On Integer Chebyshev Polynomials

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Abstract

We are concerned with the problem of minimizing the supremum norm on $[0,1]$ of a nonzero polynomial of degree at most n with integer coefficients. We use the structure of such polynomials to derive an efficient algorithm for computing them. We give a table of these polynomials for degree up to 75 and use a value from this table to answer an open problem and improve a lower bound from [3].

Sur les polynômes de Tchébychev entiers

Résumé

Nous nous intéressons à la minimisation de la norme sup sur $[0,1]$ des polynômes non-nuls de degré au plus n à coefficients entiers. Nous exploitons la structure de ces polynômes pour trouver un algorithme efficace permettant de les calculer. Nous donnons une table de ces polynômes jusqu'au degré 75 et utilisons une des valeurs de cette table pour répondre à un problème ouvert et pour améliorer une borne inférieure de [3].

ON INTEGER CHEBYSHEV POLYNOMIALS

LAURENT HABSIEGER AND BRUNO SALVY

ABSTRACT. We are concerned with the problem of minimizing the supremum norm on $[0,1]$ of a nonzero polynomial of degree at most n with integer coefficients. We use the structure of such polynomials to derive an efficient algorithm for computing them. We give a table of these polynomials for degree up to 75 and use a value from this table to answer an open problem and improve a lower bound from [3].

1. INTRODUCTION

Let d_n denote the lowest common multiple of $1, 2, \ldots, n$. The prime number theorem may be stated as

$$\lim_{n\to\infty} \frac{\log(d_n)}{n} = 1\,.$$

Let $\mathbb{Z}_n[X]$ be the set of polynomials of degree less than or equal to n with integral coefficients, and let I be the function that maps a polynomial $P(X)$ onto $\int_0^1 P(x)\,dx$. It is easy to see that

$$I\left(\mathbb{Z}_{n-1}[X]\right) = \frac{\mathbb{Z}}{d_n}. \tag{1}$$

Nair [6] used this property to show that $d_n \geq 2^n$ for $n \geq 9$, by considering the polynomial $X^n(1-X)^n$. This method may be refined as follows. Assume that $P(X)$ is a polynomial of degree $k > 0$ with integral coefficients and such that

$$||P||_\infty := \max_{t\in[0,1]} |P(t)|$$

is small. Since P is non-zero, we have $I(P^{2n}) > 0$, for any nonnegative integer n. By (1), this implies the inequality $d_{2kn+1} \geq ||P||_\infty^{-2n}$ from which we deduce

$$\liminf_{n\to\infty} \frac{\log(d_n)}{n} \geq -\frac{\log ||P||_\infty}{k}\,.$$

This motivates the study of the polynomials $P_k \in \mathbb{Z}_k[X]$ and the quantities C_k such that

$$||P_k||_\infty = \min_{P\in\mathbb{Z}_k[X]\setminus\{0\}} ||P||_\infty\,, \qquad \text{and} \qquad C_k = -\frac{1}{k}\log ||P_k||_\infty\,, \tag{2}$$

for positive integers k. According to [3], the polynomials P_k are called integer Chebyshev polynomials in the interval $[0,1]$. In [2] these polynomials are also called polynomials of minimal diophantic deviation from zero.

Much is known about these polynomials and their asymptotic structure. It was proved by Snirelman (see [1]) that the sequence $(C_k)_{k\in\mathbb{N}^*}$ converges to a limit C; and Borwein and Erdélyi [3] showed that $C \in (0.8586616, 0.8657719)$. Therefore one cannot prove the prime number theorem in this way. However the problem of finding the integer Chebyshev polynomials in the interval $[0,1]$ is interesting in itself (See [3, 5] and the references therein. In particular, Borwein and Erdélyi state in [3] that "Even computing low-degree examples is complicated.")

In this paper, we first prove two lemmas that halve the degree of the polynomials we need to look for. This step enables us to compute polynomials of larger degree but we cannot guarantee to find them all anymore. We then describe several techniques to derive an efficient algorithm for computing these polynomials for moderate degree. We give a table of these polynomials for degree up to 75 and use a value from this table (P_{70}) to answer an open problem from [3] and improve the lower bound on C.

2. Structure of the polynomials

The set

$$E_k = \{P \in \mathbb{Z}_k[X] \ : \ P(1-X) = (-1)^k P(X)\}.$$

is related to our problem by the following two lemmas.

Lemma 1. *For any nonnegative integer k, we have*

$$E_{2k} = \mathbb{Z}_k[X(1-X)] \qquad \textit{and} \qquad E_{2k+1} = (1-2X)\mathbb{Z}_k[X(1-X)].$$

Proof. We first show by induction on k that $E_{2k} = \mathbb{Z}_k[X(1-X)]$. The case $k=0$ is trivial: $E_0 = \mathbb{Z} = \mathbb{Z}_0[X(1-X)]$. Let k be a positive integer and let P be in E_{2k}. The polynomial $P(X) - P(0)$ vanishes when X equals 0, and when X equals 1, by symmetry. Therefore the quotient $Q(X) = \frac{P(X)-P(0)}{X(1-X)}$ is a polynomial in X of degree at most $2k-2$. Besides, the polynomial Q belongs to E_{2k-2}. Applying the induction hypothesis to Q then gives the desired result for P.

If P belongs to E_{2k+1}, we have $P(1/2) = -P(1/2) = 0$, which shows that $1-2X$ divides $P(X)$. The polynomial $Q(X) = P(X)/(1-2X)$ then belongs to E_{2k} and we can use the first part to complete the proof of the Lemma. □

Lemma 2. *For any positive integer k, there exists an element F of degree k in E_k for which*

$$C_k = -\frac{1}{k}\log\|F\|_\infty.$$

Proof. Let k be a positive integer and P a polynomial of degree less than or equal to k, with integral coefficients such that $C_k = -\log\|P\|_\infty/k$. We first prove that we may assume that the degree of P equals k. Let t be in $[0,1]$ such that $\|P\|_\infty = |P(t)|$, and suppose that the degree of P is less than k. If t does not belong to $\{0,1\}$, the polynomial $XP(X)$ contradicts the minimality of $\|P\|_\infty$. If t belongs to $\{0,1\}$, $|P(t)|$ is a positive integer. Therefore we have $1 \leq |P(t)|$. Since $\|X(1-X)\|_\infty = 1/4 < 1$, we get $k=1$. In this special case, we can easily show that C_1 equals 0 and that $-\log\|1-2X\|_\infty = C_1$.

Let us assume that the degree of P equals k and let us define two polynomials Q_1 and Q_2 with integral coefficients by

$$Q_1(X) = XP(X) + (-1)^k(1-X)P(1-X), \tag{3}$$

$$Q_2(X) = (1-X)P(X) + (-1)^k XP(1-X). \tag{4}$$

By construction, we have $Q_i(X) = (-1)^k Q_i(1-X)$, for $i = 1,2$. For any element t in $[0,1]$, notice that

$$|Q_i(t)| \le t\,\|P\|_\infty + (1-t)\,\|P\|_\infty = \|P\|_\infty\,,$$

which implies that $\|Q_i\|_\infty \le \|P\|_\infty$, for $i = 1,2$. We now prove that at least one of the polynomials Q_i is non-zero. Set $P(X) = \sum_{0\le i\le k} p_i X^i$ with $p_k \ne 0$, so that

$$Q_1(X) = (2p_{k-1} + (k+1)p_k)X^k + \sum_{0\le i\le k-1} q_{1,i}X^i, \tag{5}$$

$$Q_2(X) = (-2p_{k-1} - (k-1)p_k)X^k + \sum_{0\le i\le k-1} q_{2,i}X^i. \tag{6}$$

Since p_k is non-zero, there exists i in $\{1,2\}$ for which Q_i is of degree k (and therefore non-zero). We then take $F = Q_i$ to complete the proof of the lemma. $\square$

3. Computation of minimal polynomials

We now describe the techniques we use to compute a polynomial P_k of degree k satisfying (2) for k up to 75. The outline of the algorithm is as follows:

(1) Find a good upper bound for $\|P_k\|_\infty$;
(2) Use this bound to deduce polynomials that are necessarily factors of P_k;
(3) Perform an exhaustive search for the missing factors.

We now review these stages in detail.

3.1. First upper bound. A good bound is given by

$$c_k = \min_{0<p<k} \|P_p P_{k-p}\|_\infty\,.$$

For 56 out of the first 75 polynomials, c_k turns out to be optimal, which means that a minimal polynomial of degree k has been found. However, we do not have this information *a priori.*

3.2. Bounds and factors. The second stage of the algorithm is iterative. Each step attempts to prove the existence of a factor of P_k starting from c_k and a known factor F of P_k. Initially, $F = 1$ if k is even and $F = 2X - 1$ otherwise. By Lemmas 1 and 2, we concentrate on finding factors of a polynomial $G \in \mathbb{Z}[X]$, such that $P_k(X) = F(X)G(X(1-X))$. We denote by g the degree of G.

Since $|P_k(x)|$ is bounded by c_k on $[0,1]$, it follows that for all $x \in [0, 1/4]$,

$$|G(x)|\cdot|F(u(x))| \le c_k, \qquad \text{with} \qquad u(x) = \frac{1-\sqrt{1-4x}}{2}. \tag{7}$$

As G has integer coefficients, this inequality can often be used to prove the existence of factors of the form $qX-p$ (p and q integers, $0 \leq p/q \leq 1/4$) when $F(u(p/q)) \neq 0$, for then it is sufficient to check that

$$c_k < \frac{|F(u(p/q))|}{q^g},$$

which implies $q^g|G(p/q)| < 1$. This technique extends to multiple factors via Markov's inequality on the r-th derivative of any polynomial P of degree n with real coefficients:

$$\max_{a\leq x\leq b} |P^{(r)}(x)| \leq \frac{2^r}{(b-a)^r} \frac{n^2(n^2-1^2)\cdots(n^2-(r-1)^2)}{(2r-1)!!} \max_{a\leq x\leq b} |P(x)|,$$

where $(2i+1)!! = 1\cdot 3\cdot 5\cdots(2i+1)$.

In practice, we use these bounds with $p/q \in \{1/4, 1/5\}$ to find factors $(4X-1)^a$, $(5X-1)^b$ of G, corresponding to factors $(2X-1)^{2a}$, $(5X^2-5X+1)^b$ of the polynomial P_k. This technique also applies to $p/q = 0$, yielding factors $X^c(1-X)^c$ of P_k, but we rather use another bound derived from [3]. If $P_k(X) = X^{k-p}Q(X)$ with $Q(0) \neq 0$, then

$$|Q(0)| \leq \sqrt{2p+1}\binom{k+p+1}{k-p} c_k.$$

This yields factors $X^c(1-X)^c$ by Lemmas 1 and 2.

The advantage of the bounds above is that their computation can be performed rather efficiently. However, they generally fail to yield all the factors of P_k. One reason for this is that they do not really take into account the known factor F, except for its value at $u(p/q)$. To get tighter bounds on the value of G at a given x, we then turn to *Lagrange interpolation*. If $x_0,\ldots,x_g$ are $g+1$ distinct points in $[0,1/4]$ then

$$G(x) = \sum_{i=0}^{g} G(x_i) \prod_{j\neq i} \frac{x-x_j}{x_i-x_j}.$$

If the points x_j are chosen so that $F(u(x_j)) \neq 0$ for $j=0,\ldots,g$, it follows that

$$|G(x)| \leq c_k \sum_{i=0}^{g} \frac{1}{|F(u(x_i))|} \prod_{j\neq i} \left|\frac{x-x_j}{x_i-x_j}\right|. \tag{8}$$

This gives a bound on $|G(x)|$ for any $x \in \mathbb{C}$, which can be further improved by finding a set $\{x_0,\ldots,x_g\}$ which minimizes the right-hand side of (8). It turns out that it is not necessary to spend much time finding a global minimum, but that a few iterations of an optimizing scheme produce excellent results.

More generally, bounds on values of the polynomial help find factors of G of any degree. If $A(X) = a_0X^n + \cdots + a_n$ is an irreducible polynomial with integer coefficients, a necessary and sufficient condition for A to be a factor of G is that the resultant of A and G be zero. Since this resultant is an integer, denoting $\alpha_1,\ldots,\alpha_n$ the roots of A, this condition is equivalent to

$$|a_0|^g|G(\alpha_1)|\cdots|G(\alpha_n)| < 1. \tag{9}$$

Thus for each irreducible polynomial $A(X)$ such that $A(X(1-X))$ occurs as a factor of one of the P_p's, $p<k$, we compute its roots α_i's numerically and bound

the left-hand side of (9) using Lagrange interpolation as above for each $|G(\alpha_i)|$. In practice, this works well for $A(X) = 29X^2 - 11X + 1$ which occurs frequently.

During this stage of the algorithm, every time a factor is found, F and g are updated, leading to better estimates in the inequalities above, and the whole process is started over again, until no more factors are found.

3.3. Exhaustive search. For 25 out of the first 75 polynomials, the quest for factors described above is sufficient to determine all the factors of P_k. In the other cases, we still have to determine a missing factor. By plugging values of x in (7), we get linear inequalities satisfied by the coefficients of the factor G. Sufficiently many of these inequalities define a polyhedron whose interior integer points we have to determine. We have not found any reference to an efficient algorithm for doing so (except [4] in dimension 2).

We solve this problem by using a simplex method to compute bounds on each coordinate. Then if the size of the bounding polyrectangle is not too large, we check each of its points to see whether it belongs to the polyhedron. For larger polyrectangles, we select the variable with least variation and apply recursively the same technique for each of its possible values. Empirically, it appears that it is better to compute the coefficients of the reciprocal polynomials in the basis $1, (X - 4), (X-4)(X-5), \ldots$ instead of the coefficients of the polynomials themselves.

4. A new factor and its consequences

Table 1 shows the first 75 integer Chebyshev polynomials. For each degree we give only one polynomial, even when several exist. The notations are

$$A_1 = X(1-X), \quad A_2 = 1 - 2X, \quad A_3 = 5X^2 - 5X + 1,$$
$$A_4 = 6X^2 - 6X + 1, \quad A_5 = 29X^4 - 58X^3 + 40X^2 - 11X + 1,$$
$$A_6 = (13X^3 - 20X^2 + 9X - 1)(13X^3 - 19X^2 + 8X - 1),$$
$$A_7 = (31X^4 - 63X^3 + 44X^2 - 12X + 1)(31X^4 - 61X^3 + 41X^2 - 11X + 1),$$
$$A_8 = 4921X^{10} - 24605X^9 + 53804X^8 - 67586X^7 + 53866X^6$$
$$-28388X^5 + 9995X^4 - 2317X^3 + 338X^2 - 28X + 1.$$

When expressed in the variable $u = X(1-X)$, these polynomials become

$$A_1 = u, \quad A_2^2 = 4u - 1, \quad A_3 = 5u - 1, \quad A_4 = 6u - 1, \quad A_5 = 29u^2 - 11u + 1,$$
$$A_6 = 169u^3 - 94u^2 + 17u - 1, \quad A_7 = 961u^4 - 712u^3 + 194u^2 - 23u + 1,$$
$$A_8 = 4921u^5 - 4594u^4 + 1697u^3 - 310u^2 + 28u - 1.$$

Almost all these factors were already known to occur in integer Chebyshev polynomials. The most surprising result is the factor A_8 which divides P_{70}. This factor gives a negative answer to the following open problem from [3]:

> Do the integer Chebyshev polynomials on $[0,1]$ have all their zeros in $[0,1]$?

The polynomial P_{70} has four non-real zeros. The derivative of A_8 however has all its zeros in $[0,1]$.

k	C_k	Polynomial	k	C_k	Polynomial
0	$-\infty$	1	38	.8400137109	$A_1^{14}A_2^6A_3^2$
1	0	A_2	39	.8488877225	$A_1^{13}A_2^5A_3^2A_5$
2	.6931471805	A_1	40	.8404640658	$A_1^{13}A_2^6A_3^2A_5$
3	.7803552046	A_1A_2	41	.8465081502	$A_1^{14}A_2^5A_3^2A_5$
4	.6931471805	A_1^2	42	.8440344532	$A_1^{14}A_2^6A_3^2A_5$
5	.8047189562	$A_1^2A_2$	43	.8449879864	$A_1^{14}A_2^5A_3^3A_5$
6	.7803552047	$A_1^2A_2^2$	44	.8455791880	$A_1^{15}A_2^6A_3^2A_5$
7	.7991843140	$A_1^3A_2$	45	.8398268629	$A_1^{15}A_2^7A_3^2A_5$
8	.8010279578	$A_1^3A_2^2$	46	.8468722310	$A_1^{16}A_2^6A_3^2A_5$
9	.8316158874	$A_1^3A_2A_3$	47	.8430715282	$A_1^{15}A_2^5A_3^2A_5^2$
10	.8047189567	$A_1^4A_2^2$	48	.8491690644	$A_1^{16}A_2^6A_3^3A_5$
11	.8109727374	$A_1^4A_2A_3$	49	.8457300825	$A_1^{16}A_2^7A_3^2A_4A_5$
12	.8235466006	$A_1^4A_2^2A_3$	50	.8448129844	$A_1^{16}A_2^6A_3^2A_5^2$
13	.8090328223	$A_1^4A_2A_3^2$	51	.8473273518	$A_1^{17}A_2^7A_3^3A_5$
14	.8405593722	$A_1^5A_2^2A_3$	52	.8464778545	$A_1^{17}A_2^8A_3^2A_4A_5$
15	.8163003367	$A_1^5A_2^3A_3$	53	.8494236563	$A_1^{18}A_2^7A_3^3A_5$
16	.8268434981	$A_1^6A_2^2A_3$	54	.8441650118	$A_1^{18}A_2^8A_3^3A_5$
17	.8311026953	$A_1^6A_2^3A_3$	55	.8469319238	$A_1^{19}A_2^7A_3^3A_5$
18	.8316158595	$A_1^6A_2^2A_3^2$	56	.8457325337	$A_1^{19}A_2^8A_3^3A_5$
19	.8400137111	$A_1^7A_2^3A_3$	57	.8464270507	$A_1^{19}A_2^7A_3^4A_5$
20	.8288579250	$A_1^6A_2^2A_3A_5$	58	.8471145416	$A_1^{20}A_2^8A_3^3A_5$
21	.8303936176	$A_1^8A_2^3A_3$	59	.8468162432	$A_1^{19}A_2^7A_3^3A_5^2$
22	.8322820522	$A_1^8A_2^4A_3$	60	.8483301990	$A_1^{21}A_2^8A_3^3A_5$
23	.8385504326	$A_1^8A_2^3A_3^2$	61	.8462840938	$A_1^{20}A_2^7A_3^3A_5^2$
24	.8378960676	$A_1^9A_2^4A_3$	62	.8488367522	$A_1^{21}A_2^8A_3^4A_5$
25	.8448129844	$A_1^8A_2^3A_3A_5$	63	.8463191193	$A_1^{20}A_2^7A_3^3A_5A_6$
26	.8338173096	$A_1^9A_2^4A_3^2$	64	.8477264811	$A_1^{21}A_2^8A_3^3A_5^2$
27	.8434645771	$A_1^9A_2^3A_3A_5$	65	.8478630743	$A_1^{22}A_2^9A_3^4A_5$
28	.8405595853	$A_1^{10}A_2^4A_3^2$	66	.8489400289	$A_1^{22}A_2^{10}A_3^3A_4A_5$
29	.8356309576	$A_1^{11}A_2^5A_3$	67	.8492102067	$A_1^{23}A_2^9A_3^4A_5$
30	.8398858116	$A_1^{10}A_2^4A_3A_5$	68	.8468222183	$A_1^{23}A_2^{10}A_3^3A_4A_5$
31	.8358028746	$A_1^{11}A_2^5A_3^2$	69	.8471956204	$A_1^{22}A_2^9A_3^3A_5A_6$
32	.8412151163	$A_1^{11}A_2^4A_3A_5$	70	.8467991413	$A_1^{22}A_2^8A_3^3A_5A_8$
33	.8406807538	$A_1^{12}A_2^5A_3^2$	71	.8472585205	$A_1^{24}A_2^{11}A_3^3A_4A_5$
34	.8461748302	$A_1^{11}A_2^4A_3^2A_5$	72	.8499040059	$A_1^{23}A_2^8A_3^4A_5A_6$
35	.8388555719	$A_1^{11}A_2^5A_3A_4A_5$	73	.8499191960	$A_1^{24}A_2^9A_3^4A_5^2$
36	.8409740145	$A_1^{12}A_2^4A_3^2A_5$	74	.8486911214	$A_1^{23}A_2^8A_3^4A_5A_7$
37	.8431610719	$A_1^{12}A_2^5A_3^2A_5$	75	.8487246297	$A_1^{24}A_2^9A_3^4A_5A_6$

TABLE 1. Integer Chebyshev polynomials of degree up to 75

The factor A_8 can also be used to improve the bound on C. Following the lines of [3], we use a simplex method to compute $\alpha_1, \ldots, \alpha_{10}$ and c such that: the system

$$\sum_{i=1}^{10} \alpha_i \log |A_i(x_j)| \le c, \qquad j = 1, \ldots, n$$

is satisfied; c is minimal; the α_i's are nonnegative and constrained by

$$\sum_{i=1}^{10} \alpha_i \deg(A_i) = 1;$$

the polynomial A_9 is taken from [3]:

$$941[X(1-X)]^4 - 703[X(1-X)]^3 + 193[X(1-X)]^2 - 23X(1-X) + 1;$$

the polynomial A_{10} is

$$34X^4 - 68X^3 + 46X^2 - 12X + 1,$$

which was found by considering polynomials with small coefficients in the basis $1, (X-4), (X-4)(X-5), \ldots$; and the x_j's are (numerous) points in $[0, 1/2]$. We obtain

$$\begin{aligned}(\alpha_1, \ldots, \alpha_{10}) = \\ &(.3184626443, .1174461823, .0385725221, .0015929459, .0150592023, \\ &\quad .0058197886, .0025256792, .0002799383, .0057810725, .0002221637).\end{aligned}$$

From this computation we deduce a polynomial

$$\begin{aligned}Q = A_1^{3184626443} \cdot A_2^{1174461823} \cdot A_3^{385725221} \cdot A_4^{15929459} \\ \cdot A_5^{150592023} \cdot A_6^{58197886} \cdot A_7^{25256792} \cdot A_8^{2799383} \cdot A_9^{57810725} \cdot A_{10}^{2221637}\end{aligned}$$

of degree $d = 10^{10} - 9$ such that

$$-\frac{1}{d} \log \|Q\|_\infty \approx 0.8591978074438279625963 5.$$

Then since $\|P_{nd}\|_\infty \le \|Q^n\|_\infty$, we get the following improvement on the known lower bound 0.8586616.

Theorem 1. *The constant C satisfies*

$$C > 0.8591978.$$

Thus the new known interval for C has width 92% of the previous one.

5. Conclusion

All the computations have been performed using the computer algebra system Maple. By implementing the same techniques in C, one would probably find at most ten more polynomials, at the expense of a much longer programming time. However, it is clearly much more effective to look for better algorithms.

Currently, the bottleneck of the computation is the last part, which is hopeless if the degree of the missing factor is too high (our limit is 24, corresponding to thirteen undeterminate coefficients in $X(1-X)$). Sophisticated techniques from integer linear programming might help.

Also, it is crucial to find as many factors as possible before this stage. In practice, we almost always know what the best polynomial is, the problem lies in proving it. In particular, in almost all cases, the use of bounds as described in this paper is not sufficient to determine the maximal exponent of the factor $X(1-X)$. Further work on this part should help.

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