A Tridimensional Inverse Shaping Problem

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N° 2655

septembre 1995

PROGRAMME 6

Rapport de recherche

ISSN 0249-6399

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Programme 6 — Calcul scientifique, modélisation et logiciel numérique
Projet Numath

Rapport de recherche n2655 — septembre 1995 — 34 pages

Abstract: We study a question which arises in the following tridimensional magnetostatic inverse shaping problem: can one find a distribution of currents around a levitating liquid metal bubble so that it takes a given shape? It leads to the resolution of an Hamilton-Jacobi equation of eikonal type on the surface of the bubble whose solution is the norm of the magnetic induction field and which has a self-contained interest. We answer the question for closed smooth surfaces which are homeomorphic to a sphere. We give a necessary and sufficient condition on the data for existence and uniqueness of a rmC^1 solution. When the desired shape is axisymmetric and analytic, the solution is also analytic and the problem can be completely solved. But the condition mentioned above implies that not all analytic perturbed surfaces are shapable, as it can be seen by a counter-example.

Key-words: shape optimization, Hamilton-Jacobi equations, viscosity solutions, inverse problem, electromagnetic shaping, nonlinear partial differential equation on closed surfaces.

(Résumé : tsvp)

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Un problème de formage tridimensionel

Résumé : Dans cet article, nous étudions une question liée au problème inverse de formage magnétostatique tridimensionnel suivant: est-il possible de trouver une distribution de courant à placer autour d'une goutte de métal en fusion et en lévitation de sorte que celle-ci prenne une forme donnée? Ceci conduit à la résolution, sur la surface de la goutte, d'une équation d'Hamilton-Jacobi du premier ordre de type eikonal intéressante en soi et dont la solution est la norme du champ d'induction magnétique. Nous répondons à la question pour des surfaces fermées régulières homéomorphes à une sphère. Nous exhibons une condition nécessaire et suffisante sur les données pour l'existence et l'unicité d'une solution de classe C^1. Dans le cas où la surface désirée est analytique et présente une symétrie de rotation, la solution est alors elle-même analytique et le problème peut être entièrement résolu. Par contre, la condition nécessaire et suffisante obtenue implique qu'une perturbation analytique d'une forme symétrique n'est pas nécessairement formable, comme le prouve le contre-exemple que nous donnons.

Mots-clé : optimisation de formes, équations d'Hamilton-Jacobi, solutions de viscosité, problème inverse, formage électromagnétique, équation aux dérivées partielles non linéaire sur une surface fermée.

1 Introduction

Our goal is to study a question arising in a tridimensional inverse shaping problem. This question mainly consists in finding the regular closed surfaces Σ in $I\!\!R^3$ for which there exists a regular solution to the first order equation

$$(1) \qquad ||\nabla_\Sigma \Phi|| = f \qquad \text{on } \Sigma,$$

where $\Phi : \Sigma \to I\!\!R$ is the unknown function, ∇_Σ denotes the tangential gradient, $||.||$ the euclidian norm and $f : \Sigma \to [0, \infty)$ is a given function. In our situation, f depends on the mean curvature of Σ. This question arises, in particular, when looking at the inverse shaping problem in the electromagnetic levitation of liquid metal bubbles. Roughly speaking, we are given the shape of the bubble (i.e. a domain limited by a closed surface in $I\!\!R^3$). We want to find out whether it is possible to put suitable inductors and currents around the liquid metal so that its equilibrium shape be exactly the given one. More precisely, given Σ a regular closed surface in $I\!\!R^3$ and Ω its exterior, it amounts to finding a distribution of currents $j_0 : \overline{\Omega} \to I\!\!R^3$ with compact support in Ω and a vector field $B : \overline{\Omega} \to I\!\!R^3$ (the magnetic induction field) so that the following system be satisfied :

$$\begin{array}{llll}
(2) & \nabla \wedge B = j_0 & \text{in} & \Omega \\
(3) & \nabla . B = 0 & \text{in} & \Omega \\
(4) & B.n = 0 & \text{on} & \Sigma = \partial\Omega \qquad (n = \text{ unit normal to } \Sigma) \\
(5) & ||B||^2 + \tau_1 \mathcal{C} + \tau_2 z = P & \text{on} & \Sigma.
\end{array}$$

Here P is an unknown constant, $\mathcal{C}$ is the mean curvature of Σ, z denotes the vertical coordinate and τ_1, τ_2 are given nonnegative constants. Equation (1.5) states that the free boundary Σ is at equilibrium under the various forces involved, namely electromagnetic, surface tension, gravity and pressure (with density respectively $||B||^2$, $\tau_1 \mathcal{C}$, $\tau_2 z$, P). This simplified model is valid for high frequencies of the applied current: for details see [1], [2], [6], [13], [17] and for numerical computations with examples of shapes of bubbles see e.g. [10], [15], [16].

The condition "$\nabla \wedge B = 0$ around Σ", coming from (1.2) and the fact that j_0 is compactly supported in Ω, implies that $B = \nabla\Phi$ in Ω locally around each point of Σ. If Σ is homeomorphic to a sphere, then this is true globally around Σ and the nonlinear boundary condition (1.5) reduces to

$$(6) \qquad ||\nabla_\Sigma \Phi||^2 = P - \tau_1 \mathcal{C} - \tau_2 z \qquad \text{on } \Sigma$$

where the right-hand side $f^2 := P - \tau_1 \mathcal{C} - \tau_2 z$ is a given function as soon as the surface Σ is given. Indeed, τ_1, τ_2 are given, the mean curvature $\mathcal{C}$ is given as well as the vertical coordinate function. Now, Φ has to reach a maximum and a minimum on Σ, so that, if Φ is C^1, $\nabla_\Sigma \Phi$ has to vanish (at least twice) on Σ. Therefore the a priori unknown constant P is given by

$$(7) \qquad P = \max_\Sigma \{\tau_1 \mathcal{C} + \tau_2 z\}$$

since, obviously, by (1.6)

$$(8) \qquad P \geq \tau_1 \mathcal{C} + \tau_2 z \quad \text{on } \Sigma.$$

It follows that, at least when Σ is homeomorphic to a sphere, solving our inverse problem requires to first solve (1.6) which is an equation of type (1.1).

It turns out that if Σ is analytic and if (1.6) has an analytic solution Φ then Σ is "shapable", i.e. the inverse problem can be fully solved. Indeed, one can prove that the Σ-vector field $B = \nabla_\Sigma \Phi$ may be extended to the whole exterior Ω of Σ so that $\nabla . B = 0$. We then set $j_0 := \nabla \wedge B$. This part of the inverse problem, which will not be discussed here, is very similar to the analogous 2-dimensional version of this inverse problem treated in [11] and is widely discussed in [7], [8] for this 3-d problem. We just recall the two main steps that are used :

- first, given Φ analytic on Σ (itself also supposed to be analytic), the classical Cauchy-Kowaleska theorem provides an analytic extension of Φ in a neighbourhood ω of Σ so that

$$\begin{array}{lll} \nabla\Phi . n = 0 & \text{on} \quad \Sigma & (\text{i.e. } B.n = 0 \text{ as in } (1.4)) \\ \Delta\Phi = 0 & \text{in} \quad \omega & (\text{i.e. } \nabla . B = 0 \text{ in } \omega). \end{array}$$

- then choosing another closed regular surface $\tilde{\Sigma}$ surrounding Σ and included in ω, we extend $B = \nabla\Phi$ outside $\tilde{\Sigma}$ as the solution of

$$\begin{array}{lll} \nabla \wedge B = 0 & \text{outside} & \tilde{\Sigma} \\ \nabla . B = 0 & \text{outside} & \tilde{\Sigma} \\ B.n & \text{continuous across} & \tilde{\Sigma} \\ B \to 0 & \text{at infinity in some sense.} & \end{array}$$

With this construction, $\nabla . B = 0$ in the whole exterior of Σ and $j_0 = \nabla \wedge B$ is a distribution carried by $\tilde{\Sigma}$. We refer to [7], [8], [11] for more details and to [4] for auxiliary results. Note that, in dimension 2, the analyticity of the given curve is also a necessary condition for "shapability" [11]. It is very likely that it is the same in dimension 3 although only C^∞-regularity has yet been proved to be necessary (see [7]).

As announced, we will concentrate here on finding regular - sometimes even analytic - solutions to (1.6), or more generally to (1.1). For given regular surfaces Σ and continuous f, it is in general easy to write down explicit "weak solutions" to (1.1) (in the sense of "viscosity solutions" for instance). However -and this is the main point here- they are not in general C^1 even if f^2 and Σ are analytic. As we will see, this strongly depends on geometric properties of the pair (Σ, f). In the case when f vanishes only at two points, we will provide a necessary and sufficient condition for C^1-regularity.

This condition is generally easily satisfied in the case of axisymmetric data. Therefore a rather complete answer can be provided for the inverse shaping problem in that case. However, as shown by an example in Section 3, small analytic perturbations of shapable axisymmetric surfaces may not be shapable.

Let us mention that several results for this problem can be found in [7],[8]. Our approach is more systematically based on the direct analysis of the regularity of the expected solution. It can be explicitely written down and it is always a viscosity solution of (1.1). The problem is then reduced to deciding whether it is regular or not.

Note also that when Σ is homeomorphic to a torus, the situation is completely different : infinitely many solutions to the inverse problem may exist (see [7], [8]). Let us also mention [3], [14] where similar questions are discussed.

We thank M. Katz for several helpful discussions on differential geometry aspects of this problem.

2 The eikonal equation $||\nabla_\Sigma \Phi|| = f$ on a compact regular surface

Throughout this paper, we denote by Σ a surface in $I\!R^3$ such that

$$\Sigma \text{ is } C^1\text{-diffeomorphic to the unit sphere} \tag{1}$$

and by f a function from Σ into $I\!R$ such that

$$f \text{ is continuous, } f \geq 0 \text{ on } \Sigma. \tag{2}$$

These are minimal assumptions that we will have to strengthen later. We consider the eikonal equation for the unknown function $\Phi : \Sigma \to I\!R$

$$||\nabla_\Sigma \Phi|| = f \quad \text{ on } \Sigma. \tag{3}$$

Remark. We refer to the classical litterature for the definition of C^1-functions on a regular surface and for the definition of the tangential gradient ∇_Σ (see for instance [5]). Recall that if $\varphi : \Sigma \to I\!R$ is differentiable at x_0, then, for all x in Σ close enough to x_0

$$\varphi(x) - \varphi(x_0) = \nabla_\Sigma \varphi(x_0).(x - x_0) + o(||x - x_0||). \tag{4}$$

Moreover, if $\tilde{\varphi}$ is an extension of φ in a neighbourhood of x_0 in $I\!R^3$ which is differentiable at x_0, then

$$\nabla_\Sigma \varphi(x_0) = \nabla \tilde{\varphi}(x_0) - (\nabla \tilde{\varphi}(x_0).n(x_0))n(x_0) \tag{5}$$

where $n(x_0)$ is the unit normal to Σ at x_0.

We now introduce the function which will provide explicit "weak" solutions to (2.3). It is defined from $\Sigma \times \Sigma$ into $I\!R$ by :

$$(6) \qquad \forall x, y \in \Sigma \qquad L(x,y) = \inf \{ \int_0^T f(\xi(s))ds \quad ; \quad T \geq 0, \xi \in \mathcal{A}_{x,y}^T \}$$

where

$$\mathcal{A}_{x,y}^T = \{ \xi \in W^{1,\infty}(0,T;\Sigma),\ \xi(0) = x, \xi(T) = y, \text{a.e.} s \in (0,T)\ ||\dot{\xi}(s)|| \leq 1 \}.$$

<u>Remark</u>. Since Σ is C^1-diffeomorphic to a sphere, there exists at least a C^1-path joining two points x, y of Σ. Therefore the function L is everywhere defined on $\Sigma \times \Sigma$. It is a semi-distance on $\Sigma \times \Sigma$ and is a distance if f does not vanish too much. Indeed, one can easily check that for all $x, y, z \in \Sigma$: $L(x,x) = 0, L(x,y) = L(y,x),\ L(x,y) \leq L(x,z) + L(z,y)$ (see Appendix).

<u>*Proposition 2.1*</u>. Let y_0 be fixed in Σ. The function defined on Σ by

$$\Phi(x) := L(x, y_0)$$

is a Lipschitz continuous function. At each $x \in \Sigma$ where it is differentiable

$$||\nabla_\Sigma \Phi(x)|| = f(x).$$

Moreover, Φ is a viscosity solution of (2.3) on $\Sigma - \{y_0\}$ and a viscosity solution of (2.3) on Σ as soon as $f(y_0) = 0$.

The proof of this proposition is essentially similar to the case where Σ is replaced by an open subset of $I\!R^N$ (in which case it is classical, see [12]). For completeness, we indicate the details in the appendix where we also recall the definition of viscosity solutions.

<u>Remark</u>. If Φ is a C^1-function, then f must vanish at two different points. Indeed, by continuity and compactness, Φ reaches its maximum and minimum and $\nabla_\Sigma \Phi$ must vanish there.

We will now assume that we are in the "generic" case where f vanishes exactly at two points :

$$(7)\qquad \begin{array}{l} \text{There exist } x_1, x_2 \in \Sigma \quad \text{such that } x_1 \neq x_2,\ f(x_1) = f(x_2) = 0 \\ \text{and } \forall x \in \Sigma \backslash \{x_1, x_2\}, \quad f(x) > 0. \end{array}$$

Theorem 2.1. Assume (2.7) holds. If (2.3) has a C^1-solution Φ, then necessarily

$$(8)\qquad \forall x \in \Sigma \qquad L(x, x_1) + L(x, x_2) = L(x_1, x_2).$$

Moreover

$$(9)\qquad \Phi(.) = \pm L(., x_1) + \text{constant}.$$

Remark. As we will see on examples, the necessary condition (2.8) is rather restrictive. It states that all paths going from x_1 to x_2 have the same "length" relative to the distance defined by $L(.,.)$. We will study its sufficiency next. Note that this condition, although expressed differently, was already mentioned in [7]. It is also more or less classical in differential geometry.

The relation (2.9) is a uniqueness statement for the C^1-solutions of (2.3).

Proof of Theorem 2.1 Let $x \in \Sigma - \{x_1, x_2\}$. Let $T > 0$ and $\xi \in \mathcal{A}^T_{x,x_1}$. Then, since Φ is C^1, for all $t \in [0, T]$,

$$(10)\qquad \phi(\xi(t)) = \phi(x) + \int_0^t \nabla_\Sigma \phi(\xi(s)) \cdot \dot{\xi}(s) ds.$$

This implies $|\phi(x) - \phi(x_1)| \leq \int_0^T f(\xi(s)) ds$.

This last inequality holds for all $T > 0$ and all $\xi \in \mathcal{A}^T_{x,x_1}$, thus

$$(11)\qquad |\phi(x) - \phi(x_1)| \leq L(x, x_1).$$

In order to prove the converse inequality, note first that, if Φ satisfies equation (2.3), it reaches its maximum and its minimum exactly at x_1 and x_2. Indeed, by the C^1-regularity of Φ, $\nabla_\Sigma \Phi$ has to vanish there and the only points where $f = ||\nabla_\Sigma \Phi||$ vanishes, are x_1 and x_2 by (2.7). Let us, without loss of generality assume that

$$\forall x \in \Sigma \backslash \{x_1, x_2\} \quad \Phi(x_1) < \Phi(x) < \Phi(x_2). \tag{12}$$

We are now going to exhibit the optimal trajectory going from x to x_1, i.e. the one which realizes $L(x, x_1)$. Let $\xi \in C^1(0, +\infty)$ be a solution of

$$\begin{cases} \dot{\xi}(t) = -\nabla_\Sigma \phi(\xi(t)), \ \forall t > 0 \\ \xi(0) = x, \quad \forall t \geq 0 \quad \xi(t) \in \Sigma. \end{cases} \tag{13}$$

Then for all $t > 0$, we have

$$\begin{aligned} &\phi(\xi(t)) = \phi(x) + \textstyle\int_0^t \nabla_\Sigma \phi(\xi(s)) \cdot \dot{\xi}(s) ds \\ &= \phi(x) - \textstyle\int_0^t f(\xi(s)) |\dot{\xi}(s)| ds. \end{aligned} \tag{14}$$

Therefore, the function $t \mapsto \phi(\xi(t))$ is nonincreasing; thus, by (2.12), the trajectory ξ stays away from x_2.

Let $t_0 = \inf\{t > 0 / f(\xi(t)) = 0\} = \inf\{t > 0 / \xi(t) = x_1\}$ (t_0 may be infinite). Let ζ be the trajectory defined by

$$\zeta(\Psi(t)) = \xi(t) \text{ where } \Psi(t) = \int_0^t |\dot{\xi}(\sigma)| d\sigma.$$

Then, for all $t < t_0$, we have

$$\int_0^t f(\xi(\tau)) |\dot{\xi}(\tau)| d\tau = \int_0^{\Psi(t)} f(\zeta(s)) ds,$$

where $\zeta \in \mathcal{A}_{x,\xi(t)}^{\Psi(t)}$ (note that $|\dot{\zeta}(s)| = 1$ a.e.s). Thus, (2.14) becomes

$$\phi(x) - \phi(\xi(t)) = \int_0^{\psi(t)} f(\zeta(s)) ds \geq L(x, \xi(t)).$$

Now, if $t_0 < +\infty$, this last inequality implies, letting t go to t_0, that

$$\phi(x) - \phi(x_1) \geq L(x, x_1). \tag{15}$$

Otherwise, since $\phi(\xi(t))$ is nonincreasing in t, it converges as $t \to \infty$. By (2.14), so does

$$\int_0^\infty f(\xi(s))^2 ds = \int_0^\infty f(\xi(s))|\dot{\xi}(s)|ds = \phi(x) - \lim_{t\uparrow\infty} \phi(\zeta(t)).$$

Thus, there exists a sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n \to +\infty$, $f(\xi(t_n)) \to 0$ and $\xi(t_n) \to x_1$. We have

$$\phi(x) - \phi(\xi(t_n)) = \int_0^{\Psi(t_n)} f(\zeta(s))ds \geq L(x, \xi(t_n)).$$

By passing to the limit, we get the same conclusion (2.15) as in the previous case. Then, from (2.15), (2.11), we obtain :

$$\phi(x) = \phi(x_1) + L(x, x_1). \tag{16}$$

The same kind of arguments also yields

$$\phi(x_2) - \phi(x) = L(x, x_2)$$

and thus

$$\phi(x_1) + L(x, x_1) = \phi(x_2) - L(x, x_2)$$

i.e.

$$L(x, x_1) + L(x, x_2) = \phi(x_2) - \phi(x_1).$$

Finally, as x goes to x_1, $\phi(x_2) - \phi(x_1) = L(x_1, x_2)$ and the proof of (2.8) is complete.

We now look at the converse of Theorem 2.1. For simplicity, we also assume that

$$\Sigma \text{ is } C^3 - \text{diffeomorphic to the unit sphere.} \tag{17}$$

$$f \in C^2(\Sigma \backslash \{x_1, x_2\}). \tag{18}$$

Theorem 2.2 Assume (2.7), (2.17), (2.18) and the structure condition (2.8). Then $\phi = L(., x_1)$ belongs to $C^1(\Sigma)$ and is a classical solution to (2.3) (unique up to a constant and to the sign).

We decompose the proof of Theorem 2.2 into several lemmas.

Lemma 2.1 For all $x \in \Sigma - \{x_1, x_2\}$, there exists a unique $\xi_x \in \mathcal{A}^{+\infty}_{x,x_1}$ such that, for all $t > 0$,

$$L(x, \xi_x(t)) = \int_0^t f(\xi_x(s))ds, \tag{19}$$

$$L(\xi_x(t), x_1) = \int_t^\infty f(\xi_x(s))ds, \tag{20}$$

$$L(x, x_1) = \int_0^\infty f(\xi_x(s))ds. \tag{21}$$

Moreover,

$$t \to \xi_x(t) \text{ is differentiable on } [0, t_x) \tag{22}$$

where $t_x = \inf\{t \in [0, \infty]; \xi_x(t) = x_1\}$,

$$x \to \dot{\xi}_x(0) \text{ is continuous on } \Sigma \backslash \{x_1, x_2\}. \tag{23}$$

Proof. By definition of $L(.,.)$ (see (2.6)), for all $\varepsilon > 0$, there exist $T_\varepsilon > 0$ and $\xi_\varepsilon \in \mathcal{A}^{T_\varepsilon}_{x,x_1}$ such that

$$L(x, x_1) \leq \int_0^{T_\varepsilon} f(\xi_\varepsilon(s))ds \leq L(x, x_1) + \varepsilon. \tag{24}$$

We first assume that a subsequence of T_ε is bounded. Since $||\dot{\xi}_\varepsilon||_{L^\infty(0,T_\varepsilon)} \leq 1$, by Arzela-Ascoli's theorem, ξ_ϵ is relatively compact for the uniform convergence and there exists a subsequence, that we call again T_ε, which converges to some T_0 and $\xi_x \in W^{1,\infty}(0, T_0)$ such that ξ_ε converges to ξ_x in $L^\infty(0, T_0)$. Moreover $\xi_x(0) = x, \xi_x(T_0) = x_1$ and $||\dot{\xi}_x(s)|| \leq 1$ a.e. $s \in (0, T_0)$. We set $\xi_x(t) = x_1$ for all $t \in [T_0, \infty)$ to obtain $\xi_x \in \mathcal{A}^{+\infty}_{x,x_1}$.

Now, if T_ε diverges to $+\infty$, again by Arzela-Ascoli's theorem, there exists a subsequence that we call again ξ_ε and $\xi_x \in W^{1,\infty}(0, +\infty)$ such that ξ_ε converges uniformly to ξ_x on all bounded interval $[0,T]$. Moreover ξ_x satisfies $\xi_x(0)=x$ and $||\dot{\xi}_x(s)|| \leq 1$ a.e. $s \in (0,+\infty)$. What follows will show that $\lim_{t\to\infty} \xi_x(t) = x_1$.

We have for all $t \in (0,T_\varepsilon)$ (using (2.24) and the definition (2.6)),

$$L(x,\xi_\varepsilon(t)) + \int_t^{T_\varepsilon} f(\xi_\varepsilon(s))ds \leq \int_0^{T_\varepsilon} f(\xi_\varepsilon(s))ds \leq L(x,x_1)+\varepsilon$$

$$\leq L(x,\xi_\varepsilon(t)) + L(\xi_\varepsilon(t),x_1)+\varepsilon$$

so that

$$L(\xi_\varepsilon(t),x_1)+\varepsilon \geq \int_t^{T_\varepsilon} f(\xi_\varepsilon(s))ds \geq L(\xi_\varepsilon(t),x_1),$$

from which we deduce that for all $t \geq 0$

$$\int_t^{T_\varepsilon} f(\xi_\varepsilon(s))ds \to L(\xi_x(t),x_1) \text{ as } \varepsilon \to 0. \tag{25}$$

By a similar argument, we can also prove that

$$L(x,\xi_\varepsilon(t))+\varepsilon \geq \int_0^t f(\xi_\varepsilon(s))ds \geq L(x,\xi_\varepsilon(t)). \tag{26}$$

>From this last statement, we deduce (2.19) since ξ_ε converges to ξ_x uniformly on $[0,t]$. Moreover, we get (see (2.24), (2.25), (2.26)) :

$$L(x,\xi_x(t)) + L(\xi_x(t),x_1) = L(x,x_1). \tag{27}$$

The function $t \mapsto L(x,\xi_x(t))$ is nondecreasing and bounded by $L(x,x_1)$. Hence by (2.19), the integral $\int_0^\infty f(\xi_x(s))ds$ converges and there exists a sequence t_n such that $f(\xi_x(t_n)) \to 0$ when $n \to +\infty$. We can extract from the sequence $\xi_x(t_n)$ subsequences which converge either to x_1 or to x_2. If there was one converging to x_2, we would have from (2.27) at the limit

$$L(x,x_2)+L(x_2,x_1) = L(x,x_1).$$

But, by the structure assumption (2.8), this would imply

$$L(x,x_2)+L(x_2,x_1) = L(x_1,x_2) - L(x,x_2)$$

that is $L(x, x_2) = 0$, which is a contradiction with $x \neq x_2$ and $f(x) \neq 0$ (see Appendix). Therefore, necessarily, the whole sequence t_n converges to x_1.

To prove that $\xi_x(t) \to x_1$ as $t \to \infty$, assume that for a sequence s_n, $\xi_x(s_n)$ converges to some $y \in \Sigma$. Since $t \mapsto L(x, \xi_x(t))$ is nondecreasing, it has a limit as $t \uparrow \infty$. By the previous step, it is the same as $\lim\limits_{n\to\infty} L(x, \xi_x(t_n))$, that is $L(x, x_1)$. Thus, applying (2.27) with $t = s_n$ and passing to the limit lead to

$$L(x, x_1) + L(y, x_1) = L(x, x_1) \Rightarrow L(y, x_1) = 0$$

which implies that $y = x_1$ (see (A4) in the Appendix). Therefore $\xi_x(t) \to x_1$ as $t \to +\infty$ and $\xi_x \in \mathcal{A}^{+\infty}_{x,x_1}$.

Note also that

$$L(x, x_1) = \lim_{t\to+\infty} L(x, \xi_x(t)) = \int_0^{+\infty} f(\xi_x(s))ds$$

and by difference in (2.27)

$$L(\xi_x(t), x_1) = L(x, x_1) - L(x, \xi_x(t)) = \int_t^{+\infty} f(\xi_x(s))ds$$

which proves (2.20) and (2.21).

The rest of the proof of Lemma 2.1 will be a consequence of the following lemma which describes the properties of optimal trajectories in the definition of $L(.,.)$.

<u>*Lemma 2.2*</u> Let $\xi \in \mathcal{A}^T_{y,z}$ satisfying

$$(28) \qquad L(y, z) = \int_0^T f(\xi(t))dt, \quad \xi([0, T]) \subset \Sigma\backslash\{x_1, x_2\}.$$

Then, $\xi \in C^2(0, T; \Sigma)$ and for all $t \in (0, T)$

$$(29) \qquad ||\dot{\xi}(t)|| = 1,\ \nabla_\Sigma f(\xi(t)) - \frac{d}{dt}(f(\xi(t)\dot{\xi}(t)) \text{ is normal to } \Sigma \text{ at } \xi(t)$$

$$(30) \qquad \forall t \in (0, T)\ f(\xi(t))||\ddot{\xi}(t)|| \leq C(||\nabla_\Sigma f(\xi(t))|| + 1)$$

where C depends only on Σ and $||f||_\infty$.

We postpone the proof of this lemma and finish first the proof of Lemma 2.1.

Obviously, the differentiability of $t \to \xi_x(t)$ on $[0, t_x)$ follows directly from Lemma 2.2 (see the definition of t_x in (2.22)).

For the uniqueness of ξ_x, we first remark that similarly to the existence of ξ_x, we can prove the existence of $\eta_x \in \mathcal{A}^{+\infty}_{x,x_2}$ such that, for all $\tau > 0$

$$L(x, \eta_x(\tau)) = \int_0^\tau f(\eta_x(s))ds$$

$$L(\eta_x(\tau), x_2) = \int_\tau^\infty f(\eta_x(s))ds.$$

Now if $\xi \in \mathcal{A}^{+\infty}_{x,x_1}$ satisfies (2.19), (2.20), the union ζ of ξ and η defined by

$$\zeta(t) = \begin{cases} \eta_x(t) \text{ if } t \in [0, \infty] \\ \xi(-t) \text{ if } t \in [-\infty, 0] \end{cases}$$

is an optimal path from x_1 to x_2, since, by the main assumption (2.8) and the properties of ξ, η

$$L(x_1, x_2) = L(x_1, x) + L(x, x_2) = \int_{-\infty}^0 f(\zeta(s))ds + \int_0^\infty f(\zeta(s))ds. \tag{31}$$

We deduce that ζ is an optimal path from any $\xi(t)$ to $\eta_x(\tau)$. Indeed, by the triangular inequality

$$L(x_1, \xi(t)) + L(\xi(t), \eta_x(\tau)) + L(\eta_x(\tau), x_2) \geq L(x_1, x_2) = \int_{-\infty}^{+\infty} f(\zeta(s))ds.$$

But by (2.20) applied to ξ and η_x, the above inequality becomes

$$L(\xi(t), \eta_x(\tau)) \geq \int_{-t}^\tau f(\zeta(s))ds,$$

whence the optimality of ζ when going from $\xi(t)$ to $\eta_x(\tau)$. Choosing t and τ small enough so that $\zeta([-t,\tau]) \subset \Sigma - \{x_1, x_2\}$, we deduce by Lemma 2.2 that $\zeta \in C^2([-t,\tau],\Sigma)$.

In particular,

$$\dot{\xi}(0) = -\dot{\eta}_x(0).$$

Again by Lemma 2.2, ξ is solution of a second order differential equation on Σ for which $\xi(0)$ and $\dot{\xi}(0)$ are given (see (2.39), (2.40)). Thanks to the regularity of Σ and f assumed in (2.17), (2.18), all the functions in (2.39), (2.40) are at least Lipschitz-continuous with respect to $\xi, \dot{\xi}$. It follows that the solution of this differential equation is unique, whence the uniqueness of ξ satisfying (2.19), (2.20).

For the continuity of $x \to \dot{\xi}_x(0)$, let $x_n \to x$ in $\Sigma\backslash\{x_1, x_2\}$. Since $||\xi_{x_n}(t) - x|| \leq t||\dot{\xi}_{x_n}||_\infty \leq t$, $\xi_{x_n}(0,\varepsilon)$ remains at a positive distance of x_1 and x_2 for some $\varepsilon > 0$ independent of n. From (2.30), we obtain that

$$\sup_{t\in[0,\varepsilon]} ||\ddot{\xi}_{x_n}(t)|| \leq C.$$

By Arzela-Ascoli's theorem, ξ_{x_n} is relatively compact in $L^\infty(0,T)$ for all T and $\dot{\xi}_{x_n}$ is relatively compact in $L^\infty(0,\epsilon)$. The limit ξ is such that

$$\xi(0) = x,\ \dot{\xi}(0) = \lim_{n\to\infty} \dot{\xi}_{x_n}(0)$$

and passing to the limit in (2.19) and (2.27)

$$\forall t \in (0,\infty), \qquad L(x,\xi(t)) = \int_0^t f(\xi(s))ds.$$

$$\forall t \in (0,\infty), \qquad L(x,\xi(t)) + L(\xi(t),x_1) = L(x,x_1).$$

By the previous uniqueness result, to prove that $\xi = \xi_x$, it is sufficient to verify that $\lim_{t\uparrow\infty} \xi(t) = x_1$. We argue as in the beginning of the proof by using the last two relations above. Finally we have obtained that

$$\lim_{n\to\infty} \dot{\xi}_{x_n}(0) = \dot{\xi}_x(0).$$

Proof of Lemma 2.2 Let $V : (0,T) \times \Sigma \to I\!\!R^3$ be a time-dependent tangent vector field to Σ of class C^1 and such that $V(0,x) = V(T,x) = 0$ for all $x \in \Sigma$. For $t \in (0,T)$, consider the solution $\xi(.,t) : [-\varepsilon,\varepsilon] \to \Sigma$ of

$$\frac{\partial \xi}{\partial s}(s,t) = V(t,\xi(s,t)),\ \xi(0,t) = \xi(t) \tag{32}$$

(for the existence of $\xi(.,t)$ and its regularity, see e.g. [5]). By compactness of $[0,T]$ and regularity of V, for some $\varepsilon > 0$, $\xi(.,.)$ exists in $W^{1,\infty}((-\varepsilon,\varepsilon)\times(0,T);\Sigma)$. For all $s \in (-\varepsilon,\varepsilon)$, $\xi(s,0) = y, \xi(s,T) = z$, and $\dot\xi(s,t) = \dfrac{\partial}{\partial t}\xi(s,t)$ satisfies for a.e. $t \in (0,T)$

$$\frac{\partial \dot\xi}{\partial s}(s,t) = V_t(t,\xi(s,t)) + D_\Sigma V(t,\xi(s,t))\dot\xi(s,t), \dot\xi(0,t) = \dot\xi(t), \tag{33}$$

which is a linear differential equation for $\dot\xi(.,t)$ associated with a continuous matrix $D_\Sigma V(t,\xi(s,t))$.

By the minimization property of ξ (see (2.28)), for all $s \in [-\varepsilon,\varepsilon]$

$$\int_0^T f(\xi(\sigma))d\sigma \le \int_0^T f(\xi(s,\sigma))||\dot\xi(s,\sigma)||d\sigma. \tag{34}$$

Indeed, setting $\hat\sigma = \displaystyle\int_0^\sigma ||\dot\xi(s,\tau)||d\tau, \hat T = \int_0^T ||\dot\xi(s,\tau)||d\tau$, the second integral above writes

$$\int_0^{\hat T} f(\xi(s,\sigma(\hat\sigma)))d\hat\sigma \text{ where } ||\frac{\partial}{\partial\hat\sigma}\xi(s,\sigma(\hat\sigma))|| = 1 \text{ a.e.}, \tag{35}$$

so that $\xi(s,.) \in \mathcal{A}^{\hat T}_{y,z}$ and (2.33) follows from (2.28) and the definition of the function L.

Applying (2.33) for $s = 0$ gives, together with $||\dot\xi(\sigma)|| \le 1$,

$$\int_0^T f(\xi(\sigma))d\sigma \le \int_0^T f(\xi(\sigma))||\dot\xi(\sigma)||d\sigma \le \int_0^T f(\xi(\sigma))d\sigma. \tag{36}$$

We deduce that

$$\int_0^T f(\xi(\sigma))(1 - ||\dot\xi(\sigma)||)d\sigma = 0,$$

which implies $||\dot\xi(\sigma)|| = 1$ a.e. σ since $f(\xi(\sigma)) \neq 0$ owing to $\xi(\sigma) \notin \{x_1,x_2\}$.

We also deduce from (2.33) that

$$0 = \frac{\partial}{\partial s}_{|s=0} \int_0^T f(\xi(s,\sigma))||\dot{\xi}(s,\sigma)||d\sigma,$$

which gives with $\xi_0(\sigma) = \frac{\partial}{\partial s}_{|s=0}\xi(s,\sigma)$ and $\dot{\xi}_0(\sigma) = \frac{\partial}{\partial s}_{|s=0}\dot{\xi}(s,\sigma)$ (which exists a.e. $\sigma \in (0,T)$)

$$0 = \int_0^T (\nabla_\Sigma f(\xi(\sigma)).\xi_0(\sigma) + f(\xi(\sigma))\dot{\xi}(\sigma)\dot{\xi}_0(\sigma))d\sigma.$$

Using (2.31), (2.32), this writes

$$0 = \int_0^T \nabla_\Sigma f(\xi).V(t,\xi) + f(\xi)\dot{\xi}[D_\Sigma V(t,\xi).\dot{\xi} + V_t(t,\xi)]$$

or

(37) $$0 = \int_0^T \nabla_\Sigma f(\xi).V(t,\xi) + f(\xi)\ \dot{\xi}\frac{d}{dt}V(t,\xi).$$

Since this is true for any regular tangent vector field V on Σ, it says that, in a sense to be made precise

(38) $$a.e.t \quad \nabla_\Sigma f(\xi(t)) - \frac{d}{dt}(f(\xi).\dot{\xi}) \text{ is orthogonal to } \Sigma \text{ at } \xi(t).$$

To see this, let us take $\varphi : B(0,R) \subset I\!R^2 \to \Sigma$ a C^3 diffeomorphism such that $\xi(0,T) \subset \varphi(B(0,R))$ and let $\Psi \in C_0^\infty(0,T)$ arbitrary. We denote by (u,v) the variable in $I\!R^2$. We choose $V = \Psi\varphi_u$ in (2.36) to get with $\xi(t) = \varphi(u(t),v(t))$

$$0 = \int_0^T \Psi\nabla_\Sigma f(\xi)\varphi_u + f(\xi)\dot{\xi}(\Psi'\varphi_u + \Psi\frac{d}{dt}\varphi_u(u,v)).$$

This implies that in $\mathcal{D}'(0,T)$

$$\nabla_\Sigma f(\xi)\varphi_u + f(\xi)\dot{\xi}\frac{d}{dt}\varphi_u(u,v) - \frac{d}{dt}(f(\xi)\dot{\xi}\varphi_u) = 0.$$

Choosing similarly $V = \Psi\varphi_v$, we obtain

$$\nabla_\Sigma f(\xi)\varphi_v + f(\xi)\dot{\xi}\frac{d}{dt}\varphi_v(u,v) - \frac{d}{dt}(f(\xi)\dot{\xi}\varphi_v) = 0.$$

This is a second order differential equation for $t \to (u(t), v(t))$ of the form

$$f(\xi)(\varphi_u.\varphi_u\ \ddot{u} + \varphi_u.\varphi_v\ddot{v}) = H_1(u, v, \dot{u}, \dot{v}) \tag{39}$$

$$f(\xi)(\varphi_v.\varphi_u\ \ddot{u} + \varphi_v.\varphi_v\ddot{v}) = H_2(u, v, \dot{u}, \dot{v}) \tag{40}$$

where H_1, H_2 are functions of class C^1. But

$$||\varphi_u||^2||\varphi_v||^2 - (\varphi_u.\varphi_v)^2 = ||\varphi_u \wedge \varphi_v||^2 \neq 0 \text{ (since } \varphi \text{ is a diffeomorphism).}$$

Therefore the system (2.38)-(2.39) can be rewritten

$$f(\xi)\binom{\ddot{u}}{\ddot{v}} = \tilde{H}(u, v, \dot{u}, \dot{v}). \tag{41}$$

Since $\tilde{H}$ is regular and $f(\xi)$ is bounded away from 0, this implies that $u, v \in W^{2,\infty}(0, T)$ and so does ξ (recall that $\ddot{\xi} = \varphi_{uu}\dot{u}^2 + 2\varphi_{uv}\dot{u}\dot{v} + \varphi_{vv}\dot{v}^2 + \varphi_u\ddot{u} + \varphi_v\ddot{v}$). More precisely, from the expression of $H_1, H_2, \tilde{H}$, we easily obtain

$$a.e.t \quad f(\xi)||(\ddot{u}, \ddot{v})|| \leq C(||\nabla_\Sigma f(\xi)|| + 1)$$

where C depends on $||f||_\infty$, φ and its first and second derivatives. The estimate (2.30) follows by using a finite number of local charts. Then the continuity of the second derivatives comes from (2.41) since $u, v, \dot{u}, \dot{v}$ are continuous.

Lemma 2.3 If $L(., x_1)$ is differentiable at $x \neq \{x_1, x_2\}$, then

$$\nabla_\Sigma L(x, x_1) = -f(x)\dot{\xi}_x(0).$$

Proof If $L(., x_1)$ is differentiable at x, we know that (see Prop. 2.1)

$$||\nabla_\Sigma L(x, x_1)|| \leq f(x).$$

But by the chain rule and (2.20) :

$$\frac{d}{dt}L(\xi_x(t), x_1)_{|t=0} = \nabla_\Sigma L(x, x_1) \cdot \dot{\xi}_x(0) = -f(x).$$

Thus

$$f(x) = -\nabla_\Sigma L(x, x_1) \cdot \dot{\xi}_x(0) \leq \|\nabla_\Sigma L(x, x_1)\| \cdot \|\dot{\xi}_x(0)\| \leq f(x).$$

The equality of the terms above implies

$$\nabla_\Sigma L(x, x_1) = -f(x)\dot{\xi}_x(0).$$

Note that this equality holds a.e. in $\Sigma - \{x_1, x_2\}$ since $L(., x_1)$ is Lipschitz continuous.

Lemma 2.4 (see Appendix) Let $V : \Sigma \mapsto I\!R$ be Lipschitz continuous and such that there exists a continuous vector field H tangent to Σ which satisfies

(42) $$\nabla_\Sigma V(x) = H(x) \text{ a.e. } x \in \Sigma.$$

Then $V \in C^1(\Sigma)$ and (2.42) holds for all $x \in \Sigma$.

Proof of Theorem 2.2

We apply Lemma 2.4 to $V = L(., x_1)$ and H defined by

$$H(x) = \begin{cases} -f(x)\dot{\xi}_x(0) \text{ for } x \notin \{x_1, x_2\} \\ 0 \text{ if } x \in \{x_1, x_2\}. \end{cases}$$

Continuity of H comes from Lemma 2.1 for $x \notin \{x_1, x_2\}$ and can be checked directly at x_1, x_2. Now by Lemma 2.3

$$\nabla_\Sigma L(x, x_1) = H(x) \text{ a.e.}$$

$>$From Lemma 2.4, we deduce that $L(., x_1)$ is of class C^1. This completes the proof of Theorem 2.2.

3 Back to the shaping problem

3.1 The axisymmetric case

Assume Σ is an axisymmetric regular closed surface diffeomorphic to the unit sphere and assume that $f : \Sigma \to [0, \infty)$, regular, is also invariant by rotation about the same axis. Then, if the eikonal equation

(1) $$||\nabla_\Sigma \Phi|| = f \qquad \text{on } \Sigma,$$

has a regular solution, f has to vanish at the poles (even is Φ is not a priori assumed to be axisymmetric). Indeed, Φ reaches its maximum at some $M \in \Sigma$ and its minimum at some $m \in \Sigma$ and $\nabla_\Sigma \Phi$ vanishes there. By (3.1) and the axisymmetry of f, $\nabla_\Sigma \Phi$ vanishes all along the parallel lines going through M and m. If these lines accumulate at one pole, by continuity, $\nabla_\Sigma \Phi$ vanishes also at this pole. Assume now that there is no accumulation at a pole of lines along which $\nabla_\Sigma \Phi$ vanishes. Then, we look at the maximum and the minimum of the restriction of Φ to the portion of surface going from the pole to the first of these lines. If both of the maximum and the minimum of this restriction of Φ were reached on this line, then they would be equal since, by (3.1) and axisymmetry of f, $\nabla_\Sigma \Phi$ would vanish all along the line and Φ would be constant on this line. But then Φ would be constant on the considered portion of surface and this has been excluded (we are in the case of no accumulation of lines). Therefore, one of the maximum or the minimum is reached inside the portion and it can only be at the pole, since no vanishing line for $\nabla_\Sigma \Phi$ exists inside this portion.

In the generic case (2.7), the two points x_1 and x_2 must then be the North pole and the South pole of Σ. By axisymmetry of f and Σ, the structure condition (2.8) is obviously satisfied since, by construction, the function $L(.,.)$ is itself axisymmetric. Therefore, (3.1) has always a C^1-solution. Actually, in this particular case, one can write explicitely the solution Φ. Assume for instance that the surface is generated by the rotation about the vertical axis $(x = y = 0)$ of the Jordan curve described by

$$s \to (0, y(s), z(s)) \tag{2}$$

where s is the length parameter, $y(s) > 0$ for all $s \in (0, S)$ and $x_1 = (0,0,z(0)), x_2 = (0,0,z(S)), y(0) = y(S) = 0$. Then, Φ is the rotation invariant function given in the half-plane $\Pi = \{(0,y,z); y \geq 0\}$ and in terms of s by

$$\Phi(s) = \int_0^s f(0, y(\sigma), z(\sigma)) d\sigma \quad . \tag{3}$$

Obviously here, $||\nabla_\Sigma \Phi|| = |\Phi'(s)| = f(0, y(s), z(s))$ for all $s \in (0, S)$.

In the shaping problem, the data Σ and f^2 are analytic. Recall that $f^2 = P - \tau_1 \mathcal{C} - \tau_2 z$ where $P = \max_\Sigma(\tau_1 \mathcal{C} + \tau_2 z)$. Then, we directly check that

Φ, given by (3.3), is analytic for all $s \in (0, S)$ so that Φ is analytic everywhere except may be at the poles x_1, x_2. Around x_1 in the plane $\{x = 0\}$, we have $\Phi'(\sigma) = sign(\sigma) f(0, y(\sigma), z(\sigma))$ for $\sigma \in (-\epsilon, \epsilon)$ and $|\sigma| = s$. Since f^2 is analytic and vanishes at $\sigma = 0$, $f^2(\sigma) \sim a_p \sigma^{2p}$ for some $p \geq 1$ and $a_p > 0$, and $\Phi'(\sigma) \sim sign(\sigma)\sqrt{a_p}|\sigma|^p$. Therefore, Φ' is analytic around x_1 if and only if p is **odd**. The same is true around x_2.

To summarize, in the axisymmetric case, a necessary condition for shapability is that (see (1.6)) :

$$\begin{array}{rcl} \tau_1 \mathcal{C}(x_1) + \tau_2 z(x_1) & = & \tau_1 \mathcal{C}(x_2) + \tau_2 z(x_2) \\ & \geq & \tau_1 \mathcal{C}(x) + \tau_2 z(x) \quad \text{for } x \text{ in } \Sigma \end{array} \tag{4}$$

(5) the multiplicity of the maxima of $\tau_1 \mathcal{C}(.) + \tau_2 z(.)$ at x_1, x_2 is odd.

Conversely, if we assume (3.5) and a strict version of (3.4), namely

$$\begin{array}{rcl} \tau_1 \mathcal{C}(x_1) + \tau_2 z(x_1) & = & \tau_1 \mathcal{C}(x_2) + \tau_2 z(x_2) \\ & > & \tau_1 \mathcal{C}(x) + \tau_2 z(x) \text{ for all } x \in \Sigma \backslash \{x_1, x_2\}, \end{array} \tag{6}$$

by the above analysis, one can find Φ analytic solution of the shaping problem. Therefore, the surface is then "shapable".

The geometric condition (3.4) is rather strong. It implies for instance that Σ should not have "bumps" with large positive curvature. On the other hand, the condition (3.5) is generically satisfied.

3.2 About analytic perturbations of axisymmetric shapes

The next question is to understand what happens for small analytic perturbations of shapable axisymmetric surfaces. The following example shows that the necessary condition (2.8) is highly unstable under small perturbations, even in our specific geometric shaping problem.

Proposition 3.2. Let Σ be the surface of an ellipsoid defined in $\mathbb{R}^3$ by

$$a^2 x^2 + b^2 y^2 + c^2 z^2 = 1 \tag{7}$$

where we assume

$$(8) \qquad c^2 < b^2 \leq a^2 \qquad , \qquad a, b, c > 0.$$

We consider the eikonal equation on Σ

$$(9) \qquad \Phi \in C^1(\Sigma), \quad ||\nabla_\Sigma \Phi||^2 = \mathcal{C}_M - \mathcal{C}$$

where $\mathcal{C}$ is the mean curvature function and $\mathcal{C}_M = \max_\Sigma \mathcal{C}$. Then

$$(10) \qquad \text{if } a = b, \text{ (3.9) has a solution and } \Phi \text{ is analytic}$$

$$(11) \qquad \begin{array}{l} \text{if } b < a, \text{ and } b \text{ is close enough to } c, \\ \text{(3.9) does not have any solution.} \end{array}$$

<u>Remark</u>. As we will see below, the mean curvature reaches its maximum at the top and at the bottom of the ellipsoid (assuming the z-axis is vertical). As a consequence, the necessary condition (see (3.4))

$$\tau_1 \mathcal{C}(x_1) + \tau_2 z(x_1) = \tau_1 \mathcal{C}(x_2) + \tau_2 z(x_2)$$

can only be satisfied if $\tau_2 = 0$. Therefore the question of "shapability" can only be considered without gravity.

<u>*Proof of the proposition*</u>. We first compute the mean curvature $\mathcal{C}$ of the ellipsoid Σ using the formula (see e.g. [5]).

$$(12) \qquad -2\mathcal{C} = \frac{\{(1 + h_x^2)h_{yy} - 2h_x h_y h_{xy} + (1 + h_y^2)h_{xx}\}}{\{1 + h_x^2 + h_y^2\}^{3/2}},$$

where Σ is considered as the graph of $z = h(x, y)$ with

$$(13) \qquad h(x, y) = \{1 - a^2x^2 - b^2y^2\}^{1/2}/c.$$

Easy computations lead to

$$(14) \qquad 2\mathcal{C} = \frac{a^2b^2}{c} g(u, v)$$

where

$$g(u,v) = (u+v+A)/(1+a^2u+b^2v)^{3/2} \tag{15}$$

$$u = (\frac{a^2}{c^2}-1)x^2, \quad v = (\frac{b^2}{c^2}-1)y^2, \quad A = \frac{1}{a^2}+\frac{1}{b^2}. \tag{16}$$

We easily check that

- if $b^2 < \frac{3}{2}a^2$, we have $g_u(u,v) < 0$ and $g_v(0,v) < 0$,
- if $a^2 < \frac{3}{2}b^2$, we have $g_v(u,v) < 0$ and $g_u(u,0) < 0$,

so that in all cases, g reaches a strict maximum at $(u,v) = (0,0)$ and the order of the maximum is 1.

In the case of axisymmetry $a = b$, we know there exists an axisymmetric solution Φ given on $x = 0, \quad y \in (-b^{-1}, b^{-1})$ by

$$\Phi(0,y,\frac{1}{c}\sqrt{1-b^2y^2}) = \int_0^y sign(y)\{\mathcal{C}_M - \mathcal{C}(y)\}^{1/2}\sigma(y)dy$$

where $\mathcal{C}(y) = \mathcal{C}(0,y,\frac{1}{c}\sqrt{1-b^2y^2})$

$$\sigma(y) = ||\frac{d}{dy}(0,y,\frac{1}{c}\sqrt{1-b^2y^2})|| = \{\frac{c^2+b^2y^2(b^2-c^2)}{c^2(1-b^2y^2)}\}^{1/2}.$$

We check that $y \to sign(y)\{\mathcal{C}_M - \mathcal{C}(y)\}^{1/2}\sigma(y)$ is analytic, even around $y = 0$. So is the function Φ.

Assume now $a \neq b$. A necessary condition for existence of solutions to (3.9) is that

$$\forall P \in \Sigma, L(S_0,P) + L(P,N_0) = L(S_0,N_0) \tag{17}$$

where $S_0 = (0,0,-c^{-1}), N_0 = (0,0,c^{-1})$ and L is the function associated with $f(\xi) = \{\mathcal{C}_M - \mathcal{C}(\xi)\}^{1/2}$ as in section 2 (see necessary condition (2.8)). We know that, if this condition holds, then there exists a unique path $\xi_P(.)$ of velocity 1 joining any P of Σ to N_0 and such that

$$L(P,N_0)=\int_0^T f(\xi_P(t))dt.$$

For symmetry reasons, if $I=(a^{-1},0,0)$, $J=(0,b^{-1},0)$, the path joining I to N_0 is in the plane $y=0$ and the one joining J to N_0 is in the plane $x=0$. Moreover,

$$L(I,N_0)=L(S_0,I),\quad L(J,N_0)=L(S_0,J).$$

Therefore, if (2.8) is true, we have

$$L(I,N_0)=\frac{1}{2}L(S_0,N_0)=L(J,N_0). \tag{18}$$

We will see now that (3.17) is a contradiction, at least when b is close enough to c. For this, we compare the two following expressions (where $\xi_I(.),\xi_J(.)$ are the optimal paths) :

$$\int_0^{T_I}\{\mathcal{C}_M-\mathcal{C}(\xi_I(t)\}^{1/2}dt=\int_0^{1/a}\{\mathcal{C}_M-\mathcal{C}(\tilde{\xi}_I(x))\}^{1/2}||\frac{d}{dx}\tilde{\xi}_I(x)||dx \tag{19}$$

where $\tilde{\xi}_I(x)=(x,0,\frac{1}{c}\sqrt{1-a^2x^2}),T_I=\int_0^{1/a}||\frac{d}{dx}\tilde{\xi}_I(x)||dx$

$$\int_0^{T_J}\{\mathcal{C}_M-\mathcal{C}(\xi_J(t))\}^{1/2}dt=\int_0^{1/b}\{\mathcal{C}_M-\mathcal{C}(\tilde{\xi}_J(y))\}^{1/2}||\frac{d}{dy}\tilde{\xi}_J(y)||dy \tag{20}$$

where $\tilde{\xi}_J(y)=(0,y,\frac{1}{c}\sqrt{1-b^2y^2}),T_J=\int_0^{1/b}||\frac{d}{dy}\tilde{\xi}_J(y)||dy.$

The expression in (3.19) writes

$$\frac{1}{\sqrt{2}}\int_0^{1/b}\{\frac{a^2+b^2}{c}-\frac{a^2b^2(b^2-c^2)y^2+(a^2+b^2)c^2}{\{c^2+b^2y^2(b^2-c^2)\}^{3/2}}\}^{1/2}\{\frac{c^2+b^2y^2(b^2-c^2)}{c^2(1-b^2y^2)}\}^{1/2}dy.$$

Obviously, this tends to 0 as b tends to c. The expression in (3.18) writes

$$\frac{1}{\sqrt{2}}\int_0^{1/a}\{\frac{a^2+b^2}{c}-\frac{a^2b^2(a^2-c^2)x^2+(a^2+b^2)c^2}{\{c^2+a^2x^2(a^2-c^2)\}^{3/2}}\}^{1/2}\{\frac{c^2+a^2x^2(a^2-c^2)}{c^2(1-a^2x^2)}\}^{1/2}dx$$

which tends to a strictly positive number as b tends to c. Therefore, $L(J, N_0) > L(I, N_0)$ when b is close to c which contradicts (3.17).

APPENDIX

Proposition 2.1 may be deduced from the following lemmas which concern the function L defined in (2.6) and the intrinsic distance on Σ. We denote by d this distance which is defined by (see (2.6))

$$\forall x, y \in \Sigma \qquad d(x,y) = \inf\{T > 0; \exists \xi \in \mathcal{A}^T_{x,y}\}. \tag{A.1}$$

It is well known that d is a distance on Σ which is equivalent to the restriction of the euclidian distance on Σ and that

$$\lim_{\substack{y \mapsto x \\ y \neq x}} \frac{d(x,y)}{||x-y||} = 1. \tag{A.2}$$

Most of the statements below are classical when Σ is replaced by an open subset of $I\!R^N$ (see e.g. [12]). For completeness, we check here that they can easily be adapted to our context.

Throughout this appendix, we assume that Σ and f satisfy assumptions (2.1), (2.2).

<u>*Lemma A.1*</u> The function L satisfies the following properties

$$\forall x, y, z \in \Sigma \quad L(x,y) = L(y,x), \quad L(x,z) \leq L(x,y) + L(y,z) \tag{A.3}$$

$$\text{if } f(x) > 0, \quad L(x,y) = 0 \Leftrightarrow x = y. \tag{A.4}$$

<u>*Proof*</u> Obviously L is defined on $\Sigma \times \Sigma$, takes its values in $I\!R^+$ and is symmetric. Let x, y and z be in Σ; for all $\varepsilon > 0$, there exist $T_\varepsilon > 0$, $\xi_\varepsilon \in \mathcal{A}^{T_\varepsilon}_{x,y}$ and $S_\varepsilon > 0$, $\zeta_\varepsilon \in \mathcal{A}^{S_\varepsilon}_{y,z}$ such that

$$\begin{cases} L(x,y) \leq \displaystyle\int_0^{T_\varepsilon} f(\xi_\varepsilon(s))ds \leq L(x,y) + \varepsilon \\ L(y,z) \leq \displaystyle\int_0^{S_\varepsilon} f(\zeta_\varepsilon(s))ds \leq L(y,z) + \varepsilon. \end{cases}$$

Now, we define, for all $s \in [0, T_\varepsilon + S_\varepsilon]$,

$$\eta_\varepsilon(s) = \begin{cases} \xi_\varepsilon(s) \text{ if } 0 \le s \le T_\varepsilon \\ \zeta_\varepsilon(s - T_\varepsilon) \text{ if } T_\varepsilon \le s \le T_\varepsilon + S_\varepsilon. \end{cases}$$

Obviously, $\eta_\varepsilon \in \mathcal{A}_{x,z}^{T_\varepsilon + S_\varepsilon}$ and

$$L(x,z) \le \int_0^{T_\varepsilon + S_\varepsilon} f(\eta_\varepsilon(s))ds \le L(x,y) + L(y,z) + 2\varepsilon,$$

which yields

(A.5) $$L(x,z) \le L(x,y) + L(y,z).$$

Let $x,\ y \in \Sigma$ be such that $L(x,y) = 0$. Then, for all $\varepsilon > 0$, there exist $T_\varepsilon > 0$ and $\xi_\varepsilon \in \mathcal{A}_{x,y}^{T_\varepsilon}$ such that

$$0 \le \int_0^{T_\varepsilon} f(\xi_\varepsilon(s))ds \le \varepsilon.$$

If $T_{\varepsilon_n} \to 0$ for some $\epsilon_n \downarrow 0$, then

$$||y - x|| = ||\xi_{\varepsilon_n}(T_{\varepsilon_n}) - \xi_{\varepsilon_n}(0)|| \le T_{\varepsilon_n} \to 0$$

and $x = y$. Otherwise, assume $f(x) \neq 0$ and let $0 < \tau \le T_\varepsilon$ for all $\varepsilon > 0$ be such that

$$||z - x|| \le \tau \Rightarrow f(z) \ge f(x)/2$$

(recall that f is continuous). Then for all $s \le \tau$ and for all $\varepsilon > 0$, we have $||\xi_\varepsilon(s) - x|| \le s \le \tau$ and thus

$$\varepsilon \ge \int_0^\tau f(\xi_\varepsilon(s))ds \ge \tau f(x)/2$$

which is contradictory. This proves that T_ε goes to 0 when ε goes to 0 and $x = y$ as soon as $f(x) \neq 0$.

Lemma A.2 For all $y_0 \in \Sigma$, $L(., y_0)$ is Lipschitz continuous and at a point $x \in \Sigma$ where $L(., y_0)$ is differentiable

(A.6) $$||\nabla_\Sigma L(x, y_0)|| \le f(x).$$

This holds for a.e.$x \in \Sigma$. Moreover at every x where $f(x) = 0$, $L(., y_0)$ is differentiable and $\nabla_\Sigma L(x, y_0) = 0$.

Proof Fix $y_0 \in \Sigma$ and let x, y be two distinct points of Σ. By (A.3), we have

$$|L(x, y_0) - L(y, y_0)| \leq L(x, y). \tag{A.7}$$

By the definition of L (see (2.6)), for all $\xi \in \mathcal{A}_{x,y}^T$

$$L(x, y) \leq \int_0^T f(\xi(t))dt. \tag{A.8}$$

This implies in particular

$$|L(x, y_0) - L(y, y_0)| \leq ||f||_\infty \inf\{T; \exists \quad \xi \in \mathcal{A}_{x,y}^T\} = ||f||_\infty d(x, y)$$

which proves that L is Lipschitz continuous. More precisely, for all $\xi \in \mathcal{A}_{x,y}^T$ and $t \in (0, T)$, since $||\xi(t) - x|| \leq T$, we have

$$|L(x, y_0) - L(y, y_0)| \leq T \sup\{f(z); z \in \Sigma, ||z - x|| \leq T\}.$$

Applying this to a sequence T_n converging to $d(x, y)$ in the definition (A.1), we obtain

$$|L(x, y_0) - L(y, y_0)| \leq d(x, y) \sup\{f(z); z \in \Sigma, ||z - x|| \leq d(x, y)\}. \tag{A.9}$$

Consequently, if $L(., y_0)$ is differentiable at x, we have be continuity of f

$$||\nabla_\Sigma L(x, y_0)|| = \limsup_{y \to x} \frac{|L(x, y_0) - L(y, y_0)|}{d(x, y)} \leq f(x).$$

Actually, if $f(x) = 0$, from (A.9) we directly obtain that $L(., y_0)$ is differentiable at x and $\nabla_\Sigma L(x, y_0) = 0$.

One can improve the property (A.6) : as shown below, equality holds at each point where $L(., y_0)$ is differentiable. This can be essentially deduced from the fact that $L(., y_0)$ is a viscosity solution of equation (2.3). This property of L is also of interest and is proved below.

We first recall briefly the notion of viscosity solutions for equation (2.3). It is a straightforward extension to surfaces of the notion introduced in [12] for classical Hamilton-Jacobi equations. As we will see below, if f vanishes at least at one point, then (1.1) has always a viscosity solution.

Definition A continuous function $\Phi : \Sigma \to I\!R$ is a viscosity solution of (2.3) on an open subset ω of Σ if, for all φ in $C^1(\omega)$ and all x_0 in ω such that $(\Phi - \varphi)$ has a local maximum at x_0 (resp. $(\Phi - \varphi)$ has a local minimum at x_0), we have

$$||\nabla_\Sigma \varphi(x_0)|| \leq f(x_0) \quad (\text{resp. } ||\nabla_\Sigma \varphi(x_0)|| \geq f(x_0)). \tag{A.10}$$

Obviously, C^1-solutions of (2.3) are viscosity solutions on Σ. Conversely, when viscosity solutions are regular, then they are classical solutions. In particular, we have here the following :

Lemma A.3 For all $y_0 \in \Sigma$, $L(.,y_0)$ is a viscosity solution on $\Sigma \backslash \{y_0\}$. It is a viscosity solution on Σ if $f(y_0) = 0$. At each point x where $L(.,y_0)$ is differentiable, we have

$$||\nabla_\Sigma L(x,y_0)|| = f(x).$$

Proof Assume $\varphi \in C^1(\Sigma)$ is such that $L(.,y_0) - \varphi$ has a local maximum at x_0. Then for x close to x_0

$$L(x,y_0) - \varphi(x) \leq L(x_0,y_0) - \varphi(x_0).$$

Expanding φ around x_0 gives

$$\nabla_\Sigma \varphi(x_0)(x - x_0) + o(||x - x_0||) \geq L(x,y_0) - L(x_0,y_0). \tag{A.11}$$

If $\nabla_\Sigma \varphi(x_0) = 0$, (A.10) is obviously satisfied. If not, we choose a path $\xi : (0,\varepsilon) \to \Sigma$ such that

$$\forall t \in (0,\varepsilon) \quad \xi(t) = x_0 - t\nabla_\Sigma \varphi(x_0)/||\nabla_\Sigma \varphi(x_0)|| + o(t).$$

This can be done for instance by solving locally the differential equation

$$\dot{\xi}(t) = -\nabla_\Sigma\varphi(\xi(t))/||\nabla_\Sigma\varphi(\xi(t))||, \quad \xi(0) = x_0.$$

Plugging into (A.11) with $x = \xi(t)$ leads to

$$-t||\nabla_\Sigma\varphi(x_0)|| \geq L(\xi(t), y_0) - L(x_0, y_0) + o(t)$$

or also

$$||\nabla_\Sigma\varphi(x_0)|| \leq \frac{1}{t}|L(\xi(t), y_0) - L(x_0, y_0)| + o(1).$$

By letting t tend to 0 and using (A.9), we obtain

$$||\nabla_\Sigma\varphi(x_0)|| \leq f(x_0).$$

Note that this statement is true even if $x_0 = y_0$.

Assume now that $\varphi \in C^1(\Sigma\backslash\{y_0\})$ is such that $L(., y_0) - \varphi$ has a local minimum at $x_0 \neq y_0$. Then, for x close to x_0

$$L(x, y_0) - \varphi(x) \geq L(x_0, y_0) - \varphi(x_0). \tag{A.12}$$

For $\varepsilon > 0$, let T_ε, $\xi_\varepsilon \in \mathcal{A}^{T_\varepsilon}_{x_0,y_0}$ such that

$$L(x_0, y_0) \leq \int_0^{T_\varepsilon} f(\xi_\varepsilon(s))ds \leq L(x_0, y_0) + \varepsilon.$$

Since $L(\xi_\varepsilon(t), y_0) \leq \int_t^{T_\varepsilon} f(\xi_\varepsilon(s))ds$, setting $x = \xi_\varepsilon(t)$ in (A.12) gives

$$\varphi(x_0) - \varphi(\xi_\epsilon(t)) \geq L(x_0, y_0) - L(\xi_\epsilon(t), y_0) \geq \int_0^t f(\xi_\epsilon(s))ds - \epsilon.$$

If $f(x_0) = 0$, then (A.10) obviously holds. If not, it follows from the definition of T_ε that $0 < \eta = ||y_0 - x_0|| = ||\xi_\epsilon(T_\epsilon) - \xi_\epsilon(0)|| \leq T_\epsilon$. By compactness, one can assume that ξ_ε converges uniformly on $(0, \eta)$ to $\xi \in W^{1,\infty}(0, \eta)$ such that $\xi(0) = x_0$, $||\dot{\xi}(t)|| \leq 1 \quad \forall t \in (0, \eta)$. We get, for all $t \in (0, \eta)$

$$\varphi(x_0) - \varphi(\xi(t)) \geq \int_0^t f(\xi(s))ds.$$

and by expansion of φ around x_0

$$\nabla_\Sigma \varphi(x_0)(x_0 - \xi(t)) + o(||x_0 - \xi(t)||) \geq \int_0^t f(\xi(s))ds. \tag{A.13}$$

But, since $||x_0 - \xi(t)|| \leq t$, there exists $t_n \downarrow 0$ and p tangent to Σ at x_0 such that

$$\frac{x_0 - \xi(t_n)}{t_n} \overset{n \to \infty}{\to} p, \quad ||p|| \leq 1,$$
$$\nabla_\Sigma \varphi(x_0).p \geq f(x_0).$$

Using this sequence t_n in (A.13), we deduce that $f(x_0) \leq ||\nabla_\Sigma \varphi(x_0)||$.

If $x_0 = y_0$ and $f(y_0) = 0$, obviously (A.10) holds. This completes the statement about viscosity solutions.

If now $L(., y_0)$ is differentiable at x_0, we already know that $||\nabla_\Sigma L(x_0, y_0)|| \leq f(x_0)$. For the reverse inequality, we argue as above, replacing φ by $L(., y_0)$ (note that we used only the differentiability of φ at x_0).

Proof of Lemma 2.4 If H is given by the Lemma 2.4 and if $\varphi : B(0, r) \to \Sigma$ denotes a C^1-diffeomorphism onto a neighborhood of $x_0 = \varphi(0) \in \Sigma$, we define

$$\tilde{H}(\xi) := (H \circ \varphi).D\varphi \tag{A.14}$$

(i.e $\tilde{H}_1 = (H_1 \circ \varphi)\varphi_{1u} + (H_2 \circ \varphi)\varphi_{2u} + (H_3 \circ \varphi)\varphi_{3u}$, $\tilde{H}_2 = (H_1 \circ \varphi)\varphi_{1v} + (H_2 \circ \varphi)\varphi_{2v} + (H_3 \circ \varphi)\varphi_{3v}$ where $H = (H_1, H_2, H_3)$, $\quad \varphi = (\varphi_1, \varphi_2, \varphi_3)$, $\quad \xi = (u, v)$). We check that $\tilde{H}$ is continuous.

If V satisfies the assumptions of Lemma 2.4, the function $\tilde{V}$ defined on a neighborhood of 0 by $\tilde{V}(\xi) = V(\varphi(\xi))$, is Lipschitz continuous. By (2.42), (A.14) and the chain rule, it satisfies

$$\nabla \tilde{V} = \tilde{H} \text{ a.e..}$$

If ρ_n is a regularizing sequence in $I\!R^2$, $\rho_n * \tilde{V}$ is defined around 0 and is C^∞. It converges uniformly to $\tilde{V}$. Therefore $\nabla(\rho_n * \tilde{V})$ converges in the sense of distribution to $\nabla \tilde{V}$. By continuity of $\tilde{H}$, $\rho_n * \tilde{H}$ converges uniformly to $\tilde{H}$. We will check below that

$$\nabla(\rho_n * \tilde{V}) = \rho_n * \tilde{H}. \tag{A.15}$$

It will follow by passing to the limit that $\nabla\tilde{V} = \tilde{H}$ in the sense of distribution. Therefore $\tilde{V}$ is C^1 and Lemma 2.4 follows.

To prove (A.15), we have to prove that, for $i = 1, 2$ and $\forall \xi \in I\!R^2$,

$$\lim_{h \downarrow 0} \int_{I\!R^2} \frac{\tilde{V}(\xi + he_i - \eta) - \tilde{V}(\xi - \eta)}{h} \rho_n(\eta) d\eta = \int_{I\!R^2} \tilde{H}_i(\xi - \eta) \rho_n(\eta) d\eta$$

where we denote $e_1 = (1, 0), e_2 = (0, 1)$. This follows from (A.14) and the dominated convergence theorem since, thanks to the Lipschitz continuity of $\tilde{V}$, we have

$$|\tilde{V}(\xi + he_i - \eta) - \tilde{V}(\xi - \eta)| \leq h ||\tilde{V}||_{Lip}.$$

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Éditeur
INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
ISSN 0249-6399