Blow-Up in Reaction Diffusion Systems with Dissipation of Mass

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Abstract: We prove blow up in finite time of the solutions to some reaction-diffusion systems preserving nonnegativity and for which the total mass of the components is uniformly bounded (these are natural properties in applications). This is done by exhibiting explicit counterexamples constructed with the help of a formal computation software. Several partial results of global existence had been obtained before in the literature. Our counterexamples a posteriori explain why extra conditions were needed. Negative results are also provided as a by-product for linear parabolic equations in non divergence form and with discontinuous coefficients and for nonlinear Hamilton-Jacobi evolution equations.

Key-words: parabolic system, reaction-diffusion, global existence, blow-up, parabolic equation in non divergence form, Hamilton-Jacobi equation

(Résumé : tsvp)

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Explosion en temps fini pour des systèmes avec dissipation de masse

Résumé : Nous montrons la possibilité d'explosion en temps fini des solutions de systèmes de réaction-diffusion préservant la positivité et pour lesquels la masse totale des composants est uniformément bornée (propriétés naturelles dans les applications). Nous exhibons en fait des contre-exemples explicites construits avec l'aide d'un logiciel de calcul formel. Plusieurs résultats partiels d'existence globale avaient été obtenus pour cette classe de systèmes avec des hypothèses supplémentaires. Ces contre-exemples justifient a posteriori l'adjonction de ces conditions supplémentaires. Des réponses négatives à des questions indépendantes d'intérêt propre sont obtenues comme sous-produit; elles concernent des équations paraboliques linéaires à forme non divergentielle et coefficients discontinus et des équations d'évolution de Hamilton-Jacobi.

Mots-clé : système parabolique, réaction-diffusion, existence globale, explosion en temps fini, équation parabolique à forme non divergentielle, équation de Hamilton-Jacobi

1 Introduction

We are mainly interested in global existence in time or blow up in finite time of solutions to reaction-diffusion systems of the form

$$\frac{\partial u}{\partial t} - d_1 \Delta u = f(u,v) \text{ on } (0,\infty)\times\Omega \tag{1.1}$$

$$\frac{\partial v}{\partial t} - d_2 \Delta v = g(u,v) \text{ on } (0,\infty)\times\Omega \tag{1.2}$$

for which the following two main properties hold

- the positivity of the solutions is preserved with time, which is equivalent to

$$\forall u, v \geq 0 \qquad f(0,v) \geq 0, \qquad g(u,0) \geq 0 \tag{1.3}$$

- the total mass of the components u, v is nonincreasing with time which is essentially ensured by the structure condition

$$f + g \leq 0. \tag{1.4}$$

Here f, g are regular functions from $[0,\infty[^2$ into $I\!R$, d_1, d_2 are positive constants and Ω is a smooth bounded open subset of $I\!R^N$. As usual, "good" boundary conditions are supposed to be prescribed for u and v, for instance

$$u = 0, \qquad v = 0 \qquad \text{on } \partial\Omega \tag{1.5}$$

as well as initial conditions

$$u(0,.) = u_0 \geq 0, \qquad v(0,.) = v_0 \geq 0. \tag{1.6}$$

The main consequence of the two properties (1.3), (1.4) is that the solutions u, v satisfy an a priori L^1-estimate uniform in time. Indeed, integrating the sum of (1.1), (1.2) and taking into account (1.5) lead to

$$\frac{\partial}{\partial t}\int_\Omega u(t) + v(t) \leq \int_\Omega f + g \leq 0.$$

$$\int_\Omega u(t) + v(t) \leq \int_\Omega u_0 + v_0. \tag{1.7}$$

Since u, v are nonnegative, this is a uniform estimate of their L^1-norms.It is well known that local existence of nonnegative solutions holds for the system (1.1), (1.2), (1.5), (1.6) when $u_0, v_0 \in L^\infty(\Omega)$. Moreover, existence is global as soon as $u(t), v(t)$ satisfy an a priori L^∞-estimate uniform in time.

Here, the a priori estimate is only in $L^1(\Omega)$. Much work has been done to analyze how this L^1 estimate or more generally structure conditions like (1.4) help to provide global existence.

Note for instance that if $d_1 = d_2$, summing (1.1) and (1.2) leads, thanks to (1.4), to

$$\frac{\partial}{\partial t}(u+v) - d_1 \Delta(u+v) \leq 0$$

and by maximum principle

$$||(u+v)(t)||_\infty \leq ||u_0 + v_0||_\infty$$

so that global existence holds.

Note also that the two properties (1.3), (1.4) imply global existence (for nonnegative data) for the associated ordinary differential system

$$\dot{u} = f(u, v), \qquad \dot{v} = g(u, v).$$

For the complete system with different diffusion coefficients $d_1 \neq d_2$, the question is quite more delicate. One of the main result is that, in general, if one of u or v is a priori bounded, then so is the other one. This is the case if, for instance, we have the extra information

$$f \leq 0. \tag{1.8}$$

Obviously, by maximum principle, this implies $||u(t)||_\infty \leq ||u_0||_\infty$. If g is at most of polynomial growth, then global existence can be proved [2], [6], [8], [4], [7].

However, it was still an open problem to decide what happens for systems without any a priori L^∞-bound on either u or v and whether extra conditions must be added to (1.3), (1.4). This is precisely the goal of this paper where we prove that blow up in L^∞ may occur in finite time for these systems.

In order to better understand the question, let us discuss some explicit examples of "systems" that naturally appear in this class. A first one is the following:

$$u_t - d_1\Delta u = \lambda u^3 v^2 - u^2 v^3 \tag{1.9}$$

$$v_t - d_2\Delta v = -u^2 v^3 + u^3 v^2. \tag{1.10}$$

Here $f + g = (\lambda - 1)u^3 v^2 \leq 0$ if $\lambda \in [0,1]$. If $d_1 = d_2$, or more generally if d_1 is close to d_2, then (1.9), (1.10), (1.5), (1.6) has a global solution (see the remark above when $d_1 = d_2$ and [10], [1] in general).

If now d_1 and d_2 are distinct and not close to each other, the question is more serious. If $\lambda = 0$, we are in the situation described above where (1.8) also holds. Then u is a priori uniformly bounded and global existence can be proved. The case when λ is small enough can also be similarly taken care of ([10]). Now the question is open when $\lambda \in]0,1]$. Note that we even have here that

$$f + \lambda g = (\lambda - 1)u^2 v^3 \leq 0, \tag{1.11}$$

so that if $\lambda \in [0,1[$, the two inequalities (1.4), (1.11) are linearly independent. This implies for instance an a priori L^1-bound on the nonlinear terms.

We will see here that similar systems with these properties may present blow up in L^∞.

Let us also discuss another example which is specifically studied in [3], [9], [10].

$$u_t - d_1 u_{xx} = -c(x)u^\alpha v^\beta$$

$$v_t - d_2 v_{xx} = c(x)u^\alpha v^\beta$$

where $c : (-1,1) \to I\!R$ is given. Here $\Omega = (-1,1)$ and f, g depend also on the space variable x and satisfy $f + g = 0$.

If $c \equiv 1$, we are in the situation (1.8) and global existence follows. The same holds if $c(.)$ is of constant sign. Now the situation is quite different if $c(.)$ changes sign. The following specific case is analyzed in [9] :

$$c(x) > 0 \text{ on } (0,1), \quad c(0) = 0, \quad c(x) < 0 \text{ on } (-1,0).$$

It can be shown that u and v are uniformly bounded in $L^\infty_{loc}([0,\infty) \times (0,1])$ and $L^\infty_{loc}([0,\infty) \times [-1,0))$. Therefore, blow up can only occur at $x = 0$ and, if so, will occur for u and v at the same time. If c(.) vanishes fast enough at 0, then no blow up occurs (see [9]). The question with a general $c(.)$ is still open. However we exhibit here similar examples where blow up in L^∞ does happen.

It is interesting to remark that in all our examples, although solutions blow up in $L^\infty(\Omega)$ for some T, they can be extended across T as global "weak" solutions, for instance, in the sense of distributions.

Let us finally mention that the blow up examples described in this paper provide interesting by-products for two questions of independent interest related to the following equations

$$\left\{ \begin{array}{l} u_t - a(x,t)\Delta u = f \text{ on } (0,T)\times\Omega \\ u_{|\partial\Omega} = 0, \quad u(0,.) = 0 \end{array} \right. \tag{1.12}$$

where

$$0 < d_1 \le a(x,t) \le d_2, \tag{1.13}$$

and

$$\left\{ \begin{array}{l} u_t - \max(d_1\Delta u, d_2\Delta u) = f \text{ on } (0,T)\times\Omega \\ u_{|\partial\Omega} = 0, \quad u(0,.) = 0. \end{array} \right. \tag{1.14}$$

We prove that the problem (1.14) is ill-posed for $f \in L^p(Q_T)$ when p is close to 1 (although it is well-posed if $p \ge 2$). For problem (1.12), we prove there is no estimate of the form

$$||u(T)||_{L^1(\Omega)} \le C||f||_{L^p(Q_T)} \tag{1.15}$$

with a constant C depending only on d_1, d_2, T, Ω when p is close to 1. Here also, such estimates are valid if $p \ge 2$. Note also that (1.15) holds when $a(.,.)$ is continuous with C depending on its modulus of continuity. Indeed, we have an L^p-theory and $u_t, \Delta u$ are bounded in L^p. Obviously, the question concerns here a sub-class of parabolic operators of non divergence form with discontinuous coefficients.

2 The main results

We denote by B the euclidian unit ball in $I\!R^N$, $Q_T = (0,T)\times B$, $\Sigma_T = (0,T)\times\partial B$.

Theorem 2.1 :
There exist $f, g \in C^\infty([0,\infty)^2, I\!R), d_1, d_2 > 0, T > 0, u_0, v_0 \in C^\infty(\overline{B}), u_0 \geq 0, v_0 \geq 0, \alpha_1, \alpha_2 \in C^\infty([0,T]), \lambda \in]0,1[$ and $u, v \geq 0$ classical solutions of

$$(2.1) \qquad \frac{\partial u}{\partial t} - d_1 \Delta u = f(u,v) \qquad \text{on} \quad Q_T$$

$$(2.2) \qquad \frac{\partial v}{\partial t} - d_2 \Delta v = g(u,v) \qquad \text{on} \quad Q_T$$

$$(2.3) \qquad u(t,x) = \alpha_1(t), v(t,x) = \alpha_2(t) \qquad \text{on} \quad \Sigma_T$$

$$(2.4) \qquad u(0,x) = u_0(x), v(0,x) = v_0(x) \qquad \text{on} \quad B$$

such that

$$(2.5) \qquad f + g \leq 0, \qquad f + \lambda g \leq 0$$

$$(2.6) \quad \exists k > 0,\ p \geq 1,\ \forall u, v \geq 0 \qquad |f(u,v)| + |g(u,v)| \leq k(u^p + v^p + 1)$$

$$(2.7) \qquad \forall r, s \geq 0 \qquad f(0,s) \geq 0, \quad g(r,0) \geq 0$$

and

$$(2.8) \qquad \lim_{t \uparrow T} ||u(t)||_{L^\infty(B)} = \lim_{t \uparrow T} ||v(t)||_{L^\infty(B)} = +\infty.$$

Theorem 2.2 :
There exist $\alpha, \beta > 1, d_1, d_2 > 0, T > 0, u_0, v_0 \in C^\infty(\overline{B}), u_0 \geq 0, v_0 \geq 0, \alpha_1, \alpha_2 \in C^\infty([0,T]), c_1, c_2 \in C^k(\overline{Q}_T)$ with $k \geq 0$ and $u, v \geq 0$ classical solutions of

$$(2.9) \qquad \frac{\partial u}{\partial t} - d_1 \Delta u = c_1(t,x) u^\alpha v^\beta \qquad \text{on} \quad Q_T$$

$$(2.10) \qquad \frac{\partial v}{\partial t} - d_2 \Delta v = c_2(t,x) u^\alpha v^\beta \qquad \text{on} \quad Q_T$$

$$(2.11) \qquad u(t,x) = \alpha_1(t), v(t,x) = \alpha_2(t) \qquad \text{on} \quad \Sigma_T$$

$$(2.12) \qquad u(0,x) = u_0(x), v(0,x) = v_0(x) \qquad \text{on} \quad \Omega$$

such that

$$(2.13) \qquad c_1(x,t) + c_2(x,t) \leq 0 \quad \text{on } Q_T$$

and

$$(2.14) \qquad \lim_{t \uparrow T} ||u(t)||_{L^\infty(B)} = \lim_{t \uparrow T} ||v(t)||_{L^\infty(B)} = +\infty.$$

Remark 2.1 In both theorems, u and v satisfy

$$\frac{\partial u}{\partial t} - d_1 \Delta u + \frac{\partial v}{\partial t} - d_2 \Delta v \leq 0. \tag{2.15}$$

This implies, in particular, together with the boundary conditions on u, v, that the L^1-norms of $u(t), v(t)$ are bounded on $(0, T)$. Actually, as it will appear clearly on the examples (see next Section), in both cases, there exists $p^* \in (1, \infty)$ such that

$$\forall p < p^* \sup_{t \in (0,T)} (||u(t)||_{L^p(B)} + ||v(t)||_{L^p(B)}) < \infty \tag{2.16}$$

$$\forall p \geq p^* \lim_{t \uparrow T} ||u(t)||_{L^p(B)} = \lim_{t \uparrow T} ||v(t)||_{L^p(B)} = +\infty. \tag{2.17}$$

The proof of both theorems will be obtained by exhibiting explicit solutions u and v satisfying the fundamental inequality (2.15). A lot of consequences may be derived from it together with the nonnegativity of u and v. We have already mentionned the uniform L^1-bound on u, v. It also implies (see [2], [6]) that, for all $p \in (1, \infty)$, there exists $C = C(p, T, \Omega, \alpha_1, \alpha_2, ||u_0||_\infty, ||v_0||_\infty)$ such that

$$\forall t \in (0, T), \quad ||u||_{L^p(Q_T)} \leq C ||v||_{L^p(Q_T)}, ||v||_{L^p(Q_T)} \leq C ||u||_{L^p(Q_T)}. \tag{2.18}$$

Consequently, u and v can only blow up at the same time t when (2.15) holds.

Remark 2.2. Another interesting consequence of (2.15) may be obtained by setting

$$w(t,x) := u(t,x) + v(t,x), \qquad a(t,x) = \frac{d_1 u(t,x) + d_2 v(t,x)}{u(t,x) + v(t,x)}. \tag{2.19}$$

Then, (2.15) can be rewritten

$$\frac{\partial w}{\partial t} - \Delta(aw) \leq 0. \tag{2.20}$$

Here, thanks to the positivity of u, v, we have the a priori estimate

$$0 < \min(d_1, d_2) \le a(t, x) \le \max(d_1, d_2). \tag{2.21}$$

A natural question is to know whether the parabolic inequality (2.20) implies the existence of a constant C depending only on $d_1, d_2, T, \alpha_1, \alpha_2$ such that, for p large,

$$||w||_{L^p(Q_T)} \le C(||w_0||_\infty + 1). \tag{2.22}$$

By duality this is equivalent (see next section) to the existence of a similar constant C such that

$$||z(T)||_{L^1(Q_T)} \le C||\theta||_{L^q(Q_T)} \tag{2.23}$$

where $1/q + 1/p = 1$ and z is solution of the dual problem

$$z_t - b\Delta z = \theta \text{ on } Q_T, \qquad b(t) = a(T-t) \tag{2.24}$$

$$z(0,.) = 0, \qquad z_{|\partial\Omega} = 0. \tag{2.25}$$

It turns out that (2.23) is valid for all $q \ge 2$. Indeed, for $q = \infty$, it is an easy consequence of the maximum principle which holds for (2.24). For $q = 2$, we may (formally) multiply (2.24) by $-\Delta z$ to obtain

$$\frac{\partial}{\partial t}\frac{1}{2}\int_\Omega |\nabla z(t)|^2 + \int_\Omega b(\Delta z)^2 = -\int_\Omega \theta\Delta z.$$

Using the uniform estimate (2.21) and Young's inequality, we deduce

$$\frac{1}{2}\int_\Omega |\nabla z(t)|^2 + \min(d_1, d_2)\int_0^t\int_\Omega (\Delta z)^2 \le \frac{1}{2}\min(d_1, d_2)\int_0^t\int_\Omega (\Delta z)^2 + C\int_0^t\int_\Omega \theta^2$$

which implies

$$\int_{Q_T} ||\Delta z||_{L^2(Q_T)} \le C||\theta||_{L^2(Q_T)} \tag{2.26}$$

where C depends only on d_1, d_2. Therefore, the estimate (2.23) is true for $q = 2$ (and for any $q \in [2, \infty]$ by interpolation). We even have that z_t and Δz (and not only z itself) are bounded in $L^2(Q_T)$.

As a consequence of the counterexamples exhibited in Theorems 2.1 and 2.2, we will see that (2.23) is false when q is close to 1. More precisely, we have the following:

Proposition 2.3 :
There exists $p \in (1,2), b_n \in C^\infty(\overline{Q}_T), d_1, d_2 > 0, \theta_n \in C^\infty(\overline{Q}_T),\ \theta_n \geq 0$ and $z_n \in C^\infty(\overline{Q}_T)$ solution of

$$\frac{\partial z_n}{\partial t} - b_n \Delta z_n = \theta_n \text{ on } Q_T \tag{2.27}$$

$$z_n(0,.) = 0, \quad z_{n|_{\partial B}} = 0 \tag{2.28}$$

such that

$$0 < d_1 \leq b_n \leq d_2 \tag{2.29}$$

$$||\theta_n||_{L^p(Q_T)} = 1 \tag{2.30}$$

$$\lim_{n\to\infty} ||z_n||_{L^\infty(0,T;L^1(B))} = +\infty. \tag{2.31}$$

Finally, we mention a last consequence of independent interest for the following nonlinear Hamilton-Jacobi evolution equations.

Proposition 2.4 :
There exist $p \in (1,2), d_1, d_2 > 0, \theta_n \in C^\infty(\overline{Q}_T)$ and $z_n \in L^2(Q_T)$ with $\Delta_{z_n}, \partial z_n/\partial t \in L^2(Q_T)$ solution of

$$\frac{\partial z_n}{\partial t} - \max(d_1 \Delta z_n, d_2 \Delta z_n) = \theta_n \text{ on } Q_T \tag{2.32}$$

$$z_n(0,.) = 0, z_{n|_{\partial B}} = 0 \tag{2.33}$$

such that

$$||\theta_n||_{L^p(Q_T)} = 1 \tag{2.34}$$

$$\lim_{n\to\infty} ||z_n||_{L^\infty(0,T;L^1(B))} = +\infty. \tag{2.35}$$

3 The proofs

Preliminary remarks

As announced in Section 2, the proofs of Theorems 2.1, 2.2 are obtained by constructing explicit functions u and v satisfying the inequality

$$u_t - d_1\Delta u + v_t - d_2\Delta v \leq 0. \tag{3.1}$$

It will turn out that they are also solutions of systems of the form (2.1), (2.2) and (2.9), (2.10) with the properties listed in the statement of the theorems.

For reasons coming from a precise analysis of the problem and a guess of the possible singularities, we a priori look for functions u, v of the form

$$u(t,x) = \frac{a(T-t) + b|x|^2}{(T-t+|x|^2)^\gamma}, \quad v(t,x) = \frac{c(T-t) + d|x|^2}{(T-t+|x|^2)^\gamma} \tag{3.2}$$

where $|.|$ denotes the euclidian norm, $a, b, c, d > 0$ and $\gamma > 1$ are to be determined so that (3.1) holds for some $d_1, d_2 > 0$, also to be determined. This has been done with the help of the formal software Maple where the unknown coefficients can be progressively adapted "by hand" to satisfy (3.1). As a consequence, the solutions that we found in this way are not numerically simple.

We give here one solution which can be also explicity checked by direct computations. For this, we choose

$$N = 10 \text{ (for the dimension)}, \quad \gamma = 5/4 \tag{3.3}$$

$$a = 1/25, \quad b = 1, \quad c = 11/2, \quad d = 1/10 \tag{3.4}$$

$$d_1 = 1, \quad d_2 = 1/10. \tag{3.5}$$

Lengthly, but straightforward computations show that u and v given by (3.2) with the data (3.3)-(3.5) satisfy the inequality (3.1) and more precisely

$$u_t - d_1\Delta u = \frac{A_1(T-t)^2 + B_1(T-t)|x|^2 + C_1|x|^4}{(T-t+|x|^2)^{\gamma+2}} \tag{3.6}$$

$$v_t - d_2\Delta v = \frac{A_2(T-t)^2 + B_2(T-t)|x|^2 + C_2|x|^4}{(T-t+|x|^2)^{\gamma+2}} \tag{3.7}$$

$$u_t + v_t - d_1 \Delta u - d_2 \Delta v = \frac{A(T-t)^2 + B(T-t)|x|^2 + C|x|^4}{(T-t+|x|^2)^{\gamma+2}} \tag{3.8}$$

where

$$A_1 = -1899/100, \quad B_1 = -323/100, \quad C_1 = 496/100 \tag{3.9}$$

$$A_2 = 1194/80, \quad B_2 = 281/80, \quad C_2 = -427/80 \tag{3.10}$$

$$A = -1626/400, \quad B = 113/400, \quad C = -151/400. \tag{3.11}$$

We check that $B^2 - 4AC < 0$ so that (3.1) holds.

Remark. Note that other explicit solutions have also found in dimension $N = 1$. We easily show that they satisfy (2.9), (2.10) in Theorem 2.2. However the corresponding functions f, g of Theorem 2.1 are technically difficult to exhibit.

Proof of Theorem 2.1

We will show that u and v defined as above satisfy the conclusions of the theorem. Obviously $u(0), v(0) \in C^\infty(\bar{B})$ and $u_{|\partial B} = \alpha_1 \in C^\infty([0,T])$, $v_{|\partial B} = \alpha_2 \in C^\infty([0,T])$ and, since $\gamma > 1$,

$$\lim_{t \uparrow T} ||u(t)||_\infty = \lim_{t \uparrow T} ||v(t)||_\infty = +\infty. \tag{3.12}$$

More precisely, all the L^p-norms for $p \geq 20$ blow up at $t = T$.

Now we need to determine f and g. Starting from (3.6), (3.7), one looks for polynomial functions P, Q, homogeneous and of degree 5 in u and v such that

$$u_t - d_1 \Delta u = P(u,v), \quad v_t - d_2 \Delta v = Q(u,v). \tag{3.13}$$

Formal computations lead to the following expressions which can be checked directly by, again lenghtly, but straightforward computations:

$$P(u,v) = \sum_{k=0}^{5} \lambda_k u^{5-k} v^k, \qquad Q(u,v) = \sum_{k=0}^{5} \mu_k u^{5-k} v^k \tag{3.14}$$

with $\lambda_k = (229)^{-5} \tilde{\lambda}_k$, $\mu_k = (229)^{-5} \tilde{\mu}_k$ and where $\tilde{\lambda}_k, \tilde{\mu}_k$ are given in the following table:

k	$\tilde{\lambda_k}$	$\tilde{\mu_k}$	$\tilde{\nu_k}$
0	$2^{-3} \times 3^7 \times 5^8 \times 43 \times 653$	$-2^{-2} \times 3^7 \times 5^9 \times 3023$	$-2^{-3} \times 3^9 \times 5^8 \times 239$
1	$2^{-2} \times 3^5 \times 5^8 \times 7 \times 7703$	$-2^{-4} \times 3^5 \times 5^8 \times 11 \times 20717$	$-2^{-4} \times 3^5 \times 5^8 \times 12203$
2	$-2 \times 3^5 \times 5^7 \times 19 \times 359$	$2^{-3} \times 3^4 \times 5^7 \times 29 \times 61 \times 131$	$-2^{-3} \times 3^4 \times 5^7 \times 7 \times 79 \times 173$
3	$-2^4 \times 3 \times 5^6 \times 23 \times 31 \times 397$	$3 \times 5^6 \times 409 \times 9011$	$-3^2 \times 5^6 \times 281159$
4	$-3^{-1} \times 2^7 \times 5^5 \times 61 \times 4567$	$3^{-1} \times 2^3 \times 5^8 \times 28591$	$-3^{-1} \times 2^3 \times 5^5 \times 883517$
5	$-3^{-1} \times 2^{13} \times 5^3 \times 17 \times 19^2$	$3^{-1} \times 2^6 \times 5^4 \times 59 \times 2087$	$-3^{-1} \times 2^6 \times 5^3 \times 13 \times 73 \times 179$

We check that $P+Q = \sum_{k=0}^{5} \nu_k u^{5-k} v^k$ where all the $\nu_k = (229)^{-5}\tilde{\nu}_k$ are negative so that for λ close to 1, $P+\lambda Q$ has also negative coefficients. Therefore (2.5), hold for P and Q. However, P and Q do not satisfy (2.7). Since u, v are bounded from below on Q_T respectively by,

$$m_1 = \min(a,b)/(T+1)^{\gamma-1}, \quad m_2 = \min(c,d)/(T+1)^{\gamma-1}$$

one can always modify P and Q on a neighborhood of $\{0\}\times[0,\infty[\cup[0,\infty[\times\{0\}$. Let $\varphi \in C^\infty(I\!R^2)$ a function which is identically one on $[m_1,\infty)\times[m_2,\infty)$ and such that

$$\forall u,v \geq 0, \quad \varphi(0,v) = \varphi(u,0) = 0.$$

We set

$$\forall u,v \geq 0 \qquad f(u,v) := \varphi(u,v)P(u,v) \quad g(u,v) := \varphi(u,v)Q(u,v).$$

Then for the same values of λ close to 1 as above

$$f + \lambda g = \varphi(P+\lambda Q) \leq 0.$$

So that (2.1), (2.2), (2.5), (2.6), (2.7) are satisfied.

Proof of Theorem 2.2

We use the same functions u and v as above. We only have to prove that the expressions in (3.6), (3.7) can be written as in (2.9), (2.10). For this we choose $\alpha, \beta > 1$ such that

$$\alpha + \beta > (\gamma + 2)/(\gamma - 1) \tag{3.15}$$

and (see (3.2), (3.6), (3.7)).

$$c_1(t,x) := \frac{A_1(T-t)^2 + B_1(T-t)|x|^2 + C_1|x|^4}{(a(T-t)+b|x|^2)^\alpha (c(T-t)+d|x|^2)^\beta}(T-t+|x|^2)^{(\alpha+\beta)\gamma-\gamma-2}$$

$$c_2(t,x) := \frac{A_2(T-t)^2 + B_2(T-t)|x|^2 + C_2|x|^4}{(a(T-t)+b|x|^2)^\alpha (c(T-t)+d|x|^2)^\beta}(T-t+|x|^2)^{(\alpha+\beta)\gamma-\gamma-2}$$

Obviously with this choice of c_1, c_2, the relations (2.9), (2.10) hold. The sign of $c_1 + c_2$ is the same as the sign of $A(T-t)^2 + B(T-t)|x|^2 + C|x|^4$ in (3.8) which has already been checked to be negative. Now, thanks to the choice of (3.15), c_1, c_2 are at least continuous on $\overline{Q_T}$. Actually they are C^∞ except at the point $(T, 0)$ where they tend to 0. We can make them more regular by choosing α and β even larger.

Proof of Proposition 2.3

We take u, v as in (3.2) and we set

$$w_n(t,x) := u(t - \frac{1}{n}, x) + v(t - \frac{1}{n}, x) \tag{3.16}$$

$$a_n(t,x) := \left[d_1 u(t - \frac{1}{n}, x) + d_2 v(t - \frac{1}{n}, x)\right] / w_n(t,x). \tag{3.17}$$

Note that

$$0 < \min(d_1, d_2) \le a_n \le \max(d_1, d_2).$$

By (3.1),

$$\frac{\partial w_n}{\partial t} - \Delta(a_n w_n) \le 0 \qquad \text{on } Q_T. \tag{3.18}$$

For $\theta \in C^\infty(\overline{Q}_T), \theta \geq 0$, and $b_n(t) = a_n(T-t)$, let $z \in C^\infty(\overline{Q}_T), z \geq 0$ be the solution of

$$\frac{\partial z}{\partial t} - b_n \Delta z = \theta \text{ on } Q_T \tag{3.19}$$

$$z(0,.) = 0, \quad z_{|\partial B} = 0. \tag{3.20}$$

The existence of z is classical (see e.g [5]). We set $\tilde{z}(t) = z(T-t)$.
So that

$$-(\tilde{z}_t + a_n \Delta \tilde{z}) = \tilde{\theta}, \qquad \tilde{\theta}(t) = \theta(T-t) \tag{3.21}$$

Let now $\varphi \in C_0^\infty(B)$ with $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ on $B(0,1/2)$. Using (3.18)-(3.21), we have

$$\int_{Q_T} \tilde{\theta} w_n \varphi = \int_{Q_T} -w_n \varphi(\tilde{z}_t + a_n \Delta \tilde{z}) = \int_\Omega w_n(0)\varphi \tilde{z}(0) + \int_{Q_T} \tilde{z}(w_{n_t}\varphi - \Delta(a_n w_n \varphi))$$

$$= \int_\Omega w_n(0)\varphi z(T) + \int_{Q_T} \tilde{z}\varphi(w_{n_t} - \Delta(a_n w_n)) - 2\nabla\varphi\nabla(a_n w_n) - a_n w_n \Delta\varphi.$$

Since $\nabla\varphi, \Delta\varphi$ are identically zero around the origin, the terms $\nabla\varphi\nabla(a_n w_n)$, $a_n w_n \Delta\varphi$ are uniformly bounded independently of n. Using also the inequality (3.18), we obtain

$$\begin{aligned} \int_{Q_T} \tilde{\theta} w_n \varphi \ &\leq \int_\Omega w_n(0) z(T) + C \int_{Q_T} z \\ &\leq (||w_n(0)||_\infty + CT)||z||_{L^\infty(0,T,L^1(B))}. \end{aligned} \tag{3.22}$$

If we had

$$||z||_{L^\infty(0,T,L^1(B))} \leq k||\theta||_{L^p(Q_T)} \tag{3.23}$$

for some $p \in (1,2)$ and some $k = k(d_1, d_2, p, T)$ and all $\theta \in C^\infty(\overline{Q}_T)$, then from (3.22) we would deduce by duality that, for some C independent of n

$$||w_n \varphi||_{L^{p'}(Q_T)} \leq C.$$

This is false for p' large enough (that is for p small enough) by the construction of u and v and the definition (3.16) of w_n. Therefore the solutions of (3.19), (3.20) do not satisfy the estimate (3.23) (see(3.22)) for some p close to 1, whence the statement of Proposition 2.3.

Proof of Proposition 2.4

For $\theta_n \in C^\infty(\overline{Q}_T)$, it is classical (see e.g. [10], [1]) that there exists a solution of (2.32), (2.33) whose regularity is at least such that

$$z_n \in L^\infty(Q_T), \quad z_{n_t}, \Delta z_n \in L^2(Q_T)$$

and (2.32) is satisfied at least a.e $x, t \in Q_T$. We choose θ_n as in the statement of Proposition (2.3). We obviously have that

$$\max(d_1 \Delta z_n, d_2 \Delta z_n) \geq b_n \Delta z_n \quad a.e.$$

so that

$$\frac{\partial z_n}{\partial t} - b_n \Delta z_n \geq \theta_n.$$

By the maximum principle applied to the operator $\dfrac{\partial}{\partial t} - b_n \Delta$, this solution z_n is greater than the one defined in Proposition 2.3. As a consequence, (2.35) follows from (2.31).

References

[1] M. Drissi-Zelladji, *Méthodes d'agrégation et méthodes des familles*, thèse de doctorat, Université de Franche-Comté (1994).

[2] S. Hollis, R.H. Martin, M. Pierre, *Global existence and boundedness in reaction-diffusion systems*, SIAM J. Math. Anal. 18, p. 744-761 (1987).

[3] S. Hollis, J. Morgan, *Interior estimate for a class of reaction-diffusion systems from L^1 a priori estimate* J.D.E. 98, p. 260-276 (1992).

[4] A. Haraux, Youkana, *On a result of K. Masuda concernig reaction-diffusion equation*, Tohoku Math. J. 40, p.159-163 (1988).

[5] O.A. Ladyzenskaja, V.A. Solonnikov and N.N. Vral'ceva, *Linear and quasilinear equations of parabolic type*, Volume 23 of Translation of Mathematical Monographs AMS (1968).

[6] R. H. Martin, M. Pierre, *Nonlinear reaction diffusion systems*, Nonlinear Equations in the Applied Sciences, ed. W.F. Ames and C. Rogers, Academic Press (1991), Notes and Reports in Mathematics in Sciences Engineering.

[7] K. Masuda, *On the global existence and asymptotic behavior of reaction-diffusion equations*, Hokkaido Math. J. 12, p.360-370 (1983).

[8] J. Morgan, *Global existence for semilinear parabolic systems*, SIAM J. Math. Anal. 20, N^{0}5, p. 1128-1144 (1989).

[9] M. Pierre, D. Schmitt, *Global existence for a reaction-diffusion system with a balance law*, In G.R. Goldstein and G.A. Goldstein editors, Proceedings of the Curaçao conference on semigroups and nonlinear operators and applications (1993).

[10] D. Schmitt, *Existence globale ou explosion pour les systèmes de réactio-diffusion avec contrôle de masse*, thèse d'université, Université Henri Poincaré, Nancy I (1995).

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