Multifractional Brownian Motion: Definition and Preliminary Results

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INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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N° 2645

Août 1995

PROGRAMME 5

Rapport de recherche

ISSN 0249-6399

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Programme 5 — Traitement du signal, automatique et productique
Projet Fractales

Rapport de recherche n° 2645 — Août 1995 — 39 pages

Abstract: We generalize the definition of the fractional Brownian motion of exponent H to the case where H is no longer a constant, but a function of the time index of the process. This allows us to model non stationary continuous processes, and we show that $H(t)$ and $2 - H(t)$ are indeed respectively the local Hölder exponent and the local box and Hausdorff dimension at point t. Finally, we propose a simulation method and an estimation procedure for $H(t)$ for our model.

Key-words: fractional Brownian motion, fractal, Hölder exponent, Hausdorff dimension, Gaussian process, continuous process.

(Résumé : tsvp)

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INRIA ROCQUENCOURT

Mouvement Brownien multifractionnaire: définition et résultats préliminaires

Résumé : Nous généralisons la définition du mouvement Brownien fractionnaire de paramètre H au cas où H n'est plus une constante mais une fonction de l'index de temps du processus. Cela nous permet de modéliser des processus continus non stationnaires, et nous montrons que $H(t)$ et $2 - H(t)$ sont en effet respectivement l'exposant de Hölder et les dimensions de boîtes et d'Hausdorff locales au point t. Enfin, nous proposons une méthode de simulation et une procédure d'estimation de $H(t)$ pour notre modèle.

Mots-clé : mouvement Brownien fractionnaire, fractale, exposant d'Hölder, dimension d'Hausdorff, processus Gaussien, processus continu.

1. Introduction

The fractional Brownian motion (fBm) of index H $(0 < H < 1)$ was defined by Mandelbrot and Van Ness (1968) as the stochastic integral, for $t \geq 0$

$$B_H(t) = \frac{1}{\Gamma(H+1/2)} \left\{ \int_{-\infty}^{0} [(t-s)^{H-1/2} - (-s)^{H-1/2}] dW(s) + \int_0^t (t-s)^{H-1/2} dW(s) \right\}.$$

Where W denotes a Wiener process defined on $(-\infty, \infty)$.
They showed that this process has self-similar and stationary increments and continuous sample paths with probability one. It may be shown (see e.g. Mandelbrot *et al* (1968) p. 425) that with probability one, its graph has Hausdorff and box dimension equal to $2 - H$.

In this paper, we define the *multifractional Brownian motion* (mBm) which generalizes the fBm with $t \in [0, \infty)$ by substituting to the parameter H a Hölder function $H(t)$, such that $0 < H(t) < 1$.

We believe that this new process does provide a useful model for a host of continuous and non stationary natural signals. It is well known that fBm's are self-similar processes[1], allowing to conveniently describe irregular signals which arise in many situations (see e.g. Mandelbrot *et al* (1968)). However, the pointwise irregularity of an fBm is the same all along its path. This last property is sometimes undesirable, since it restricts the field of application. For instance, fBm have frequently been used for synthesizing artificial mountains (see e.g. Voss R.F. (1985)). Such a modeling assumes that the irregularity of a mountain is everywhere the same. It appears that in reality this assumption is too strong because in particular, one does not take into account erosion phenomena. Consequently, it should be convenient to relax the constraint of stationarity and be instead able to control the local irregularity. The model proposed in this work seems to be the simplest generalization of fBm that fulfills this requirement.
The remainder of our paper is organized as follows. In Section 2, we define the process as a stochastic integral and we show that it is continuous with probability one. We then prove a homotopy property of the fBm. In Section 3, we study the local Hölder properties and show that with probability one, the sample process has at each point $t_0 \geq 0$ local *box and Hausdorff dimension* equal to $2 - H_{t_0}$ and *punctual Hölder exponent* equal to H_{t_0} (see definitions in Section 3). In Section 4, we propose a simulation procedure and describe a natural method for estimating the function $H(t)$. We conclude this section with some experimental results.

2. Definition of multifractional Brownian motion

Before introducing the mBm, we set the following definition of a Hölder function:

Definition 1 (Hölder function of exponent β).
Let (X, d_X) and (Y, d_Y) be two metric spaces. A function $f : X \to Y$ is called a Hölder function of exponent $\beta > 0$, *if for each $x, y \in X$ such that $d_X(x, y) < 1$ we have:*

$$d_Y(f(x), f(y)) \leq c.d_X(x, y)^\beta$$

[1] to be quite rigorous, these processes are rather self-affine than self-similar, but this last term is more widely used.

for some constant $c > 0$.

Definition 2 (Multifractional Brownian Motion).
Let $H : [0, \infty) \to (0, 1)$ be a Hölder function of exponent $\beta > 0$. For $t \geq 0$ the following random function, denoted by $W_H(t)$ or $W_{H_t}(t)$, is called reduced ***multifractional Brownian motion*** *with functional parameter H:*

$$W_{H_t}(t) = \frac{1}{\Gamma(H_t + 1/2)} \left\{ \int_{-\infty}^{0} [(t-s)^{H_t - 1/2} - (-s)^{H_t - 1/2}] dW(s) + \int_0^t (t-s)^{H_t - 1/2} dW(s) \right\},$$

where W denotes the ordinary Brownian motion and the integration is taken in the mean square sense.

This natural extension of fBm results in some sense in a "loss" of properties: the increments of mBm indeed are non stationary and the process is no more self-similar. However, we show that the assumption that H is a Hölder function entails the continuity of the mBm. Note that this condition is a sufficient one, which might be relaxed to include more general functions.

Before proving the continuity of the mBm, we recall some well known results:

(1) **Fact.** *Let X denote a random variable following an $N(0, \sigma^2)$ law. Then for any $r > 0$ we have (see e.g. Papoulis (1991) p. 110):*

$$E[|X|^r] = \frac{2^{r/2}\Gamma\left(\frac{r+1}{2}\right)}{\Gamma(\frac{1}{2})}\sigma^r. \tag{2.0}$$

(2) **Theorem [Kolmogorov criterion]**(see e.g. Wong *et al* (1985) p. 57).
Let X_t, $t \in T$ be a separable process and let T be a finite interval. If there exist strictly positive constants α, $\mathcal{B}$, C such that $E|X_{t+h} - X_t|^\alpha \leq Ch^{1+\mathcal{B}}$ then

$$\sup_{\substack{t,s \in T \\ |t-s|<h}} |X_t - X_s| \xrightarrow[h \to 0]{a.s.} 0.$$

The assumption of separability of the process is not strong, because if T indeed is a real interval then every process may be assumed separable (see e.g. Adler (1981) p. 15).

(3) **Theorem [Isometry property]** (see e.g. Wong *et al* (1985) p. 144).
Let $\{W_t, \mathcal{A}_t\}$ be a Brownian motion and let $\phi(\omega, t)$ and $\psi(\omega, t)$ be random functions jointly measurable in (ω, t) (with respect to $\mathcal{A}$ in ω and the Lebesgue measure in t) and such that for any $(a, b) \in \mathbb{R}^2$, $a < b$, we have $\int_a^b E(|\phi_t|^2)dt < \infty$ and $\int_a^b E(|\psi_t|^2)dt < \infty$. Then

$$E[\int_a^b \phi_t dW(t). \int_a^b \psi_t dW(t)] = \int_a^b E(\phi_t \psi_t) dt.$$

Proposition 3. *With probability one, $W_{H_t}(t)$, $(0 \le t < \infty)$ is a continuous function of t.*

Proof.

For convenience, we introduce the following notation.
For each $t \ge 0$ set $W_{H_t}(t) = \frac{1}{\Gamma(H_t)}\{P_1(t) + P_2(t)\}$ where $P_1(t) = \int_{-\infty}^0 f(s,t)dW(s)$, $P_2(t) = \int_0^t g(s,t)dW(s)$. For each $s < t$, let $g(s,t) = (t-s)^{H_t - \frac{1}{2}}$ and for each $t > 0$, $s < 0$, $f(s,t) = g(s,t) - (-s)^{H_t - \frac{1}{2}}$.
We prove that $P_1(t)$ and $P_2(t)$ are continuous functions at any $t \ge 0$.

Step 1. We show that, for $0 \le t < \infty$, $P_1(t)$ has almost all paths continuous.
It is sufficient to show the continuity on each compact interval $[a,b] \subset [0,\infty)$ with $b - a < 1$. For the sake of notational simplicity, we assume that $a \le t < t' \le b$ and $0 < H_t < H_{t'} < 1$, and we set $N = P_1(t') - P_1(t)$. N is a Gaussian random variable. Then we know, by the isometry property, that $N \sim N(0, \int_{-\infty}^0 (f(s,t') - f(s,t))^2 ds)$.
Let $h_1(u) = (u-s)^{H_{t'} - \frac{1}{2}}$ and $h_2(v) = (t-s)^{v-\frac{1}{2}} - (-s)^{v-\frac{1}{2}}$. Then:
$f(s,t') - f(s,t) = [h_1(t') - h_1(t)] + [h_2(H_{t'}) - h_2(H_t)]$.

We apply the theorem on finite increments to the functions h_1 (for $u \in [t,t']$) and h_2 (for $v \in [H_{t'}, H_t]$). We obtain the following inequality (we have used the following obvious fact $(a+b)^2 \le 2(a^2+b^2)$):

$$(f(s,t') - f(s,t))^2 \le 2[\{(t'-t)^2 (H_{t'} - \frac{1}{2})^2 (\tau - s)^{2H_{t'} - 3}\} + \{(H_{t'} - H_t)^2 \left(\log(t-s)(t-s)^{\iota - \frac{1}{2}} - \log(-s)(-s)^{\iota - \frac{1}{2}}\right)^2\}], \tag{2.1}$$

with $\tau \in (t,t')$ and $\iota \in (H_t, H_{t'})$.

We now check the Kolmogorov criterion for $P_1(t)$, $(t \in [a,b])$. We proceed in three steps:

- We have $2[(t'-t)^2 (H_{t'} - \frac{1}{2})^2 (\tau - s)^{2H_{t'} - 3}] \le \frac{1}{2}(t'-t)^2 (t-s)^{2H_{t'} - 3}$, therefore there exists $\sigma_1 > 0$ such that

$$\int_{-\infty}^0 2[(t'-t)^2 (H_{t'} - \frac{1}{2})^2 (\tau - s)^{2H_{t'} - 3}] ds \le \sigma_1^2 (t'-t)^2 < \infty. \tag{2.2}$$

- By definition of the function H we have the inequality

$$(H_{t'} - H_t)^2 \le c|t'-t|^{2\beta}, \qquad c > 0, \tag{2.3}$$

which will be useful in the sequel.

We now show that the following inequality holds:

$$\sup_{t,\iota} \int_{-\infty}^0 \left(\log(t-s)(t-s)^{\iota - \frac{1}{2}} - \log(-s)(-s)^{\iota - \frac{1}{2}}\right)^2 ds < \infty. \tag{2.4}$$

The Hölder function H is continuous on $[a,b]$, therefore there exist $\mu,\nu \in (0,1)$ such that for any $t \in [a,b]$ we have $H_t \in [\mu,\nu]$.

- In the first place, we show that

$$\sup_{t,\iota} \int_{-1}^{0} \Big(\log(t-s)(t-s)^{\iota-\frac{1}{2}} - \log(-s)(-s)^{\iota-\frac{1}{2}}\Big)^2 ds < \infty. \tag{2.5}$$

We have

$$\begin{aligned}
&\int_{-1}^{0} \Big(\log(t-s)(t-s)^{\iota-\frac{1}{2}} - \log(-s)(-s)^{\iota-\frac{1}{2}}\Big)^2 ds \\
&\leq 2\Big[\int_{-1}^{0} (\log(t-s))^2 (t-s)^{2\iota-1} ds + \int_{-1}^{0} (\log(-s))^2(-s)^{2\iota-1} ds\Big] \\
&\leq 2\Big[\int_{-1}^{0} (\log(b-s))^2 (t-s)^{2\iota-1} ds + \int_{-1}^{0} (\log(-s))^2(-s)^{2\mu-1} ds\Big] \\
&\leq 2(\log(b+1))^2 \frac{(b+1)^{2\nu}-a}{2\mu} + 2\int_{-1}^{0} (\log(-s))^2(-s)^{2\mu-1} ds < \infty,
\end{aligned}$$

which entails (2.5).

- In the second place, we show that

$$\sup_{t,\iota} \int_{-\infty}^{-1} \Big(\log(t-s)(t-s)^{\iota-\frac{1}{2}} - \log(-s)(-s)^{\iota-\frac{1}{2}}\Big)^2 ds < \infty. \tag{2.6}$$

For $s < -1$, the theorem on finite increments applied to the function $\tau \longmapsto \log(\tau-s)(\tau-s)^{\iota-\frac{1}{2}} - \log(-s)(-s)^{\iota-\frac{1}{2}}$ defined on $[0,t]$ with $t \in [a,b]$, gives

$$\begin{aligned}
&\Big(\log(t-s)(t-s)^{\iota-\frac{1}{2}} - \log(-s)(-s)^{\iota-\frac{1}{2}}\Big)^2 \leq \Big(b[1+\log(b-s)(\iota-\frac{1}{2})](-s)^{\iota-\frac{3}{2}}\Big)^2 \\
&\leq 2b^2[1+(\nu-\frac{1}{2})^2(\log(b-s))^2](-s)^{2\nu-3},
\end{aligned}$$

and the last member is integrable on $(-\infty,-1]$. This proves (2.6).

Combining (2.5) and (2.6), we obtain (2.4).

- By combining the above statements (2.1), (2.2), (2.3) and (2.4), we infer that there exists $\gamma_1 > 0$ such that $Var(N) \leq \gamma_1 |t'-t|^{2\beta}$.
From (2.0) we get that for any $\alpha > 0$, there exists $\gamma_2 > 0$ such that $E(|P_1(t') - P_1(t)|^{\alpha}) \leq \gamma_2 |t'-t|^{\beta\alpha}$. Taking $\alpha > \frac{1}{\beta}$ and using Kolmogorov criterion, we complete Step 1.

Step 2. Next, for $0 \leq t < \infty$, we show that $P_2(t)$ has almost all paths continuous.
As above, it is sufficient to show the continuity on each compact interval $[a,b] \subset [0,\infty)$ with $b-a<1$.
For the sake of notational simplicity, we set $a \leq t < t' \leq b$ and $0 < H_t < H_{t'} < 1$ and with the notation of Step 1, we assume that $H_t \in [\mu,\nu] \subset (0,1)$.
For convenience, we introduce the following notation:
$N_1 = \int_0^{t'} [(t'-s)^{H_{t'}-\frac{1}{2}} - (t'-s)^{H_t-\frac{1}{2}}] dW(s)$; $N_2 = \int_t^{t'} (t'-s)^{H_t-\frac{1}{2}} dW(s)$;
$N_3 = \int_0^t [(t'-s)^{H_t-\frac{1}{2}} - (t-s)^{H_t-\frac{1}{2}}] dW(s)$.
We have the following relation:

$E(|P_2(t') - P_2(t)|^2) = E(|N_1 + N_2 + N_3|^2) \leq 4(E(|N_1|^2) + E(|N_2|^2) + E(|N_3|^2)).$

We now check the Kolmogorov criterion for $P_2(t)$, for $t \in [a, b]$.

- We show that there exists $c_1 > 0$ independent of t and t' such that

$$E(|N_1|^2) \leq c_1 |t' - t|^{2\beta}. \tag{2.7}$$

By the theorem on finite increments applied to the function, defined on $[H_t, H_{t'}]$, $h \longmapsto (t' - s)^{h - \frac{1}{2}}$, there exists $\tau_s \in (H_t, H_{t'})$ such that
$(t' - s)^{H_{t'} - \frac{1}{2}} - (t' - s)^{H_t - \frac{1}{2}} = (H_{t'} - H_t) \log(t' - s)(t' - s)^{\tau_s - \frac{1}{2}}$.
Using the isometry property,

$$\begin{aligned} E(|N_1|^2) &= \int_0^{t'} [(H_{t'} - H_t) \log(t' - s)(t' - s)^{\tau_s - \frac{1}{2}}]^2 ds \qquad \text{(a)} \\ &= (H_{t'} - H_t)^2 \int_0^{t'} (\log(t' - s))^2 (t' - s)^{2\tau_s - 1} ds. \end{aligned}$$

Using the Hölder property of H_t,

$$(H_{t'} - H_t)^2 \leq c^2 (t' - t)^{2\beta}. \tag{b}$$

For $0 < s < t'$ the function $\tau \longmapsto (\log(t' - s))^2 (t' - s)^{2\tau - 1}$ is clearly positive and monotonic on $[\mu, \nu]$.
Thus:

$$\begin{aligned} \int_0^{t'} (\log(t' - s))^2 (t' - s)^{2\tau_s - 1} ds &= \int_0^{t'} [(\log u)^2 u^{2\mu - 1} + (\log u)^2 u^{2\nu - 1}] du \\ &\leq \int_0^{b} [(\log u)^2 u^{2\mu - 1} + (\log u)^2 u^{2\nu - 1}] du < \infty \end{aligned}$$

which combined with (a) and (b) yields (2.7).

- We show that there exists $c_2 > 0$ such that

$$E(|N_2|^2) \leq c_2 |t' - t|^{2\mu}. \tag{2.8}$$

Indeed we have

$$E(|N_2|^2) = \int_t^{t'} (t' - s)^{2H_t - 1} ds = \frac{(t' - t)^{2H_t}}{2H_t},$$

which entails (2.8).

- We show that there exists $c_3 > 0$ such that

$$E(|N_3|^2) \leq c_3 |t' - t|^{\min\{1, 2\mu\}}. \tag{2.9}$$

Let for convenience, $h(s) = ((t'-s)^{H_t-\frac{1}{2}} - (t-s)^{H_t-\frac{1}{2}})^2$, for $0 \le s < t$, . We have the following equalities:

$$\begin{aligned}
\int_0^t h(s)ds &= \int_0^t ((t-s)^{H_t-\frac{1}{2}} - (t'-s)^{H_t-\frac{1}{2}})^2 ds \\
&= \int_0^t (t-s)^{2H_t-1} + (t'-s)^{2H_t-1} - 2((t-s)^{H_t-\frac{1}{2}}(t'-s)^{H_t-\frac{1}{2}} ds \\
&= \frac{t^{2H_t}}{2H_t} + \frac{(t')^{2H_t}}{2H_t} - \frac{(t'-t)^{2H_t}}{2H_t} - 2\int_0^t ((t-s)^{H_t-\frac{1}{2}}(t'-s)^{H_t-\frac{1}{2}} ds.
\end{aligned}$$

We study three cases:

– **Case 1**. $H_t < \frac{1}{2}$. We have the following inequalities:

$$\begin{aligned}
\int_0^t h(s)ds &\le \frac{t^{2H_t}}{2H_t} + \frac{(t')^{2H_t}}{2H_t} - \frac{(t'-t)^{2H_t}}{2H_t} - 2\int_0^t (t'-s)^{2H_t-1} ds \\
&\le \frac{t^{2H_t} - (t')^{2H_t}}{2H_t} + \frac{(t'-t)^{2H_t}}{2H_t} \\
&\le \frac{(t'-t)^{2H_t}}{2H_t}.
\end{aligned}$$

Therefore there exists $c_{3.1} > 0$ such that

$$E(|N_3|^2) = \int_0^t h(s)ds \le c_{3.1}|t'-t|^{2\mu}.$$

– **Case 2**. $H_t > \frac{1}{2}$.

$$\begin{aligned}
\int_0^t h(s)ds &\le \frac{t^{2H_t}}{2H_t} + \frac{(t')^{2H_t}}{2H_t} - \frac{(t'-t)^{2H_t}}{2H_t} - 2\int_0^t (t-s)^{2H_t-1} ds \\
&\le \frac{(t')^{2H_t} - t^{2H_t}}{2H_t} - \frac{(t'-t)^{2H_t}}{2H_t} \\
&\le \frac{(t'-t+t)^{2H_t} - t^{2H_t}}{2H_t} \\
&= \frac{t^{2H_t}[(1+(t'-t)/t)^{2H_t} - 1]}{2H_t}
\end{aligned}$$

Therefore there exists $c_{3.2} > 0$ such that

$$E(|N_3|^2) = \int_0^t h(s)ds \le c_{3.2}|t'-t|.$$

– **Case 3**. $H_t = \frac{1}{2}$. In this case (2.9) is obvious.

- Combining (2.7), (2.8), (2.9), we obtain: there exists $\gamma_1 > 0$ such that $E[(P_2(t') - P_2(t))^2] \le \gamma_1 |t'-t|^{2min(1/2,\beta,\mu)}$.
  From (2.0) we get that for any $\alpha > 0$, there exists $\gamma_2 > 0$ such that $E(|P_2(t') - P_2(t)|^\alpha) \le \gamma_2 |t'-t|^{min(1/2,\beta,\mu)\alpha}$. This inequality, when combined with Kolmogorov criterion and $\alpha > \frac{1}{min(1/2,\beta,\mu)}$, completes the proof of Proposition 3.

□

A consequence of the proof of the previous proposition is the following one. Let us consider the function:

$$\begin{aligned}
B : (0,1) \times [0,K] &\longrightarrow \mathbb{R} \\
(H,t) &\longmapsto B_H(t)
\end{aligned}$$

for $K>0$. The following result essentially proves the continuity of B w.r.t. H uniformly in t.

Theorem 4. *Let $(B_H(t))_{t\geq 0}$ be an fBm of index H $(0<H<1)$. Then for any interval $[a,b]\subset(0,1)$ and $K>0$ we have almost surely*

$$\lim_{h\to 0}\ \sup_{\substack{a\leq H,H'\leq b\\ |H'-H|<h}}\ \sup_{t\in[0,K]}|B_H(t)-B_{H'}(t)|=0.$$

Proof. We consider the process defined for $t\geq 0$ by $D_{H,H'}(t)=B_{H'}(t)-B_H(t)$ which is obviously continuous. We set for each $H\in[a,b]$ and $0<h<1$,

$$\begin{aligned}
\sigma_{H,H'}(t) &= \sqrt{Var[D_{H,H'}(t)]},\\
A_h(H) &= \sup_{\substack{a\leq H'\leq b\\ |H'-H|<h}}\ \sup_{t\in[0,K]}|D_{H,H'}(t)|, \qquad A_h=\sup_{a\leq H\leq b}A_h(H),\\
f_H(s,t) &= (t-s)^{H-1/2}-(-s)^{H-1/2}, \qquad g_H(s,t)=(t-s)^{H-1/2},\\
F(s,t) &= f_{H'}(s,t)-f_H(s,t) \text{ and } G(s,t)=g_{H'}(s,t)-g_H(s,t).
\end{aligned}$$

We will make use of the following five steps.
Step 1. We show that there exists $C>0$ such that

$$\sup_{t\in[0,K]}Var(D_{H,H'}(t))\leq C|H'-H|^2. \tag{2.10}$$

We have

$$\begin{aligned}
&Var(D_{H,H'}(t)) = E[(B_{H'}(t)-B_H(t))^2]\\
&= E\left[\left(\frac{\int_{-\infty}^0 f_{H'}(s,t)dW(s)+\int_0^t g_{H'}(s,t)dW(s)}{\Gamma[H'+1/2]}-\frac{\int_{-\infty}^0 f_H(s,t)dW(s)+\int_0^t g_H(s,t)dW(s)}{\Gamma[H+1/2]}\right)^2\right]\\
&\leq 2E\left[\left(\frac{\int_{-\infty}^0 F(s,t)dW(s)+\int_0^t G(s,t)dW(s)}{\Gamma[H'+1/2]}\right)^2\right]+2\left(\frac{\Gamma(H+1/2)}{\Gamma(H'+1/2)}-1\right)^2E((B_H(t))^2)\\
&\leq \frac{4}{(\Gamma[H'+1/2])^2}\left\{\int_{-\infty}^0(F(s,t))^2ds+\int_0^t(G(s,t))^2ds\right\}+2V_Ht^{2H}\left(\frac{\Gamma(H+1/2)}{\Gamma(H'+1/2)}-1\right)^2,
\end{aligned}$$

where V_H is defined in Mandelbrot *et al* (1968) p. 425, by
$V_H=[\Gamma(H+1/2)]^{-2}\{\int_{-\infty}^0[(1-s)^{H+1/2}-(-s)^{H+1/2}]^2ds+\frac{1}{2H}\}$. $H\longmapsto\Gamma_H$ is clearly continuous. Therefore there exists $C_0>0$ such that for $H,H'\in[a,b]$,

$$Var(D_{H,H'}(t))\leq C_0\left\{\int_{-\infty}^0(F(s,t))^2ds+\int_0^t(G(s,t))^2ds+\left(\frac{\Gamma(H+1/2)}{\Gamma(H'+1/2)}-1\right)^2\right\}.$$

We study separately the three terms of the above inequality

(1) The function $\Gamma(x)$ has continuous derivatives of all orders for $x>0$ (see e.g. Fikhtengol'ts (1965) (II) p. 171). Thus, the theorem on finite increments entails that there exists $C_1>0$ such that :

$$C_0\left(\frac{\Gamma(H+1/2)}{\Gamma(H'+1/2)}-1\right)^2\leq C_1(H'-H)^2. \tag{2.11}$$

(2) We use again the theorem on finite increments applied to the function $\tau \longmapsto (t-s)^{\tau-1/2}$, for $\iota \in [H,H']$. There exists $\iota_{s,t} \in (H,H')$ such that

$$\begin{aligned}\int_0^t (G(s,t))^2 ds &= (H'-H)^2 \int_0^t [\log(t-s)]^2 (t-s)^{2\iota_{s,t}-1} ds \\ &\leq (H'-H)^2 [\int_0^1 [\log u]^2 u^{2\iota_{s,t}-1} du + \int_1^{\max\{1,t\}} [\log u]^2 u^{2\iota_{s,t}-1} du] \\ &\leq (H'-H)^2 [\int_0^1 [\log u]^2 u^{2a-1} du + \int_1^{\max\{1,t\}} [\log u]^2 u^{2b-1} du].\end{aligned}$$

Let $C_2 = \{\int_0^1 [\log u]^2 u^{2a-1} du + \int_1^{\max\{1,K\}} [\log u]^2 u^{2b-1} du\}$. Then we get

$$\sup_{t\in[0,K]} \left\{ \int_0^t (G(s,t))^2 ds \right\} \leq C_2 (H'-H)^2. \tag{2.12}$$

The proof of the last inequality is longer. We split it into two claims.

(3) We show that there exists $C_3 > 0$ such that

$$\sup_{t\in[0,K]} \left\{ \int_{-\infty}^0 (F(s,t))^2 ds \right\} \leq C_3 (H'-H)^2. \tag{2.13}$$

Making use of the theorem on finite increments for the function $\iota \longmapsto (t-s)^{\iota-1/2} - (-s)^{\iota-1/2}$, for $\iota \in [H,H']$, there exists $\iota_{s,t} \in (H,H')$ such that

$$\begin{aligned}\int_{-\infty}^0 (F(s,t))^2 ds &= (H'-H)^2 \int_{-\infty}^0 (\log(t-s)(t-s)^{\iota_{s,t}-1/2} - \log(-s)(-s)^{\iota_{s,t}-1/2})^2 ds \\ &\leq 2(H'-H)^2 \int_{-\infty}^0 (\log(t-s))^2 ((t-s)^{\iota_{s,t}-1/2} - (-s)^{\iota_{s,t}-1/2})^2 ds + \\ &2(H'-H)^2 \int_{-\infty}^0 (\log(t-s) - \log(-s))^2 (-s)^{2\iota_{s,t}-1} ds.\end{aligned}$$

It is thus enough to prove that the two integrals above can be bounded by constants independent of t, H and H'.

- Let us show first that

$$\sup_{H,H'\in[a,b]} \sup_{t\in[0,K]} \int_{-\infty}^0 (\log(t-s))^2 ((t-s)^{\iota_{s,t}-1/2} - (-s)^{\iota_{s,t}-1/2})^2 ds < \infty. \tag{a}$$

We assume first that $K \geq 1$ and $t \in [1,K]$. We use the theorem on finite increments applied to the function $\tau \longmapsto (\tau - s)^{\iota_{s,t}-1/2}$, for $\tau \in [0,t]$ and

$s \in (-\infty, 0)$.

$$\int_{-\infty}^{0} (\log(t-s))^2((t-s)^{\iota_{s,t}-1/2} - (-s)^{\iota_{s,t}-1/2})^2 ds$$
$$\leq \int_{-\infty}^{-1} (\log(K-s))^2((t-s)^{\iota_{s,t}-1/2} - (-s)^{\iota_{s,t}-1/2})^2 ds +$$
$$\int_{-1}^{0} (\log(K-s))^2((t-s)^{\iota_{s,t}-1/2} - (-s)^{\iota_{s,t}-1/2})^2 ds$$
$$\leq \int_{-\infty}^{-1} (\log(K-s))^2 t^2(\iota_{s,t}-1/2)^2(\tau_{s,t}-s)^{2\iota_{s,t}-3} ds +$$
$$\int_{-1}^{0} 2(\log(K-s))^2((t-s)^{2\iota_{s,t}-1} + (-s)^{2\iota_{s,t}-1}) ds$$
$$\leq \int_{-\infty}^{-1} (\log(K-s))^2(K/2)^2(-s)^{2b-3} ds + 2(\log(1+K))^2 \times$$
$$\int_{-1}^{0} ((1-s)^{2a-1} + (1-s)^{2b-1} + (K-s)^{2a-1} + (K-s)^{2b-1} + (-s)^{2a-1} + (-s)^{2b-1}) ds < \infty.$$

We now assume that $K \geq 1$ and $t \in [0,1]$, or $K < 1$ and $t \in [0,K]$.

$$\int_{-\infty}^{0} (\log(t-s))^2((t-s)^{\iota_{s,t}-1/2} - (-s)^{\iota_{s,t}-1/2})^2 ds$$
$$\leq \int_{-\infty}^{-1} (\log(K-s))^2((t-s)^{\iota_{s,t}-1/2} - (-s)^{\iota_{s,t}-1/2})^2 ds +$$
$$\int_{-1}^{t-1} (\log(K-s))^2((t-s)^{\iota_{s,t}-1/2} - (-s)^{\iota_{s,t}-1/2})^2 ds +$$
$$\int_{t-1}^{0} (\log(-s))^2((t-s)^{\iota_{s,t}-1/2} - (-s)^{\iota_{s,t}-1/2})^2 ds.$$

With the above notation we get

$$\int_{-\infty}^{0} (\log(t-s))^2((t-s)^{\iota_{s,t}-1/2} - (-s)^{\iota_{s,t}-1/2})^2 ds$$
$$\leq \int_{-\infty}^{-1} (\log(K-s))^2 t^2(\iota_{s,t}-1/2)^2(\tau_{s,t}-s)^{2\iota_{s,t}-3} ds +$$
$$\int_{-1}^{t-1} 2(\log(K-s))^2((t-s)^{2\iota_{s,t}-1} + (-s)^{2\iota_{s,t}-1}) ds +$$
$$\int_{t-1}^{0} 2(\log(-s))^2((t-s)^{2\iota_{s,t}-1} + (-s)^{2\iota_{s,t}-1}) ds$$
$$\leq \int_{-\infty}^{-1} (\log(K-s))^2(K/2)^2(-s)^{2b-3} ds +$$
$$2(\log(1+K))^2 \int_{-1}^{0} [(1-s)^{2a-1} + (1-s)^{2b-1} + 2((-s)^{2a-1} + (-s)^{2b-1})] ds +$$
$$\int_{-1}^{0} 2(\log(-s))^2((1-s)^{2a-1} + (1-s)^{2b-1} + 2((-s)^{2a-1} + (-s)^{2b-1})) ds < \infty,$$

which ends the proof of (a).

- We now show that

$$\sup_{H,H' \in [a,b]} \sup_{t \in [0,K]} \{\int_{-\infty}^{0} (\log(t-s) - \log(-s))^2(-s)^{2\iota_{s,t}-1} ds\} < \infty. \quad \text{(b)}$$

We use the theorem on finite increments applied to the function $\tau \longmapsto \log(\tau - s)$, for $\tau \in [0,t]$ and $s \in (-\infty, -1]$.

$$\begin{aligned}
&\int_{-\infty}^{0} [\log(t-s) - \log(-s)]^2 (-s)^{2\iota_{s,t}-1} ds \\
&\leq \int_{-\infty}^{-1} [\log(t-s) - \log(-s)]^2 (-s)^{2b-1} ds + \int_{-1}^{0} [\log(t-s) - \log(-s)]^2 (-s)^{2a-1} ds \\
&\leq \int_{-\infty}^{-1} [t/(\tau_{s,t} - s)]^2 (-s)^{2b-1} ds + \int_{-1}^{0} [\log(K-s) - \log(-s)]^2 (-s)^{2a-1} ds \\
&\leq K^2 \int_{-\infty}^{-1} (-s)^{2b-3} ds + \int_{-1}^{0} [\log(K-s) - \log(-s)]^2 (-s)^{2a-1} ds < \infty,
\end{aligned}$$

which ends the proof of (b).

Therefore (2.13) holds.

Finally, (2.10) results from combining (2.11), (2.12) and (2.13). Therefore Step 1 is completed.

Step 2. We recall the following theorem (see e.g. Adler (1990) p. 43) which will be useful in the sequel.

Theorem: *Let $\{X_t\}_{t\in T}$ be a centered Gaussian process with sample paths bounded a.s. Let $||X|| = \sup_{t\in T} X_t$ and $\sigma_T^2 = \sup_{t\in T} E(X_t^2)$. Then $E||X|| < \infty$, and for all $u > 0$,*

$$P\{|\ ||X|| - E||X||\ | > u\} \leq 2e^{-\frac{1}{2}u^2/\sigma_T^2}, \tag{2.14}$$

and for all $u > E||X||$,

$$P\{||X|| > u\} \leq 2e^{-\frac{1}{2}(u - E||X||)^2/\sigma_T^2}. \tag{2.15}$$

With the assumption of the theorem above, let $|||X||| = \sup_{t\in T} |X_t|$. It follows from (2.15) that for all $u > E||X||$,

$$P\{|||X||| > u\} \leq 4e^{-\frac{1}{2}(u - E||X||)^2/\sigma_T^2}. \tag{2.16}$$

We will prove that

$$0 \leq \sup_{0<H,H'<1} E\{\sup_{t\in[0,K]} D_{H,H'}(t)\} < \infty. \tag{2.17}$$

$D_{H,H'}(0) = 0$ yields the first inequality. To prove the second inequality, we will use the above theorem with $T = [0,K]$ and $X_t = D_{H,H'}(t)$. Integrating by parts and using (2.16) yields:

$$\begin{aligned}
E||X|| &= \int_0^\infty P(||X|| > u) du \\
&\leq 2 \int_0^\infty e^{-\frac{(u - E||X||)^2}{2\sigma_T^2}} du \\
&= 2\sigma_T \int_{-E||X||/\sigma_T}^{\infty} e^{-\frac{v^2}{2}} dv \\
&\leq 2\sqrt{2\pi}\sigma_T,
\end{aligned}$$

which, combined with (2.10), yields (we denote $\kappa = 2\sqrt{2\pi}$):

$$E||X|| \leq \kappa\sqrt{C}|H' - H|. \tag{2.18}$$

This inequality entails (2.17).

Step 3. With the notation defined at the beginning of the proof of Theorem 4, we will prove for any $\epsilon > 0$ there exists $\mathcal{C} > 0$ such that

$$P\left\{A_h \geq \nu h\right\} \leq \frac{\mathcal{C}}{h} e^{-\frac{(\nu - \sqrt{C}\kappa)^2}{2C+\epsilon}} \tag{2.19}$$

holds for every $\nu > 0$ and $h < 1$ where $\kappa = 2\sqrt{2\pi}$.

Set $0 < H < 1$ and let r be an integer. Let $\lfloor x \rfloor$ denote the integer part of x and let $H_r = \frac{\lfloor 2^r H \rfloor}{2^r} = \sum_{j=1}^{r} \frac{\epsilon_j(H)}{2^j}$ ($\epsilon_j(H) = 0, 1$; $j = 1, 2, \ldots$ and $\epsilon_j(H)$ should not be identically 1 from some j on). For each H, H', r, t fixed, we have almost surely

$$\begin{aligned}
|B_{H'}(t) - B_H(t)| &\leq |B_{H'_r}(t) - B_{H_r}(t)| + |B_{H'}(t) - B_{H'_r}(t)| + |B_{H_r}(t) - B_H(t)| \\
&\leq |B_{H'_r}(t) - B_{H_r}(t)| + \sum_{j=0}^{\infty} |B_{H'_{r+j+1}}(t) - B_{H'_{r+j}}(t)| + \sum_{j=0}^{\infty} |B_{H_{r+j+1}}(t) - B_{H_{r+j}}(t)| \\
&\leq \sup_{t\in[0,K]} |B_{H'_r}(t) - B_{H_r}(t)| + \sum_{j=0}^{\infty} \sup_{t\in[0,K]} |B_{H'_{r+j+1}}(t) - B_{H'_{r+j}}(t)| + \\
&\quad \sum_{j=0}^{\infty} \sup_{t\in[0,K]} |B_{H_{r+j+1}}(t) - B_{H_{r+j}}(t)|.
\end{aligned}$$

Therefore,

$$\sup_{t\in[0,K]} |D_{H,H'}(t)| \leq \sup_{t\in[0,K]} |D_{H_r,H'_r}(t)| + \sum_{j=0}^{\infty} \sup_{t\in[0,K]} |D_{H_{r+j+1},H_{r+j}}(t)| + \sum_{j=0}^{\infty} \sup_{t\in[0,K]} |D_{H_{r+j+1},H_{r+j}}(t)|. \tag{2.20}$$

The sequel is an adaptation of the proofs of the lemma and theorem in Csörgö and Révész (1981) p. 24-26.

From Step 1, $D_{H'_r,H_r}(t)$ follows a normal law of mean zero and variance lower than or equal to $C|H'_r - H_r|^2$. Write $R = 2^r$. It is easy to show that: $\sup_{0<|H'-H|<h} |H'_r - H_r| \leq h + R^{-1}$. Thus, for any sufficiently large positive u, x_j and $0 < h < 1$, and for any integer r, j, we have by (2.16) and (2.18) of Step 2:

$$\begin{aligned}
&P\left\{ \sup_{\substack{a\leq H_r,H'_r\leq b \\ |H'_r - H_r|<h}} \sup_{t\in[0,K]} |D_{H_r,H'_r}(t)| \geq (u+\kappa)\sqrt{C}(h + R^{-1}) \right\} \\
&\leq P\left\{ \sup_{\substack{a\leq H_r,H'_r\leq b \\ |H'_r - H_r|<h}} \sup_{t\in[0,K]} |D_{H_r,H'_r}(t)| \geq u \sup_{t\in[0,K]} \sigma_{H'_r,H_r}(t) + E[\sup_{t\in[0,K]} D_{H_r,H'_r}(t)] \right\} \\
&\leq 4e^{-\frac{u^2}{2}} \times \frac{R}{Rh+1}.
\end{aligned}$$

And since $\sup_{0<|H'-H|<h} |H'_{r+j+1} - H'_{r+j}| \le 2^{-(r+j+1)}$, we have

$$P\left\{ \sup_{\substack{a\le H_r, H'_r \le b \\ |H'_r - H_r|<h}} \sup_{t\in[0,K]} |D_{H'_{r+j+1}, H'_{r+j}}(t)| \ge (x_j+\kappa)\frac{\sqrt{C}}{2^{r+j+1}} \right\} \le 4e^{-\frac{x_j^2}{2}} 2^{r+j+1},$$

since $\sup_{0<|H'-H|<h} |H_{r+j+1} - H_{r+j}| \le 2^{-(r+j+1)}$, we have similarly

$$P\left\{ \sup_{\substack{a\le H_r, H'_r \le b \\ |H'_r - H_r|<h}} \sup_{t\in[0,K]} |D_{H_{r+j+1}, H_{r+j}}(t)| \ge (x_j+\kappa)\frac{\sqrt{C}}{2^{r+j+1}} \right\} \le 4e^{-\frac{x_j^2}{2}} 2^{r+j+1}.$$

Whence, by (2.20) we have :

$$\begin{aligned} &P\left\{ \sup_{\substack{a\le H, H' \le b \\ |H' - H|<h}} \sup_{t\in[0,K]} |D_{H,H'}(t)| \ge (u+\kappa)\sqrt{C}(h+R^{-1}) + 2\sum_{j=0}^{\infty} \frac{(x_j+\kappa)\sqrt{C}}{2^{r+j+1}} \right\} \\ &\le 4\frac{R}{Rh+1} e^{-\frac{u^2}{2}} + 16R\sum_{j=0}^{\infty} 2^j e^{-\frac{x_j^2}{2}}. \end{aligned} \tag{2.21}$$

Put $x_j = \sqrt{2j+u^2}$ and R such that $2R > K/h \ge R$, where K is a positive constant to be specified later. Then

$$16R\sum_{j=0}^{\infty} 2^j e^{-x_j^2/2} \le \frac{16K}{h}\sum_{j=0}^{\infty}(2/e)^j e^{-u^2/2} = \frac{AK}{h}e^{-u^2/2},$$

where

$$A = 16\sum_{j=0}^{\infty}(2/e)^j$$

and

$$\begin{aligned} &(u+\kappa)\sqrt{C}(h+R^{-1}) + 2\sum_{j=0}^{\infty}\frac{(x_j+\kappa)\sqrt{C}}{2^{r+j+1}} \\ &\le (u+\kappa)\sqrt{C}(h+R^{-1}) + 2\sqrt{C}\frac{h}{K}\left[\sum_{j=0}^{\infty}\frac{\sqrt{2j}+\kappa}{2^j} + u\sum_{j=0}^{\infty}\frac{1}{2^j}\right] \\ &\le (u+\kappa)\sqrt{C}(h+\frac{2h}{K}) + 2\sqrt{C}\frac{h}{K}[B+uG] \\ &= h\left\{u[\sqrt{C}(1+2/K+2G/K)] + \sqrt{C}(\kappa+2\kappa/K+2B/K)\right\}, \end{aligned}$$

where

$$B = \sum_{j=0}^{\infty}\frac{\sqrt{2j}+\kappa}{2^j} \quad \text{and} \quad G = \sum_{j=0}^{\infty}\frac{1}{2^j}.$$

Letting now $\nu = \left\{u[\sqrt{C}(1+2/K+2G/K)] + \sqrt{C}(\kappa + 2\kappa/K + 2B/K)\right\}$ we get by (2.21) that

$$P\left\{\sup_{\substack{a\leq H,H'\leq b\\ |H'-H|<h}} \sup_{t\in[0,K]} |D_{H,H'}(t)| \geq h\nu\right\} \leq \frac{4K}{h}(K/2+1)^{-1}e^{-u^2/2} + \frac{AK}{h}e^{-u^2/2}$$
$$= \frac{1}{h}e^{-u^2/2}[4K(K/2+1)^{-1} + AK]$$
$$\leq \frac{\mathcal{C}}{h}e^{-(\nu-\sqrt{C}\kappa)^2/(2C+\epsilon)},$$

where the last inequality follows from

$$u = \frac{\nu - \sqrt{C}(\kappa + 2\kappa/K + 2B/K)}{\sqrt{C}(1+2/K+2G/K)} \geq \frac{\nu - \sqrt{C}\kappa}{\sqrt{C+\epsilon/2}} \geq 0,$$

which, in turn, is true for instance for all $\nu \geq 2\kappa\sqrt{C}$ and any given $\epsilon > 0$ provided K is large enough. This proves (2.19) with $\nu \geq 2\kappa\sqrt{C}$, while it is trivially true for $\nu \in (0, 2\kappa\sqrt{C})$, since, in the latter case, the right hand side of (2.19) is larger than one for $\mathcal{C}$ big enough.

Step 4. We end the proof of Theorem 4 by showing that

$$\overline{\lim_{h\to 0}} \frac{A_h - \sqrt{C}\kappa h}{h\sqrt{\log(1/h)}} \leq \sqrt{2\lambda C} \qquad \text{a.s.} \tag{2.22}$$

where $\lambda = \max\{\sqrt{C}, 1/\sqrt{C}\}$.
The proof is similar to that of Lévy's theorem (see e.g. Csörgö and Révész (1981) p. 26). For each $\epsilon > 0$, we have

$$P\left\{\frac{A_h - \sqrt{C}\kappa h}{h\sqrt{\lambda\log(1/h)}} \geq \sqrt{2C} + \epsilon\right\} = P\{A_h \geq h(\sqrt{C}\kappa + \sqrt{\lambda}(\sqrt{2C}+\epsilon)\sqrt{\log(1/h)})\}.$$

We apply inequality (2.19) of Step 3 with $\nu = \sqrt{C}\kappa + \sqrt{\lambda}(\sqrt{2C}+\epsilon)\sqrt{\log(1/h)}$ and we obtain

$$P\{A_h \geq h\nu\} \leq \frac{\mathcal{C}}{h}\exp\left\{-\frac{\lambda(\sqrt{2C}+\epsilon)^2}{2C+\epsilon}\log(1/h)\right\}$$
$$\leq \mathcal{C}h^{\epsilon},$$

which is true as soon as $\epsilon > 0$ is sufficiently small ($\frac{\lambda(\sqrt{2C}+\epsilon)^2}{2C+\epsilon} \geq 1+\epsilon$ for ϵ sufficiently small). Take $T > 1/\epsilon$ and let $h = h_n = n^{-T}$. Then we have

$$\sum_{n=1}^{\infty} P\left\{\frac{A_{h_n} - \sqrt{C}\kappa h_n}{h_n\sqrt{\lambda\log(1/h_n)}} \geq \sqrt{2C} + \epsilon\right\} \leq \sum_{n=1}^{\infty} \mathcal{C}n^{-T\epsilon} < \infty$$

and the Borel-Cantelli lemma implies that

$$\overline{\lim_{n\to\infty}} \frac{A_{h_n} - \sqrt{C}\kappa h_n}{h_n\sqrt{\lambda\log(1/h_n)}} \leq \sqrt{2C} + \epsilon \qquad \text{a.s.}$$

Let us take $h_{n+1} < h \leq h_n$. Then, almost surely

$$\begin{aligned}\overline{\lim_{h\to 0}} \frac{A_h - \sqrt{C}\kappa h}{h\sqrt{\lambda \log(1/h)}} &\leq \overline{\lim_{n\to\infty}} \frac{A_{h_n} - \sqrt{C}\kappa h_{n+1}}{h_{n+1}\sqrt{\lambda\log(1/h_n)}} \\ &= \overline{\lim_{n\to\infty}} \left\{ \frac{A_{h_n} - \sqrt{C}\kappa h_n}{h_n\sqrt{\lambda\log(1/h_n)}} \frac{h_n}{h_{n+1}} + \frac{\sqrt{C}\kappa(h_n - h_{n+1})}{h_{n+1}\sqrt{\lambda\log(1/h_n)}} \right\} \\ &\leq \sqrt{2C} + \epsilon \qquad \text{a.s.}\end{aligned}$$

for all $\epsilon > 0$ and whence we have (2.22).

Note : in fact, we proved a little more than Theorem 4, since we obtained a rate of convergence, namely $A_h = O(h\log(1/h))$ for $h \to 0^+$. □

3. Local Hölder properties of MBM

In this section, we consider an mBm such that the Hölder function H verifies:
$0 < H(t) < \min(1, \beta)$ for each $t \geq 0$.

Since mBm is a non stationary process, we are not interested in obtaining global properties, but rather local ones, especially concerning the Hölder exponent at each $t \geq 0$. For the sake of notational simplicity, let us from now on denote by (Y_t) the reduced mBm $(W_H(t)) = (W_{H_t}(t))$ for $t \geq 0$.

Proposition 5. *There exists a positive continuous function defined for $t \geq 0$, $t \longmapsto \sigma_t$, such that for any $t \geq 0$*

$$\frac{Y_{t+h} - Y_t}{h^{H_t}} \xrightarrow[h\to 0]{\mathcal{L}} N(0, \sigma_t^2).$$

Proof. Since the process $(Y_t)_{t\geq 0}$ is Gaussian, by the theorem of Levy (see e.g. Grimmett and Stirzaker (1992) p.172), we prove the convergence in distribution by showing the following two statements.

(1) $E(\frac{Y_{t+h}-Y_t}{h^{H_t}}) \xrightarrow[h\to 0]{} 0,$

(2) $E\left(\frac{(Y_{t+h}-Y_t)^2}{h^{2H_t}}\right) \xrightarrow[h\to 0]{} \sigma_t^2.$

(1) This statement is obvious.

(2) By splitting the expectation into three members and denoting $\sigma_t^2 = Var\left(\frac{W_{H_t}(t+h)-W_{H_t}(t)}{h^{H_t}}\right)$ (which is independent of h), we have

$$E\left(\frac{(Y_{t+h} - Y_t)^2}{h^{2H_t}}\right) = E\left(\frac{(W_{H_{t+h}}(t+h) - W_{H_t}(t+h))^2}{h^{2H_t}}\right) + \sigma_t^2 + \\ 2E\left(\frac{(W_{H_{t+h}}(t+h) - W_{H_t}(t+h))(W_{H_t}(t+h) - W_{H_t}(t))}{h^{2H_t}}\right).$$

We show that the two expectations tends to 0.

- Setting $b>0$, we claim: *there exists $\beta_0>0$ such that for each $h\in(0,b)$ and $t\in[0,b-h]$, we have*

$$E\left(\frac{[W_{H_{t+h}}(t+h)-W_{H_t}(t+h)]^2}{h^{2H_t}}\right)\le \beta_0 h^{2\beta-2H_t}. \tag{3.1}$$

  With the notation of Theorem 4 and by (2.10), we have

$$\begin{aligned}E([W_{H_{t+h}}(t+h)-W_{H_t}(t+h)]^2) &= Var(D_{H_t,H_{t+h}}(t+h))\\ &\le C|H_{t+h}-H_t|^2.\end{aligned}$$

  By definition of H we get (3.1).

- We have $E((W_{H_t}(t+h)-W_{H_t}(t))^2)=V_{H_t}h^{2H_t}$, where $H\longmapsto V_H$ is defined in Step 1 of the proof of Theorem 4. Thus, by Schwarz Inequality and (3.1) we have

$$\begin{aligned}&E\left(\frac{(W_{H_{t+h}}(t+h)-W_{H_t}(t+h))(W_{H_t}(t+h)-W_{H_t}(t))}{h^{2H_t}}\right)\\ &\le \frac{\sqrt{E\left((W_{H_{t+h}}(t+h)-W_{H_t}(t+h))^2\right)E\left((W_{H_t}(t+h)-W_{H_t}(t))^2\right)}}{h^{2H_t}}\\ &\le \sqrt{\beta_0 V_{H_t}}h^{\beta-H_t}\longrightarrow 0 \text{ as } h\to 0,\end{aligned}$$

  which completes the proof of (2).

(3) To end the proof we show that $t\longmapsto\sigma_t$ is continuous. It is sufficient to prove that $t\longmapsto\sigma_t^2$ is continuous.
Fix $t\ge0$ and $h>0$. Let k be such that $t+k\ge0$. We shall consider the function $H' : t\longmapsto H_{t-h}$. H' is a Hölder function with exponent β. Let for convenience ${\sigma'_{t'}}^2 = Var\left(\frac{W_{H'_{t'}}(t'+h)-W_{H'_{t'}}(t')}{h^{H'_{t'}}}\right)$ (which is independent of h). We have the following equalities.

$$\begin{aligned}\sigma_{t+k}^2-\sigma_t^2 &= E\left(\frac{(W_{H_{t+k}}(t+h+k)-W_{H_{t+k}}(t+k))^2}{h^{2H_{t+k}}}-\frac{(W_{H_t}(t+h)-W_{H_t}(t))^2}{h^{2H_t}}\right)\\
&= E\left(\frac{(W_{H_{t+k}}(t+h+k)-W_{H_{t+k}}(t+k))^2-(W_{H_t}(t+h)-W_{H_t}(t))^2}{h^{2H_{t+k}}}\right)-\\
&\quad E\left((W_{H_t}(t+h)-W_{H_t}(t))^2\right)\left(\frac{1}{h^{2H_{t+k}}}-\frac{1}{h^{2H_t}}\right)\\
&= E\left(\frac{(W_{H_{t+k}}(t+h+k)-W_{H_{t+k}}(t+k)-W_{H_t}(t+h)+W_{H_t}(t))(W_{H_{t+k}}(t+h+k)-W_{H_{t+k}}(t+k)+W_{H_t}(t+h)-W_{H_t}(t))}{h^{2H_{t+k}}}\right)\\
&\quad -\sigma_t^2h^{2H_t}\left(\frac{1}{h^{2H_{t+k}}}-\frac{1}{h^{2H_t}}\right)\\
&= E\left(\frac{(W_{H'_{t'+k}}(t'+k)-W_{H_{t+k}}(t+k)-W_{H'_{t'}}(t')+W_{H_t}(t))(W_{H'_{t'}}(t'+k)-W_{H_{t+k}}(t+k)+W_{H'_{t'}}(t')-W_{H_t}(t))}{h^{2H_{t+k}}}\right)\\
&\quad -\sigma_t^2h^{2H_t}\left(\frac{1}{h^{2H_{t+k}}}-\frac{1}{h^{2H_t}}\right).\end{aligned}$$

By Schwarz Inequality we obtain:

$$\begin{aligned}|\sigma_{t+k}^2-\sigma_t^2| &\le \frac{1}{h^{2H_{t+k}}}\sqrt{E\left((W_{H'_{t'+k}}(t'+k)-W_{H_{t+k}}(t+k)-W_{H'_{t'}}(t')+W_{H_t}(t))^2\right)}\times\\
&\quad\sqrt{E\left((W_{H'_{t'}}(t'+k)-W_{H_{t+k}}(t+k)+W_{H'_{t'}}(t')-W_{H_t}(t))^2\right)}+|\sigma_t^2h^{2H_t}\left(\frac{1}{h^{2H_{t+k}}}-\frac{1}{h^{2H_t}}\right)|.\end{aligned}$$

Using (2.10), there exist $C_t > 0$ and $C_{t'} > 0$ such that $Var(D_{H,H'}(t)) \leq C_t(H'-H)^2$ and $Var(D_{H,H'}(t')) \leq C_{t'}(H'-H)^2$. Thus,

$$|\sigma_{t+k}^2 - \sigma_t^2| \leq \frac{2}{h^{2H_{t+k}}}\sqrt{E\left((W_{H'_{t'+k}}(t'+k) - W_{H'_{t'}}(t'))^2\right) + E\left((W_{H_{t+k}}(t+k) - W_{H_t}(t))^2\right)} \times$$
$$\sqrt{E\left((W_{H'_{t'}}(t'+k) - W_{H'_{t'}}(t'))^2\right) + E\left((W_{H_{t+k}}(t+k) + W_{H_t}(t) - 2W_{H'_{t'}}(t'))^2\right)}$$
$$+ |\sigma_t^2 h^{2H_t}(\frac{1}{h^{2H_{t+k}}} - \frac{1}{h^{2H_t}})|$$
$$\leq \frac{2\sqrt{2}}{h^{2H_{t+k}}}\sqrt{{\sigma'_{t'}}^2|k|^{2H'_{t'+k}} + C_{t'}(H'_{t'+k} - H'_{t'})^2 + \sigma_t^2|k|^{2H_{t+k}} + C_t(H_{t+k} - H_t)^2} \times$$
$$\sqrt{{\sigma'_{t'}}^2|k|^{2H_{t'}} + 2[E\left((W_{H_{t+k}}(t+k) - W_{H_t}(t))^2]\right) + E\left((2W_{H'_{t'}}(t') - 2W_{H_t}(t))^2\right)]}$$
$$+ |\sigma_t^2 h^{2H_t}(\frac{1}{h^{2H_{t+k}}} - \frac{1}{h^{2H_t}})|$$
$$\leq \frac{2\sqrt{2}}{h^{2H_{t+k}}}\sqrt{{\sigma'_{t'}}^2|k|^{2H'_{t'+k}} + C_{t'}|k|^{2\beta} + \sigma_t^2 k^{2H_{t+k}} + C_t|k|^{2\beta}} \times$$
$$\sqrt{{\sigma'_{t'}}^2 k^{2H_{t'}} + 4\sigma_t^2 k^{2H_{t+k}} + 4C_t|k|^{2\beta} + 16[E\left((W_{H'_{t'}}(t'))^2\right) + E((W_{H_t}(t))^2)]} +$$
$$|\sigma_t^2 h^{2H_t}(\frac{1}{h^{2H_{t+k}}} - \frac{1}{h^{2H_t}})|.$$

Since the functions $t \longmapsto H_t$ and $t \longmapsto H'_t$ are continuous, we have therefore $|\sigma_{t+k}^2 - \sigma_t^2| \underset{k\to 0}{\longrightarrow} 0$.

□

We can now define the standard mBm:

Definition 6 (Standard Multifractional Brownian Motion).
Let $(Y_t)_{(t\geq 0)}$ be a Multifractional Brownian Motion and let $t \longmapsto H_t$ be its Hölder functional parameter of exponent $\beta > 0$, such that for any $t \geq 0$, $0 < H(t) < \min(1,\beta)$. Then there exists a unique continuous positive function $t \longmapsto \sigma_t$ such that the process $(Z_t)_{(t\geq 0)}$ defined by $Z_t = \frac{Y_t}{\sigma_t}$ is continuous and verifies the following property

$$Var\left(\frac{Z_{t+h} - Z_t}{h^{H_t}}\right) \underset{h\to 0}{\longrightarrow} 1.$$

The process $(Z_t)_{(t\geq 0)}$ is called Standard Multifractional Brownian Motion.

We recall a lemma which will be useful in the sequel (see Peltier *et al* (1994)).
Whenever $X : [0,1] \to \mathbb{R}$ is a continuous function, there exists a sequence $X_n : [0,1] \to \mathbb{R}$, $n = 1, 2, ...$ of polygonal functions which converges uniformly to X on $[0,1]$. Moreover, we may and do assume that the vertices of the graph of each X_n are of the form $\{(\frac{k}{n}, X_n(\frac{k}{n})),\ 0 \leq k \leq n\}$, with $X_n(0) = X(0)$. Setting $\mathcal{X}_{k,n} = X_n\left(\frac{k}{n}\right)$ for $0 \leq k \leq n$, we will set for convenience $\mathcal{X}_{(n)} = (\mathcal{X}_{1,n}, \dots, \mathcal{X}_{n,n})$.
Given a polygonal function X_n as above, we set for each $n \geq 2$,

$$L_n = \sum_{i=1}^{n-1} |\mathcal{X}_{i+1,n} - \mathcal{X}_{i,n}|. \tag{3.2}$$

Lemma *Let $X : [0,1] \to \mathbb{R}$ be a continuous function with box dimension D, and let $(X_n)_{n\geq 1}$*

be a sequence of polygonal functions as above which converges uniformly to X on $[0,1]$. The following results hold:

(1) *If X is constant, then $D = 1$;*
(2) *If X is non constant, then*

$$D = 1 + \lim_{n\to\infty} \frac{\log L_n}{\log(n-1)}. \tag{3.3}$$

We recall the following properties of Hausdorff dimension, denoted by dim_H (see e.g. Falconer (1990) p. 29):
If $F_1, F_2, \ldots$ is a (countable) sequence of sets, then

$$dim_H \left\{ \bigcup_{i=1}^{\infty} F_i \right\} = \sup_{1\le i<\infty} dim_H\{F_i\}. \tag{3.4}$$

For any set $F \subset I\!R^n$ $(n \ge 1)$, we have

$$dim_H\{F\} \le \underline{dim_B}\{F\} \le \overline{dim_B}\{F\}, \tag{3.5}$$

where $\underline{dim_B}\{F\}$ (respectively $\overline{dim_B}\{F\}$), denotes the lower (respectively the upper) box dimension of F. For each $t_0 \ge 0$ and $\eta > 0$, let for convenience
$m_{t_0}(\eta) = \min_{|t-t_0|\le\eta} H_t$, $M_{t_0}(\eta) = \max_{|t-t_0|\le\eta} H_t$ and $F_{t_0}(\eta) = Graph\{Y_t, |t-t_0| \le \eta\}$. The following lemma will be useful for establishing the next proposition.

Lemma 7. *With probability one, the graph of the mBm $(Y_t)_{t\ge0}$ verifies the following property: for any $t_0 \ge 0$ and $\eta > 0$,*

$$2 - M_{t_0}(\eta) \le dim_H\{F_{t_0}(\eta)\} \le \underline{dim_B}\{F_{t_0}(\eta)\} \le \overline{dim_B}\{F_{t_0}(\eta)\} \le 2 - m_{t_0}(\eta). \tag{3.6}$$

Proof. We first prove the last inequality of (3.6).
Step 1. Without loss of generality, we assume that $t_0 = 1/2$, $\eta = 1/2$, and for each integer $n > 0$, we set

$$Y_{i,n} = Y\left(\frac{i}{n}\right) \text{ for } i = 0, 1, \ldots, n. \tag{3.7}$$

We show that for each $0 < H' < m_{t_0}(\eta)$, we have

$$\lim_{n\to\infty}\left((n-1)^{H'} \max_{1\le i\le n-1}(|Y_{i+1,n} - Y_{i,n}|)\right) = 0 \qquad \text{a.s.} \tag{3.8}$$

We start by observing that, for any $\epsilon > 0$ and $p \ge 1$, the Markov Inequality entails that

$$\begin{aligned} P[(n-1)^{H'} \max_{1\le i\le n-1}(|Y_{i+1,n} - Y_{i,n}|) > \epsilon] &\le \sum_{i=1}^{n-1} P[(n-1)^{H'}|Y_{i+1,n} - Y_{i,n}| > \epsilon] \\ &\le \frac{(n-1)}{\epsilon^p} \max_{1\le i\le n-1} E((n-1)^{pH'}|Y_{i+1,n} - Y_{i,n}|^p). \end{aligned}$$

By (3.1) and (2.0), there exist $\mathcal{B}_p > 0$ and $\mathcal{B}'_p > 0$ such that

$$\begin{aligned}
&P[(n-1)^{H'} \max_{1\le i\le n-1}(|Y_{i+1,n}-Y_{i,n}|) > \epsilon] \\
&\le \frac{(n-1)2^p}{\epsilon^p} \max_{1\le i\le n-1} E[(n-1)^{pH'}|W_{H_{\frac{i+1}{n-1}}}(\frac{i+1}{n-1}) - W_{H_{\frac{i}{n-1}}}(\frac{i+1}{n-1})|^p \\
&\qquad\qquad + (n-1)^{pH'}|W_{H_{\frac{i}{n-1}}}(\frac{i+1}{n-1}) - W_{H_{\frac{i}{n-1}}}(\frac{i}{n-1})|^p] \\
&\le \frac{2^p}{\epsilon^p}\{\frac{\mathcal{B}_p}{(n-1)^{(\beta-H')p-1}} + \frac{\mathcal{B}'_p}{(n-1)^{(m_{t_0}(\eta)-H')p-1}}\} \\
&\le \frac{2^p}{\epsilon^p}.\frac{\mathcal{B}_p+\mathcal{B}'_p}{(n-1)^{(m_{t_0}(\eta)-H')p-1}}.
\end{aligned}$$

By setting $p > \frac{2}{m_{t_0}(\eta)-H'}$, (3.8) follows readily from the Borel-Cantelli lemma.

Step 2. By Step 1, we easily see that for each $0 < H' < m_{t_0}(\eta)$ we have almost surely

$$\lim_{n\to\infty}(n-1)^{H'-1}\sum_{i=1}^{n-1}|Y_{i+1,n}-Y_{i,n}| = 0,$$

which entails, with the notation of the above Lemma when applied to $X = Y$, that almost surely, for all n sufficiently large

$$(n-1)^{H'-1}L_n \le 1.$$

Whence, for each $0 < H' < m_{t_0}(\eta)$

$$1 + \overline{\lim_{n\to\infty}} \frac{\log L_n}{\log(n-1)} \le 2 - H',$$

which yields $\overline{dim_B}\{F_{t_0}(\eta)\} \le 2 - m_{t_0}(\eta)$, and ends the proof of the last inequality of (3.6). Let us now prove the first inequality of (3.6).

Step 3. We assume, as in Step 1, that $t_0 = 1/2$ and $\eta = 1/2$. We show that there exists $c > 0$ such that, for any t and h which satisfy $|t-t_0| \le \eta$ and $|t+h-t_0| \le \eta$,

$$\sigma_{t,h}^2 = E[(Y_{t+h}-Y_t)^2] \ge c.h^{2M_{t_0}(\eta)}. \tag{3.9}$$

We denote by $H \longmapsto V_H$ the continuous function defined in Step 1 of the proof of Theorem 4, and by β_0 the constant defined in (3.1). We easily have

$$\begin{aligned}
\sigma_{t,h}^2 &= E[(W_{H_{t+h}}(t+h) - W_{H_{t+h}}(t) + W_{H_{t+h}}(t) - W_{H_t}(t))^2] \\
&\ge V_{H_{t+h}}|h|^{2H_{t+h}} - 2E[\ |\ (W_{H_{t+h}}(t+h) - W_{H_{t+h}}(t))(W_{H_{t+h}}(t) - W_{H_t}(t))\ |\ ] \\
&\ge V_{H_{t+h}}|h|^{2H_{t+h}} - 2\sqrt{E[(W_{H_{t+h}}(t+h) - W_{H_{t+h}}(t))^2].E[(W_{H_{t+h}}(t) - W_{H_t}(t))^2]},
\end{aligned}$$

and finally,

$$\sigma_{t,h}^2 \ge V_{H_{t+h}}|h|^{2H_{t+h}} - 2\sqrt{V_{H_{t+h}}\beta_0}|h|^{H_{t+h}+\beta}, \tag{3.10}$$

which entails (3.9).

Step 4. Set $1 < s < 2$. We show that there exists $c_1 > 0$ such that for any t and h which satisfy $|t-t_0| \le \eta$ and $|t+h-t_0| \le \eta$

$$E[(|Y_{t+h}-Y_t|^2 + h^2)^{-s/2}] \le c_1 h^{1-M_{t_0}(\eta)-s}. \tag{3.11}$$

We have

$$
\begin{aligned}
E[(|Y_{t+h}-Y_t|^2+h^2)^{-s/2}] &= \sqrt{\frac{2}{\pi}}\sigma_{t,h}\int_0^\infty (r^2+h^2)^{-s/2} exp\left(\frac{-r^2}{2\sigma_{t,h}^2}\right) dr \\
&= \sqrt{\frac{1}{2\pi}}\int_0^\infty (\sigma_{t,s}^2 u+h^2)^{-s/2}u^{-1/2} exp\left(\frac{-u}{2}\right) du \\
&\le \sqrt{\frac{1}{2\pi}}\int_0^{h^2\sigma_{t,h}^{-2}} (h^2)^{-s/2}u^{-1/2}du + \sqrt{\frac{1}{2\pi}}\int_{h^2\sigma_{t,h}^{-2}}^\infty (\sigma_{t,h}^2 u)^{-s/2}u^{-1/2}du \\
&= \sqrt{\frac{2}{\pi}}h^{1-s}\sigma_{t,h}^{-1} + \sqrt{\frac{2}{\pi}}\frac{1}{s-1}h^{1-s}\sigma_{t,h}^{-1} \\
&\le c_1 h^{1-M_{t_0}(\eta)-s},
\end{aligned}
$$

by Step 3.

Step 5. The end of the proof follows the one presented in Falconer (1990) p. 244 for computing the Hausdorff dimension of a Brownian motion. We then obtain
$2-\max_{|t-t_0|\le\eta} H_t \le dim_H\{F_{t_0}(\eta)\} \le \underline{dim_B}\{F_{t_0}(\eta)\}$. □

Proposition 8. *With probability one, for each interval $[a,b]\subset I\!R^+$, the graph of the mBm $(Y_t)_{t\in[a,b]}$ verifies the following property :*

$$dim_H\{Y_t, t\in[a,b]\} = dim_B\{Y_t, t\in[a,b]\} = 2-\min\{H_t, t\in[a,b]\},$$

Proof. Let $t_0\in[a,b]$ be such that $H_{t_0}=\min\{H_t, t\in[a,b]\}$. We consider the two following sequences of intervals defined, for each $n\ge 0$, by $E_n=[t_0+\frac{b-t_0}{2^{n+1}}, t_0+\frac{b-t_0}{2^n}]$ and $G_n=[t_0-\frac{t_0-a}{2^n}, t_0-\frac{t_0-a}{2^{n+1}}]$. We easily have $\left\{\bigcup_{n\ge0} E_n\right\}\bigcup\left\{\bigcup_{n\ge0} G_n\right\}=[a,b]-\{t_0\}$. It follows from Lemma 7 that, on the one hand,

$$2-M_{t_0+\frac{3}{4}\frac{b-t_0}{2^n}}(\frac{b-t_0}{2^{n+2}}) \le dim_H\{F_{t_0+\frac{3}{4}\frac{b-t_0}{2^n}}(\frac{b-t_0}{2^{n+2}})\} = dim_H\{Y_t, t\in E_n\} \le 2-m_{t_0+\frac{3}{4}\frac{b-t_0}{2^n}}(\frac{b-t_0}{2^{n+2}}),$$

on the other hand,

$$2-M_{t_0+\frac{3}{4}\frac{t_0-a}{2^n}}(\frac{t_0-a}{2^{n+2}}) \le dim_H\{F_{t_0-\frac{3}{4}\frac{t_0-a}{2^n}}(\frac{t_0-a}{2^{n+2}})\} = dim_H\{Y_t, t\in G_n\} \le 2-m_{t_0+\frac{3}{4}\frac{t_0-a}{2^n}}(\frac{t_0-a}{2^{n+2}}).$$

This entails, by (3.4)

$$\sup_{n\ge0}\left\{2-M_{t_0+\frac{3}{4}\frac{b-t_0}{2^n}}(\frac{b-t_0}{2^{n+2}})\right\} \le dim_H\{Y_t, t\in(t_0,b]\} \le \sup_{n\ge0}\left\{2-m_{t_0+\frac{3}{4}\frac{b-t_0}{2^n}}(\frac{b-t_0}{2^{n+2}})\right\},$$

and similarly

$$\sup_{n\ge0}\left\{2-M_{t_0+\frac{3}{4}\frac{t_0-a}{2^n}}(\frac{t_0-a}{2^{n+2}})\right\} \le dim_H\{Y_t, t\in[a,t_0)\} \le \sup_{n\ge0}\left\{2-m_{t_0+\frac{3}{4}\frac{t_0-a}{2^n}}(\frac{t_0-a}{2^{n+2}})\right\},$$

and using the continuity of the function H_t, we obtain

$$2-H_{t_0}\le dim_H\{Y_t, t\in[a,b]\} = \max\{dim_H(Y_t, t\in(t_0,b]), dim_H(Y_t, t\in(a,t_0))\} \le 2-H_{t_0}.$$

Using $\overline{dim_B}\{Y_t, t\in[a,b]\}=2-\min\{H_{t_0}\}$ and (3.5), the desired result follows. □

We next study the Hölder property of the multifractional Brownian motion. We first recall the following definition:

Definition 9 (Hölder exponent).
A real function f is said to have a Hölder exponent $0 < H_{t_0} < 1$ at point t_0 iff:

(1) *for every real γ such that $\gamma < H_{t_0}$:*

$$\lim_{h\to 0}\frac{|f(t_0+h)-f(t_0)|}{|h|^\gamma}=0,$$

(2) *for every real $\gamma > H_{t_0}$:*

$$\limsup_{h\to 0}\frac{|f(t_0+h)-f(t_0)|}{|h|^\gamma}=\infty.$$

Proposition 10. *With probability one, the Hölder exponent at point $t_0 \geq 0$ of a multifractional Brownian motion is H_{t_0}.*

Proof. **Step 1.** We show that with probability one for each $\gamma > 0$ such that $0 < \gamma < H_{t_0}$:

$$\lim_{h\to 0}\frac{|Y_{t_0+h}-Y_{t_0}|}{|h|^\gamma}=0. \tag{3.12}$$

We have

$$\begin{aligned}\frac{|Y_{t_0+h}-Y_{t_0}|}{|h|^\gamma} &= \frac{|W_{H_{t_0+h}}(t_0+h)-W_{H_{t_0}}(t_0)|}{|h|^\gamma}\\ &\leq \frac{|W_{H_{t_0+h}}(t_0+h)-W_{H_{t_0}}(t_0+h)|}{|h|^\gamma}+\frac{|W_{H_{t_0}}(t_0+h)-W_{H_{t_0}}(t_0)|}{|h|^\gamma}.\end{aligned}$$

We show that each term of the right member of the above inequality tends almost surely to 0. Let $[a,b]$ be an interval such that $t_0 \in [a,b] \subset \mathbb{R}^+$ and let $t \in [a,b]$. We consider the process $\mathcal{X}_0 = 0$ and $\mathcal{X}_h = \frac{W_{H_{t_0+h}}(t)-W_{H_{t_0}}(t)}{|h|^\gamma}$ defined almost surely for $|h|$ sufficiently small. Our aim is to show that for each $0 < \gamma < H_{t_0}$, there exists $c_\gamma > 0$ independent of t such that for each h, h' sufficiently small, we have

$$E[(\mathcal{X}_h-\mathcal{X}_{h'})^2] \leq c_\gamma |h'-h|^{2(\beta-\gamma)}. \tag{3.13}$$

Case 1: $h' > 0$.
Without loss of generality we assume $0 < |h| < h'$, and by (2.10) combined with the Hölder property of the function H, there exists $c > 0$ independent of t such that

$$\begin{aligned}E[(\mathcal{X}_h-\mathcal{X}_{h'})^2] &\leq 2E\left(\frac{(W_{H_{t_0+h'}}(t)-W_{H_{t_0+h}}(t))^2}{|h'|^{2\gamma}}\right)+2\left(\frac{1}{|h'|^\gamma}-\frac{1}{|h|^\gamma}\right)^2 E\left((W_{H_{t_0+h}}(t)-W_{H_{t_0}}(t))^2\right)\\ &\leq 2c\frac{|h'-h|^{2\beta}}{|h'|^{2\gamma}}+2c\frac{(|h'|^\gamma-|h|^\gamma)^2}{|hh'|^{2\gamma}}|h|^{2\beta}\\ &\leq 2^{2\gamma+1}c|h'-h|^{2(\beta-\gamma)}+2c|h|^{2(\beta-\gamma)}\left(1-\left|\frac{h}{h'}\right|^\gamma\right)^2\\ &\leq 2^{2\gamma+1}c|h'-h|^{2(\beta-\gamma)}+(2c+8\gamma^2)|h'-h|^{2(\beta-\gamma)},\end{aligned}$$

which results by the following facts: if $h \in (-h', h'/2]$ then $|h| \leq |h'-h|$; if $h \in (h'/2, h')$ then $|h|^{2(\beta-\gamma)}\left(1-\left|\frac{h}{h'}\right|^\gamma\right)^2 \leq |h'|^{2(\beta-\gamma)}4\gamma^2(1-\frac{h}{h'})^2 \leq 4\gamma^2|h'-h|^{2(\beta-\gamma)}$, which yields (3.13).
Case 2: $h' < 0$. The proof goes along the same line.

We know that $\mathcal{X}_h - \mathcal{X}_{h'}$ follows a normal law. Thus, combining (3.13), (2.0) and the Kolmogorov criterion we obtain that the above first member tends almost surely to 0.

Let us now study the second term. The fractional Brownian motion satisfies a stochastic Hölder condition (see e.g. Adler (1981) p. 202) in the following sense: *For each $\gamma' < H_{t_0}$ there exists $A > 0$ such that almost surely:*

$$\sup_{t \geq 0} |W_{H_{t_0}}(t+h) - W_{H_{t_0}}(t)| \leq A|h|^{\gamma'},$$

which ends the proof of (3.12).

Step 2. Let $(h_n)_{n\geq 1}$ be a sequence such that $\lim_{n\to\infty} h_n = 0$. We show that for each $\gamma > 0$ such that $\beta > \gamma > H_{t_0}$:

$$\frac{|h_n|^\gamma}{|Y_{t_0+h_n} - Y_{t_0}|} \xrightarrow[n\to\infty]{P} 0. \tag{3.14}$$

For each $t > 0$ we have, using (3.10) and the Mean Value Theorem

$$\begin{aligned} P(\frac{|h_n|^\gamma}{|Y_{t_0+h_n} - Y_{t_0}|} > t) &= P(|Y_{t_0+h_n} - Y_{t_0}| < |h_n|^\gamma/t) \\ &\leq P\left(\frac{|Y_{t_0+h_n} - Y_{t_0}|}{\sigma_{t_0,h_n}} < \frac{|h_n|^\gamma}{t\sqrt{V_{H_{t_0+h_n}}|h_n|^{2H_{t_0+h_n}} - 2\sqrt{V_{H_{t_0+h_n}}}\beta_0|h|^{H_{t_0+h_n}+\beta}}}\right) \\ &= O(|h_n|^{\gamma - H_{t_0} - h_n}) \longrightarrow 0, \end{aligned}$$

as $h_n \to 0$.

Step 3. We know that (3.14) entails that there exists a subsequence which converges almost surely to 0 (see e.g. Lukacs (1968) p. 46) and thus, we have almost surely

$$\limsup_{h\to 0} \frac{|Y_{t_0+h} - Y_{t_0}|}{|h|^\gamma} = \infty.$$

□

Remark 1: Take $H(t) = t$ on $[a,b] \subset (0,1)$. Then it is easy to see that the Hölder multifractal spectrum[2] of the associated mBm is: $f(\alpha) = 0$ for $\alpha \in [a,b]$; $f(\alpha) = -\infty$ otherwise. More general spectra may be obtained by using more complex H functions. This topic will be developed in a forthcoming paper.

At last, we establish a property of multifractional Brownian motion related to its box-dimension or more precisely to its Minkowski content (see e.g. Falconer (1990) p. 42; Mattila (1995) p. 79 for definitions). We assume w.l.o.g. that $0 < \beta \leq 1$. Fix $t_0 > 0$ and $1 < \zeta < \frac{\beta}{H_{t_0}}$. For sufficiently small values of $\delta > 0$, let $F_\delta(t_0) = \{(t, Z(t)) : t \in [t_0 - \delta/2, t_0 + \delta/2]\}$ denotes the local graph around $t_0 > 0$ of the standard multifractional motion $(Z(t))_{t\geq 0}$. For $i = 1 \ldots 5$, we denote by $\underline{c_\delta}^{(i)}(t_0)$ and $\overline{c_\delta}^{(i)}(t_0)$ the random variables defined almost surely by

$$\underline{c_\delta}^{(i)}(t_0) = \underline{\lim}_{\delta\to 0} \delta^{\zeta(2-H_{t_0})} N^{(i)}_{\delta^\zeta}(F_\delta(t_0)) \quad \text{and} \quad \overline{c_\delta}^{(i)}(t_0) = \overline{\lim}_{\delta\to 0} \delta^{\zeta(2-H_{t_0})} N^{(i)}_{\delta^\zeta}(F_\delta(t_0)),$$

[2]For the definitions and properties of multifractal spectra see for instance: Lévy Véhel *et al* (1995) - Olsen (1994) - Falconer (1994) - Brown *et al* (1992)

where $N^{(i)}_{\delta^\zeta}(F_\delta(t_0))$ is respectively one of the following (see e.g. Falconer (1990) p. 41):

(1) For $i = 1$, the number of δ^ζ-mesh squares that intersect $F_\delta(t_0)$;

(2) For $i = 2$, the smallest number of squares of size δ^ζ that cover $F_\delta(t_0)$;

(3) For $i = 3$, the smallest number of closed balls of diameter δ^ζ that cover $F_\delta(t_0)$;

(4) For $i = 4$, the smallest number of sets of diameter δ^ζ that cover $F_\delta(t_0)$;

(5) For $i = 5$, the largest number of disjoint balls of diameter δ^ζ with centres in $F_\delta(t_0)$.

Theorem 11. *There exists $m_{H_{t_0}} > 0$ such that we have with probability one*

(1) $\sqrt{\frac{2}{\pi}} \leq \underline{c}^{(1)} = \overline{c}^{(1)} = m_{H_{t_0}}$;

(2) $\frac{1}{\sqrt{2\pi}} \leq \underline{c}^{(2)} \leq \overline{c}^{(2)} \leq m_{H_{t_0}}$;

(3) *For $i = 3, 4$ and 5,* $\frac{1}{\sqrt{2\pi}} \leq \underline{c}^{(i)} \leq \overline{c}^{(i)} \leq m_{H_{t_0}}\sqrt{2}$.

For $0 < H_{t_0} < 1$,

$$m_{H_{t_0}} \leq 4\sqrt{2\pi}.$$

Furthermore, if $1/2 \leq H_{t_0} < 1$ then

$$m_{H_{t_0}} \leq 2\sqrt{\frac{2}{\pi}}$$

and if $0 < H_{t_0} \leq 1/2$ then

$$2\sqrt{\frac{2}{\pi}} \leq m_{H_{t_0}}.$$

Proof.

The proof uses the following theorem, which provides a similar result in the case of fractional Brownian motion (see Peltier - Lévy Véhel (1994)).

First, we recall some notation. Let $F = \{(t, X_H(t)) : t \in [0,1]\}$ denote the graph of a fractional Brownian motion, with fractal (Hausdorff or box) dimension $s = 2 - H$. Let further $\sigma > 0$ be defined as the usual scale factor. For $i = 1 \ldots 5$, we denote by $\underline{\mathfrak{c}}^{(i)}$ and $\overline{\mathfrak{c}}^{(i)}$ the random variables defined almost surely by

$$\underline{\mathfrak{c}}^{(i)} = \underline{\lim}_{\delta\to 0} \delta^s N^{(i)}_\delta(F) \quad \text{and} \quad \overline{\mathfrak{c}}^{(i)} = \overline{\lim}_{\delta\to 0} \delta^s N^{(i)}_\delta(F),$$

where $N^{(i)}_\delta(F)$ is respectively one of the following alternative definitions:

(1) For $i = 1$, the number of δ-mesh squares that intersect F;

(2) For $i = 2$, the smallest number of squares of size δ that cover F;

(3) For $i = 3$, the smallest number of closed balls of diameter δ that cover F;

(4) For $i = 4$, the smallest number of sets of diameter δ that cover F;

(5) For $i = 5$, the largest number of disjoint balls of diameter δ with centres in F.

Theorem A [Peltier - Lévy Véhel (1994)]: *There exists $m_H > 0$ such that we have with probability one*

(1) $\sqrt{\frac{2}{\pi}} \leq \underline{\mathfrak{c}}^{(1)} = \overline{\mathfrak{c}}^{(1)} = m_H$;

(2) $\frac{\sigma}{\sqrt{2\pi}} \leq \underline{\mathfrak{c}}^{(2)} \leq \overline{\mathfrak{c}}^{(2)} \leq m_H$;

(3) *For $i = 3, 4$ and 5* $\frac{\sigma}{\sqrt{2\pi}} \leq \underline{\mathfrak{c}}^{(i)} \leq \overline{\mathfrak{c}}^{(i)} \leq m_H\sqrt{2}$.

For $0 < H < 1$,

$$m_H \leq 4\sigma\sqrt{2\pi}.$$

Furthermore, if $1/2 \leq H < 1$ *then*

$$m_H \leq 2\sigma\sqrt{\frac{2}{\pi}}$$

and if $0 < H \leq 1/2$ *then*

$$2\sigma\sqrt{\frac{2}{\pi}} \leq m_H.$$

For convenience the proof of this theorem is recalled in ANNEX B.

We now move to the proof of Theorem 11.

Write $\zeta = \frac{\alpha}{\gamma}$ with the condition:

$$\gamma + \max\{1, \frac{1}{2(1-H_{t_0})}\} < \alpha < \frac{\gamma\beta}{H_{t_0}} \tag{3.15}$$

• (1) is equivalent to:

$$\lim_{\delta\to 0} \delta^{\alpha(2-H_{t_0})} N^{(1)}_{\delta^\alpha}(F_{\delta^\gamma}(t_0)) = 2\sqrt{\frac{2}{\pi}}.$$

Set for convenience
$R_{i,n} = \lfloor n^\alpha \rfloor^{H_{t_0}} \sup_{(i-1)/\lfloor n^\alpha\rfloor \leq s,t \leq i/\lfloor n^\alpha \rfloor} Z(t_0+t-1/(2n^\gamma)) - Z(t_0+s-1/(2n^\gamma))$ for each $1 \leq i \leq \lfloor n^{\alpha-\gamma}\rfloor - 1$. $R_{i,n}/\lfloor n^\alpha\rfloor^{H_{t_0}}$ are the maximum oscillations defined on intervals corresponding to the subdivision of the interval $[t_0 - 1/(2n^\gamma), t_0 + 1/(2n^\gamma)]$ in time steps of size $1/\lfloor n^\alpha \rfloor$. We show that there exists $m_{H_{t_0}} > 0$ such that:

$$\lim_{n\to\infty} \frac{1}{\lfloor n^{\alpha-\gamma}\rfloor - 1} \sum_{i=1}^{\lfloor n^{\alpha-\gamma}\rfloor - 1} R_{i,n} = m_{H_{t_0}} \qquad \text{a.s.} \tag{3.16}$$

Write $B_{H_t}(t) = Z(t)$ for any $t \geq 0$ and:
$S_{i,n} = \lfloor n^\alpha\rfloor^{H_{t_0}} \sup_{(i-1)/\lfloor n^\alpha\rfloor \leq s,t \leq i/\lfloor n^\alpha \rfloor} B_{H_{t_0}}(t_0+t-1/(2n^\gamma)) - B_{H_{t_0}}(t_0+s-1/(2n^\gamma))$ for each $1 \leq i \leq \lfloor n^{\alpha-\gamma}\rfloor - 1$. We have

$$\frac{1}{\lfloor n^{\alpha-\gamma}\rfloor - 1} \sum_{i=1}^{\lfloor n^{\alpha-\gamma}\rfloor - 1} R_{i,n} =$$

$$\frac{\lfloor n^\alpha\rfloor^{H_{t_0}}}{\lfloor n^{\alpha-\gamma}\rfloor - 1} \sum_{i=1}^{\lfloor n^{\alpha-\gamma}\rfloor - 1} \sup_{(i-1)/\lfloor n^\alpha\rfloor \leq s,t \leq i/\lfloor n^\alpha \rfloor} \{B_{H_{t_0}}(t_0+t-1/(2n^\gamma)) - B_{H_{t_0}}(t_0+s-1/(2n^\gamma)) +$$
$$Z(t_0+t-1/(2n^\gamma)) - B_{H_{t_0}}(t_0+t-1/(2n^\gamma)) + B_{H_{t_0}}(t_0+s-1/(2n^\gamma)) - Z(t_0+s-1/(2n^\gamma))\}$$
$$\leq \frac{1}{\lfloor n^{\alpha-\gamma}\rfloor - 1} \sum_{i=1}^{\lfloor n^{\alpha-\gamma}\rfloor - 1} S_{i,n} + 2\lfloor n^\alpha\rfloor^{H_{t_0}} \sup_{(i-1)/\lfloor n^\alpha\rfloor \leq t \leq i/\lfloor n^\alpha \rfloor} |Z(t_0+t-1/(2n^\gamma)) - B_{H_{t_0}}(t_0+t-1/(2n^\gamma))|$$

We derive from (K) in ANNEX B that:

$$\lim_{n\to\infty} \frac{1}{\lfloor n^{\alpha-\gamma}\rfloor - 1} \sum_{i=1}^{\lfloor n^{\alpha-\gamma}\rfloor - 1} S_{i,n} = m_{H_{t_0}} \qquad \text{a.s.} \tag{3.17}$$

as soon as $\alpha > \gamma + \max\{1, \frac{1}{2(1-H_{t_0})}\}$.

By (2.22) of Step 4 of the proof of Theorem 4 we derive that for any $1 \leq i \leq \lfloor n^{\alpha-\gamma} \rfloor - 1$,

$$\lfloor n^\alpha \rfloor^{H_{t_0}} \sup_{(i-1)/\lfloor n^\alpha \rfloor \leq t \leq i/\lfloor n^\alpha \rfloor} |Z(t_0+t-1/(2n^\gamma))-B_{H_{t_0}}(t_0+t-1/(2n^\gamma))| \leq O(\lfloor n^\alpha \rfloor^{H_{t_0}} n^{-\gamma} \log n^\gamma).$$

Thus, if $\zeta H_{t_0} < 1$ we have almost surely:

$$\lim_{n\to\infty} \frac{1}{\lfloor n^{\alpha-\gamma} \rfloor - 1} \sum_{i=1}^{\lfloor n^{\alpha-\gamma} \rfloor - 1} \lfloor n^\alpha \rfloor^{H_{t_0}} \sup_{(i-1)/\lfloor n^\alpha \rfloor \leq t \leq i/\lfloor n^\alpha \rfloor} |Z(t_0+t-1/(2n^\gamma))-B_{H_{t_0}}(t_0+t-1/(2n^\gamma))| = 0,$$

which, combined with (3.17), entails:

$$\lim_{n\to\infty} \frac{1}{\lfloor n^{\alpha-\gamma} \rfloor - 1} \sum_{i=1}^{\lfloor n^{\alpha-\gamma} \rfloor - 1} R_{i,n} \leq m_{H_{t_0}} \qquad \text{a.s.} \tag{3.18}$$

We have

$$\frac{1}{\lfloor n^{\alpha-\gamma} \rfloor - 1} \sum_{i=1}^{\lfloor n^{\alpha-\gamma} \rfloor - 1} R_{i,n} \geq \frac{1}{\lfloor n^{\alpha-\gamma} \rfloor - 1} \sum_{i=1}^{\lfloor n^{\alpha-\gamma} \rfloor - 1} S_{i,n} + \lfloor n^\alpha \rfloor^{H_{t_0}} \inf_{(i-1)/\lfloor n^\alpha \rfloor \leq s,t \leq i/\lfloor n^\alpha \rfloor}$$
$$\{Z(t_0+t-1/(2n^\gamma)) - B_{H_{t_0}}(t_0+t-1/(2n^\gamma)) + B_{H_{t_0}}(t_0+s-1/(2n^\gamma)) - Z(t_0+s-1/(2n^\gamma))\}$$
$$= \frac{1}{\lfloor n^{\alpha-\gamma} \rfloor - 1} \sum_{i=1}^{\lfloor n^{\alpha-\gamma} \rfloor - 1} S_{i,n}.$$

(3.17) entails that

$$\lim_{n\to\infty} \frac{1}{\lfloor n^{\alpha-\gamma} \rfloor - 1} \sum_{i=1}^{\lfloor n^{\alpha-\gamma} \rfloor - 1} R_{i,n} \geq m_{H_{t_0}} \qquad \text{a.s.}$$

which, combined with (3.18), ends the proof of (3.16).
The proof of (1) is completed following the same line as in Step 5 of the proof of (1) in Theorem A (Annex B).

• (2) and (3). Set for convenience
$V_{i,n} = \lfloor n^\alpha \rfloor^{H_{t_0}}(Z(t_0 + i/\lfloor n^\alpha \rfloor - 1/(2n^\gamma)) - Z(t_0 + (i-1)/\lfloor n^\alpha \rfloor - 1/(2n^\gamma)))$ for each $1 \leq i \leq \lfloor n^{\alpha-\gamma} \rfloor - 1$. $V_{i,n}/\lfloor n^\alpha \rfloor^{H_{t_0}}$ are the increments corresponding to the subdivision of the interval $[t_0 - 1/(2n^\gamma), t_0 + 1/(2n^\gamma)]$ in time steps of size $1/\lfloor n^\alpha \rfloor$. We show that:

$$\lim_{n\to\infty} \frac{1}{\lfloor n^{\alpha-\gamma} \rfloor - 1} \sum_{i=1}^{\lfloor n^{\alpha-\gamma} \rfloor - 1} |V_{i,n}| = \sqrt{\frac{2}{\pi}} \qquad \text{a.s.} \tag{3.19}$$

Write $B_{H_t}(t) = Z(t)$ for any $t \geq 0$ and:
$W_{i,n} = \lfloor n^\alpha \rfloor^{H_{t_0}}(B_{H_{t_0}}(t_0 + i/\lfloor n^\alpha \rfloor - 1/(2n^\gamma)) - B_{H_{t_0}}(t_0 + (i-1)/\lfloor n^\alpha \rfloor - 1/(2n^\gamma)))$ for each $1 \leq i \leq \lfloor n^{\alpha-\gamma} \rfloor - 1$.

$$\frac{1}{\lfloor n^{\alpha-\gamma} \rfloor - 1} \sum_{i=1}^{\lfloor n^{\alpha-\gamma} \rfloor - 1} |V_{i,n}| = \frac{1}{\lfloor n^{\alpha-\gamma} \rfloor - 1} \sum_{i=1}^{\lfloor n^{\alpha-\gamma} \rfloor - 1} (|V_{i,n}| - |W_{i,n}|) + \frac{1}{\lfloor n^{\alpha-\gamma} \rfloor - 1} \sum_{i=1}^{\lfloor n^{\alpha-\gamma} \rfloor - 1} |W_{i,n}|.$$

- In ANNEX D, we show that :

$$\lim_{n\to\infty} \frac{1}{\lfloor n^{\alpha-\gamma}\rfloor - 1} \sum_{i=1}^{\lfloor n^{\alpha-\gamma}\rfloor-1} |W_{i,n}| = \sqrt{\frac{2}{\pi}} \qquad \text{a.s.} \tag{3.20}$$

as soon as $\alpha > \gamma + \max\{1, \frac{1}{4(1-H_{t_0})}\}$, which is true when (3.15) holds.

- We show that

$$\lim_{n\to\infty} \frac{1}{\lfloor n^{\alpha-\gamma}\rfloor - 1} \sum_{i=1}^{\lfloor n^{\alpha-\gamma}\rfloor-1} (|V_{i,n}| - |W_{i,n}|) = 0 \qquad \text{a.s.} \tag{3.21}$$

Using (2.0), the notation of Theorem 4 and (3.1), for each $\epsilon > 0$ and for each $p \geq 1$ we have

$$P\left(\left|\frac{1}{\lfloor n^{\alpha-\gamma}\rfloor - 1} \sum_{i=1}^{\lfloor n^{\alpha-\gamma}\rfloor-1} (|V_{i,n}| - |W_{i,n}|)\right| > \epsilon\right) \leq \frac{1}{\epsilon^p} \max_{1\leq i\leq \lfloor n^{\alpha-\gamma}\rfloor-1} E|(|V_{i,n}| - |W_{i,n}|)^p|$$
$$= \frac{2^{p/2}\Gamma(\frac{p+1}{2})}{\epsilon^p\Gamma(\frac{1}{2})} \max_{1\leq i\leq \lfloor n^{\alpha-\gamma}\rfloor-1} [Var(V_{i,n} - W_{i,n})]^{p/2}$$
$$= \frac{2^{p/2}\Gamma(\frac{p+1}{2})}{\epsilon^p\Gamma(\frac{1}{2})} \max_{1\leq i\leq \lfloor n^{\alpha-\gamma}\rfloor-1} \lfloor n^{\alpha}\rfloor^{pH_{t_0}} \times$$
$$[Var(B_{H_{t_0+i/\lfloor n^\alpha\rfloor-1/(2n^\gamma)}}(t_0 + i/\lfloor n^\alpha\rfloor - 1/(2n^\gamma)) - B_{H_{t_0}}(t_0 + i/\lfloor n^\alpha\rfloor - 1/(2n^\gamma))$$
$$- B_{H_{t_0+(i-1)/\lfloor n^\alpha\rfloor-1/(2n^\gamma)}}(t_0 + (i-1)/\lfloor n^\alpha\rfloor - 1/(2n^\gamma)) + B_{H_{t_0}}(t_0 + (i-1)/\lfloor n^\alpha\rfloor - 1/(2n^\gamma))]^{p/2}$$
$$\leq \frac{2^{p}\Gamma(\frac{p+1}{2})}{\epsilon^p\Gamma(\frac{1}{2})} \lfloor n^{\alpha}\rfloor^{pH_{t_0}} [\max_{1\leq i\leq \lfloor n^{\alpha-\gamma}\rfloor-1} Var(D_{H_{t_0+i/\lfloor n^\alpha\rfloor-1/(2n^\gamma)}, H_{t_0}}(t_0 + i/\lfloor n^\alpha\rfloor - 1/(2n^\gamma)))$$
$$+ \max_{1\leq i\leq \lfloor n^{\alpha-\gamma}\rfloor-1} Var(D_{H_{t_0+(i-1)/\lfloor n^\alpha\rfloor-1/(2n^\gamma)}, H_{t_0}}(t_0 + (i-1)/\lfloor n^\alpha\rfloor - 1/(2n^\gamma)))]^{p/2}$$
$$\leq \frac{2^{(p+1)}\beta_0^{p/2}\Gamma(\frac{p+1}{2})}{\epsilon^p\Gamma(\frac{1}{2})} \lfloor n^{\alpha}\rfloor^{pH_{t_0}} \max_{0\leq i\leq \lfloor n^{\alpha}\rfloor-1} |i/\lfloor n^\alpha\rfloor - 1/(2n^\gamma)|^{\beta p}$$
$$\leq \frac{2^{(p+1)}\beta_0^{p/2}\Gamma(\frac{p+1}{2})}{\epsilon^p\Gamma(\frac{1}{2})} \lfloor n^{\alpha}\rfloor^{pH_{t_0}} n^{-\gamma\beta p}.$$

$\gamma\beta - (\alpha - \gamma)H_{t_0}$ is positive by condition (3.15). Setting $p > \frac{1}{\gamma\beta-(\alpha-\gamma)H_{t_0}}$, (3.21) follows readily by the Borel-Cantelli lemma.

The end of the proof follows the same line as the proof of the corresponding statement (2) presented in ANNEX B. □

Remark 2: The upper bound for ζ in Theorem 11 have the following intuitive interpretation : assume that H has a sharp maximum at t_0. Then, on any small neighbourhood of t_0, the contributions to the local irregularity will be dominated by the boxes which do not contain t_0. Thus, to correctly evaluate the irregularity around t_0, the neighbourhood should not be divided into "too many" boxes. That is why, for high values of $H(t)$, the rate of growth of the number of boxes should be kept as small as possible.

4. Simulation. - Estimation.

We present in this section a method for simulating an mBm. This method is based upon Theorem 4: we assume given a function $t \longmapsto H(t)$ and an integer N representing the desired size of the sample. For each value of $H(i/N)$, $1 \leq i \leq N$, an fBm $B_{H(i/N)}$ of exponent $H(i/N)$ is generated. The mBm W_H is then obtained by setting:

$$W_H(\frac{i}{N}) = B_{H_{\frac{i}{N}}}(\frac{i}{N}) \qquad 1 \leq i \leq N.$$

In practice, any method may be used for generating each fBm. In the following simulations the so called "Random Midpoints Displacement" method (see e.g. Barnsley *et al*) have been used. Although this method does not give exact results, it is widely used because of its simplicity.

Annex A gives an algorithm for the simulation of mBm which is an adaptation of the method FM1D presented in Barnsley *et al* (p. 86).

We have also performed some estimations of the function H, based upon the estimator presented in Peltier - Lévy Véhel (1994). We recall the main result of this paper:

Let $(X_H(t))_{t\in[0,1]}$ be a standard fractional Brownian motion and $\{X_{i,n} = X_H\left(\frac{i}{n}\right), 0 \leq i \leq n\}$ be the sampled process. Let

$$S_n = \frac{1}{n-1}\sum_{i=1}^{n-1} |X_{i+1,n} - X_{i,n}|,$$

and

$$H_n = -\frac{\log\left[\sqrt{\frac{\pi}{2}}S_n\right]}{\log(n-1)}.$$

Then

$$\lim_{n\to\infty} H_n = H \qquad \text{a.s.}$$

In the following, we make no attempt to justify rigorously the use of this estimator for the functional parameter H of the mBm. Instead, we just give some intuitive arguments to explain why this method performs reasonably well.

The continuity of $H(t)$ together with Theorem 4 yields that, for each $t_0 \in [0,1]$, there exists a neighbourhood $\mathcal{V}$ of t_0 in which we may estimate the function $H(t)$ "as if" it were a constant H_{t_0} on $\mathcal{V}$. Thus, we will assume that there exists an optimal neighbourhood for which the estimation method performs well. We will not investigate the problem of the determination of the optimal neighbourhood, and will restrict ourselves in the sequel to functional estimations with a fixed interval length.

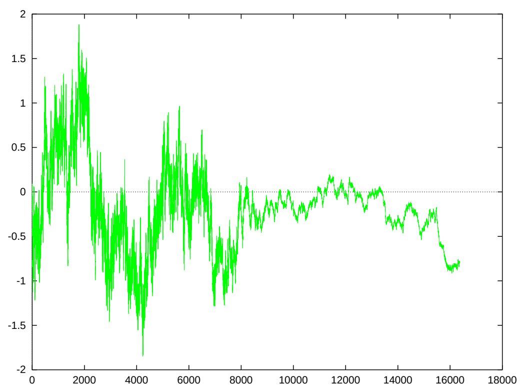
Let n be the number of data of a sample mBm. Let $1 < k < n$ be the length of the neighbourhood used for estimating the functional parameter. We will estimate $H(t)$ only for t in $[k/n, 1-k/n]$. Without loss of generality, we assume $m = n/k$ to be an integer. Then, our estimator of $H(i/(n-1))$ is the following

$$\hat{H}_{i/(n-1)} = -\frac{\log[\sqrt{\pi/2}S_{k,n}(i)]}{\log(n-1)}$$

where

$$S_{k,n}(i) = \frac{m}{n-1} \sum_{j \in [i-k/2, i+k/2]} |X_{j+1,n} - X_{j,n}|.$$

We have considered four different examples, where $H(t)$ evolves towards situations of increasing complexity. Figure 1 to 4 show that the method performs reasonably well compared for instance to that of Flandrin *et al* (1993 , 1994).



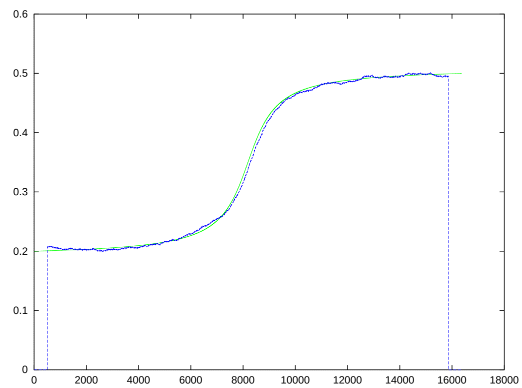


FIGURE 1. Sample path of an mBm (left, $n = 16384$) with an arctangent functional parameter (right, theoretical : dotted line ; estimated : continuous line, $k = 1024$)

To introduce a possible generalization of mBm, the last two examples deal with a case where the functional parameter $H(t)$ is not continuous.

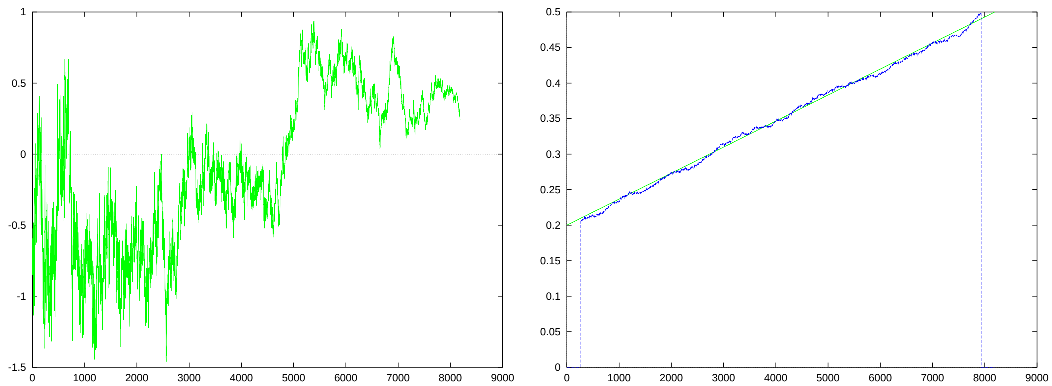


FIGURE 2. Sample path of an mBm (left, $n = 8192$) with a linear functional parameter (right, theoretical : dotted line ; estimated : continuous line, $k = 512$)

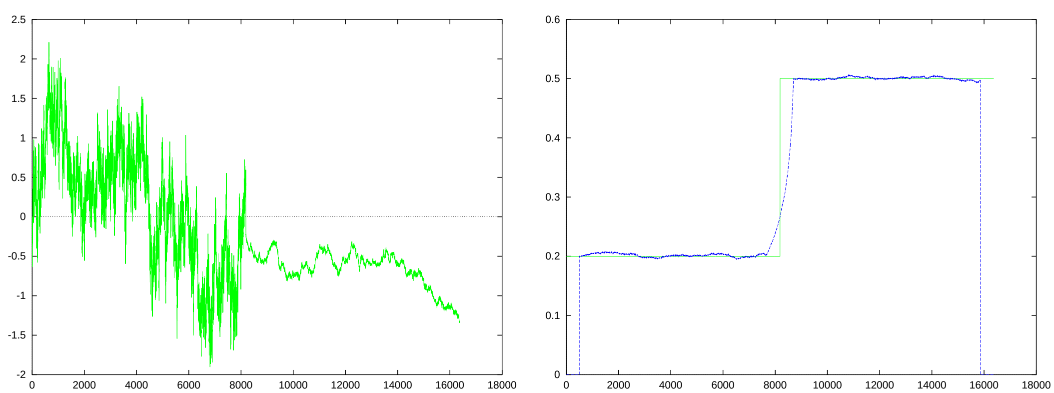


FIGURE 3. Sample path of an mBm (left, $n = 16384$) with a step functional parameter (right, theoretical : dotted line ; estimated :continuous line, $k = 1024$)

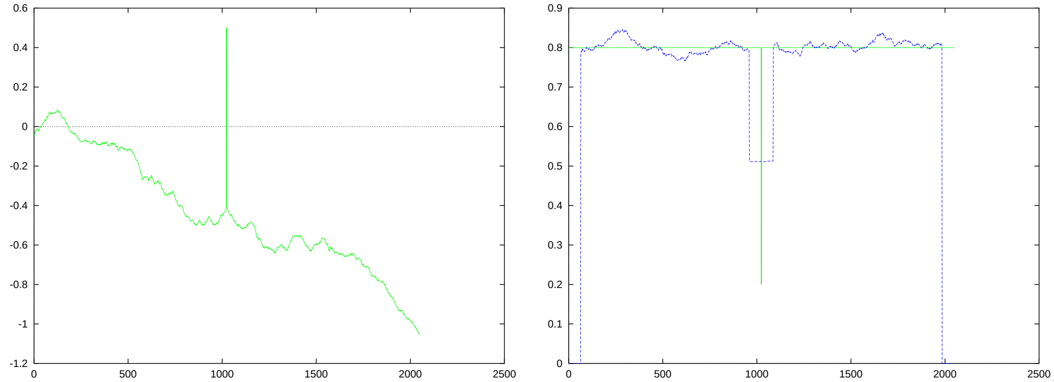


FIGURE 4. Sample path of an mBm (left, $n = 2048$) with a Dirac$=0.8\mathbb{1}_{\{t\neq 0.5\}}(t) + 0.2\mathbb{1}_{\{0.5\}}(t)$ (right, theoretical : dotted line ; estimated :continuous line, $k = 127$)

ACKNOWLEDGEMENT

The authors would like to thank professors Broniatowski and Deheuvels for their valuable comments on earlier versions of this paper.

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A N N E X A

ALGORITHM mBm *(Y,maxlevel,sigma,H(),seed)*

Title	*One-dimensional*	*multifractal motion via successive random additions*
Arguments	*Y[ ]:*	*real array of size* $2^{maxlevel}+1$
	maxlevel:	*maximal number of recursions*
	sigma:	*initial standard deviation*
	H(), $(0 < H(t) < 1)$*:*	*is the functional parameter of mBm*
	seed:	*seed value for random number generator*
	Globals	*delta: array holding standard deviations* Δ_i
	Variables	*i,N,d,D: integers*
		level: integer
		h: real
		X[ ]: real array of size $2^{maxlevel}+1$

```
BEGIN
    N:=power(2,maxlevel)
    For t:=1 to N do
        InitGauss(seed)
        h:=H(t/N)
        For i:=1 to maxlevel do
            delta[i]:=(sigma*power(0.5,i*h)*sqrt(0.5)* sqrt(1-power(2,2*h-2)));
        END FOR
        X[0]:=0
        X[N]:=sigma * Gauss ()
        D:=N
        d:=D/2
        level:=1
        WHILE(level<=maxlevel) DO
            FOR i:=d to N-d STEP D DO
                IF (ABS(t-i)<D) THEN X[i]:=0.5*(X[i-d]+X[i+d])
            END FOR
            FOR i:=0 to N STEP d DO
                X[i]:=X[i]+delta[level]*Gauss();
            END FOR
            D:=D/2
            d:=d/2
            level:=level + 1
        END WHILE
        Y[t]:=X[t]
    END FOR
END
```

ANNEX B

Proof of Theorem A.

(1) We split the proof into five steps.

Step 1.

- We recall (see e.g. Billingsley (1968) p. 72) that if $\{W(s), -\infty < s < \infty\}$ denotes a Wiener process extended to the real line, we have

$$P\left[\sup_{t\in[0,1]} W(t) \leq \alpha\right] = \frac{2}{\sqrt{2\pi}}\int_0^\alpha e^{-\frac{1}{2}u^2}du, \qquad \alpha \geq 0. \tag{A}$$

Thus, we easily obtain

$$E\left(\sup_{0\leq s,t\leq 1} |W(t)-W(s)|\right) = 2\sqrt{\frac{2}{\pi}}. \tag{B}$$

- We recall Slepian's Inequality (see e.g. Adler (1990) p. 49). *If X and Y are a.s bounded, centered Gaussian processes on T such that $E(X_t^2) = E(Y_t^2)$ for all $t \in T$, and*

$$E(X_t - X_s)^2 \leq E(Y_t - Y_s)^2 \quad \textit{for all } s,t \in T,$$

then for all real λ

$$P\left[\sup_{t\in T} X_t > \lambda\right] \leq P\left[\sup_{t\in T} Y_t > \lambda\right], \tag{C}$$

and then,

$$E\left[\sup_{t\in T} X_t\right] \leq E\left[\sup_{t\in T} Y_t\right]. \tag{D}$$

- We recall a proposition on the box dimension of graphs (see e.g. Falconer (1990) p. 146). Given a function f and an interval $[t_1, t_2]$, we write R_f for the *maximum range* of f over an interval,

$$R_f[t_1,t_2] = \sup_{t_1<t,u<t_2} |f(t)-f(u)|.$$

Proposition: *Let $f : [0,1] \to I\!R$ be continuous. Let $0 < \delta < 1$, and q be the least integer greater than or equal to $1/\delta$. Then, if N_δ is the number of squares of the δ-mesh that intersect graph f,*

$$\delta^{-1}\sum_{i=0}^{q-1} R_f[i\delta,(i+1)\delta] \leq N_\delta \leq 2q + \delta^{-1}\sum_{i=0}^{q-1} R_f[i\delta,(i+1)\delta]. \tag{E}$$

Step 2. We show that for each $0 < H < 1$

$$\sigma\sqrt{\frac{2}{\pi}} \leq E(R_{X_H}[0,1]) = m_H \leq 4\sigma\sqrt{2\pi}. \tag{F}$$

For $\frac{1}{2} \leq H < 1$,

$$m_H \leq 2\sigma\sqrt{\frac{2}{\pi}} \tag{F'}$$

and for $0 < H \leq \frac{1}{2}$,

$$2\sigma\sqrt{\frac{2}{\pi}} \leq m_H. \tag{F"}$$

The first inequality of the first statement results from (2.0) and the following obvious inequality:

$$|X_H(1)| \leq R_{X_H}[0,1].$$

(2.14) yields that $m_H < \infty$.
Integrating by parts and using (2.15) we have:

$$\begin{aligned} E(R_{X_H}[0,1]) &= 2E\left(\sup_{0\leq t\leq 1} X_H(t)\right) \\ &\leq 2\int_0^{\infty} 2e^{-\frac{(x-m_H/2)^2}{2\sigma^2}}dx = 4\sigma\int_{-m_H/(2\sigma)}^{\infty} e^{-\frac{u^2}{2}}du \\ &\leq 4\sigma\int_{\mathbb{R}} e^{-\frac{u^2}{2}}du = 4\sigma\sqrt{2\pi}, \end{aligned}$$

which completes the proof of (F).
We now establish both inequalities in (F') and (F"):
Let $Z_t^{1,H} = \frac{1}{\sigma}X_H(t^{\frac{1}{2H}})$ and $Z_t^{2,H} = \frac{1}{\sigma}X_H(t^{-\frac{1}{2H}})$ be Gaussian processes defined respectively for $t \in [0,1]$ and $t \in [1,\infty)$. We have the obvious equalities:

$$\sup_{0\leq t\leq 1} Z_t^{1,H} = \frac{1}{\sigma}\sup_{0\leq t\leq 1} X_H(t) \text{ and } \sup_{1\leq t<\infty} Z_t^{2,H} = \frac{1}{\sigma}\sup_{0\leq t\leq 1} X_H(t). \tag{G}$$

We easily have that for each $0 \leq s \leq t \leq 1$:

$$E[(Z_t^{1,H})^2] = t = E[(Z_t^{1,\frac{1}{2}})^2]$$

and

$$E[(Z_t^{1,H} - Z_s^{1,H})^2] = (t^{\frac{1}{2H}} - s^{\frac{1}{2H}})^{2H}.$$

- If $0 < H \leq \frac{1}{2}$ then $E[(Z_t^{1,H} - Z_s^{1,H})^2] \geq (t-s) = E[(Z_t^{1,\frac{1}{2}} - Z_s^{1,\frac{1}{2}})^2]$.
- If $\frac{1}{2} \leq H < 1$ then $E[(Z_t^{1,H} - Z_s^{1,H})^2] \leq (t-s) = E[(Z_t^{1,\frac{1}{2}} - Z_s^{1,\frac{1}{2}})^2]$.

Combining Slepian's Inequality (D) applied to $Z^{1,H}$ and $Z^{1,\frac{1}{2}}$, (G) and (B) we derive that:

- if $0 < H \leq \frac{1}{2}$ then $E\left[\sup_{0\leq s,t\leq 1} |X_H(t) - X_H(s)|\right] \geq 2\sigma\sqrt{\frac{2}{\pi}}$
- if $\frac{1}{2} \leq H < 1$ then $E\left[\sup_{0\leq s,t\leq 1} |X_H(t) - X_H(s)|\right] \leq 2\sigma\sqrt{\frac{2}{\pi}}$.

Thus, we finally obtain (F') and (F").
Step 3. For the sake of notational simplicity, let for each $i \geq 1$,
$R_{X_H}^{i,n} = (n-1)^H R_{X_H}[i/(n-1),(i+1)/(n-1)]$. $R_{X_H}^{i,n}$ is a stationary process whose law is independent of n. By Step 2, $E(R_{X_H}^{i,n}) = 2\sigma\sqrt{\frac{2}{\pi}}$.
We show that for each $1 \leq i < j \leq n-1$, we have

$$Cov(R_{X_H}^{i,n}, R_{X_H}^{j,n}) = O\left(\frac{1}{|j-i|^{2(1-H)}}\right). \tag{H}$$

Let $\mathcal{F}_1^n$ be the σ algebra generated by $\{X_H(t)-X_H(s)\,;s,t\in\left[\frac{1}{n-1},\frac{2}{n-1}\right]\}$. Let $\mathcal{Y}(t,u)$ (respectively $\mathcal{Y}'(t,u)$) be the random variable defined by:

$$\mathcal{Y}(t,u)=\sigma\frac{X_H(t)-E(X_H(t)\mid\mathcal{F}_1^n)-(X_H(u)-E(X_H(u)\mid\mathcal{F}_1^n))}{\sqrt{Var(X_H((j+2)/(n-1))-X_H((j+1)/(n-1))\mid\mathcal{F}_1^n)}}$$

(respectively

$$\mathcal{Y}'(t,u)=E(X_H(t)\mid\mathcal{F}_1^n)-E(X_H(u)\mid\mathcal{F}_1^n).)$$

We have

$$E(\sup_{\frac{j+1}{n-1}<t,u<\frac{j+2}{n-1}}\mathcal{Y}(t,u)\mid\mathcal{F}_1^n)=E(R_{X_H}^{1,n}).$$

By *e.g.* Mandelbrot (1968) p. 434 and Grimmet *et al* (1992) p. 384 we have:

$$Var(X_H((j+2)/(n-1))-X_H((j+1)/(n-1))\mid\mathcal{F}_1^n)=\frac{\sigma^2}{(n-1)^{2H}}(1-O((\frac{1}{|j|^{2(1-H)}})^2))$$

and

$$\mathcal{Y}'(t,u)=O(\frac{1}{|j|^{2(1-H)}})\mathcal{Y}"(t,u)$$

where $\mathcal{Y}"(t,u)$ is a Gaussian random variable with mean and variance bounded by constants independent of j. We have

$$\begin{aligned}
&Cov(R_{X_H}^{1,n},R_{X_H}^{j+1,n})=E[R_{X_H}^{1,n}R_{X_H}^{j+1,n}]-(E(R_{X_H}^{1,n}))^2=E[R_{X_H}^{1,n}E(R_{X_H}^{j+1,n}\mid\mathcal{F}_1^n)]-(E(R_{X_H}^{1,n}))^2\\
&=E\left[R_{X_H}^{1,n}E(R_{X_H}^{j+1,n}-\sup_{\frac{j+1}{n-1}<t,u<\frac{j+2}{n-1}}|\mathcal{Y}(t,u)|\mid\mathcal{F}_1^n)\right]\\
&\leq E\left[R_{X_H}^{1,n}E(R_{X_H}^{j+1,n}-(R_{X_H}^{j+1,n}-\sup_{\frac{j+1}{n-1}<t,u<\frac{j+2}{n-1}}|\mathcal{Y}'(t,u)|)(1+O((\frac{1}{|j|^{2(1-H)}})^2))\mid\mathcal{F}_1^n)\right]\\
&=E\left[R_{X_H}^{1,n}\sup_{\frac{j+1}{n-1}<t,u<\frac{j+2}{n-1}}|\mathcal{Y}'(t,u)|(1+O((\frac{1}{|j|^{2(1-H)}})^2))\right]=E[R_{X_H}^{1,n}O(\frac{1}{|j|^{2(1-H)}})\sup_{\frac{j+1}{n-1}<t,u<\frac{j+2}{n-1}}|\mathcal{Y}"(t,u)|]\\
&=O(\frac{1}{|j|^{2(1-H)}}),
\end{aligned}$$

and similarly we have $Cov(R_{X_H}^{1,n},R_{X_H}^{j+1,n})\geq O(c_0(j))$, which yields (H).

Step 4. Let $(\nu'_n)_{n\in\mathbb{N}}$ be a sequence defined by $\nu'_n=\lfloor n^{\frac{1}{1-H}-b'_H}\rfloor$ and $0<b'_H<\min\{\frac{H}{1-H},\frac{1}{2(1-H)}\}$ be fixed. We show that

$$\lim_{n\to\infty}N_{\delta_{\nu'_n}}\delta_{\nu'_n}^s=m_H. \tag{I}$$

- **Case $H=\frac{1}{2}$.**

  With the notation of Step 2, From (E) we first derive that

  $$\frac{1}{n-1}\sum_{i=1}^{n-1}R_{X_H}^{i,n}\leq N_{\delta_n,n}\delta_n^s\leq\frac{1}{n-1}\sum_{i=1}^{n-1}R_{X_H}^{i,n}+\delta_n^{1-H}. \tag{J}$$

  Combining the strong law of large numbers, (B) and (J), (I) holds and we have $m_{\frac{1}{2}}=2\sigma\sqrt{\frac{2}{\pi}}$.

- **Case $H \neq \frac{1}{2}$.**

  We split the proof of this statement into two steps.

  Sub-step 1: We denote by $u_n \approx v_n$ (resp. $u_n \sim v_n$) the fact that $u_n/v_n \to \lambda$ as $n \to \infty$ for some $\lambda \in (0,\infty)$ (resp. for $\lambda = 1$) . Through elementary analysis, we have: *For each $x \in \mathbb{R}$, we have*

$$\frac{1}{n^x}\sum_{i=1}^{n}\frac{1}{i} \approx \frac{\log n}{n^x},$$

$$-\infty < y < 1, \qquad \frac{1}{n^x}\sum_{i=1}^{n}\frac{1}{i^y} \approx \frac{1}{n^{x+y-1}},$$

$$y > 1, \qquad \frac{1}{n^x}\sum_{i=1}^{n}\frac{1}{i^y} \approx \frac{1}{n^x}.$$

  Sub-step 2: Fix any $\epsilon > 0$ and set $q_n = P\left(\frac{1}{\nu'_n-1}|\sum_{i=1}^{\nu'_n-1}(R_{X_H}^{i,\nu'_n} - E(R_{X_H}^{i,\nu'_n})| > \epsilon\right)$.

  We have

$$\begin{aligned}
q_n &\leq \frac{1}{(\nu'_n-1)^2\epsilon^2}E\left(|\sum_{i=1}^{\nu'_n-1}(R_{X_H}^{i,\nu'_n} - E(R_{X_H}^{i,\nu'_n})|^2\right)\\
&= \frac{1}{(\nu'_n-1)^2\epsilon^2}\{\sum_{i=1}^{\nu'_n-1} Var(R_{X_H}^{i,\nu'_n}) + 2\sum_{1\leq i<j\leq \nu'_n-1} Cov(R_{X_H}^{i,\nu'_n}, R_{X_H}^{j,\nu'_n})\}\\
&= \frac{1}{(\nu'_n-1)^2\epsilon^2}\{(\nu'_n-1)v_{\mathfrak{R}} + 2\sum_{i=1}^{\nu'_n-2}(\nu'_n-i-1)Cov(R_{X_H}^{1,\nu'_n}, R_{X_H}^{i+1,\nu'_n})\}\\
&= \frac{v_{\mathfrak{R}}}{(\nu'_n-1)\epsilon^2} + \frac{2}{(\nu'_n-1)\epsilon^2}\sum_{i=1}^{\nu'_n-2}Cov(R_{X_H}^{1,\nu'_n}, R_{X_H}^{i+1,\nu'_n}) - \frac{2}{(\nu'_n-1)^2\epsilon^2}\sum_{i=1}^{\nu'_n-2} iCov(R_{X_H}^{1,\nu'_n}, R_{X_H}^{i+1,\nu'_n}).
\end{aligned}$$

  In view of Step 1 we obtain $q_n = O(\frac{\log \nu'_n}{\nu'_n}) + O(\frac{1}{(\nu'_n)^{2(1-H)}})$, which combined with (H), (F) and the Borel-Cantelli lemma, yields:

$$\lim_{n\to\infty}\frac{1}{\nu'_n-1}\sum_{i=1}^{\nu'_n-1}R_{X_H}^{i,\nu'_n} = m_H \qquad \text{a.s.} \tag{K}$$

  Combining (K) and (J) we obtain (I).

Step 5. Next, we show that (1) holds for any $\delta > 0$. For any $\varepsilon > 0$, there exists $N_1 > 0$ such that, for each $n > N_1$,

$$|N_{\delta_{\nu'_n}}\delta_{\nu'_n}^s - c^{(1)}| \leq \varepsilon/6, \tag{a}$$

and there exists an $N_2 > 0$ such that, for each $n > N_2$,

$$|N_{\delta_{\nu'_n}}\delta_{\nu'_n}^s - N_{\delta_{\nu'_{n+1}}}\delta_{\nu'_{n+1}}^s| \leq \varepsilon/6. \tag{b}$$

We conclude by showing that there exists an $N_3 > 0$ such that, for each $n > N_3$,

$$N_{\delta_{\nu'_{n+1}}}(\delta_{\nu'_n} - \delta_{\nu'_{n+1}}) \leq \varepsilon/6. \tag{c}$$

To prove this last inequality, the following arguments are needed.

The study of the variations of the function $\phi(\eta) = (H-\eta)(H-\eta-b'_H(1-\eta)(1-H))$

on a neighbourhood of H shows that there exists η, such that $0 < \eta < H < \eta + b'_H(1-\eta)(1-H)$. Hence, by Lemma 2.1 and definition (2.0), there exists $N_3 > 0$ such that, for each $n > N_3$, we have

$$\frac{\log(N_{\delta_{\nu'_{n+1}}})}{-\log(\delta_{\nu'_{n+1}})} \leq 2-\eta.$$

Therefore, as $n \to \infty$, we have

$$\begin{aligned} N_{\delta_{\nu'_{n+1}}}(\delta_{\nu'_n} - \delta_{\nu'_{n+1}}) &\leq \frac{\delta_{\nu'_n} - \delta_{\nu'_{n+1}}}{(\delta_{\nu'_{n+1}})^{2-\eta}} \\ &\sim \frac{n^{b'_H - \frac{1}{1-H} - 1}}{n^{(b'_H - \frac{1}{1-H})(2-\eta)}} \times (\frac{1}{1-H} - b'_H) \\ &= \frac{1}{n^{1+(b'_H - \frac{1}{1-H})(1-\eta)}} \times (\frac{1}{1-H} - b'_H) \\ &= \frac{1}{n^{\frac{1-H+b'_H(1-H)(1-\eta)-1+\eta}{1-H}}} \times (\frac{1}{1-H} - b'_H) \\ &= \frac{1}{n^{\frac{\eta-H+b'_H(1-H)(1-\eta)}{1-H}}} \times (\frac{1}{1-H} - b'_H) \longrightarrow 0. \end{aligned}$$

which suffices for our needs.

By combining the above statements (a), (b) and (c), we see that, for sufficiently small values of $\delta > 0$, there exists $n > max\{N_1, N_2, N_3\}$, for which $\delta_{\nu'_{n+1}} < \delta < \delta_{\nu'_n}$, and,

$$\begin{aligned} |N_\delta \delta^s - c^{(1)}| &\leq |N_\delta \delta^s_{\nu'_n} - c^{(1)}| + |N_\delta \delta^s_{\nu'_{n+1}} - c^{(1)}| \\ &\leq |N_{\delta_{\nu'_n}} \delta^s_{\nu'_n} - c^{(1)}| + |N_{\delta_{\nu'_{n+1}}} \delta^s_{\nu'_n} - c^{(1)}| + |N_{\delta_{\nu'_n}} \delta^s_{\nu'_{n+1}} - c^{(1)}| + |N_{\delta_{\nu'_{n+1}}} \delta^s_{\nu'_{n+1}} - c^{(1)}| \\ &\leq 2|N_{\delta_{\nu'_n}} \delta^s_{\nu'_n} - c^{(1)}| + 2|N_{\delta_{\nu'_{n+1}}} \delta^s_{\nu'_{n+1}} - c^{(1)}| + N_{\delta_{\nu'_n}}(\delta^s_{\nu'_n} - \delta^s_{\nu'_{n+1}}) + N_{\delta_{\nu'_{n+1}}}(\delta^s_{\nu'_n} - \delta^s_{\nu'_{n+1}}) \\ &\leq \varepsilon. \end{aligned}$$

(2) **Step 1.** Let $\nu_n = \lfloor n^{\frac{1}{1-H} - b_H} \rfloor$ be a sequence such that $0 < b_H < \min\{\frac{H}{2(1-H)}, \frac{3}{4(1-H)}\}$. For each $n \geq 2$ and $0 \leq i \leq n-1$, set $\mathbb{X}_{i,n} = X_H(\frac{i}{n-1})$.
Set

$$Y_{i,n} = (n-1)^H(\mathbb{X}_{i+1,n} - \mathbb{X}_{i,n}), \qquad 0 \leq i \leq n-1.$$

Let $X_n : [0,1] \to \mathbb{R}$, $n = 1, 2, ...$ be a sequence of polygonal functions which converges uniformly to X_H on $[0,1]$ such that the vertices of the graph of each X_n are of the form $\{(\frac{k}{n-1}, X_n(\frac{k}{n-1})),\ 0 \leq k \leq n-1\}$, with $X_n(0) = X(0)$. Let $L_n = \sum_{i=1}^{n-1} |X_n(\frac{i+1}{n-1}) - X_n(\frac{i}{n-1})| = \sum_{i=1}^{n-1} |\mathbb{X}_{i+1,n} - \mathbb{X}_{i,n}|$. For each $n \geq 2$, we denote by $N_{\delta,n}$ the number of δ-mesh squares that intersect the graphs of X_n. Let $\delta_{\nu_n} = \frac{1}{\nu_n}$. We have the obvious inequalities

$$L_{\nu_n} \leq N_{\delta_{\nu_n},\nu_n} \delta_{\nu_n} \leq (L_{\nu_n} + 1).$$

This yields

$$L_{\nu_n} \delta^{1-H}_{\nu_n} \leq N_{\delta_{\nu_n},\nu_n} \delta^s_{\nu_n} \leq (L_{\nu_n} + 1)\delta^{1-H}_{\nu_n},$$

or equivalently

$$\frac{1}{\nu_n - 1}\sum_{i=1}^{\nu_n - 1} |Y_{i,\nu_n}| \le N_{\delta_{\nu_n},\nu_n}\delta_{\nu_n}^s \le \frac{1}{\nu_n - 1}\sum_{i=1}^{\nu_n - 1} |Y_{i,\nu_n}| + \delta_{\nu_n}^{1-H}.$$

We show in Peltier - Lévy Véhel (1994) that :

$$\lim_{n\to\infty} N_{\delta_{\nu_n},\nu_n}\delta_{\nu_n}^s = \sigma\sqrt{\frac{2}{\pi}}. \tag{L}$$

We recall in ANNEX D the proof of this result.

(L) combined with (I), entails the following remarkable result :

$$N_{\delta_{\nu_n}} = (2 + o(1))N_{\delta_{\nu_n},\nu_n} \quad \text{a.s.} \tag{M}$$

Following the same line of proof of (1) Step 5, denoting by $N_{\delta,\lfloor 1/\delta\rfloor}$ the number of δ-mesh squares that intersect the graph of $X_{\lfloor 1/\delta\rfloor}$, we have when the positive real δ tends to zero :

$$N_\delta = (2 + o(1))N_{\delta,\lfloor 1/\delta\rfloor} \quad \text{a.s.} \tag{N}$$

Step 2. Recall the definition of $N_\delta^{(1)}(F)$ and $N_\delta^{(2)}(F)$. For all $0 < \delta < 1$, set $n = \lfloor 1/\delta\rfloor$. With the notation of Lemma 2.2, by choosing $X = X_H$ in this lemma, and defining X_n accordingly, we denote respectively by $N_{\delta,n}^1(F)$ and $N_{\delta,n}^2(F)$ the number of δ-mesh squares and the smallest number of squares of size δ that intersect the graph of X_n. We will establish the following inequalities

$$N_\delta^1(F) \ge N_\delta^2(F) \ge N_{\delta,n}^2(F) \ge \frac{1}{2}N_{\delta,n}^1(F). \tag{O}$$

The two first inequalities in (3.27) are obvious. For the last inequality we assume without loss of generality that $(n-1)/2 \in \mathbb{N}$. For $1 \le i \le (n-1)/2$, we denote by F_i, the graph of X_n defined on $[\frac{2i-1}{n}, \frac{2i+1}{n}]$. We have the obvious equality

$$N_{\delta_n}^{1,n}(F) = \sum_{i=1}^{(n-1)/2} N_{\delta_n}^1(F_i).$$

We show in Annex C that the number N_i of any set of squares of size δ covering F_i satisfies the inequality

$$N_i \ge \frac{1}{2}N_\delta^1(F_i). \tag{P}$$

By (P), we have

$$\sum_{i=1}^{(n-1)/2} N_i \ge \frac{1}{2}N_{\delta_n}^1(F_i),$$

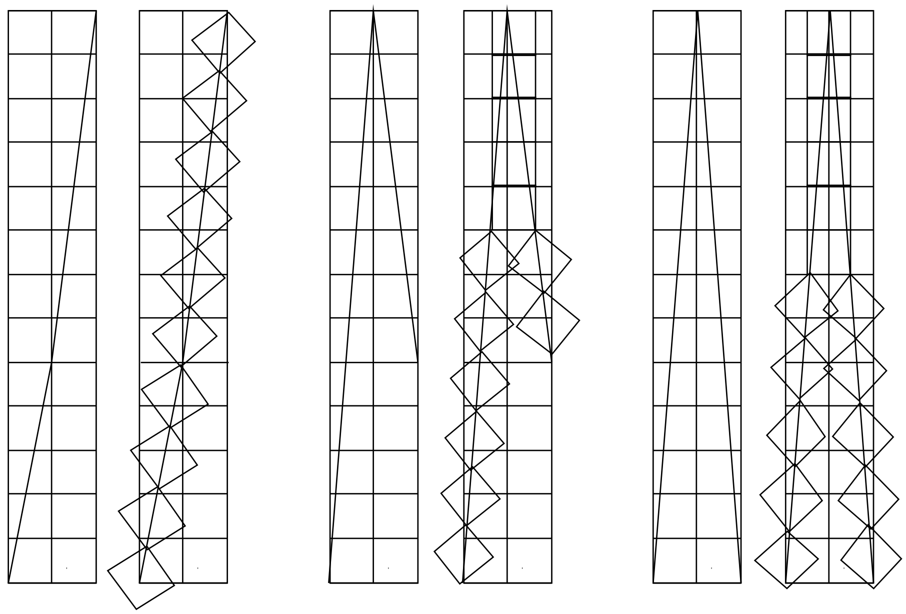
which completes the proof of the inequalities in (O).

Combining (N) and (O), we obtain (2).

(3) The proof of the first inequality is obtained by the same arguments as the ones used for (2). The last inequality is involved by the fact that *any square of size δ is covered by a closed ball of diameter $\sqrt{2}\delta$.* $\square$

ANNEX C

In view of proving (P), we consider three cases as shown in Figure 1 below. These cases show the typical aspects of the graph of X_n on three consecutive time indices, together with the corresponding possible coverings by squares.



CASE 1 **CASE 2** **CASE 3**

Figure 1

Elementary but lengthy geometric considerations allow to say that (P) holds in all cases (case 1 is obviously true and case 3 represents the worst situation which corresponds to the largest variation between $N^1_{\delta_n}(F_i)$ and N_i).

ANNEX D

Let $\nu_n = \lfloor n^{\frac{1}{1-H} - b_H} \rfloor$ be a sequence such that $0 < b_H < \min\{\frac{H}{2(1-H)}, \frac{3}{4(1-H)}\}$. For each $n \geq 2$ and $0 \leq i \leq n-1$, set $\mathbb{X}_{i,n} = X_H(\frac{i}{n-1})$.
Set

$$Y_{i,n} = (n-1)^H(\mathbb{X}_{i+1,n} - \mathbb{X}_{i,n}), \qquad 0 \leq i \leq n-1.$$

We show that almost surely we have:

$$\frac{1}{\nu_n - 1} \sum_{i=1}^{\nu_n - 1} |Y_{i,\nu_n}| = \sigma\sqrt{\frac{2}{\pi}}.$$

Step 1: We denote by $u_n \approx v_n$ (resp. $u_n \sim v_n$) the fact that $u_n/v_n \to \lambda$ as $n \to \infty$ for some $\lambda \in (0, \infty)$ (resp. for $\lambda = 1$) . Through elementary analysis, we have: *For each* $x \in \mathbb{R}$, *we have*

$$\frac{1}{n^x}\sum_{i=1}^{n}\frac{1}{i} \approx \frac{\log n}{n^x},$$

$$-\infty < y < 1, \qquad \frac{1}{n^x}\sum_{i=1}^{n}\frac{1}{i^y} \approx \frac{1}{n^{x+y-1}},$$

$$y > 1, \qquad \frac{1}{n^x}\sum_{i=1}^{n}\frac{1}{i^y} \approx \frac{1}{n^x}.$$

Step 2: The following result hlods: $Cov(|Y_{i,n}|, |Y_{j,n}|)$ does not depend on n, and

$$Cov(|Y_{i,n}|, |Y_{j,n}|) = O(\frac{1}{|j-i|^{4(1-H)}})$$

as $|j - i| \to \infty$ (see Lemma 3.3 of Peltier - Lévy Véhel (1994) for the proof).

Step 3: For each $1 \leq i \leq n$ we easily have $E(|Y_{i,n}|) = \sigma\sqrt{\frac{2}{\pi}}$ and $Var(|Y_{i,n}|) = v$ which is independent of i and n.
We have

$$\begin{aligned}
q_n &\leq \frac{1}{(\nu_n-1)^2\epsilon^2} E\left(|\sum_{i=1}^{\nu_n-1}(|Y_{i,\nu_n}| - E(|Y_{i,n}|)|^2 \right) \\
&= \frac{1}{(\nu_n-1)^2\epsilon^2}\{\sum_{i=1}^{\nu_n-1} Var(|Y_{i,\nu_n}|) + 2\sum_{1\leq i<j\leq \nu_n-1} Cov(|Y_{i,\nu_n}|, |Y_{j,\nu_n}|)\} \\
&= \frac{1}{(\nu_n-1)^2\epsilon^2}\{(\nu_n-1)v + 2\sum_{i=1}^{\nu_n-2}(\nu_n - i - 1)Cov(|Y_{1,\nu_n}|, |Y_{i+1,\nu_n}|)\} \\
&= \frac{v}{(\nu_n-1)\epsilon^2} + \frac{2}{(\nu_n-1)\epsilon^2}\sum_{i=1}^{\nu_n-2} Cov(|Y_{1,\nu_n}|, |Y_{i+1,\nu_n}|) - \frac{2}{(\nu_n-1)^2\epsilon^2}\sum_{i=1}^{\nu_n-2} iCov(|Y_{1,\nu_n}|, |Y_{i+1,\nu_n}|).
\end{aligned}$$

In view of Step 1 and Step 2, we obtain $q_n = O(\frac{\log \nu_n}{\nu_n}) + O(\frac{1}{\nu_n^{4(1-H)}})$, which combined with the Borel-Cantelli lemma, yields the result.

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ISSN 0249-6399