K). \quad (15)$$

**Definition 3.1**  $(K, \sum_K, RT_0(K))$  is a F.E. of Raviart-Thomas.

**Remark 3.1** Using the degrees of freedom previously described, it is possible to define a local interpolation operator  $\pi_K(\vec{v}_K)$ , for all  $\vec{v}_K \in H(\text{div}, K)$ , provided  $\vec{v}_K$  is slightly smoother than merely belonging to  $H(\text{div}, K)$ . In general, it is not possible to compute expressions like  $\int_{\partial K} \vec{v} \cdot \vec{n} w \, ds$ , where  $w \in R_0(\partial K)$ , as  $\vec{v}_K \cdot \vec{n}$  is only defined in  $H^{-1/2}(\partial K)$ . However, it is easy to check that if  $\vec{v}_K$  belongs to the space:

$$W(K) = \{\vec{v}_K \in (L^s(K))^2 : \text{div} \vec{v}_K \in L_2(K)\}, \quad (16)$$

for  $s$  fixed  $> 2$ , then such a construction is possible. In that case the interpolation operator  $\pi_K : W(K) \mapsto RT_0(K)$  is defined by

$$\int_{\partial K} (\vec{v}_K - \pi_K(\vec{v}_K)) \cdot \vec{n} \, ds = \mathbf{0}. \quad (17)$$

It is clear that the spaces defined previously can be used to define an internal approximation of  $H(\text{div}, \Omega)$ . At this point, let us consider

$$X_h = \{\vec{v} \in X : \vec{v}|_K \in RT_0(K) \forall K \in \mathcal{T}_h\},$$

and

$$Y_h = \{q \in Y : q|_K \in P_0(K) \forall K \in \mathcal{T}_h\},$$

A global interpolation operator from

$$W(\Omega) = H(\text{div}, \Omega) \cap (L^s(\Omega))^2, \quad (18)$$

( $s$  fixed  $> 2$ ) into  $X_h$  can be defined by simply setting

$$(\Pi_h \vec{v})|_K = \pi_K(\vec{v}|_K). \quad (19)$$

Clearly  $\text{div}(X_h) = Y_h$ .

Now, due to boundary conditions, the following spaces are considered:

$$X_{0h} = X_0 \cap X_h,$$

$$Z_h = X_h \times Y_h \text{ and } Z_{0h} = Z_0 \cap Z_h.$$

The discretization of the mixed variational problem (4) is given by:

$$\begin{cases} \text{Find } U_h = (\vec{u}_h, p_h) \in Z_{0h} \text{ so that :} \\ \mathcal{A}(U_h, V_h) = \langle \mathcal{F}, V_h \rangle_{Z', Z} \quad \forall V_h \in Z_{0h}. \end{cases} \quad (20)$$

The proof of existence and uniqueness of the previous discrete problem and the following convergence results are given in [3].

**Theorem 3.1** *Let  $(\vec{u}_h, p_h)$  be the solution of the mixed continuous problem (4). Then, the approximation error can be estimated by the inequalities*

$$(i) \quad \|p - p_h\|_{L^2(\Omega)} \leq Ch \|p\|_{H^2(\Omega)},$$

$$(ii) \quad \|\vec{u} - \vec{u}_h\|_{L^2(\Omega)} \leq Ch \|p\|_{H^2(\Omega)},$$

$$(iii) \quad \|\text{div}(\vec{u} - \vec{u}_h)\|_{L^2(\Omega)} \leq Ch^s \|p\|_{H^{s+2}(\Omega)}, \quad 0 \leq s \leq 1.$$

**Remark 3.2** *We observe that the former theorem proves convergence in  $X_0 \times Y$ , at an optimal rate and with minimal smoothness requirements on the solution.*

## 4 ‘A posteriori’ error estimator for the linear equations.

As problem (5) is well-posed, that is, there exists a unique solution for it and the suitable choice of approximation spaces imply the existence of a constant, independent of  $h$ ,  $\beta > 0$  so that:

$$\sup_{V \neq 0} \frac{A(U - U_h, V)}{\|V\|_Z} \geq \beta(\|\vec{u} - \vec{u}_h\|_X + \|p - p_h\|_Y) \quad (21)$$

$$\begin{aligned} A(U - U_h, V) &= \int_{\Omega} a(x)(\vec{u} - \vec{u}_h) \cdot \vec{v} \, dx + \int_{\Omega} \operatorname{div} \vec{v}(p - p_h) \, dx \\ &\quad - \int_{\Omega} \vec{b}(x) \cdot \vec{v}(p - p_h) \, dx + \int_{\Omega} \operatorname{div}(\vec{u} - \vec{u}_h)q \, dx \\ &\quad - \int_{\Omega} c(x)(p - p_h)q \, dx \\ &= \sum_K \left( \int_K a(x)(\vec{u} - \vec{u}_h) \cdot \vec{v} \, dx + \int_K \operatorname{div} \vec{v}(p - p_h) \, dx \right. \\ &\quad - \int_K \vec{b}(x) \cdot \vec{v}(p - p_h) \, dx + \int_K \operatorname{div}(\vec{u} - \vec{u}_h)q \, dx \\ &\quad \left. - \int_K c(x)(p - p_h)q \, dx \right). \end{aligned}$$

Applying now the Green’s formula to the term:

$$\int_K \operatorname{div} \vec{v}(p - p_h) \, dx,$$

and using the boundary conditions, we obtain

$$\begin{aligned} A(U - U_h, V) &= \sum_K \left( \int_K a(x)(\vec{u} - \vec{u}_h) \cdot \vec{v} \, dx - \int_K \vec{v} \cdot \nabla(p - p_h) \, dx \right. \\ &\quad + \langle \vec{v} \cdot \vec{n}, p - p_h \rangle_{\partial K} - \int_K \vec{b}(x) \cdot \vec{v}(p - p_h) \, dx \\ &\quad \left. + \int_K \operatorname{div}(\vec{u} - \vec{u}_h)q \, dx - \int_K c(x)(p - p_h)q \, dx \right) \end{aligned}$$

Let us define  $S(K)$  as the set of edges of  $K$ ,  $F_i$ ,  $i = 1, 2, 3$ , so that  $F_i \not\subset \partial\Omega$ . Then, the following expression is obtained:

$$\begin{aligned}
A(U - U_h, V) &= \sum_K \left( \int_K (\vec{f} - a(x)\vec{u}_h + \nabla p_h + \vec{b}(x)p_h) \cdot \vec{v} \, dx \right. \\
&\quad + \int_K (-g - \operatorname{div} \vec{u}_h + c(x)p_h) q \, dx \\
&\quad \left. + \sum_{F \in S(K)} \langle -p_h, \vec{v} \cdot \vec{n} \rangle_F + \sum_{F \subset \partial K \cap \partial \Omega_D} \langle g_1 - p_h, \vec{v} \cdot \vec{n} \rangle_F \right).
\end{aligned}$$

Thus, for all  $V \neq 0 \in Z_0$ , the following estimation is derived:

$$\begin{aligned}
\frac{A(U - U_h, V)}{\|V\|_{Z_0}} &\leq \sum_K \left( \|\vec{f} - a(x)\vec{u}_h + \nabla p_h + \vec{b}(x)p_h\|_{L^2(K)} \right. \\
&\quad + \|-g - \operatorname{div} \vec{u}_h + c(x)p_h\|_{L^2(K)} \\
&\quad \left. + \frac{1}{2} \sum_{F \in S(K)} h_F^{-\frac{1}{2}} \|[p_h]\|_{L^2(F)} + \sum_{F \subset \partial K \cap \partial \Omega_D} h_F^{-\frac{1}{2}} \|g_1 - p_h\|_{L^2(F)} \right).
\end{aligned}$$

Noting

$$\begin{aligned}
\eta(K) &= \|\vec{f} - a(x)\vec{u}_h + \nabla p_h + \vec{b}(x)p_h\|_{L^2(K)} \\
&\quad + \|-g - \operatorname{div} \vec{u}_h + c(x)p_h\|_{L^2(K)} \\
&\quad + \frac{1}{2} \sum_{F \in S(K)} h_F^{-\frac{1}{2}} \|[p_h]\|_{L^2(F)} + \sum_{F \subset \partial K \cap \partial \Omega_D} h_F^{-\frac{1}{2}} \|g_1 - p_h\|_{L^2(F)}
\end{aligned} \tag{22}$$

the following result is obtained:

**Proposition 4.1** *Let  $(\vec{u}, p)$  be the solution of the problem (4) and  $(\vec{u}_h, p_h)$  be the solution of the discrete problem (20), then the following ‘a posteriori’ estimation is obtained:*

$$\|\vec{u} - \vec{u}_h\|_X + \|p - p_h\|_Y \leq C \left( \sum_{K \in \mathcal{T}_h} \eta(K)^2 \right)^{\frac{1}{2}}, \tag{23}$$

where  $C$  is a positive constant depending on  $\beta$ .

**Remark 4.1** *If we note*

$$\eta = \left( \sum_{K \in \mathcal{T}_h} \eta(K)^2 \right)^{\frac{1}{2}},$$

*then  $\eta$  is named an ‘a posteriori’ estimator computed with the residual of the equation and it can be used in isotropic mesh adaptation procedures.*

In practice, equation (22) is replaced by

$$\begin{aligned}
 \eta_R(K) &= \|\vec{f}_{mh} - a(x)\vec{u}_h + \nabla p_h + \vec{b}(x)p_h\|_{L^2(K)} \\
 &+ \|-g_{mh} - \operatorname{div}\vec{u}_h + c(x)p_h\|_{L^2(K)} \\
 &+ \frac{1}{2} \sum_{F \in S(K)} h_F^{-\frac{1}{2}} \|[p_h]\|_{L^2(F)} + \sum_{F \in \partial K \cap \partial \Omega_D} h_F^{-\frac{1}{2}} \|i_h(g_1) - p_h\|_{L^2(F)}
 \end{aligned} \tag{24}$$

where  $g_{mh}$  (respectively  $\vec{f}_{mh}$ ) is an approximation of  $g$  (respectively  $\vec{f}$ ),  $i_h$  is an interpolation operator and  $m$  is an integer greater than zero. Trivially, an analogous estimation to (23) can be derived:

**Proposition 4.2** *Let us consider the same hypotheses as in proposition 4.1, then the following estimation is obtained:*

$$\begin{aligned}
 \|\vec{u} - \vec{u}_h\|_X + \|p - p_h\|_Y &\leq C \left( \sum_{K \in \mathcal{T}_h} \eta_R(K)^2 + \|g - g_{mh}\|_{L^2(K)}^2 \right. \\
 &+ \|\vec{f} - \vec{f}_{mh}\|_{L^2(K)}^2 \\
 &+ \left. \sum_{F \in \partial K \cap \partial \Omega_D} h_F^{-1} \|g_1 - i_h(g_1)\|_{L^2(F)}^2 \right)^{\frac{1}{2}}
 \end{aligned} \tag{25}$$

**Remark 4.2** *We are principally interested in deducing an ‘a posteriori’ error estimation for the initial non linear problem. Thus, the error indicator  $\eta(K)$  given in equation (22) can be considered as an approximation of first order of the following estimator*

$$\begin{aligned}
 \eta(K) &= \|-a(x, p_h)\vec{u}_h + \nabla p_h\|_{L^2(K)} \\
 &+ \|-div\vec{u}_h + c(x, p_h)\|_{L^2(K)} \\
 &+ \frac{1}{2} h_F^{-\frac{1}{2}} \sum_{F \in S(K)} \|[p_h]\|_{L^2(F)} + h_F^{-\frac{1}{2}} \sum_{F \in \partial K \cap \partial \Omega_D} \|g_1 - p_h\|_{L^2(F)}.
 \end{aligned} \tag{26}$$

*Up to now, no theoretical proof of this estimator has been done. It will be considered in future works.*

**Interpretation:** *The two first terms of estimator (22) correspond to the residuals of the two equations of the discretized mixed variational problem. Thus, if the equations are assumed to be numerically well-solved, then these two quantities must be small. The third term controls the discontinuity of the primal variable  $p$  across*



the mesh edges. Finally, the last term controls if the imposition of the boundary condition over the device ohmic contacts is verified. The first and the two last terms cannot be estimated in an optimal way. This is due to the anisotropy of  $H(\text{div}, \Omega)$ . Moreover, the traces of  $H(\text{div}, \Omega)$ -functions are only in  $H^{-1/2}(\partial\Omega)$ . In order to avoid the negative exponent, different norms must be used (in particular mesh-dependent norms).

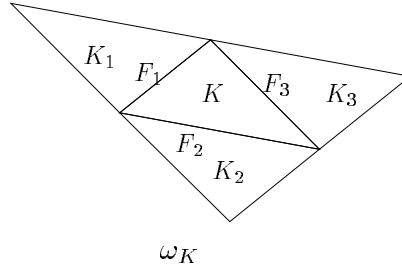


Figure 1: Definition of  $\omega_K$ .

Let us prove a reciprocal result to proposition 4.1.

**Proposition 4.3** *With the definition (22), there is a constant  $C$ , which only depends on the minimal angle in the triangulation such that*

$$\eta(K) \leq C(\|\vec{u} - \vec{u}_h\|_{H(\text{div}, \omega_K)} + h_K^{-1} \|p - p_h\|_{L^2(\omega_K)}), \quad (27)$$

where  $\omega_K = \{K_0 \in \mathcal{T}_h : K_0 \cap K = F_i \text{ edges of } K, i = 1, 2, 3\}$ .

**Proof:**

The proof of this proposition is quite technique and it is based on the study of a function  $\mathcal{E}$  defined by

$$\mathcal{E}(V) = \mathcal{A}(U - U_h, V), \quad \forall V \in Z_0,$$

then

$$\begin{aligned} \mathcal{E}(V) &= \sum_K \int_K (\vec{f} - a(x)\vec{u}_h + \nabla p_h + \vec{b}(x)p_h) \cdot \vec{v} \, dx \\ &+ \int_K (-g - \text{div} \vec{u}_h + c(x)p_h)q \, dx \end{aligned}$$

$$+ \sum_{F \in \mathcal{S}(K)} \langle -p_h, \vec{v} \cdot \vec{n} \rangle_F + \sum_{F \in \partial K \cap \partial \Omega_D} \langle g_1 - p_h, \vec{v} \cdot \vec{n} \rangle_F.$$

Let us introduce the following notations:

$$\begin{aligned} \text{I} &= \|\vec{f} - a(x)\vec{u}_h + \nabla p_h + \vec{b}(x)p_h\|_{L^2(K)}, \\ \text{II} &= \|\ -g - \operatorname{div} \vec{u}_h + c(x)p_h\|_{L^2(K)}, \\ \text{III} &= \frac{1}{2} h_F^{-\frac{1}{2}} \sum_{F \in \mathcal{S}(K)} \|[p_h]\|_{L^2(F)}, \\ \text{IV} &= h_F^{-\frac{1}{2}} \sum_{F \in \partial K \cap \partial \Omega_D} \|g_1 - p_h\|_{L^2(F)}. \end{aligned}$$

For all triangle  $K \in \mathcal{T}_h$ , let us consider a function  $\varphi_K$  defined as:

$$\varphi_K(x) = \begin{cases} \hat{\varphi}_{\hat{K}} \circ F_K^{-1}(x) & \text{if } x \in \bar{K} \\ 0 & \text{otherwise,} \end{cases}$$

where  $\hat{K}$  is the reference element and  $\hat{\varphi}_{\hat{K}}$  is real function verifying that:

- $0 \leq \hat{\varphi}_{\hat{K}}(\hat{x}) \leq 1$ , for all  $\hat{x} \in \hat{K}$ , with  $\hat{\varphi}_{\hat{K}}(\hat{x}) = 0$ , for all  $\hat{x} \in \partial \hat{K}$  and
- $\hat{\varphi}_{\hat{K}} \in C^1(\hat{K})$  and  $\nabla \hat{\varphi}_{\hat{K}} \in L^\infty(\hat{K})^2$ .

Clearly, for all  $K \in \mathcal{T}_h$  and for all  $\vec{v}_K$  element of a finite dimensional subspace of  $L^2(K)$ , using the fact that norms are equivalent in finite dimensional spaces and passing through the reference element  $\hat{K}$ , we obtain

$$C_0 \|\vec{v}_K\|_{L^2(K)} \leq \|\vec{v}_K \varphi_K(x)^{\frac{1}{2}}\|_{L^2(K)} \leq C_1 \|\vec{v}_K\|_{L^2(K)}, \quad (28)$$

where  $C_0$  and  $C_1$  are independent of  $K$  and  $\vec{v}_k$ . We can also obtain that

$$\|\vec{v}_K \varphi_K(x)\|_{L^2(K)} \leq \|\vec{v}_K\|_{L^2(K)} \quad (29)$$

**Majoration of I.** Let us consider the following test function

$$\vec{v}_K = \varphi_K(x)(\vec{f}(x) - a(x)\vec{u}_h + \nabla p_h + \vec{b}(x)p_h).$$

From its definition  $\vec{v}_K$  is an element of  $H(\operatorname{div}, \Omega)$  ( $\vec{v}_K \in H(\operatorname{div}, K)$ ) and we have the continuity of the normal component  $\vec{v}_K \cdot \vec{n}_{F_i}$ , where  $F_i$  are the interior edges of  $K$ ,

in fact their are all 0). Denoting by  $V = (\vec{v}_K, 0)$  then

$$\begin{aligned} \mathcal{E}(V) &= \int_K (\vec{f} - a(x)\vec{u}_h + \nabla p_h + \vec{b}(x)p_h)^2 \varphi_K dx \\ &= \|(\vec{f} - a(x)\vec{u}_h + \nabla p_h + \vec{b}(x)p_h)\varphi_K^{\frac{1}{2}}\|_{L^2(K)}^2 \\ &\geq C_1^I \|\vec{f} - a(x)\vec{u}_h + \nabla p_h + \vec{b}(x)p_h\|_{L^2(K)}^2, \end{aligned}$$

thus,

$$\begin{aligned} \text{(I)}^2 &\leq \mathcal{E}(V) \\ &\leq C_2^I \|U - U_h\|_{Z(K)} \|V\|_{Z(K)} \\ &\leq C_2^I (\|\vec{u} - \vec{u}_h\|_{H(\text{div},K)} + \|p - p_h\|_{L^2(K)}) \\ &\quad \|(\vec{f} - a(x)\vec{u}_h + \nabla p_h + \vec{b}(x)p_h)\varphi_K\|_{H(\text{div},K)}, \end{aligned}$$

now, using the equivalence of norms in finite dimensional spaces in the reference configuration and an inverse inequality for the div-norm we obtain

$$I \leq C_3^I (1 + h_K^{-1}) (\|\vec{u} - \vec{u}_h\|_{H(\text{div},K)} + \|p - p_h\|_{L^2(K)}).$$

**Majoration of II.** Let us consider the following test function

$$q_K = \varphi_K (-g - \text{div} \vec{u}_h + c(x)p_h).$$

As in the previous estimation, it is clear that  $q_K$  is an element of  $L^2(\Omega)$ . Noting by  $V = (0, q_K)$  then

$$\begin{aligned} \mathcal{E}(V) &= \int_K (-g - \text{div} \vec{u}_h + c(x)p_h)^2 \varphi_K dx \\ &= \|(-g - \text{div} \vec{u}_h + c(x)p_h)\varphi_K^{\frac{1}{2}}\|_{L^2(K)}^2 \\ &\geq C_1^{II} \|-g - \text{div} \vec{u}_h + c(x)p_h\|_{L^2(K)}^2, \end{aligned}$$

therefore,

$$\begin{aligned} \text{(II)}^2 &\leq \mathcal{E}(V) \\ &\leq C_2^{II} \|U - U_h\|_{Z(K)} \|V\|_{Z(K)} \\ &\leq C_2^{II} (\|\vec{u} - \vec{u}_h\|_{H(\text{div},K)} + \|p - p_h\|_{L^2(K)}) \\ &\quad \|(-g - \text{div} \vec{u}_h + c(x)p_h)\varphi_K\|_{L^2(K)} \\ &\leq C_3^{II} (\|\vec{u} - \vec{u}_h\|_{H(\text{div},K)} + \|p - p_h\|_{L^2(K)}) \\ &\quad \|-g - \text{div} \vec{u}_h + c(x)p_h\|_{L^2(K)}, \end{aligned}$$

thus,  $\Pi$  is upper bounded by

$$\Pi \leq C_3^{II} (\|\vec{u} - \vec{u}_h\|_{H(\text{div}, K)} + \|p - p_h\|_{L^2(K)}).$$

**Remark 4.3** *To establish similar the majorations of I and II, we can also work with polynomial interpolations  $f_h, g_h, c_h, a_h$  and  $b_h$  of  $f, g, c, a$  and  $b$ .*

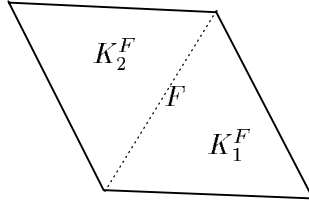


Figure 2: Definition of  $\omega_F = K_1^F \cup K_2^F$ .

**Majoration of III.** Let  $F$  be an internal edge, therefore, there exists two triangles  $K_1^F$  and  $K_2^F$  so that  $F = K_1^F \cap K_2^F$ . Let us define  $\omega_F = K_1^F \cup K_2^F$  (see fig. 2). Observing that  $[p_h]|_F$  is constant, then a suitable function  $P_F$  is constructed by:

$$P_F([p_h])(x) = \begin{cases} \vec{v}_1^F(x) & \text{if } x \in K_1^F \\ \vec{v}_2^F(x) & \text{otherwise} \end{cases} \quad (30)$$

where  $\vec{v}_1^F$  (respectively  $\vec{v}_2^F$ ) is the unique polynomial defined over  $K_1^F$  (respectively  $K_2^F$ ) in  $RT_0$  so that each degree of freedom is 0 except the one in  $F$  that it is imposed to be  $[p_h]$ . It is clear from the definition of  $P_F([p_h])$  that it belongs to  $H(\text{div}, \omega_K)$ . Similar constructions can be considered if  $[p_h]$  is polynomial over  $F$  using the appropriated generalized Raviart-Thomas F.E. It is not very difficult to prove that  $P_F$  is linear and verifies

$$\|P_F([p_h])\|_{H(\text{div}, \omega_F)} \leq Ch_F^{\frac{1}{2}} \|[p_h]\|_{L^2(F)},$$

Let us consider the test function

$$V = (P_F([p_h]), 0),$$

then

$$\| [p_h] \|_{L^2(F)}^2 \leq \left| \begin{array}{l} \mathcal{E}(V) - \sum_{K \subset K_1^F \cup K_2^F} \int_K (\vec{f} - a(x)\vec{u}_h) \\ + \nabla p_h + \vec{b}(x)p_h) P_F([p_h]) dx \end{array} \right|,$$

thus

$$\begin{aligned} \| [p_h] \|_{L^2(F)}^2 &\leq \|A\| \|U - U_h\|_{Z(\omega_F)} \|P_F([p_h])\|_{Z(\omega_F)} \\ &+ \| \vec{f} - a(x)\vec{u}_h + \nabla p_h + \vec{b}(x)p_h \|_{L^2(\omega_F)} \|P_F([p_h])\|_{L^2(\omega_F)}, \end{aligned}$$

using now the majoration of  $P_F$  we obtain

$$h_F^{-\frac{1}{2}} \| [p_h] \|_{L^2(F)} \leq C_h^{III} \|U - U_h\|_{Z(\omega_F)}.$$

**Majoration of IV.** Finally, let  $F$  be a boundary edge satisfying  $F \in \partial\Omega_D$  and let  $K_1^F$  be the triangle containing  $F$ . For the sake of simplicity, let us suppose  $g_1$  constant over  $F$ . In this case the function  $P_F$  is constructed:

$$P_F(g_1 - p_h)(x) = \vec{v}_1^F(x) \quad (31)$$

where  $\vec{v}_1^F$  is the unique polynomial defined over  $K_1^F$  in  $RT_0$  so that each degree of freedom are 0 except the one in  $F$  that it is imposed to be  $g_1 - p_h$ . As in previous case,  $P_F$  is linear and verifies

$$\| P_F(g_1 - p_h) \|_{H(\text{div}, K_1^F)} \leq Ch_F^{\frac{1}{2}} \|g_1 - p_h\|_{L^2(F)},$$

Let us consider the test function

$$V = (P_F(g_1 - p_h), 0),$$

then

$$\|g_1 - p_h\|_{L^2(F)}^2 \leq \left| \begin{array}{l} \mathcal{E}(V) - \int_{K_1^F} (\vec{f} - a(x)\vec{u}_h + \nabla p_h + \vec{b}(x)p_h) \\ P_F(g_1 - p_h) dx \end{array} \right|,$$

thus

$$\begin{aligned} \|g_1 - p_h\|_{L^2(F)}^2 &\leq \|A\| \|U - U_h\|_{Z(K_1^F)} \|P_F(g_1 - p_h)\|_{Z(K_1^F)} \\ &+ \| \vec{f} - a(x)\vec{u}_h + \nabla p_h + \vec{b}(x)p_h \|_{L^2(K_1^F)} \\ &\| P_F(g_1 - p_h) \|_{L^2(K_1^F)}, \end{aligned}$$

using the majoration of  $P_F$  we obtain

$$h_F^{-\frac{1}{2}} \|g_1 - p_h\|_{L^2(F)} \leq C_h^{IV} \|U - U_h\|_{Z(K_1^F)}.$$

In the general case, where  $g_1$  is not constant over  $F$ , then a suitable polynomial approximation  $g_{1,s}$  is considered. In that case the following estimation is obtained:

$$h_F^{-\frac{1}{2}} \|g_1 - p_h\|_{L^2(F)} \leq C_h^{IV} (\|U - U_h\|_{Z(K_1^F)} + h^{-\frac{1}{2}} \|g_1 - g_{1,s}\|_{L^2(F)}).$$

Finally, if the estimator given by the equation (24) is considered, then the following proposition can be derived:

**Proposition 4.4** *With the definition (24) the following estimation is obtained*

$$\begin{aligned} \eta_R(K) \leq C & \left( \|\vec{u} - \vec{u}_h\|_{H(\text{div}, \omega_K)} + h_K^{-1} \|p - p_h\|_{L^2(\omega_K)} \right) \\ & + \|\vec{f} - \vec{f}_{m,h}\|_{L^2(\omega_K)} + \|g - g_{m,h}\|_{L^2(\omega_K)} + h_K^{-\frac{1}{2}} \|g_1 - i_h(g_1)\|_{L^2(\partial K \cap \partial \Omega_D)} \end{aligned} \quad (32)$$

where  $\omega_K = \{K_0 \in \mathcal{T}_h : K_0 \cap K = F_i \text{ edges of } K, i = 1, 2, 3\}$  (see fig. 1).

**Remark 4.4** *A comparison with the upper bound (4.1) shows that the estimate (4.4) is non optimal since it involves a negative power of  $h$ .*

## 5 Conclusion.

The paper has outlined an ‘a posteriori’ local error estimator based on the residual of the linearized transport equation for electrons. Two theoretical results gives an idea about the efficacy of the error estimator, nevertheless up to now, no numerical results have been obtained. Future works will consider the construction of an anisotropic estimator based on local corrections of numerical solution as well as numerical experiences comparing the isotropic and anisotropic estimators.

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