Ergodic Theorems for Stochastic Operators and Discrete Event Networks

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Abstract

We present a survey of the main ergodic theory techniques which are used in the study of iterates of monotone and homogeneous stochastic operators. It is shown that ergodic theorems on discrete event networks (queueing networks and/or Petri nets) are a generalization of these stochastic operator theorems. Kingman's subadditive ergodic Theorem is the key tool for deriving what we call first order ergodic results. We also show how to use backward constructions (also called Loynes schemes in network theory) in order to obtain second order ergodic results. We will propose a review of systems entering the framework insisting on two models, precedence constraints networks and Jackson type networks.

Introduction Many systems appearing in manufacturing, communication or computer science accept a description in terms of discrete event systems. A usual characteristic of these systems is the existence of some sources of randomness affecting their behaviour. Hence a natural framework to study them is the one of stochastic discrete event systems.

In this survey paper, we are concerned with two different types of models. First, we consider the study of the iterates $T_n \circ T_{n-1} \circ \cdots \circ T_0$, where $T_i : \mathbb{R}^k \times \Omega \to \mathbb{R}^k$ is a random monotone and homogeneous operator. Second, we introduce and study stochastic discrete event networks entering the so-called monotone-separable framework. A subclass of interest is that of stochastic open discrete event networks.

It will appear that these models, although they have been studied quite independently in the past years, have a lot of common points. They share the same kind of assumptions and properties : monotonicity, homogeneity and non-expansiveness. In fact, we are going to show that monotone-separable discrete event networks are a generalization of monotone-homogeneous operators. However, when a system can be modelled as an operator, it provides a more precise description and stronger results.

In both types of models, we are working with daters. Typically, we have to study a random process $X(n) \in \mathbb{R}^k$, where $X(n)_i$ represents the $n-th$

occurrence of some event in the system. We are going to propose two types of asymptotic results :

1. First order results, concerning the asymptotic rates $\lim_n X(n)_i/n$.
2. Second order results, concerning the asymptotic behaviour of differences such as $X(n)_i - X(n)_j$.

The main references for the results proposed in the paper are the following ones. First order results for operators appear in Vincent [43]. Second order results for operators are new. First and second order results for open discrete event networks are proved in Baccelli and Foss [5]. First order results for general discrete event networks are new. A more complete presentation will be done in a forthcoming paper [7].

The paper is organized as follows. In Part I, we treat first order results and in Part II, second order ones. In each part, we consider operators and discrete event networks separately. In a last part, we propose a review of systems entering the frameworks insisting on two models, precedence constraints networks and Jackson type networks.

We aim at emphasizing how theorems on stochastic systems are obtained as an interaction between structural properties of deterministic systems and probabilistic tools. In order to do so, we introduce first the probabilistic tools (§1 and 5). Then we present some properties on deterministic systems. At last, we prove the main theorem for stochastic systems.

Part I

First Order Ergodic Results

1 Probabilistic Tools

We consider a probability space $(\Omega, \mathcal{F}, P)$. We consider a bijective and bi-measurable shift operator $\theta : \Omega \to \Omega$. We assume that θ is stationary and ergodic with respect to the probability P.

Lemma 1.1 (Ergodic lemma). *If $\mathcal{A} \in \mathcal{F}$ is such that $\theta(\mathcal{A}) \subset \mathcal{A}$ then $P\{\mathcal{A}\} = 0$ or 1.*

Theorem 1.2 (Kingman's subadditive ergodic theorem [32]). *Let us consider $X_{l,n}, l < n \in \mathbb{Z}$, a doubly-indexed sequence of integrable random variables such that*

- **stationarity** *: $X_{n,n+p} = X_{0,p} \circ \theta^n, \ \forall n, p, \ p > 0$.*

- **boundedness** : *$E[X_{0,n}] \geqslant -Cn, \quad \forall n > 0$, for some finite constant $C > 0$.*
- **subadditivity** : *$X_{l,n} \leqslant X_{l,m} + X_{m,n}, \quad \forall l < m < n$.*

Then there exists a constant γ such that the following convergence holds both in expectation and a.s.

$$\lim_{n\to\infty} \frac{E[X_{0,n}]}{n} = \gamma, \quad \lim_{n\to\infty} \frac{X_{0,n}}{n} = \gamma \quad P \text{ a.s.} \tag{1.1}$$

Remark 1.1. The convergence in expectation is straightforward. In fact, we have by subadditivity, $E(X_{0,n}) \leqslant E(X_{0,m}) + E(X_{m,n})$. By stationarity, it implies $E(X_{0,n}) \leqslant E(X_{0,m}) + E(X_{0,n-m})$. The real sequence $u_n = \{E(X_{0,n})\}$ is subadditive, hence u_n/n converges in $\mathbb{R} \cup \{-\infty\}$. Because of the boundedness assumption, we conclude that the limit is finite.

Remark 1.2. If we have additivity instead of subadditivity, then the previous theorem reduces to the following result:

$$\lim_{n\to\infty} \frac{\Sigma_{i=0}^{n} X_{i,i+1}}{n} \overset{n\to\infty}{\longrightarrow} E(X_{0,1}) \;\; P \text{ a.s.}$$

When the sequence $\{X_{n,n+1}, n \in \mathbb{N}\}$ is i.i.d., this is simply the Strong Law of Large Numbers. More generally, when the sequence $\{X_{n,n+1}, n \in \mathbb{N}\}$ is stationary ergodic (i.e. $X_{n,n+1} = X_{0,1} \circ \theta^n$), it is Birkhoff's ergodic theorem.

2 Application to Operators

2.1 Subadditivity

We call (deterministic) operator a map $T : \mathbb{R}^k \to \mathbb{R}^k$ which is measurable with respect to $\mathcal{B}$, the Borel σ-field of $\mathbb{R}^k$. Let $\{T_n, n \in \mathbb{N}\}$ be a sequence of operators. We associate with it and an initial condition $x_0 \in \mathbb{R}^k$, a sequence on $\mathbb{R}^k$:

$$\begin{cases} x(n+1) = T_n(x(n)) = T_n \circ \cdots \circ T_0(x(0)) \\ \qquad\qquad x(0) = x_0 \,. \end{cases} \tag{2.1}$$

We will sometimes use the notation $x(n, x_0)$ to emphasize the value of the initial condition.

We consider a probability space $(\Omega, \mathcal{F}, P, \theta)$ as defined above. We call random (or stochastic) operator a map $T : \mathbb{R}^k \times \Omega \to \mathbb{R}^k$ which is measurable with respect to $\mathcal{B} \times \mathcal{F}$. As usual, we will often write $T(x)$ for $T(x, \omega), x \in \mathbb{R}^k, \omega \in$

Ω. A stationary and ergodic sequence of random operators is a sequence $\{T_n, n \in \mathbb{N}\}$ verifying $T_n(x,\omega) = T_0(x, \theta^n \omega)$. In the same way as in Equation (2.1), we associate with $\{T_n, n \in \mathbb{N}\}$ and a (possibly random) initial condition x_0, a random process $\{x(n), n \in \mathbb{N}\}$ taking its values in $\mathbb{R}^k$.

In what follows, definitions apply to deterministic *and* random operators. For random operators, the properties have to be verified with probability 1.

Definition 2.1.

1. **Homogeneity** *T is homogeneous if for all $x \in \mathbb{R}^k$ and λ in $\mathbb{R}$, $T(x + \lambda\vec{1}) = \lambda\vec{1} + T(x)$, where $\vec{1}$ is the vector of $\mathbb{R}^k$ with all its coordinates equal to* 1.
2. **Monotonicity** *T is monotone if $x \leqslant y$ implies $T(x) \leqslant T(y)$ coordinatewise.*

For a "physical" interpretation of these conditions, see Remark 2.1. The next theorem is a key tool in understanding the importance of homogeneity and monotonicity in what follows.

Theorem 2.2 (Crandall-Tartar [19]). *We consider an operator $T : \mathbb{R}^k \to \mathbb{R}^k$ and the following properties*

- H. *T is homogeneous.*
- M. *T is monotone.*
- NE. *T is non-expansive with respect to the sup-norm, i.e $\forall x, y \in \mathbb{R}^k$, we have $||T(x) - T(y)||_\infty \leqslant ||x - y||_\infty$.*

If H *holds, then there is equivalence between* M *and* NE. *Such operators will be referred to as monotone-homogeneous operators.*

Corollary 2.3. *Let us consider a sequence $T_n : \mathbb{R}^k \to \mathbb{R}^k$, $n \in \mathbb{N}$, of monotone-homogeneous operators. If $\exists x \in \mathbb{R}^k$, $\exists i \in \{1, \ldots, k\}$ such that $\lim_n T_n \circ \cdots \circ T_0(x)_i / n$ exists then :*

$$\forall y \in \mathbb{R}^k, \ \lim_n \frac{T_n \circ \cdots \circ T_0(y)_i}{n} = \lim_n \frac{T_n \circ \cdots \circ T_0(x)_i}{n} . \tag{2.2}$$

Proof. Straightforward from non-expansiveness

$$\lim_n \frac{||T_n \circ \cdots \circ T_0(y) - T_n \circ \cdots \circ T_0(x)||_\infty}{n} \leqslant \lim_n \frac{||x-y||_\infty}{n} .$$

□

Proposition 2.4. *Let $T_n : \mathbb{R}^k \to \mathbb{R}^k$ be a sequence of monotone-homogeneous operators. We define $e = (0, \dots, 0)'$ and for $l < n$, $x_{l,n} = T_{n-1} \circ \cdots \circ T_l(e)$. The maximal (resp. minimal) coordinate of $x_{l,n}$ forms a subadditive (resp. super-additive) process, i.e.*

$$\begin{aligned} \forall l < m < n \in \mathbb{N}, \quad \max_i (x_{l,n})_i \leqslant & \quad \max_i (x_{l,m})_i + \max_i (x_{m,n})_i \\ \min_i (x_{l,n})_i \geqslant & \quad \min_i (x_{l,m})_i + \min_i (x_{m,n})_i \, . \end{aligned} \tag{2.3}$$

Proof. We have $\forall l < m < n \in \mathbb{N}$,

$$\begin{aligned} x_{l,n} &= T_{n-1} \circ \cdots \circ T_m \circ T_{m-1} \circ \cdots \circ T_l(e) = T_{n-1} \circ \cdots \circ T_m \left(x_{l,m} \right) \\ &\leqslant T_{n-1} \circ \cdots \circ T_m \left(e + (\max_i (x_{l,m})_i) \vec{1} \right) \quad \text{(monotonicity)} \\ &\leqslant T_{n-1} \circ \cdots \circ T_m(e) + (\max_i (x_{l,m})_i) \vec{1} \quad \text{(homogeneity)}. \end{aligned}$$

Therefore,

$$\max_i (x_{l,n})_i \leqslant \max_i (x_{l,m})_i + \max_i (x_{m,n})_i.$$

The proof of the super-additivity of the minimal coordinate is equivalent. □

We are now ready to prove the following theorem on stochastic operators.

Theorem 2.5 (Vincent [43]). *Let $\{T_n, n \in \mathbb{N}\}$ be a stationary ergodic sequence of monotone-homogeneous random operators. We define the process $x(n, y), y \in \mathbb{R}^k$, as in Equation (2.1). If, for all n, the random variable $T_n \circ \cdots \circ T_1(0)$ is integrable and such that $E(T_n \circ \cdots \circ T_1(0)) > -Cn$, for some positive C, then $\exists \overline{\gamma}, \underline{\gamma} \in \mathbb{R}$ such that $\forall y \in \mathbb{R}^k$,*

$$\lim_n \frac{\max_i x(n, y)_i}{n} = \overline{\gamma} \;\; P \text{ a.s.}, \quad \lim_n \frac{E(\max_i x(n,y)_i)}{n} = \overline{\gamma} \tag{2.4}$$

$$\lim_n \frac{\min_i x(n, y)_i}{n} = \underline{\gamma} \;\; P \text{ a.s.}, \quad \lim_n \frac{E(\min_i x(n,y)_i)}{n} = \underline{\gamma} \tag{2.5}$$

Proof. We define as previously the doubly-indexed sequence $x_{l,n} = T_{n-1} \circ \cdots \circ T_l(e)_i$, $l < n$. Using Prop. 2.4, the sequences $\max_i (x_{n,m})_i$ and $-\min_i (x_{n,m})_i$ are subadditive. Hence they satisfy the conditions of Theorem 1.2. So Equation (2.4) holds for $y = e = (0, \dots, 0)'$. For any other initial condition y, we obtain $\lim_n x(n, y)/n = \lim_n x(n, e)/n$ using the non-expansiveness as in Corollary 2.3. □

The convergence for the maximal and minimal rates does not imply that of the coordinates. Here is a counter-example borrowed from [43].

Example 2.6. We consider a random operator $T_0 : \mathbb{R}^3 \to \mathbb{R}^3$ verifying:

$$x = (x_1, x_2, x_3)', \; T_0(x) = (x_1 + 1, x_2 + 2, U_0 x_1 + (1 - U_0) x_2)' \, ,$$

where U_0 is a $[0,1]$-uniform random variable. We have $\liminf(T_n \dots T_0(x)_3)/n = 1$ and $\limsup(T_n \dots T_0(x)_3)/n = 2$.
Here is another example of the same kind:

$$\begin{aligned} T_0(x) &= (\delta_0(\max(x_1,x_2)+2)+(1-\delta_0)(\min(x_1,x_2)+1),(1-\delta_0) \\ &\quad (\max(x_1,x_2)+2)+\delta_0(\min(x_1,x_2)+1), U_0x_1+(1-U_0)x_2)', \end{aligned}$$

where U_0 is a $[0,1]$-uniform random variable and δ_0 is a $(0,1)$ Bernouilli random variable. The random variables U_0 and δ_0 are independent.

2.2 Projective boundedness

In order to complete Proposition 2.4 or Theorem 2.5, the two main questions are :

i. Does a limit exist for $(T_n \circ \cdots \circ T_0(y)_1/n, \dots, T_n \circ \cdots \circ T_0(y)_k/n)$?

ii. Is this limit equal to a constant $(\gamma, \dots, \gamma)$?

The general answers to these questions are not known (even for deterministic operators). We are going to propose a sufficient condition to answer positively *i.* and *ii.* Let us introduce some definitions.

Definition 2.7 ($\mathbb{PR}^k$). *We consider the parallelism relation :*

$$u, v \in \mathbb{R}^k \quad u \simeq v \iff \exists a \in \mathbb{R} \text{ such that } \forall i,\ u_i = a + v_i .$$

We define the projective space $\mathbb{PR}^k$ as the quotient of $\mathbb{R}^k$ by this parallelism relation. Let π be the canonical projection of $\mathbb{R}^k$ into $\mathbb{PR}^k$.

Definition 2.8. *Let T be an operator of $\mathbb{R}^k$ into $\mathbb{R}^k$.*

1. *T is projectively bounded if $\exists K$ a compact of $\mathbb{PR}^k$ such that the image of T is included in K, i.e. $\pi(\mathrm{Im}(T)) \subset K$.*

2. *T has a generalized fixed point if $\exists \gamma \in \mathbb{R}, x_0 \in \mathbb{R}^k$ such that $T(x_0) = \gamma\vec{1} + x_0$. It is equivalent to say that T has a fixed point in the projective space (see Def. 2.7).*

Proposition 2.9. *Let us consider $T : \mathbb{R}^k \to \mathbb{R}^k$ a monotone-homogeneous operator. Let us consider the following assumptions.*
A. T is projectively bounded.
B. T has a generalized fixed point.
C. $\forall x,\ \lim_n T^n(x)/n = (\gamma, \dots, \gamma)'$.
The following implications hold : $A \Rightarrow B \Rightarrow C$. The other implications are false, $C \not\Rightarrow B \not\Rightarrow A$ and $C \not\Rightarrow A$.

Proof. 1. $A \Rightarrow B$. Let K be a compact of $\mathbb{PR}^k$ such that $\pi(T(\mathbb{R}^k)) \subset K$. It implies that $\pi(T): K \to K$. Hence $\pi(T)$ is continuous on a compact and has a fixed point by application of Brouwer's Theorem.

2. $B \Rightarrow C$. Let $x \in \mathbb{R}^k$ be a generalized fixed point of T, i.e $T(x) = \gamma\vec{1} + x$. It implies $T^n(x) = n\gamma\vec{1} + x$ and $\lim_n T^n(x)/n = (\gamma, \ldots, \gamma)'$. From Corollary 2.3, we have $\forall y \in \mathbb{R}^k, \lim_n T^n(y)/n = (\gamma, \ldots, \gamma)'$.

3. $B \not\Rightarrow A$ and $C \not\Rightarrow A$. An easy counter-example is obtained by considering the identity operator $I : \mathbb{R}^k \to \mathbb{R}^k$, $I(x) = x$.

4. $C \not\Rightarrow B$ There exist counter-examples of dimension 2, [27].

$\square$

This Proposition has an interesting application for stochastic operators.

Theorem 2.10. *Let $\{T_n, n \in \mathbb{N}\}$ be a stationary and ergodic sequence of random operators. We assume that there exist $l \in \mathbb{N}$ and K a compact of $\mathbb{PR}^k$ such that :*

$$\mathcal{E} = \{\pi(\mathrm{Im}(T_{l-1} \circ \cdots \circ T_0)) \subset K\}\,, \quad P(\mathcal{E}) > 0\,. \tag{2.6}$$

Then $\exists \gamma \in \mathbb{R}$, such that

$$\forall x \in \mathbb{R}^k, \quad \lim_n \frac{T_n \circ \cdots \circ T_0(x)}{n} = (\gamma, \ldots, \gamma)'\,.$$

Proof. We define recursively the random variables

$$\begin{aligned} N_1 &= \min\{n \in \mathbb{N} \mid T_{n+l-1} \circ \cdots \circ T_n \in \mathcal{E}\}, \\ N_{i+1} &= \min\{n \in \mathbb{N} \mid n \geqslant N_i + l,\ T_{n+l-1} \circ \cdots \circ T_n \in \mathcal{E}\}\,. \end{aligned}$$

First of all, let us prove that the random variables N_i are almost surely finite. Let us consider the event $\mathcal{A}_1 = \{N_1 < +\infty\}$. It is easy to see that $\mathcal{A}_1$ is invariant by the shift θ. In fact $N_1(\theta^{-1}\omega) = N_1(\omega) + 1$ or 0. Hence $\{N_1(\omega) < +\infty\} \Rightarrow \{N_1(\theta^{-1}\omega) < +\infty\}$, i.e. $\theta(\mathcal{A}) \subset \mathcal{A}$. By Lemma 1.1, it implies that $\mathcal{A}$ is of probability 0 or 1. But $(\{N_1 = 0\} = \mathcal{E}) \subset \mathcal{A}$ and by assumption $P(\mathcal{E}) > 0$. We conclude that $P(\mathcal{A}) = 1$. A similar argument can now be applied to N_2. For $\mathcal{A} \in \mathcal{F}$, we define the indicator function $\mathbf{1}_{\mathcal{A}} : \Omega \to \Omega$, $\mathbf{1}_{\mathcal{A}}(\omega) = 1$ iff $\omega \in \mathcal{A}$. We have

$$\begin{aligned} P(N_2 < +\infty) &= E(\mathbf{1}_{\{N_2<+\infty\}}) = E(\sum_k \mathbf{1}_{\{N_1=k\}}\mathbf{1}_{\{N_2<+\infty\}}) \\ &= E(\sum_k \mathbf{1}_{\{N_1=k\}}\mathbf{1}_{\{N_1\circ\theta^{k+l}<+\infty\}}) = E(\sum_k \mathbf{1}_{\{N_1=k\}}) = 1\,. \end{aligned}$$

We conclude the proof by induction.

Let $\overline{\gamma}$ and $\underline{\gamma}$ be the maximal and minimal rates as defined in Prop. 2.4. Let us assume that $\overline{\gamma} \neq \underline{\gamma}$. It implies, $\forall x \in \mathbb{R}^k$,

$$\liminf_n (\max_i x(n)_i - \min_i x(n)_i) = +\infty \, . \tag{2.7}$$

But we also have that $\forall i \in \mathbb{N}, \pi(x(N_i + l)) \subset K$. It implies that $(\max_j x(N_i + l)_j - \min_j x(N_i + l)_j) \subset K'$ where K' is a compact of $\mathbb{R}$. Hence there exists a subsequence $N_{\sigma(i)}$ such that $(\max_j x(N_{\sigma(i)} + l)_j - \min_j x(N_{\sigma(i)} + l)_j)$ converges to a finite limit. This is in contradiction with (2.7). □

Remark 2.1. In many applications, the operator will be applied on a vector of dates for a physical system. The vectors $x(n)$ and $x(n+1) = T_n(x(n))$ will represent the dates of the n^{th} and $(n+1)^{th}$ occurrences of some events in a system. In such a case, the homogeneity property can be interpreted as the fact that changing the absolute origin of times does not modify the dynamic of the system. Hence it becomes a very natural assumption. The monotonicity is interpreted as the fact that delaying an event delays all following events.

3 Application to Discrete Event Networks

3.1 Discrete event networks

A *discrete event network* is characterized by

1. A sequence
$$N = N_{[-\infty,\infty]} = \{\sigma(k), M(k),\ k \in \mathbb{Z}\},$$
where $\sigma(k) \in \mathbb{R}^+$ and $\{M(k)\}$ is a sequence of F-valued variables, where F is some measurable space. With N and $n \leqslant m \in \mathbb{Z}$, we associate the sequence $N_{[n,m]}$ defined by:
$$N_{[n,m]} \stackrel{\text{def}}{=} \{\sigma_{[n,m]}(k), M(n+k),\ k \in \mathbb{N}\},$$
where $\sigma_{[n,m]}(k) \stackrel{\text{def}}{=} \sigma(n+k)$, for $0 \leqslant k \leqslant m-n$, and $\sigma_{[n,m]}(k) \stackrel{\text{def}}{=} \infty$, for $k > m-n$.

2. Measurable operators $\Phi(k,.)$ and $\Psi(.)$: $(\mathbb{R}^+ \times F)^{\mathbb{N}} \to \mathbb{R} \cup \{\infty\}$, $k \in \mathbb{N}^*$, through which are defined
$$X_{[n,m]} = \Psi(N_{[n,m]}), \quad n \leqslant m, \quad X^-_{[n,m]}(k) = \Phi(k, N_{[n,m]}), \quad k \geqslant 1.$$

Remark These variables receive the following interpretations: $X^-_{[n,m]}(k)$ is the initiation date of the k-th event on some reference node, for the *driving sequence* $N_{[n,m]}$.

$$X^+_{[n,m]}(k) \stackrel{\text{def}}{=} X^-_{[n,m]}(k) + \sigma_{[n,m]}(k), \quad n \leqslant m,\ k \geqslant 0$$

is the completion date of this event. $X^+_{[n,m]}(k)$ and $X^-_{[n,m]}(k)$ are called *internal daters*. $X_{[n,m]}$ is the *maximal dater*, i.e. the date of the last event in the network, for the sequence $N_{[n,m]}$.

3.2 The monotone–separable framework

Let N and $\widetilde{N}$ be two driving sequences such that $\sigma(k) \leqslant \widetilde{\sigma}(k) < \infty$ for all k, and with $M(k) = \widetilde{M}(k)$ for all k. We denote $X^-_{[1,m]}(k)$, $X^+_{[1,m]}(k)$ and $X_{[1,m]}$ the daters associated with $N_{[1,m]}$ and $\widetilde{X}^-_{[1,m]}(k)$, etc. those associated with $\widetilde{N}_{[1,m]}$.

A network is said to be *monotone-separable* if it satisfies the following properties for all $m \geqslant 1$, $k \geqslant 1$ and for all N and $\widetilde{N}$ as above:

- **causality** $X^-_{[1,m]}(m+1) \leqslant X_{[1,m]} < \infty$.
- **monotonicity** $X^-_{[1,m]}(k) \leqslant \widetilde{X}^-_{[1,m]}(k)$ and $X_{[1,m]} \leqslant \widetilde{X}_{[1,m]}$.
- **non-expansiveness** [1] $\widetilde{X}^-_{[1,m]}(k) - X^-_{[1,m]}(k) \leqslant x$ and $\widetilde{X}_{[1,m]} - X_{[1,m]} \leqslant x$, if $\widetilde{\sigma}(k) = \sigma(k)$ for all $k \neq l$, and $\widetilde{\sigma}(l) = \sigma(l) + x$, $x > 0$.
- **separability** For $1 \leqslant l < m$, if $X_{[1,l]} \leqslant X^+_{[1,m]}(l+1)$ then $X_{[1,m]} \leqslant X^-_{[1,m]}(l+1) + X_{[l+1,m]}$.

Proposition 3.1. *Under the above assumptions, the sequence $X_{[m,n]}$ satisfies the sub-additive inequality*

$$X_{[m,n]} \leqslant X_{[m,l]} + X_{[l+1,n]}, \quad \forall m \leqslant l < n.$$

Proof. It is enough to prove the property for $m = 1$, since the general relation will then be obtained by applying the relation for $m = 1$ to the variables associated with some adequate sequence. Let $1 \leqslant l < n$. There are two cases:

[1] If one sees $(\Psi(.), \Phi(k,.),\ k \geqslant 1)$ as an operator: $(\mathbb{R}^+)^{\mathbb{N}} \to (\mathbb{R} \cup \{\infty\})^{\mathbb{N}}$ – the sequence $\{M(k)\}$ being fixed – this is indeed non-expansiveness when taking a L^1 norm on $(\mathbb{R}^+)^{\mathbb{N}}$ and a L^∞ norm on $(\mathbb{R} \cup \{\infty\})^{\mathbb{N}}$.

Case 1: $X_{[1,l]} \leqslant X^+_{[1,n]}(l+1)$. Then, in view of separability

$$\begin{aligned} X_{[1,n]} &\leqslant X^-_{[1,n]}(l+1) + X_{[l+1,n]} \\ &\leqslant X_{[1,l]} + X_{[l+1,n]}, \end{aligned}$$

where we used the fact that $X_{[1,l]} \geqslant X^-_{[1,l]}(l+1) \geqslant X^-_{[1,n]}(l+1)$, which follows from causality and monotonicity ($X^-_{[1,l]}(l+1) = \widetilde{X}^-_{[1,n]}(l+1)$ with $\widetilde{\sigma}(k) = \sigma(k)$ for $1 \leqslant k \leqslant l$ and $\widetilde{\sigma}(k) = \infty$ for $k > l$).

Case 2: $X_{[1,l]} > X^+_{[1,n]}(l+1)$.

Consider the two sequences $\{\sigma(k)\}$ and $\{\widetilde{\sigma}(k)\}$, which only differ in their $(l+1)$-st coordinate, for which we take $\widetilde{\sigma}(l+1) = \sigma(l+1) + x$, $x > 0$. In view of monotonicity, $X_{[1,n]} \leqslant \widetilde{X}_{[1,n]}$. In particular, if we take $x = x^*$ with $x^* = X_{[1,l]} - X^+_{[1,n]}(l+1) > 0$, then

$$\begin{aligned} \widetilde{X}^+_{[1,n]}(l+1) &= \widetilde{X}^-_{[1,n]}(l+1) + \sigma(l+1) + x^* \\ &= \widetilde{X}^-_{[1,n]}(l+1) + \sigma(l+1) + X_{[1,l]} - X^+_{[1,n]}(l+1) \\ &= X_{[1,l]} + \widetilde{X}^-_{[1,n]}(l+1) - X^-_{[1,n]}(l+1). \end{aligned} \tag{3.1}$$

But $X_{[1,l]}$ does not depend on $\sigma(l+1)$, and so $X_{[1,l]} = \widetilde{X}_{[1,l]}$. Therefore

$$\begin{aligned} \widetilde{X}^+_{[1,n]}(l+1) &= \widetilde{X}_{[1,l]} + \widetilde{X}^-_{[1,n]}(l+1) - X^-_{[1,n]}(l+1) \\ &\geqslant \widetilde{X}_{[1,l]} \text{ (monot.)}. \end{aligned}$$

We finally obtain that, for $x = x^*$

$$\begin{aligned} \widetilde{X}_{[1,n]} &\leqslant \widetilde{X}^-_{[1,n]}(l+1) + \widetilde{X}_{[l+1,n]}, \quad \text{(separability)} \\ &= \widetilde{X}^+_{[1,n]}(l+1) + X^-_{[1,n]}(l+1) - X_{[1,l]} + \widetilde{X}_{[l+1,n]}, \quad \text{(Equation (3.1))} \\ &\leqslant \widetilde{X}^+_{[1,n]}(l+1) + X^-_{[1,n]}(l+1) - X_{[1,l]} + x^* + X_{[l+1,n]}, \quad (\text{non}-\text{exp.}) \\ &= \widetilde{X}^+_{[1,n]}(l+1) + X^-_{[1,n]}(l+1) - X_{[1,l]} + X_{[1,l]} - X^+_{[1,n]}(l+1) + X_{[l+1,n]} \\ &= \widetilde{X}^+_{[1,n]}(l+1) - X^+_{[1,n]}(l+1) + X^-_{[1,n]}(l+1) + X_{[l+1,n]} \\ &\leqslant x^* + X^-_{[1,n]}(l+1) + X_{[l+1,n]}, \quad (\text{non}-\text{exp.}) \\ &= X^-_{[1,n]}(l+1) - X^+_{[1,n]}(l+1) + X_{[1,l]} + X_{[l+1,n]} \\ &\leqslant X_{[1,l]} + X_{[l+1,n]}. \end{aligned}$$

□

Remark 3.1. Under the additional assumption that $X^-_{[1,m]}(l+1)$ is a function of $\{\sigma(k),\ 1 \leqslant k \leqslant l,$ and $M(p),\ 1 \leqslant p \leqslant m\}$ only, non-expansiveness can be replaced by the following property:

- **sub-homogeneity** $\widetilde{X}_{[1,m]} \leqslant X_{[1,m]} + \lambda$, if $\widetilde{\sigma}(1) = \sigma(1) + \lambda$ and $\widetilde{\sigma}(k) = \sigma(k)$ for all $k > 1$, $\lambda > 0$ and $m \geqslant 1$.

The proof is exactly the same for case 1. For case 2, taking x^* as in the proof of Proposition 3.1 gives $\widetilde{X}^+_{[1,n]}(l+1) = X_{[1,l]}$ and

$$\begin{aligned}
\widetilde{X}_{[1,n]} &\leqslant \widetilde{X}^-_{[1,n]}(l+1) + \widetilde{X}_{[l+1,n]}, \quad \text{(separability)} \\
&= X^-_{[1,n]}(l+1) + \widetilde{X}_{[l+1,n]} \\
&\leqslant X^-_{[1,n]}(l+1) + X_{[l+1,n]} + X_{[1,l]} - X^+_{[1,n]}(l+1), \quad (\text{sub} - \text{homog.}) \\
&\leqslant X_{[1,l]} + X_{[l+1,n]}.
\end{aligned}$$

Remark 3.2. Some generalizations of the framework, with internal daters, will be proposed in [7]. See also the Jackson network example of §8.2. The comments on the physical interpretation of homogeneity or monotonicity made in Remark 2.1 also apply to discrete event networks.

3.3 Open discrete event networks

A discrete event network is said to be *open* if the following additional assumption holds for all $m \geqslant 1$:

$$\forall 1 \leqslant k \leqslant m,\ X^-_{[1,m]}(k+1) = X^-_{[1,\infty]}(k+1) = X^+_{[1,m]}(k),\ \text{and } X^-_{[1,m]}(1) = 0.$$

One can then define a point process $\{A_k\}_{k \geqslant 1}$ by

$$A_k = A_1 + X^-_{[1,\infty]}(k).$$

The origin of this point process is arbitrary. It is then possible to interpret $\{A_k\}$ as an external arrival process, the inter-arrival times being the sequence $\{\sigma(k)\}$. To summarize, an open discrete network is described by a sequence $N = N_{[-\infty,\infty]} = \{A_k, M(k),\ k \in \mathbb{Z}\}$.

The conditions of the monotone separable framework take the following form for an open network (which corresponds to the conditions of [5]) : for all $m \geqslant 1$, the following properties hold:

- **causality** $A_m \leqslant A_1 + X_{[1,m]} < \infty$.

- **monotonicity** $\widetilde{X}_{[1,m]} \geqslant X_{[1,m]}$, for $\widetilde{N}$ and N with $\widetilde{\sigma}(k) \geqslant \sigma(k)$ for all k.

- **homogeneity** Let $\widetilde{N}$ be the point process obtained by shifting the points of N A_k, $k \geqslant 1$, of $\lambda > 0$ to the right. Then $\widetilde{X}_{[1,n]} = X_{[1,n]}$.

- **separability** $A_1 + X_{[1,m]} \leqslant A_{l+1} + X_{[l+1,m]}$ for all $1 \leqslant l < m$ such that $A_1 + X_{[1,l]} \leqslant A_{l+1}$.

For an open network, monotonicity can be interpreted as the fact that delaying an arrival delays all forthcoming events in the network. For a possible interpretation of separability, see Remark 6.1.

3.4 Stochastic discrete event networks

We consider a probability space $(\Omega, \mathcal{F}, P, \theta)$ as in §1. The following stochastic assumptions are made:

- **compatibility** $(\sigma(k), M(k)) = (\sigma(0), M(0)) \circ \theta^k$ for all $k \in \mathbb{Z}$.
- **integrability** $\exists C > 0$, $-Cm \leqslant E[X_{[1,m]}] < \infty$ for all $m \geqslant 0$.

Theorem 3.2. *For all discrete event network which satisfies the monotone-separable assumptions and the above stochastic assumptions, we have*

$$\lim_{n\to\infty} \frac{X_{[1,n]}}{n} = \gamma \quad \text{a.s.} \quad \text{and} \lim_{n\to\infty} \frac{E[X_{[1,n]}]}{n} = \gamma \tag{3.2}$$

for some finite constant γ.

Proof. We have $X_{[m,m+p]} = X_{[0,p]} \circ \theta^m$, for all $m \in \mathbb{Z}$ and $p \geqslant 0$. For $m \leqslant n$, define $Y_{[m,n+1]} = X_{[m,n]}$. From Proposition 3.1, for all $m \leqslant l < n$,

$$Y_{[m,n+1]} \leqslant Y_{[m,l+1]} + Y_{[l+1,n+1]}.$$

So $\{Y_{[m,n]}\}$, $m < n$, satisfies all the assumptions of Theorem 1.2. $\square$

4 Relations Between Operators and Networks

Let us investigate the relation between the operator framework considered in §2 and the monotone-separable framework considered above. Let $\{T_n\}$ be a sequence of monotone-homogeneous operators. Let $\sigma(n) \equiv 0$ and $M(n) = T_n$. Let $x(n,0)$ be the variables associated with the operator recurrence equation (2.1) with initial condition $x_0 = 0$. With these variables, we associate

$$X^-_{[0,n]}(k) = X^+_{[0,n]}(k) = \max_i x(k-1,0)_i, \quad k \geqslant 1, \quad \text{and} \quad X_{[0,n]} = \max_i x(n,0)_i,$$

Note that these variables are functions of $\{M(l)\}$. We have

- $X^-_{[0,n]}(n+1) = X_{[0,n]} < \infty$, so that causality holds.

- Monotonicity and non-expansiveness trivially hold as neither $X^-_{[0,n]}(k)$ nor $X_{[0,n]}$ depend upon $\{\sigma(l)\}$.
- Separability holds because it is always true that $X_{[0,l]} = X^-_{[0,m]}(l+1)$ and

$$\begin{aligned}
X_{[0,m]} &= \max_i \left(T_m \circ \ldots \circ T_{l+1}(x(l,0))\right)_i \\
&= \max_i \left(T_m \circ \ldots \circ T_{l+1}(x(l,0) + (X_{[0,l]} - X_{[0,l]})\vec{1})\right)_i \\
&= X_{[0,l]} + \max_i \left(T_m \circ \ldots \circ T_{l+1}(x(l,0) - X_{[0,l]}\vec{1})\right)_i, \quad \text{(homog.)} \\
&\leqslant X_{[0,l]} + \max_i \left(T_m \circ \ldots \circ T_{l+1}(0)\right)_i, \quad \text{(monotonicity)} \\
&= X^-_{[0,m]}(l+1) + X_{[l+1,m]}.
\end{aligned}$$

Hence, monotone separable operators are a special case of monotone separable discrete event networks. On the other hand, it should be remarked that an operator can *not* be represented as an *open* discrete event network. A representation in terms of operators is interesting as it is more precise than the corresponding discrete event network one. In particular, we will see that we are able to obtain second order results for operators, §7, and not for non-open discrete event networks, §6.2.

Part II

Second Order Ergodic Results

We will introduce a construction which is known as the Loynes scheme. This type of construction will be used for both types of models, discrete event networks and operators, but in a rather different way.

5 Basic Example and Probabilistic Tools

The basic construction was introduced by Loynes in [34] to study the stability of the $G/G/1/\infty$ queue. A G/G arrival process is a stationary and ergodic marked point process $N = \{(\tau_n, \sigma_n)), n \in \mathbb{Z}\}$, where $\sigma_n \in \mathbb{R}^+$ is the service time required by customer n and $\tau_n = A_{n+1} - A_n$ the inter-arrival time between customers n and $n+1$. The $1/\infty$ part describes the queueing mechanism. There is a single server and an infinite waiting room or buffer. Upon arrival at instant A_n, customer n is served immediately if the server is idle at A_n^- and is queued in the buffer otherwise. The server operates at unit rate until all customers present in the buffer have been served. Let $X_{[l,n]}$ be the time of last activity in the system, i.e. the departure of the last customer, for the restriction $N_{[l,n]}$. Here are two equivalent ways to describe the system :

- As a stochastic operator,

$$\begin{pmatrix} A_{n+1} \\ X_{[l,n+1]} \end{pmatrix} = \begin{pmatrix} \tau_n + A_n \\ \max(\tau_n + \sigma_{n+1} + A_n \, , \, \sigma_{n+1} + X_{[l,n]}) \end{pmatrix} \tag{5.1}$$

$$= \begin{pmatrix} \tau_n & \varepsilon \\ \tau_n \otimes \sigma_{n+1} & \sigma_{n+1} \end{pmatrix} \otimes \begin{pmatrix} A_n \\ X_{[l,n]} \end{pmatrix} \tag{5.2}$$

Equation (5.1) can be written $X_{[l,n+1]} = \max(A_{n+1}, X_{[l,n]}) + \sigma_{n+1}$. The meaning is that the server starts working on customer $n+1$ as soon as this customer has arrived (A_{n+1}) *and* the server has completed the services of the previous customers ($X_{[l,n]}$). Equation (5.2) is just a re-writing using the (max,+) notations, see also §8.1. It is easy to verify that this operator is monotone and homogeneous.

- As an open network, by means of the function Ψ of §3.

$$X_{[l,n]} = \Psi(\tau_i, \sigma_i, i \in \{l, \ldots, n\})$$

$$= (A_n - A_l) + \sigma_n + \max(0, \max_{k=1}^{n-l} \sum_{i=1}^{k} (\sigma_{n-i} - \tau_{n-i})) \,. \tag{5.3}$$

The easiest way to understand Equation (5.3) is to look at Figure 1. Function Ψ is monotone, homogeneous and separable.

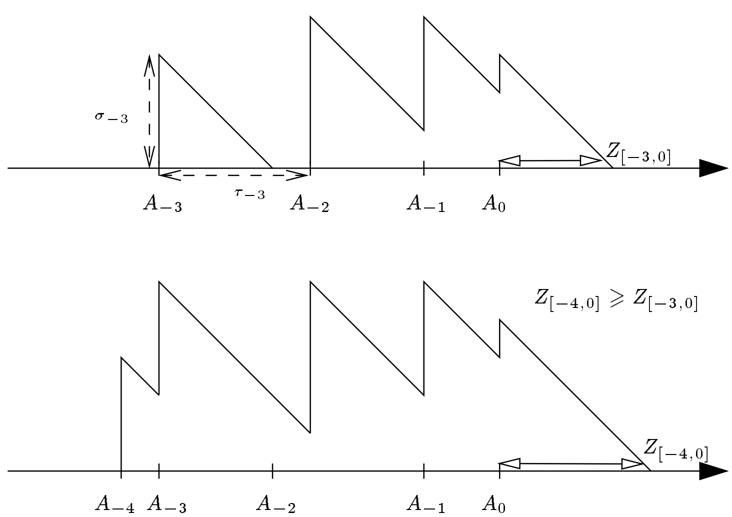

Figure 1: Loynes scheme for the $G/G/1/\infty$ queue.

Let us consider the sequence of variables $\{Z_{[l,n]},\ l \leqslant n \in \mathbb{Z}\}$ defined by $Z_{[l,n]} = X_{[l,n]} - (A_n - A_l)$. The variables $Z_{[l,n]}$ verify Lindley's equation[2] $Z_{[l,n+1]} = (Z_{[l,n]} - \tau_n)^+ + \sigma_{n+1}$.

Theorem 5.1 (Loynes [34]). *The sequence $Z_{[-n,0]}$ is increasing in n, i.e. $Z_{[-n-1,0]} \geqslant Z_{[-n,0]}$. The limit $Z = \lim_n Z_{[-n,0]}$ verifies $P\{Z < +\infty\} = 0$ or 1. Furthermore Z is a stationary solution of Lindley's equation, i.e. $Z(\theta\omega) = (Z(\omega) - \tau_0)^+ + \sigma_1$. When $P\{Z < +\infty\} = 1$, the sequence $\{Z_{[0,n]}, n \in \mathbb{N}\}$ couples in finite time with the stationary sequence $\{Z \circ \theta^n\}$.*

Proof. The monotonicity of $Z_{[-n,0]}$ is easy to obtain from Equation (5.3). It is also illustrated in Figure 1. Hence the limit $Z = \lim_n Z_{[-n,0]}$ exists. Let us denote $\mathcal{A} = \{Z < +\infty\}$. From $Z_{[-n,1]} = (Z_{[-n,0]} - \tau_0)^+ + \sigma_1$ and the fact that σ_1 is a.s. finite, we obtain

$$Z(\omega) < +\infty \Leftrightarrow \exists K\ \forall n,\ Z_{[-n,0]}(\omega) < K \Rightarrow \exists K'\ \forall n,\ Z_{[-n,1]}(\omega) < K' \ .$$

But we also have

$$Z_{[-n,1]}(\omega) = Z_{[-n-1,0]}(\theta\omega) \ . \tag{5.4}$$

We conclude that $Z(\theta\omega) < +\infty$. We have proved that $\theta(\mathcal{A}) \subset \mathcal{A}$ which implies, Ergodic Lemma 1.1, that $P\{\mathcal{A}\} = 0$ or 1. From Equation (5.4), letting n go to ∞, we deduce that $Z(\theta\omega) = (Z(\omega) - \tau_0)^+ + \sigma_1$. For a proof of the remaining point, see [34] or [1]. $\square$

The limit Z is usually referred to as Loynes variable. We can obtain, using Equation (5.3), $P\{Z < +\infty\} = 1 \Leftrightarrow E(\sigma) < E(\tau)$. The condition $E(\sigma) < E(\tau)$ is called the stability condition and is usually written under the form $\rho = E(\sigma)/E(\tau) < 1$. We will see a similar type of stability condition in Theorem 6.2.

6 Application to Discrete Event Networks

6.1 Open discrete event networks

The assumptions and notations are those of §3.3 but we replace the separability assumption by

- **strong separability** For $1 \leqslant l < m$, if $A_1 + X_{[1,l]} \leqslant A_{l+1}$ then $A_1 + X_{[1,m]} = A_{l+1} + X_{[l+1,m]}$.

[2] It is more classical, but equivalent, to work with the workload variable $W_n = X_{[0,n]} - A_n - \sigma(n)$, yielding equation $W_{n+1} = (W_n + \sigma_n - \tau_n)^+$.

Remark 6.1. Strong separability can be interpreted as follows. If the arrival of customer $l+1$ takes place later than the last activity for the arrival process $[1,l]$, then the evolution of the network after time A_{l+1} is the same as in the network which starts "empty" at this time.

We define $\lambda = E(A_{n+1} - A_n)^{-1}$ interpreted as the arrival rate and

$$Z_{[l,n]} = X_{[l,n]} - (A_n - A_l), \quad l \leqslant n \,. \tag{6.1}$$

Proposition 6.1 (Internal monotonicity). *Under the above assumptions, we have*

$$Z_{[l-1,n]} \geqslant Z_{[l,n]}, \quad l \leqslant n.$$

Proof. Consider the point process $\widetilde{N}$ with $\widetilde{\sigma}(l-1) = \sigma(l-1) + Z_{[l-1,l-1]}$ and $\widetilde{\sigma}(k) = \sigma(k)$ everywhere else. For $\widetilde{N}_{[l-1,n]}$, we have separability in l so that

$$\begin{aligned}
\widetilde{X}_{[l-1,n]} &= \widetilde{X}_{[l,n]} + \widetilde{A}_l - \widetilde{A}_{l-1} \\
&= X_{[l,n]} + \widetilde{A}_l - \widetilde{A}_{l-1} \quad (\text{strong} - \text{separability}) \\
&= X_{[l,n]} + A_l - A_{l-1} + Z_{[l-1,l-1]} \,.
\end{aligned} \tag{6.2}$$

Therefore

$$\begin{aligned}
Z_{[l-1,n]} &= X_{[l-1,n]} - (A_n - A_{l-1}) \\
&= X_{[l-1,n]} - (A_n - A_l) - (A_l - A_{l-1}) \\
&= X_{[l-1,n]} - (A_n - A_l) + X_{[l,n]} - \widetilde{X}_{[l-1,n]} + Z_{[l-1,l-1]} \quad (\text{by (6.2)}) \\
&= Z_{[l,n]} + X_{[l-1,n]} - \widetilde{X}_{[l-1,n]} + Z_{[l-1,l-1]} \\
&\geqslant Z_{[l,n]}, \quad (\text{non} - \text{expansiveness}).
\end{aligned}$$

□

Let $Z = \lim_n Z_{[-n,0]}(N)$, which exists by internal monotonicity of $Z_{[-n,0]}(N)$. We define a c-scaling of the arrival point process N in the following way :

$$0 \leqslant c < +\infty, \quad cN = \{cA_n, M(n), n \in \mathbb{Z}\} \,.$$

From Equation (6.1) and Prop. 3.1 , we obtain that $Z_{[1,n]}$ is subadditive. Applying Theorem 3.2, we obtain the existence of the limits

$$\lim_n \frac{Z_{[1,n]}(cN)}{n} = \lim_n \frac{Z_{[-n,0]}(cN)}{n} = \gamma(c) \,.$$

From Equation (6.1), we obtain

$$\lim_n \frac{X_{[1,n]}(cN)}{n} = \lim_n \frac{X_{[-n,0]}(cN)}{n} = \gamma(c) + \frac{c}{\lambda} \,.$$

For $c \geqslant \tilde{c}$, we have $cN \geqslant \tilde{c}N$. We obtain by internal monotonicity and by monotonicity respectively :

1. $Z_{[-n,0]}(cN)$ is decreasing in $c \Longrightarrow \gamma(c)$ is decreasing in c.

2. $X_{[0,n]}(cN)$ is increasing in $c \Longrightarrow \gamma(c) + c/\lambda$ is increasing in c.

We deduce the existence of a constant $\gamma(0)$ defined by :

$$\lim_{c\to 0} \searrow \gamma(c) + \frac{c}{\lambda} = \gamma(0) = \lim_{c\to 0} \nearrow \gamma(c) . \tag{6.3}$$

The intuitive interpretation is that $\gamma(0)^{-1}$ is the throughput of the network when we saturate the input, i.e. when $A_n = 0, \forall n$. It is the maximal possible throughput.

Theorem 6.2. *Let $N = \{A_n, M_n, n \in \mathbb{Z}\}$ be a stationary ergodic point process. We set $\rho = \lambda\gamma(0)$. If $\rho > 1$, then $P(Z = +\infty) = 1$. If $\rho < 1$, then $P(Z < +\infty) = 1$ and $\{Z_{[0,n]}, n \in \mathbb{N}\}$ couples in finite time with the stationary sequence $\{Z \circ \theta^n\}$.*

Proof. The first part of the theorem is immediate. In fact relation (6.3) implies $\gamma(1) + 1/\lambda \geqslant \gamma(0)$. We have :

$$\left(\lim_n \frac{Z_{[-n,0]}}{n} = \gamma(1)\right) \geqslant \left(\gamma(0) - \frac{1}{\lambda} = \frac{\rho - 1}{\lambda}\right) .$$

Therefore $\rho - 1 > 0$ implies $P(Z = +\infty) = 1$. For a complete proof of the result, the reader is referred to [5]. □

Remark 6.2. For $\rho < 1$, Z is the smallest stationary regime for the response time of the system (which is defined as the time to the last activity under the restriction $[-\infty, 0]$ of N). Intuitively it is the stationary regime corresponding to an "empty" initial condition as it is the limit of the systems starting "empty" and fed up with the restrictions $[-n, 0]$ of N. In many cases, there will be multiple stationary regimes depending on the initial condition. A simple example of a monotone and separable open network having multiple stationary regimes is proposed in [1], p.83. It is a $G/G/2/\infty$ queue with a "shortest workload" allocation rule (see also Theorem 7.5).

6.2 General discrete event networks

For discrete event networks which are not open, there are no general results. The reason is the absence of internal monotonicity of the variables $Z_{[-n,0]} = X_{[-n,0]} - X^-_{[-n,0]}$. We illustrate the phenomenon on Figure 2 where we compare the case of a general network and the case of an open network.

For open and general networks, we consider successively the restrictions $[-n, 0]$ and $[-n-1, 0]$. In the open case, the internal monotonicity has been illustrated in Figure 2. In the general case, the variables X^- are internal

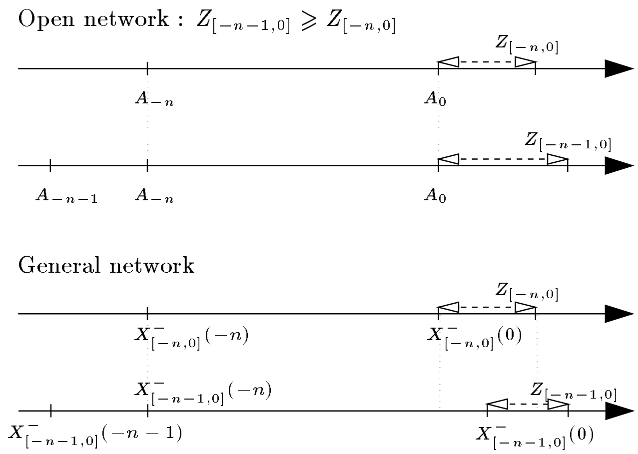

Figure 2: Loynes scheme for monotone-separable networks.

variables, hence their value are modified when we go from the restriction $[-n,0]$ to $[-n-1,0]$. As a consequence, there is no internal monotonicity. On Figure 2, for the ease of comparison, we have assumed that $X^-_{[-n-1,0]}(-n) = X^-_{[-n-1,0]}(-n)$ (these quantities are defined up to an additive constant).

7 Application to Operators

We propose in Sections 7.1 and 7.2 two very different approaches. They correspond to two different types of operators, see Remark 7.1. The first approach is directly based on the Loynes scheme. The second one uses fixed points results.

7.1 Monotonicity

Definition 7.1. *We say that the operator $T : \mathbb{R}^k \to \mathbb{R}^k$ has a minimal value if there exists $x_0 \in \mathbb{R}^k$ such that $\forall y \geqslant x_0,\ T(y) \geqslant x_0$.*

Let us consider a sequence of monotone operators $\{T_n, n \in \mathbb{Z}\}$. If all the operators have a common minimal value x_0, then we are able to construct a Loynes scheme, in the same way as in §5. In fact, we have $T_0(x_0) \geqslant x_0$ and $T_0 \circ T_{-1}(x_0) \geqslant T_0(x_0) \geqslant x_0$ using monotonicity. We obtain that

$$\exists Z \in (\mathbb{R} \cup \{+\infty\})^k,\ \lim_n T_0 \circ T_{-1} \circ \cdots \circ T_{-n}(x_0) = Z\ . \tag{7.1}$$

The main question is whether the limit Z is finite or not, the finite case being the interesting one. In particular, if we consider a sequence of monotone-homogeneous operators, then the limits $\overline{\gamma}$ and $\underline{\gamma}$ as defined in Proposition 2.4

exist. Because of the existence of the minimal value x_0, we have $\overline{\gamma} \geqslant \underline{\gamma} \geqslant 0$. If $\overline{\gamma} > 0$ then there exists i such that $Z_i = +\infty$ (the proof is immediate).

For this reason, it is usually not interesting to construct a Loynes scheme directly on the sequence of operators T_n. For example, in the case of the operator of the G/G/1 queue, see Equation (5.1), the Loynes scheme was not built on $(A_n, X_{[l,n]})'$ but on the differences $Z_{[l,n]} = X_{[l,n]} - A_n$. In order to generalize the construction, the good approach is to consider the operators T_n in a projective space.

We have already defined the projective space $\mathbb{PR}^k$ in Definition 2.7. The space $\mathbb{PR}^k$ is isomorphic to $\mathbb{R}^{k-1}$. Here are different possible ways to map $\mathbb{PR}^k$ onto $\mathbb{R}^{k-1}$. Let $i \in \{1, \dots, k\}$, we define :

$$\begin{array}{ll} \pi_i : & \mathbb{R}^k \longrightarrow \mathbb{R}^{k-1}, \quad \pi_i(x) = (x_1 - x_i, \dots, x_{i-1} - x_i, x_{i+1} - x_i, \dots, x_k - x_i)' \\ \phi_i : & \mathbb{PR}^k \rightarrow \mathbb{R}^{k-1}, \quad \phi_i = \pi_i \circ \pi^{-1}, \end{array}$$

where π was defined in Definition 2.7. It is easy to verify that ϕ_i is defined without ambiguity and is bijective.

Definition 7.2. *Let $x \in \mathbb{R}^k$. We define $|x|_{\mathcal{P}} = \max_i x_i - \min_i x_i$. Let $u \in \mathbb{PR}^k$ (resp. $u \in \mathbb{R}^{k-1}$) and x be a representative of u, i.e. $\pi(x) = u$ (resp. $\pi_i(x) = u$) We define $|u|_{\mathcal{P}} = \max_i x_i - \min_i x_i$.*

The function $|.|_{\mathcal{P}}$ is a semi-norm on $\mathbb{R}^k$ as $|x|_{\mathcal{P}} = 0 \Rightarrow x_i = \lambda, \forall i$. On the other hand, it defines a norm on $\mathbb{PR}^k$ or $\mathbb{R}^{k-1}$. We call it the projective norm. We use the same notation for the semi-norm on $\mathbb{R}^k$ and the norms on $\mathbb{PR}^k$ and $\mathbb{R}^{k-1}$ in order not to carry too many notations.

Form now on, we are going to work on $\mathbb{R}^{k-1}$ equipped with the projective norm. Without loss of generality, we will restrict our attention to π_1, ϕ_1. Working on $\mathbb{R}^{k-1}$ rather than on $\mathbb{PR}^k$ enables us to have a natural partial order. The projective norm is indeed compatible with the coordinatewise partial ordering on $\mathbb{R}^{k-1}$, i.e. $u, v \in \mathbb{R}^{k-1}, u \geqslant v \Rightarrow |u|_{\mathcal{P}} \geqslant |v|_{\mathcal{P}}$.

Let $T : \mathbb{R}^k \rightarrow \mathbb{R}^k$ be an homogeneous operator. We define

$$\tilde{T} : \ \mathbb{R}^{k-1} \rightarrow \mathbb{R}^{k-1} \quad \tilde{T}(u) = \pi_1(T(x)), \ x \in \pi_1^{-1}(u) .$$

Because of homogeneity, $\tilde{T}(u)$ is unambiguously defined. We can write with abbreviated notations $\tilde{T} = \pi_1 \circ T \circ \pi_1^{-1}$.

Lemma 7.3. *We consider an homogeneous operator $T : \mathbb{R}^k \rightarrow \mathbb{R}^k$ and the associated operator $\tilde{T} : \mathbb{R}^{k-1} \rightarrow \mathbb{R}^{k-1}$, satisfying the following assumptions:*

A. T is monotone.

B. $T(x)_1 - x_1$ *is independent of* $x \in \mathbb{R}^k$.

C. $\exists x_0$ *such that* $T(x_0)_1 - (x_0)_1 = \min_i(T(x_0)_i - (x_0)_i)$.

Under Assumption A, $\tilde{T}$ *is non-expansive. Under Assumptions* $A+B$, $\tilde{T}$ *is monotone. Under Assumptions A+B+C,* $\tilde{T}$ *has minimal value* $\tilde{x}_0 = \pi_1(x_0)$.

Proof. We consider $u, v \in \mathbb{R}^{k-1}$ verifying $u \geqslant v$. Let $x, y \in \mathbb{R}^k$ be such that $\pi_1(x) = u, \pi_1(y) = v$ and $x_1 = y_1$.

1. $A \Rightarrow \tilde{T}$ is non-expansive. The representatives x and y are such that $|u - v|_{\mathcal{P}} = |x-y|_{\mathcal{P}} = ||x-y||_\infty$. By monotonicity of T, we have $T(x) \geqslant T(y)$, hence $|T(x) - T(y)|_{\mathcal{P}} \leqslant ||T(x) - T(y)||_\infty$. By non-expansiveness of T (Theorem 2.2), we have $||T(x) - T(y)||_\infty \leqslant ||x-y||_\infty$. We conclude that :

$$
\begin{aligned}
|\tilde{T}(u) - \tilde{T}(v)|_{\mathcal{P}} = |T(x) - T(y)|_{\mathcal{P}} &\leqslant ||T(x) - T(y)||_\infty \\
&\leqslant ||x-y||_\infty = |u-v|_{\mathcal{P}} \, .
\end{aligned}
$$

2. $A + B \Rightarrow \tilde{T}$ is monotone. Let the representatives x and y verify $x_1 = y_1$. Hence by Assumption B, we have $T(x)_1 = T(y)_1$. We conclude that $T(x) \geqslant T(y) \Rightarrow \tilde{T}(u) \geqslant \tilde{T}(v)$.

3. $A + B + C \Rightarrow \tilde{T}$ has minimal value $\tilde{x}_0 = \pi_1(x_0)$. We have

$$
\begin{aligned}
\tilde{T}_0(\tilde{x}_0)_i &= T(x_0)_i - T(x_0)_1 = T(x_0)_i - (x_0)_i + (x_0)_i - T(x_0)_1 \\
&\geqslant T(x_0)_1 - (x_0)_1 + (x_0)_i - T(x_0)_1 = (\tilde{x}_0)_i \, .
\end{aligned}
$$

   We conclude with the monotonicity of $\tilde{T}$ that $\forall y \in \mathbb{R}^{k-1}, y \geqslant \tilde{x}_0 \Rightarrow \tilde{T}(y) \geqslant \tilde{x}_0$.

□

The operator $\tilde{T}$ is not homogeneous in general. Hence the conditions ensuring monotonicity and non-expansiveness are not the same (to be compared with Theorem 2.2).

Remark 7.1. Assumption B. can be easily weakened and replaced by :

B′. $\forall x, y \in \mathbb{R}^k, \; x_1 - y_1 = \min_i x_i - y_i \Rightarrow T(x)_1 - T(y)_1 = \min_i T(x)_i - T(y)_i \, .$

In Lemma 7.3, we have presented the assumptions which appear naturally in physical systems. In particular, Assumption B is verified when the first coordinate of T is the dater of an exogeneous arrival process. Assumption C is verified if the other coordinates of T correspond to events which are induced by the arrivals (hence occur later on). It was the case for the operator associated with the $G/G/1/\infty$ queue, see Equation (5.1). In that example, the minimal value was $e = (0, \ldots, 0)'$.

These assumptions are of course restrictive. Roughly speaking, they will apply only to some operators associated with 'open systems'. For operators associated with 'closed systems', the conditions and results of Section 7.2 are more appropriate.

Theorem 7.4. *Let $\{T_n, n \in \mathbb{N}\}$ be a stationary and ergodic sequence of homogeneous random operators on $\mathbb{R}^k$ and $\{\tilde{T}_n, n \in \mathbb{N}\}$ the associated sequence on $\mathbb{R}^{k-1}$. We assume that Assumptions A,B and C of Lemma 7.3 are verified with probability 1 by the operators $\{T_n\}$ (in particular they have a constant minimal value x_0). We set $\tilde{x}_0 = \pi_1(x_0)$. Then the limit $Z = \lim_n \tilde{T}_0 \circ \cdots \circ \tilde{T}{-n}(\tilde{x}_0)$ exists and verifies $P\{Z < +\infty\} = 0$ or 1. Furthermore Z is a stationary solution, i.e. $Z(\theta\omega) = \tilde{T}_1(Z(\omega))$. When $P\{Z < +\infty\} = 1$, the sequence $\{T_n \circ \cdots \circ T_1(x_0)\}$ couples in finite time with the stationary sequence $\{Z \circ \theta^n\}$.*

Proof. It is exactly similar to the one of Loynes Theorem 5.1. □

The main difficulty is often to prove the finiteness of Z. Moreover, when finite, Z is usually not the unique stationary solution. Indeed, we have that $\forall \lambda \in \mathbb{R},\ \tilde{x}_0 + \lambda\vec{1}$ is a minimal value for the operators $\tilde{T}_n$. Hence by Theorem 7.4, the limits

$$Z^\lambda = \lim_n \tilde{T}_0 \circ \cdots \circ \tilde{T}{-n}(\tilde{x}_0 + \lambda\vec{1})$$

exist and are stationary solutions. The variables Z^λ are increasing in λ by monotonicity of $\tilde{T}_n$. Hence we can define the limit

$$Z^\infty = \lim_{\lambda \to +\infty} Z^\lambda . \tag{7.2}$$

Next Theorem was originally proved by Brandt for a special operator associated with the $G/G/k/\infty$ queue.

Theorem 7.5 (Brandt [14]). *We have $P\{Z^\infty < +\infty\} = 0$ or 1. If we have $P\{Z^\infty < +\infty\} = 1$, then Z^∞ is the maximal finite stationary solution, i.e $Z(\theta\omega) = \tilde{T}_1(Z(\omega))$ and*

$$Y(\theta\omega) = \tilde{T}_1(Y(\omega)),\ P\{Y < +\infty\} = 1 \Rightarrow P\{Z^\infty \geqslant Y\} = 1 .$$

Proof. The essential ingredient is the non-expansiveness of $\tilde{T}_n$. For more details, the reader is referred to [14] or [15], Theorem 1.3.2. □

Remark 7.2. The results presented in this section §7.1 are just a specialization to operators of finite dimension of more general results. Let $(E, \mathcal{E})$ be a Polish space (complete separable metric space) equipped with its Borel σ-field. We consider $\{\phi_n, n \in \mathbb{Z}\}$ a stationary and ergodic sequence of measurable random functions $\phi_n : E \times \Omega \to E$. The recursive equations

$x(n+1) = \phi_n(x(n)), x(0) = x_0$ define a Stochastic Recursive Sequence, following the terminology of Borovkov [12]. If the functions ϕ_n are monotone and verify $\phi_n(x_0) \geqslant x_0$ then the results of Theorem 7.4 hold (replace just T_n by ϕ_n). If we assume furthermore that the functions ϕ_n are non-expansive (with respect to the metric of E) then the results of Theorem 7.5 hold. For a detailed presentation of this framework, see [15] [13].

7.2 Fixed point

We will see, in this section, a rather different use of Loynes backward construction.

Here is a result generalizing Proposition 2.9. The proof of $A \Rightarrow B$ in Prop. 2.9 was using only the continuity of the operator T. In fact, using the non-expansiveness of T, we can get stronger results.

Theorem 7.6 (Weller [44], Sine [42]). *Let C be a compact of $\mathbb{R}^k$. We consider an operator $T : C \to C$, non-expansive with respect to the sup-norm $||.||_\infty$. Then we have :*

$$\forall x \in C, \exists p \in \mathbb{N}, \exists u \in C \; : \; \lim_{n\to\infty} T^{np}(x) = u \text{ and } T^p(u) = u \,. \tag{7.3}$$

The following corollary is the essential result in what follows.

Corollary 7.7. *Let T be defined as in Theorem 7.6. We assume that $\forall n \geqslant 1$, T^n has a unique fixed point u. Then $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that*

$$\forall n \geqslant N \; : \; \sup_{x\in C} ||T^n(x) - u||_\infty \leqslant \varepsilon \,. \tag{7.4}$$

In other words, there is uniform convergence of T^n to u.

Proof. Let us prove first that T^n converges simply to u. Let x belong to C. As u is the unique fixed point of the powers of T, we obtain by application of Theorem 7.6 :

$$\forall x \in C,\ \exists p \in \mathbb{N},\ \forall \varepsilon > 0,\ \exists M(x,\varepsilon) \in \mathbb{N}, \forall n \geqslant M(x,\varepsilon) : \;\; ||T^{np}(x) - u||_\infty \leqslant \varepsilon.$$

By non-expansiveness, we have

$$\left(||T \circ T^{np}(x) - T(u)||_\infty = ||T^{np+1}(x) - u||_\infty\right) \leqslant ||T^{np}(x) - u||_\infty$$

and by induction, $\forall q \in \mathbb{N}, ||T^{np+q}(x) - u||_\infty \leqslant ||T^{np}(x) - u||_\infty$. It implies that :

$$\forall x \in C,\ \forall \varepsilon > 0,\ \exists N(x,\varepsilon) \in \mathbb{N}, \forall n > N(x,\varepsilon), \quad ||T^n(x) - u||_\infty \leqslant \varepsilon \,.$$

We are now ready to prove that the convergence is uniform. Let us denote by $\mathcal{B}(x,\varepsilon)$ the open ball of center x and radius ε for the sup-norm. Using non-expansiveness, we have that $\forall y \in \mathcal{B}(x,\varepsilon), \forall n \geqslant N(x,\varepsilon)$,

$$||T^n(y) - u||_\infty \leqslant ||T^n(y) - T^n(x)||_\infty + ||T^n(x) - u||_\infty \leqslant 2\varepsilon \,. \tag{7.5}$$

Using Borel-Lebesgue's characterization of compact sets, there exists a finite number of points x_i such that $C \subset \bigcup_i \mathcal{B}(x_i,\varepsilon)$. Using Equation (7.5), we obtain :

$$\forall \varepsilon > 0,\ \forall n \geqslant \max_i N(x_i,\varepsilon),\ \forall x \in C \ : \quad ||T^n(x) - u||_\infty \leqslant 2\varepsilon \,.$$

This completes the proof. □

We are now ready to prove the main theorem of this section which generalizes the approach proposed in [35] [36].

Theorem 7.8. *Let $\{T_n, n \in \mathbb{N}\}$ be a stationary ergodic sequence of monotone homogeneous random operators on $\mathbb{R}^k$ and $\{\tilde{T}_n\}$ the associated sequence on $\mathbb{R}^{k-1}$. We assume that there exists a deterministic monotone homogeneous operator S on $\mathbb{R}^k$ ($\tilde{S}$ on $\mathbb{R}^{k-1}$) such that*

i. *$\tilde{S}$ is bounded i.e. there exists a compact K of $\mathbb{R}^{k-1}$ such that $\mathrm{Im}(\tilde{S}) \subset K$.*

ii. *$\forall n \geqslant 1$, $\tilde{S}^n$ has a unique fixed point.*

iii. *There exists a deterministic constant l such that $\tilde{S}$ belongs to the support of the random operator $\tilde{T}_l \circ \cdots \circ \tilde{T}_1$ and $\forall n > 0, \tilde{S}^n$ belongs to the support of $\tilde{T}_{nl} \circ \cdots \circ \tilde{T}_1$, with the following precise meaning :*

$$\forall \varepsilon > 0, \quad P\{\sup_{x\in\mathbb{R}^{k-1}} |\tilde{T}_l \ldots \tilde{T}_1(x) - \tilde{S}(x)|_{\mathcal{P}} \leqslant \varepsilon\} > 0,$$
$$P\{\sup_{x\in\mathbb{R}^{k-1}} |\tilde{T}_{nl} \ldots \tilde{T}_1(x) - \tilde{S}^n(x)|_{\mathcal{P}} \leqslant \varepsilon\} > 0 \,.$$

Then $\forall x \in \mathbb{R}^{k-1}$, $\tilde{x}(n) = \tilde{T}_{n-1} \circ \cdots \circ \tilde{T}_0(x)$ converges weakly to a unique stationary distribution.

Proof. We first prove the theorem when replacing Assumption *iii.* by the stronger assumption :

iv. $\exists l$ s.t. $P\{\tilde{T}_l \circ \cdots \circ \tilde{T}_1 = \tilde{S}\} > 0$ and $\forall n > 0$, $P\{\tilde{T}_{nl} \circ \cdots \circ \tilde{T}_1 = \tilde{S}^n\} > 0$.

For $x \in \mathbb{R}^{k-1}$, we define the variables :

$$Z_{-n,0}(x) = \tilde{T}_0 \circ \cdots \circ \tilde{T}_{-n}(x) = \tilde{x}(n,x) \circ \theta^{-n} \,. \tag{7.6}$$

We now prove that $Z_{-n,0}(x)$ admits $P.a.s.$ a limit which is independent of x.

The compact K of Assumption $i.$ is stable by $\tilde{S}$, and from Assumption $ii.$, there is a unique fixed point $u \in \mathbb{R}^{k-1}$ for the powers of $\tilde{S}$. From Lemma 7.3, $\tilde{S}$ is non-expansive with respect to the projective norm. Hence Corollary 7.7 can be applied to $\tilde{S}$ on $(\mathbb{R}^{k-1}, |.|_{\mathcal{P}})$. It implies

$$\forall \varepsilon > 0,\ \exists N(\varepsilon),\ \forall n \geqslant N(\varepsilon),\ \forall x \in \mathbb{R}^{k-1}, \quad |\tilde{S}^n(x) - u|_{\mathcal{P}} \leqslant \varepsilon \,. \tag{7.7}$$

We define the random variables

$$\forall \varepsilon > 0,\ M(\varepsilon) = \min\{n \geqslant N(\varepsilon)l \mid \tilde{T}_{-n} \circ \cdots \circ \tilde{T}_{-n-N(\varepsilon)l+1} = \tilde{S}^{N(\varepsilon)}\} \,, \tag{7.8}$$

where $N(\varepsilon)$ and l are defined in Equation (7.7) and in Assumption $iv.$ respectively. Assumption $iv.$ also implies that $P\{M(\varepsilon) < +\infty\} > 0$. We obtain

$$P\{M(\varepsilon) < +\infty\} = 1 \,, \tag{7.9}$$

in exactly the same way as we obtained $P\{N_1 < +\infty\} = 1$ in the proof of Theorem 2.10.

Let us fix $\varepsilon = 1$. We define the events $\mathcal{A}_n = \{M(1,\omega) = n\}$ which form a countable partition of Ω.

Let us work for a moment on the event $\mathcal{A} = \mathcal{A}_m$ for a given integer m. Let us consider the variables $Z_{-n,-m}(x) = \tilde{T}_{-m} \circ \ldots \tilde{T}_{-n}(x),\ n > m$. We have

$$\forall n \geqslant m + N(1)l,\ Z_{-n,-m}(x) = \tilde{S}^{N(1)}(\tilde{T}_{-m-N(1)l} \circ \cdots \circ \tilde{T}_{-n}(x)) \,. \tag{7.10}$$

Hence on $\mathcal{A}_m$, the image of $Z_{-n,-m}$ is included in the closed ball of center u and radius $\varepsilon = 1$ (Equation (7.7)) that we denote by $K(1)$,

$$\forall n \geqslant m + N(1)l,\ \mathrm{Im}(Z_{-n,-m}) \subset K(1) \,. \tag{7.11}$$

We consider the sequence of random variables $\{M(1/i), i \in \mathbb{N}\}$. By definition of the variables $M(\varepsilon)$, (7.8), the sequence $M(1/i)$ is increasing in i in particular $M(1/i) \geqslant M(1)$. We have, for all $n \geqslant M(1/i) + N(1/i)l$ (note that we consider the variables Z with respect to an unchanged ending point $-m$).

$$\begin{aligned} Z_{-n,-m}(x) &= \tilde{T}_{-m} \circ \cdots \circ \tilde{T}_{-n}(x) \\ &= \tilde{T}_{-m} \circ \cdots \circ \tilde{T}_{-M(1/i)+1} \circ\ \tilde{S}^{N(1/i)}\ \circ \tilde{T}_{-M(1/i)-N(1/i)l} \circ \cdots \circ \tilde{T}_{-n}(x) \,. \end{aligned}$$

Using Equation (7.7), we have that $\tilde{S}^{N(1/i)} \circ \tilde{T}_{-M(1/i)-N(1/i)l} \circ \cdots \circ \tilde{T}_{-n}(x)$ is included in the closed ball of center u and radius $1/i$. Using the non-expansiveness of the operators, we obtain the existence of a compact set, denoted $K(1/i)$ such that

$$\forall n \geqslant M(\frac{1}{i}) + N(\frac{1}{i})l, \ \ \mathrm{Im}(Z_{-n,-m}) \subset K(\frac{1}{i}) \,. \qquad (7.12)$$

We have built a decreasing sequence of compact sets $K(1/i)$ whose radius goes to zero. By a classical theorem on decreasing sequences of compact sets (Borel-Lebesgue Theorem), the intersection of the sets $K(1/i)$ is a single point. It means precisely that the limit of $Z_{-n,-m}(x)$, $n \to +\infty$, exists and is independent of x. We define the following notations

$$\forall \omega \in \mathcal{A}_m, \ \forall x \in \mathbb{R}^{k-1}, \ \ \lim_{n\to\infty} Z_{-n,-m}(x) = Z_{\infty,m}, \ \ Z = \tilde{T}_0 \circ \cdots \circ \tilde{T}_{-m+1}(Z_{\infty,m}) \,.$$

It is straightforward to prove that $Z = \lim_{n\to+\infty} Z_{-n,0}(x)$. By applying the same construction to all the events $\mathcal{A}_m, m \in \mathbb{N}$, we prove the a.s. existence of $Z = \lim_n Z_{-n,0}(x)$, the limit being independent of x. By analogy with §5, we call Z the Loynes variable.

We are now going to prove the existence of the Loynes variable Z under the weaker Assumption *iii.*

We define the random variables $N(\varepsilon)$ as previously, (7.7). On the other hand, the definition of the variables $M(\varepsilon)$ is modified

$$\forall \varepsilon, \ M(\varepsilon) = \min\{n \geqslant N(\varepsilon)l \mid \sup_{x\in\mathbb{R}^{k-1}} |\tilde{T}_{-n+N(\varepsilon)l} \ldots \tilde{T}_{-n}(x) - \tilde{S}^{N(\varepsilon)}(x)|_{\mathcal{P}} \leqslant \varepsilon\} \,. \qquad (7.13)$$

From Assumption *iii.* and the Ergodic Lemma 1.1, we obtain $P\{M(\varepsilon) < +\infty\} = 1$.

We define the variable $M(1)$, then the partition $\mathcal{A}_n$, the event $\mathcal{A}$ and the variables $M(1/i)$ as before. We define the variables :

$$\hat{Z}^i_{-n,-m}(x) = \tilde{T}_{-m} \ldots \tilde{T}_{-M(1/i)+1} \circ \ \left(\tilde{S}^{N(1/i)}\right) \circ \tilde{T}_{-M(1/i)-1-N(1/i)l} \ldots \tilde{T}_{-n}(x) \qquad (7.14)$$

There exists a sequence of compacts $\hat{K}(1/i)$ of radius $1/i$ such that (see the first part of the proof)

$$\forall n \geqslant M(1/i) + N(1/i)l, \ \mathrm{Im}(\hat{Z}^i_{-n,-m}) \subset \hat{K}(1/i) \,. \qquad (7.15)$$

From the definition of $M(1/i)$, Equation (7.13), we get

$$\forall n \geqslant M(1/i) + N(1/i)l, \forall x \in \mathbb{R}^{k-1}, \ |Z_{-n,-M(1/i)}(x) - \hat{Z}^i_{-n,-M(1/i)}(x)|_{\mathcal{P}} \leqslant \frac{1}{i} \,.$$

Using non-expansiveness, we obtain

$$\forall n \geqslant M(1/i)+N(1/i)l, \forall x \in \mathbb{R}^{k-1},\ |Z_{-n,-m}(x) - \hat{Z}^i_{-n,-m}(x)|_{\mathcal{P}} \leqslant \frac{1}{i}\,.$$

We conclude that $\forall n \geqslant M(1/i)+N(1/i)l, \forall x, y \in \mathbb{R}^{k-1}$

$$\begin{aligned} |Z_{-n,-m}(x) - & \ Z_{-n,-m}(y)|_{\mathcal{P}} \leqslant |Z_{-n,-m}(x) - \hat{Z}^i_{-n,-m}(x)|_{\mathcal{P}} + \\ & |\hat{Z}^i_{-n,-m}(x) - \hat{Z}^i_{-n,-m}(y)|_{\mathcal{P}} + |\hat{Z}^i_{-n,-m}(y) - Z_{-n,-m}(y)|_{\mathcal{P}} \leqslant \frac{3}{i}\,. \end{aligned}$$

Hence there exists a sequence of compacts $K(1/i)$ of radius $3/i$ such that $\forall n \geqslant M(1/i)+N(1/i)l, \mathrm{Im}(Z_{-n,-m}) \subset K(1/i)$. We conclude as in the first part of the proof.

Our aim is now to prove that we have weak convergence of the process $\tilde{x}(n) = \tilde{T}_n \circ \cdots \circ \tilde{T}_0(x(0))$ to the stationary distribution of Z. We consider a function $f : \mathbb{R}^k \to \mathbb{R}$, continuous and bounded. We have, using the stationarity of $\{\tilde{T}_n\}$

$$\begin{aligned} E\,(\ f(x(n,x(0)))\ ) &= E(\ f(\tilde{T}_{n-1} \circ \cdots \tilde{T}_0(x(0))\ ) \\ &= E(\ f(\tilde{T}_0 \circ \cdots \tilde{T}_{-n+1}(x(0))\ ) \xrightarrow{n} E\ f(Z) \quad (7.16) \end{aligned}$$

The convergence in (7.16) is obtained from Lebesgue's dominated convergence theorem (f is bounded). It proves weak convergence. □

Remark 7.3. It would be nice to replace Assumption *iii.* by the following weaker Assumption

$$\begin{aligned} v. \quad \forall \varepsilon > 0,\ \forall K \text{ compact}\ , \quad & P\{\sup_{x\in K} |\tilde{T}_l \ldots \tilde{T}_1(x) - \tilde{S}(x)|_{\mathcal{P}} \leqslant \varepsilon\} > 0, \\ & P\{\sup_{x\in K} |\tilde{T}_{nl} \ldots \tilde{T}_1(x) - \tilde{S}^n(x)|_{\mathcal{P}} \leqslant \varepsilon\} > 0\,. \end{aligned}$$

Assumption $v.$ means precisely that $\tilde{S}$ is in the support of $\tilde{T}_0$ for the topology of weak convergence on the functional space $C_0(\mathbb{R}^{k-1}, \mathbb{R}^{k-1})$ (continuous functions of $\mathbb{R}^{k-1}$).

However, Theorem 7.8 is not true under Assumption $v.$ Here is a counterexample. We consider $a, b \in \mathbb{R}^+$ and we define the monotone homogeneous operators on $\mathbb{R}^2$:

$$T_A(x) = \begin{pmatrix} x_1 \\ x_2 + a \end{pmatrix} \qquad (7.17)$$

$$\forall i \in \mathbb{N}^+,\ T_{B_i}(x) = \begin{pmatrix} x_1 \\ \max(x_2 - ib, x_1) \end{pmatrix} \qquad (7.18)$$

We consider a sequence of i.i.d. random operators $\{T_n, n \in \mathbb{N}\}$ with the following distribution :

$$P\{T_0 = T_A\} = \frac{1}{2},\ P\{T_0 = T_{B_i}\} = \frac{1}{2^{i+1}}, i \in \mathbb{N}^+ .$$

We define the monotone homogeneous operator $S : \mathbb{R}^2 \to \mathbb{R}^2$, $S(x) = (x_1, x_1)'$. It is clear that $\tilde{S}$ verifies the Assumptions *i.* and *ii.* as $\tilde{S}$ is constant. Let K be a compact set of $\mathbb{R}$ and n be such that $K \subset [-n, n]$. We obtain immediately that $\forall x \in K$, $\tilde{T}_{B_i}(x) = \tilde{S}(x)$ as soon as $ib \geqslant n$. Hence $\tilde{S}$ verifies also Assumption *v.*

The description of the process $\tilde{x}(n) = \tilde{T}_{n-1} \circ \cdots \circ \tilde{T}_0(0)$ is very easy. It is a random walk on the real line with an absorbing barrier at 0. The drift of the random walk is

$$\delta = \frac{a}{2} - \sum_{i=1}^{\infty} \frac{ib}{2^{i+1}} = \frac{a}{2} - b\,.$$

We conclude that the process $\tilde{x}(n)$ is transient if $a > 2b$ which provides the announced counter-example.

Practically speaking, the main difficulty consists in finding a deterministic operator S verifying the assumptions of Theorem 7.8. We discuss this point for some specific models in §8.1.

8 Models Entering the Framework

8.1 Operators

Let $\mathcal{A}$ and $\mathcal{B}$ be two arbitrary sets. We define applications ($\mathbb{M}_k$ denotes the set of matrices of dimension $k \times k$)

$$P : \mathcal{A} \times \mathcal{B} \to \mathbb{M}_k(\mathbb{R}),\ A : \mathcal{A} \times \mathcal{B} \to \mathbb{M}_k(\mathbb{R} \cup \{-\infty, +\infty\})\,,$$

where the matrices $P(\alpha, \beta)$ are "Markovian", i.e. verify

$$\forall i \in \{1, \ldots, k\},\ p_{ij}(\alpha, \beta) \geqslant 0,\ \sum_{j=1}^{k} p_{ij}(\alpha, \beta) = 1\,. \tag{8.1}$$

Let us consider the following "(min,max,+,×)" function

$$x \in \mathbb{R}^k, i \in \{1, \ldots, k\},\ T(x)_i = \inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} \sum_{j=1}^{k} p_{ij}(\alpha, \beta)\,(x_j + a_{ij}(\alpha, \beta))\ . \tag{8.2}$$

Equation (8.2) arises in stochastic control of dynamic games, see [11]. If $T(x)_i$ is finite $(\forall x \forall i)$ then it defines a monotone-homogeneous operator. For example, let us prove homogeneity. We have for $x \in T^k$, $\lambda \in \mathbb{R}$,

$$
\begin{aligned}
T(x+\lambda\vec{1})_i &= \inf_{\alpha} \sup_{\beta} \sum_{j=1}^{k} p_{ij}(\alpha,\beta)\,(x_j + \lambda + a_{ij}(\alpha,\beta)) \\
= \inf_\alpha \sup_\beta &\quad (\sum_{j=1}^{k} p_{ij}(\alpha,\beta)\lambda) + \sum_{j=1}^{k} p_{ij}(\alpha,\beta)\,(x_j + a_{ij}(\alpha,\beta)) = \lambda + T(x)_i \,.
\end{aligned}
$$

The following representation theorem provides a precise idea of the degree of generality of the class of monotone-homogeneous operators.

Theorem 8.1 (Kolokoltsov [33]). *Let $T : \mathbb{R}^k \to \mathbb{R}^k$ be a monotone and homogeneous operator. Then it can be represented in the form of Equation (8.2).*

The next lemma which is based on this representation, is proved in [33]. It can be coupled with Theorem 7.8 to obtain second order results for some stochastic operators.

Lemma 8.2. *Let $T : \mathbb{R}^k \to \mathbb{R}^k$ be a monotone-homogeneous operator, written in the form of Equation (8.2). Let us assume that*

$$
\exists \eta > 0 :\ \forall i, j \exists l : \forall \alpha, \beta,\ p_{il}(\alpha,\beta) > \eta, p_{jl}(\alpha,\beta) > \eta \,.
$$

Then the operators $T^n, n \in \mathbb{N}$, have a unique generalized fixed point.

From the point of view of applications, the interesting case is when the sets $\mathcal{A}$ and $\mathcal{B}$ are finite. Here are some specializations of Equation (8.2).

$(+,\times)$ linear systems The operator T is just a markovian matrix P, see Equation (8.1). We have $T(x) = Px$ (matrix-vector multiplication in the usual algebra). Matrix P can be interpreted as the matrix of transition probabilities of a Markov Chain (MC) having state space $\{1, \ldots, k\}$. The most interesting operator for a MC is $S(y) = yP$ where y is a row vector. It is well known that the limit of $S^n(y), y \geqslant 0, \sum_i y_i = 1$ is the stationary distribution of the MC. But the operator $T(x) = Px$ is also interesting from the point of view of applications. It appeared in [21] to model the problem of reaching agreement on subjective opinions. More generally, it has been studied as a special case of the general theory of products of non-negative matrices, see for example [41], Chapter 4.6.

For any markovian matrix P, we have $T(\vec{1}) = P\vec{1} = \vec{1}$. Hence the vector $\vec{1}$ is a generalized fixed point (Def. 2.8) of operator T. By application of the

Perron-Frobenius Theorem, it is the only one. Hence, applying the ergodic results of this paper to a stochastic sequence of matrices P_n, is going to yield trivial results (the convergence of $\pi(P_n \dots P_0 x)$ to $\pi(\vec{1})$). In fact much stronger results are known for such models. The necessary and sufficient conditions of convergence of $\pi(P_n \dots P_0 x)$ to $\pi(\vec{1})$, are known for a general sequence of matrices P_n, without any stochastic assumptions, see [41], Th. 4.18.

(max,+) linear systems Such operators have the following form

$$x \in \mathbb{R}^k, i \in \{1, \dots, k\}, \quad T(x)_i = \max_j (x_j + a_{ij}) \; , \tag{8.3}$$

$$T(x) = A \otimes x \; . \tag{8.4}$$

Equation (8.3) can be interpreted as a matrix-vector product in the (max,+) algebra. Equation (8.4) is simply a rewriting of Equation (8.3) using (max,+) notations. The (min,+) linear case boils down to the (max,+) case by switching to operator $-T$.

Such systems appear in many domains of applications, under various forms. For example (without any kind of exhaustivity)

- Computer science : parallel algorithms, shared memory systems, PERT graphs, see [43] or [23].
- Queueing theory : $G/G/1/\infty$ queue (see §5), queues in series, queues in series with blocking, fork-join networks [3].
- Operations research and manufacturing : Job-shop models, event graphs (a subclass of Petri nets), see [17] [28] and [3].
- Economy or control theory : dynamic optimization, see [46].
- Physics of crystal structures : Frenkel-Kontorova model, see [24].

Among the very large and complete literature on the theoretical aspects of deterministic (max,+) systems, let us quote only [3] [37] and the references therein. As far as we know, the first references on stochastic (max,+) linear systems are [18] and [39]. Thanks to the rich deterministic theory, Theorems 2.5, 7.8 become very operational for (max,+) systems. The different assumptions in these theorems can be interpreted as properties of the underlying graph structure of the model. For more details, see [35].

(min,max,+) linear systems These systems can be represented in one of the following dual forms. We use the symbol $\otimes$ for the (max,+) matrix-vector product, see (8.4), and the symbol $\odot$ for the (min,+) matrix-vector product.

$$\begin{aligned} x \in \mathbb{R}^k, \quad T(x) &= \min\left(A_1 \otimes x, A_2 \otimes x, \ldots, A_l \otimes x\right) , \\ T(x) &= \max\left(B_1 \odot x, B_2 \odot x, \ldots, B_p \odot x\right) . \end{aligned}$$

Here are some domains of application where such systems appear

- Minimax control in dynamic game theory, see [11].
- Study of timed digital circuits, see [26]. The (min,max) structure arises from the (and,or) operations of logical circuits.
- Queueing theory. G/G/s/∞ file, resequencing file, see for example [1]. Parallel processing systems [9] : there are k processors. A customer requires to use concurrently p out of the k processors to be executed.
- Motion of interfaces in particle systems [22]. As an illustration, let us describe a little bit more precisely a special case known as the marching soldier model. There is a row of k soldiers which advance in the same direction. In order to try to keep a common pace, they adopt the following strategy. At regular instants of time, each soldier checks the position of his right and left neighbours. He advances of 1 if they both are ahead of him and stays at the same position otherwise. Let $x \in \mathbb{R}^k$ denote the position of the soldiers at instant 0. Their position at instant 1, will be (with the convention $x_0 = x_{k+1} = +\infty$)

$$T(x)_i = \max\left(\min(x_{i-1}, x_i, x_{i+1}) + 1, x_i\right) .$$

The study of deterministic (min,max,+) systems (existence of generalized fixed points, projective boundedness,...) has been considered in several papers [38] [25]. However, it is far from being complete. For this reason, the only references on stochastic (min,max,+) systems concern first order results [22] [30].

(max,+,$\times$) linear systems These systems can be represented under the following form

$$x \in \mathbb{R}^k, \; T(x)_i = \max_{\alpha \in \mathcal{A}} \sum_{j=1}^{k} p_{ij}(\alpha)\left(x_j + a_i(\alpha)\right) . \qquad (8.5)$$

Equation (8.5) appears in many domains of applications like operational research, management science and engineering. It is in fact one of the optimality equation of stochastic[3] dynamic programming in discrete time, on a finite

[3]The term stochastic refers here to the markovian interpretation of matrices $P(\alpha)$. According to our terminology, Equation (8.5) is that of a deterministic operator.

state space and with undiscounted rewards. A controller observes a system which evolves in a state space $\{1, \ldots, k\}$. The set of possible decisions for the controller is $\mathcal{A}$. Under decision $\alpha \in \mathcal{A}$, the system evolves from a state i to a state j according to the transition probabilities $p_{ij}(\alpha)$. Also, under decision $\alpha \in \mathcal{A}$, there is an immediate reward for being originally in state i which is $a_i(\alpha)$. It is well known that the optimal decision and the reward vector are obtained as $\lim_n T^n(x)$, see for example [45], Chapter 3.2.

There is a very important literature on deterministic operators of type (8.5), see [40] or [45] and the references there. The next theorem is classical, for a proof see for example [45] Chapter 4.3.

Theorem 8.3. *Let T be an operator verifying Equation (8.5). A sufficient condition for the existence of a unique generalized fixed point for T is :*
$\forall \alpha \in \mathcal{A}$, matrix $P(\alpha)$ is ergodic, i.e. the graph of the non-zero terms of $P(\alpha)$ is strongly connected and aperiodic.

Remark 8.1. A (max,+) system can be viewed as a (max,+,$\times$) system with $\mathcal{A} = \{1, \ldots, k\}$ and $P(\alpha)$ is defined by $P_{ij}(\alpha) = 1$ if $j = \alpha$ and $P_{ij}(\alpha) = 0$ otherwise. Such matrices do not verify the assumption of Theorem 8.3.

The theorems presented in this paper when coupled with results like Theorem 8.3, can be used in an efficient way for systems verifying (8.5) when the rewards $a(\alpha)$ and/or the transition matrices $P(\alpha)$ become random. The authors do not know of any reference on the subject.

8.2 Discrete event networks

We are now going to review some classes of discrete event networks. We restrict our attention to systems which can not be modeled as monotone-homogeneous operators. The references that are quoted are only the ones using the monotone separable framework or similar approaches.

- Precedence constraints models. Their study has been motivated by database systems. Different variations are considered in [8] [10] [20].
- Polling models. A wide class of polling models with general routing policies and stationary ergodic inputs enters the monotone separable framework, see [16].
- Free choice Petri nets. event graphs, which are represented as (max,+) linear operators, see §8.1, or Jackson networks, see below, are subclasses of free choice Petri nets. Free choice Petri nets enter the monotone separable framework, see [6] [4] [7].

Let us detail two of these models. First we propose a simple example of precedence constraint system and second Jackson networks.

Precedence constraints models There is a stream of customers $j(n), n \in \mathbb{N}$. Each customer $j(n)$ has a service time requirement $t(n)$ and precedence constraints under the form of a list $L(n)$ of customers. More precisely, we have $L(n) = \{j(i_1), j(i_2), \ldots j(i_{l_n})\}$ with $n > i_1 > i_2 > \cdots > i_{l_n} \geqslant 0$. Job j_n starts its execution as soon as all the customers of the list $L(n)$ have completed their execution. The execution of customer $j(n)$ takes $t(n)$ units of time.

Let us distinguish two cases.

1. We assume that the length of the precedence list is uniformly bounded by k, i.e. $\forall n \in \mathbb{N}, l_n \leqslant k$. We define the vector $x(n) \in \mathbb{R}^k$ such that $x(n)_i$ is the instant of completion of customer $j(n-i)$. From the dynamic described above, we have $x(n+1) = T_n(x(n))$, where the operator $T_n : \mathbb{R}^k \to \mathbb{R}^k$ is defined as follows

$$\left\{ \begin{array}{r} T_n(x)_1 = \max_{\{i \mid j(n-i) \in L(n)\}} x_i + t(n) \\ T_n(x)_i = x_{i-1}, \ i = \{2, \ldots, k\} \end{array} \right.$$

   This operator is monotone homogeneous. It is in fact a (max,+) linear system, see §8.1.

2. Let us assume now that the length l_n is not uniformly bounded. It is not possible to describe the system as an operator of finite dimension. Let $X_{[1,n]}$ be the last instant of completion of one of the customers $j(i), i \in \{1, \ldots, n\}$. It is easy to verify that $X_{[1,n]}$ verifies the properties of the monotone-separable framework for discrete event networks, see §3.

In both cases, when $\{t(n), L(n), n \in \mathbb{N}\}$ forms a stationary ergodic sequence of random variables, we can apply the ergodic theorems presented in this paper.

Jackson networks A Jackson network (introduced in [29]) is a queueing network with I nodes, where each node is a single server FIFO queue (cf §5). Customers move from node to node in order to receive some service there. The data are $(2I)$ sequences

$$\{\sigma^i(n), n \in \mathbb{N}\}, \quad \{\nu^i(n), n \in \mathbb{N}\}, \quad i \in \{1, \ldots, I\},$$

where $\sigma^i(n) \in \mathbb{R}^+$ and $\nu^i(n) \in \{1, \ldots, I, I+1\}$.

In the nominal network, the n-th, $n \geqslant 1$, customer to be served by node i after the origin of time requires a service time $\sigma^i(n)$; after completion of its service there, it moves to node $\nu^i(n)$, where $I+1$ is the exit. We say that $\nu^i(n)$ is the n-th routing variable on node i.

We are going to describe the closed (resp. open) Jackson network as a discrete event network (resp. open discrete event network), using the notations of §3.

1. **Closed case:** the state at the origin of time is that with all customers in node 1, and service 1 is just starting on node 1. There are no external arrivals and $\nu^i(n) \in \{1, \dots, I\}$, for all i and n. The total number of customers in the network is then a constant. We take

$$\sigma(n) \stackrel{\text{def}}{=} \sigma^1(n).$$

The *internal daters* $X^{i-}_{[1,\infty]}(n)$ and $X^{i+}_{[1,\infty]}(n)$, $n \geqslant 1$, $i \in \{1, \dots, I\}$, are the initiation and completion instants of the n-th service on node i. We take

$$X^{-}_{[1,\infty]}(n) \stackrel{\text{def}}{=} X^{1-}_{[1,\infty]}(n),$$

so that $X^{-}_{[1,\infty]}(1) = 0$.

2. **Open case:** the state at the origin of time is that with all queues empty and a customer is just arriving in the network. There is an external arrival point process $\{A_n, n \geqslant 1\}$, with $A_1 = 0$, or equivalently an additional saturated node (numbered 0), which produces customers with inter-arrival times $\sigma^0(n) = A_{n+1} - A_n$, $n \geqslant 1$, regardless of the state of the network. The n-th external arrival is routed to node $\nu^0(n) \in \{1, \dots, I\}$. We take

$$\sigma(n) \stackrel{\text{def}}{=} \sigma^0(n).$$

We can extend the definition of internal daters, which is the same as above, to $i = 0$ by taking $X^{0-}_{[1,\infty]}(n) = A_n$ and $X^{0+}_{[1,\infty]}(n) = A_n + \sigma(n) = A_{n+1}$. We take

$$X^{-}_{[1,\infty]}(n) \stackrel{\text{def}}{=} X^{0-}_{[1,\infty]}(n)\,,$$

so that $X^{-}_{[1,\infty]}(1) = 0$.

In both cases, the restrictions $[1, m]$ of the process are obtained by modifying the $\{\sigma(n), n \in \mathbb{N}\}$ sequence in the following way

$$\sigma^i_{[1,m]}(n) = \begin{cases} \sigma^i(n) & \text{for all } n \geqslant 1 \text{ and } i \neq 1 \text{ (resp. } i \neq 0); \\ \sigma^i(n) & \text{for all } 1 \leqslant n \leqslant m \text{ and } i = 1 \text{ (resp. } i = 0); \\ \infty & \text{for all } n > m \text{ and } i = 1 \text{ (resp. } i = 0). \end{cases}$$

The corresponding variables are denoted $X^{-}_{[1,m]}(n), X^{+}_{[1,m]}(n)$. In both cases, the maximal dater is defined as

$$\begin{aligned} X_{[1,m]} \quad = \quad \max \Bigg(& \sup_{i,n} \left\{ X^{i-}_{[1,m]}(n) \text{ s. t. } X^{i-}_{[1,m]}(n) < \infty \right\}, \\ & \sup_{i,n} \left\{ X^{i+}_{[1,m]}(n) \text{ s. t. } X^{i+}_{[1,m]}(n) < \infty \right\} \Bigg)\,, \end{aligned}$$

where the supremum bears on $n \geqslant 1$ and $i \in \{1, \dots, I\}$ (resp. $i \in \{0, \dots, I\}$) in the closed (resp. open) case.

The following lemma follows from results proved in [2].

Lemma 8.4. *For all $i \in \{1, \dots, I\}$ and $l \geqslant 1$, there exist finite sets $\mathcal{A}(i,l) \subset \mathbb{N}$, $\mathcal{B}(i,l,p) \subset \mathbb{N}$ where $p \in \mathcal{A}(i,l)$ and $\mathcal{C}(i,l,p,q) \subset \mathbb{N} \times \mathbb{N}$ where $q \in \mathcal{B}(i,l,p)$, which depend on the routing sequences only (not on the service sequences). These sets are such that*

$$\forall m, n \geqslant 1,\ X^{i-}_{[1,m]}(n) = \inf_{l \in \mathcal{A}(i,n)} \max_{p \in \mathcal{B}(i,n,l)} \sum_{(i_q,n_q) \in \mathcal{C}(i,n,l,p)} \sigma^{i_q}_{[1,m]}(n_q). \tag{8.6}$$

A pair (i,n) appears at most once in each set $\mathcal{C}(i,n,l,p)$.

This lemma has to be interpreted as the fact that Jackson networks have a (min,max,+) structure, although a very complicated one. Hence, it should come as no surprise that they enter the monotone separable framework. Let us prove it.

Causality In both cases, the assumption is that $X_{[1,m]}$ is a.s. finite for all m. Note that this implies *causality* as defined in § 3.

Lemma 8.5. *Causality is satisfied whenever the routing sequences $\{\nu^i(n)\}_{n\in\mathbb{N}}$ are i.i.d. and independent of the service times, and the routing matrix*

$$\mathbb{P} = (p_{ij}), \quad p_{ij} = P(\nu^i(1) = j), \quad i, j \in \{1, \dots, I\}$$

is without capture in the open case, and irreducible in the closed case.

Proof. The proof is based on the following coupling idea: consider a Kelly network (i.e. a route is attached to a customer, see [31]) where the routes are independent and sampled according to the stopped Markov chain with transition matrix $\mathbb{P}$. By this we mean that in the $[1,m]$-network, the route of the first customer to leave node 1 (resp. 0) is

$$\begin{array}{rl} \{N_0 = 1, N_1, \dots, N_{U_1}\} & \text{in the closed case} \\ \{N_0 = D, N_1, \dots, N_{U_{I+1}}\} & \text{in the open case,} \end{array}$$

where $\{N_p\}$ is a path of the Markov chain $\mathbb{P}$, U_i is the return time to state i, and D is an independent random variable on $\{1, \dots, I\}$, with distribution $\pi(i) = P(\nu^0(1) = i)$. The routes of the m first customers to be served at node 1 (resp. to arrive from node 0) are assumed to be independent and identically distributed. In this Kelly network, the routes of these m customers are not affected by the service times (in contrast with what happens in the initial network). Thus, in the closed (resp. open) case, all m customers eventually return to node 1 (resp. leave) provided $\mathbb{P}$ is irreducible (resp. $\mathbb{P}$ is without capture). In addition, such a Kelly network is identical in law to the $[1,m]$ restriction of the original network. So $P(X_{[1,m]} < \infty) = 1$. ◻

In what follows, we will adopt the assumptions of Lemma 8.5 and assume in addition that the service times are integrable.

Monotonicity As an immediate corollary of Lemma 8.4, for all fixed routing sequences, for all $m, n \geqslant 1$ and i, the variable $X^{i-}_{[1,m]}(n)$ (and therefore $X^{i+}_{[1,m]}(n)$ as well) is a monotone non-decreasing function of $\{\sigma^j(n),\ j \in [2,\dots,I], n \geqslant 1,\ \sigma^1(n),\ 1 \leqslant n \leqslant m\}$ (resp. $\{\sigma^j(n),\ j \in [1,\dots,I], n \geqslant 1,\ \sigma^0(n),\ 1 \leqslant n \leqslant m\}$). This monotonicity extends to the maximal dater as well.

Non-expansiveness Let $j \leqslant I$ and $l \geqslant 1$ be fixed. Consider $\sigma^j(l)$ as a variable and all other service times as constants. Then, it follows from Lemma 8.4 that $X^{i-}_{[1,m]}(n)$ is a $(\min, \max)$ function of $\sigma^j(l)$. Thus non-expansiveness as defined in § 3 holds.

Separability Let $\varphi^i_{[1,m]} = \sup\{n \geqslant 1 \mid X^{i+}_{[1,m]}(n) < \infty\}, \quad m \geqslant 1$, (the total number of events which ever complete on station i in the $[1,m]$-network). Of course $\varphi^1_{[1,m]} = m$ in the closed case, and $\varphi^0_{[1,m]} = m$ in the open case.
The following two properties hold:

1. For all i and m, $\varphi^i_{[1,m]}$ does not depend on the (finite) values of the variables $\{\sigma^j(n),\ j \in [2,\dots,I], n \geqslant 1,\ \sigma^1(n),\ 1 \leqslant n \leqslant m\}$ (resp. $\{\sigma^j(n),\ j \in [1,\dots,I], n \geqslant 1,\ \sigma^0(n),\ 1 \leqslant n \leqslant m\}$) –this follows from Lemma 8.4.

2. For all $m \geqslant 1$, the random variables $\{\varphi^i_{[1,m]},\ i \leqslant I\}$ form a stopping time of the sequences $\{\nu^i(n),\ i \leqslant I,\ n \geqslant 1\}$ in the sense that

$$\{\varphi^i_{[1,m]} \leqslant n^i,\ i \leqslant I\} \in \mathcal{F}\{\nu^i(l),\ l \leqslant n^i,\ i \leqslant I\},$$

   where $\mathcal{F}(u)$ denotes the σ-algebra generated by the random variable u.

We are now in a position to complete the definition of $N = \{\sigma(n), M(n),\ n \in \mathbb{N}^*\}$ (see §3) for this network, by taking

$$M(n) \stackrel{\text{def}}{=} \{\sigma^i(l),\ \nu^i(l),\ l = \varphi^i_{[1,n-1]} + 1, \dots, \varphi^i_{[1,n]},\ i \leqslant I\}, \quad n \geqslant 1,$$

with the convention $\varphi^i_{[1,0]} = 0$.

With this definition, the $[m,\infty]$-network, $1 \leqslant m$, is a Jackson network as defined above, but with the driving sequences

$$\begin{aligned} \sigma^i_{[m,\infty]}(n) &= \sigma^i(n + \varphi^i_{[1,m-1]}), \quad n \geqslant 1, \\ \nu^i_{[m,\infty]}(n) &= \nu^i(n + \varphi^i_{[1,m-1]}), \quad n \geqslant 1. \end{aligned}$$

From the i.i.d. assumptions on the sequences $\{\sigma^i(n), \nu^i(n), n \in \mathbb{N}\}$ and the fact that the r. v. $\varphi^i_{[1,m-1]}$ are stopping times, we obtain that the $[m,\infty]$-network is equal in distribution to the original $[1,\infty]$-network. Separability is now clear:

- **Open case:** if $A_{l+1} \geqslant A_1 + X_{[1,l]}$, then from monotonicity, for all i,

  $$A_{l+1} \geqslant A_1 + X^{i+}_{[1,l]}(\varphi^i_{[1,l]}) \geqslant A_1 + X^{i+}_{[1,n]}(\varphi^i_{[1,l]}),$$

  and so, the $(l+1)$-st external arrival finds an empty network (we know that if there are l external arrivals and $\varphi^i_{[1,l]}$ departures from node i, then the network is empty). In addition, the next customer to be served on node i is that with index $\varphi^i_{[1,l]} + 1$, $i \leqslant I$. Thus $A_1 + X_{[1,m]} = A_{l+1} + X_{[l+1,m]}$.

- **Closed case:** if $X^+_{[1,m]}(l+1) \geqslant X_{[1,l]}$, then

  $$X^+_{[1,m]}(l+1) \geqslant X^{i+}_{[1,l]}(\varphi^i_{[1,l]}) \geqslant X^{i+}_{[1,m]}(\varphi^i_{[1,l]}),$$

  and so, by the same argument as above, when the $(l+1)$-st service ends on node 1, all customers are present in node 1. Separability follows in a way which is similar to that of the previous case.

First order ergodic theorem

Compatibility is immediate from Property 2 of $\{\varphi^i_{[1,m]}\}$. To prove Integrability, it is enough to prove that $X_{[1,1]}$ is integrable. This follows from the fact that the stopping times U_1 (resp. U_{I+1}) of $\mathbb{P}$ are integrable and from the assumption that service times are integrable.

Therefore, Theorem 3.2 applies and

$$\lim_{m\to\infty} \frac{X_{[1,m]}}{m} = \gamma, \quad \text{a.s.}$$

for some positive and finite constant, both in the open and closed case. More generally, it can be shown that the above limit implies that there exist finite constants rates γ^i such that

$$\lim \frac{X^{i-}_{[1,m]}}{m} = \gamma^i, \quad \text{a.s.,} \quad i \leqslant I,$$

both in the open and closed cases. For more details on the computation of these rates see [4] and [7].

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