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Pascale Charpin

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# Tools for coset weight enumerators of some codes<sup>1</sup>

## Éléments de calcul pour les distributions de poids des translatés de certains codes

PASCALE CHARPIN<sup>2</sup>

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# Eléments de calcul pour les distributions de poids des translatés de certains codes

## Résumé

Soient  $K$  et  $G$  deux corps finis de même caractéristique, où  $G$  est le corps d'ordre  $p^m$ . Tout code linéaire  $C$ , de longueur  $p^m$  sur  $K$ , peut être considéré comme un sous-espace d'une algèbre de groupe de type  $K[G, +]$ . Nous nous intéressons ici aux polynômes des poids des translatés d'un tel code. Nous voulons montrer que dans ce contexte les opérations de l'algèbre de groupe peuvent être efficacement utilisées. Nous présentons deux types de relations qui lient l'ensemble des éléments orthogonaux à un mot  $x$ , qui n'est pas dans  $C$ , avec les valeurs des produits  $xy$ , où  $y$  est dans  $C$ . Nous donnons quelques applications, essentiellement lorsque  $C$  est cyclique étendu.

## Tools for coset weight enumerators of some codes

### Abstract

Every extended primitive code  $C$  can be viewed in a group algebra  $K[G, +]$ , where  $K$  and  $G$  are finite fields of same characteristic. Our purpose is to show that the use of the multiplication of these algebra can provide, in some situations, some tools which apply to the determination of weight distributions of cosets of codes. Actually we will explain two formulae which provide relations between the set of elements orthogonal to a codeword  $x$  and the values of products  $xy$ ,  $y \in C^\perp$ . We give some applications.

**Keywords:** cosets, cyclic codes, group algebra, equations on finite fields

# 1 Introduction

The  $p$ -ary Reed-Muller codes (RM-codes) can be seen as *polynomials* codes or as extended cyclic codes [13]. Moreover BERMAN [4] proved that they are the powers of the radical of the group algebra  $\mathcal{A} = K[\{\mathbf{G}, +\}]$ ,  $K = GF(p)$  and  $\mathbf{G} = GF(p^m)$  ( $m > 1$ ,  $p$  a prime). The Reed-Muller codes have remarkable properties and all results on them involve results on some codes of length  $p^m$  over  $K$ , the so-called extended primitive codes. For instance KASAMI deduced weight distributions of two and triple error-correcting binary BCH codes from the weight distributions of binary Reed-Muller codes of order 1 and 2 [14].

Set  $N = p^m$ . In this paper we treat only linear codes of length  $N$  over  $K$ . Such a code  $C$  is viewed as a  $K$ -subspace of  $\mathcal{A}$  and we are mainly interested by the weight distributions of cosets of  $C$ . Let  $D$  be the code generated by  $C$  and a coset  $x + C$  of  $C$ . We will assume that the weight distribution of  $C^\perp$  is known. As  $C \subset D$ ,  $D^\perp \subset C^\perp$  so that the weight distribution of  $x + C$  can be deduced from the weight distribution of  $C^\perp$  and of  $D^\perp$ . Hence the problem consists of the determination, for any nonzero weight  $\lambda$  of  $C^\perp$ , of the number of elements  $y$  of  $C^\perp$  of weight  $\lambda$  which are orthogonal to  $x$ . We will study the zeros of the products  $xy$ , for a fixed  $x$ . Through Formulae (I) and (II), we will establish some relations between the possible values of  $xy$  and the set of codewords orthogonal to  $x$ .

The formulae are most interesting for codes  $C$  which contain the RM-code of order  $j$  and are contained in the RM-code of order  $j + 1$ , for some  $j$ . In this case the code  $C$  is an ideal of  $\mathcal{A}$ . Moreover we often will suppose that  $C$  is an extended cyclic code; then we can use together the properties of RM-codes and the permutations which conserve the code  $C$ .

The principal results are obtained in the binary case, when  $C^\perp$  is a subcode of the Reed-Muller code of order 2. For instance we were able to verify the conjecture of CAMION, COURTEAU and MONTPETIT [6]: for  $m$  even there are eight distinct weight distributions for the cosets of any 2-error-correcting extended BCH codes of length  $2^m$  [10]. We give here general tools we applied to the special case of 2-error-correcting extended BCH codes; we explain the possible applications and give some new results.

## 2 Terminology and basic properties

### 2.1 Extended cyclic codes in a group algebra

We denote by  $\mathcal{A}$  the group algebra  $K[\mathbf{G}, +]$ , which is the set of formal polynomials

$$x = \sum_{g \in \mathbf{G}} x_g X^g, \quad x_g \in K,$$

with

$$0 = \sum_{g \in \mathbf{G}} 0 X^g, \quad 1 = \sum_{g \in \mathbf{G}} X^g, \quad aX^g + bX^g = (a + b)X^g.$$

and

$$\sum_{g \in \mathbf{G}} x_g X^g \sum_{g \in \mathbf{G}} y_g X^g = \sum_{h \in \mathbf{G}} \left( \sum_{g \in \mathbf{G}} x_g y_{h-g} \right) X^h. \quad (1)$$

A code of  $\mathcal{A}$  is a  $K$ -subspace of  $\mathcal{A}$ . Let  $C$  be such a code; we will denote by  $C^\perp$  the dual of  $C$ :

$$C^\perp = \{ y \in \mathcal{A} \mid \langle x, y \rangle = 0, \text{ for all } x \in C \}$$

where  $\langle x, y \rangle = \sum_{g \in \mathbf{G}} x_g y_g$ .

The algebra  $\mathcal{A}$  has only one maximal ideal, its so-called *radical*

$$\mathcal{P} = \{ x \in \mathcal{A} \mid \sum_{g \in \mathbf{G}} x_g = 0 \} = \{ x \in \mathcal{A} \mid x^p = 0 \}. \quad (2)$$

For every  $j$ , we denote by  $\mathcal{P}^j$  the  $j$ th power of  $\mathcal{P}$ ; it is the ideal of  $\mathcal{A}$  generated by the products  $\prod_{i=1}^j x_i$ ,  $x_i \in \mathcal{P}$ . One obtains the decreasing sequence of ideals of  $\mathcal{A}$ :

$$\{0\} = \mathcal{P}^{m(p-1)+1} \subset \mathcal{P}^{m(p-1)} \subset \dots \subset \mathcal{P}^2 \subset \mathcal{P},$$

where  $\mathcal{P}^{m(p-1)} = \{ a1 \mid a \in K \}$ . The position of an element or a subset of  $\mathcal{A}$  in this sequence will appear as a useful parameter we define now:

**Definition 1** Let  $x \in \mathcal{A}$ ; let  $U$  be a subset of  $\mathcal{A}$ . Let  $j \in [0, m(p-1)]$ .

- We will say that the depth of  $x$  equals  $j$  if and only if  $x$  is in  $\mathcal{P}^j$  and not in  $\mathcal{P}^{j+1}$ .
- We will say that the depth of  $U$  equals  $j$  if and only if  $U$  is included in  $\mathcal{P}^j$  and not included in  $\mathcal{P}^{j+1}$ .

By convention,  $\mathcal{P}^0 = \mathcal{A}$ .

Let  $\mathcal{R}$  be the quotient algebra  $K[Z]/(Z^n - 1)$ , where  $n = p^m - 1$ . A cyclic code  $C^*$  of length  $n$  over  $K$  is a principal ideal of  $\mathcal{R}$ . It is generated by a polynomial of  $K[Z]$  whose roots are called the zero's of  $C^*$ . We consider the extension  $C$  of the code  $C^*$  in the algebra  $\mathcal{A}$ . Let  $\alpha$  be a primitive root of unity in  $\mathbf{G}$ . The extension is the usual one: each codeword  $c^* \in \mathcal{R}$ ,  $c^* = \sum_{i=0}^{n-1} c_i^* Z^i$ , is as follows extended to  $c \in \mathcal{A}$ .

$$c = c_0 X^0 + \sum_{i=0}^{n-1} c_{\alpha^i} X^{\alpha^i}, \quad c_0 = -\left(\sum_{i=0}^{n-1} c_i^*\right), \quad c_{\alpha^i} = c_i^*.$$

Now consider the set  $I$  of those  $k$  such that  $\alpha^k$  is a zero of the cyclic code  $C^*$ . Then the codeword  $c$  is an element of  $C$  if and only if it satisfies:

$$\sum_{g \in \mathbf{G}} c_g = 0 \quad \text{and} \quad \sum_{i=0}^{n-1} c_{\alpha^i} (\alpha^k)^i = 0, \quad \text{for all } k \in I.$$

Therefore we can identify precisely an extended cyclic code in  $\mathcal{A}$ .

**Definition 2** Let  $S = [0, n]$ . An extended cyclic code  $C$  in  $\mathcal{A}$  is uniquely defined by a subset  $T$  of  $S$  such that  $0 \in T$  and  $T$  is a union of cyclotomic cosets of  $p$  modulo  $n$ . Let us define for all  $s \in S$ :

$$\phi_s : x \in \mathcal{A} \mapsto \sum_{g \in \mathbf{G}} x_g g^s \in \mathbf{G}.$$

In particular,  $\phi_0(x) = \sum_{g \in \mathbf{G}} x_g$ . Then we have that

$$C = \{ x \in \mathcal{A} \mid \phi_s(x) = 0, \text{ for all } s \in T \}$$

We say that  $T$  is the defining set of the code  $C$ .

The  $p$ -ary Reed-Muller codes are extended cyclic codes in  $\mathcal{A}$  [13]. Their defining sets are related with the  $p$ -weights of the elements of the interval  $S = [0, n]$ : for any  $s \in S$ , let  $\sum_{i=0}^{m-1} s_i p^i$ ,  $s_i \in [0, p-1]$ , be the  $p$ -ary expansion of  $s$ ; then the  $p$ -weight of  $s$  is

$$\omega_p(s) = \sum_{i=0}^{m-1} s_i.$$

**Definition 3** The  $p$ -ary Reed-Muller code of order  $r$ , denoted by  $R_p(r, m)$ , is the extended cyclic code in  $\mathcal{A}$ , whose defining set is

$$I_p(j, m) = \{ s \in S \mid \omega_p(s) < j \}, \quad j = m(p-1) - r.$$

BERMAN proved in [4] that the  $p$ -ary Reed-Muller codes are the powers of the radical of  $\mathcal{A}$ . More precisely:

$$\mathcal{P}^j = R_p(m(p-1) - j, m), \quad \text{for all } j \in [1, m(p-1)] \quad (3)$$

Therefore, we can deduce from a wellknown property of the generalized Reed-Muller codes that  $(\mathcal{P}^j)^\perp = \mathcal{P}^{m(p-1)-j+1}$ . In the following we will identify the  $p$ -ary RM-codes with the codes  $\mathcal{P}^j$ . Note that

$$\dim \mathcal{P}^j = \text{card} \{ s \in S \mid \omega_p(s) \geq j \} \quad (4)$$

**Cosets of extended cyclic codes and equations over finite fields:** Let  $C$  be an extended cyclic code with defining set  $T$ . Let  $x \in \mathcal{A}$  and consider the coset  $C_x = x + C$ . We have

$$\phi_s(z) = \phi_s(x), \quad \text{for all } z \in C_x \text{ and for all } s \in T.$$

So every coset  $C_x$  is uniquely determined by the sequence of elements of  $\mathbf{G}$ :

$$S(x) = (\phi_s(x) \mid s \in T), \quad (5)$$

the so-called *syndrome* of  $x$  (or of the coset containing  $x$ ).

Let  $x \in \mathcal{A}$  with syndrome  $(\beta_s \mid s \in T)$ . To find the number of codewords of weight  $\lambda$  in  $C_x$  consists in solving the following problem: *find the number of codewords  $z = \sum_{g \in \mathbf{G}} z_g X^g$  of weight  $\lambda$  satisfying*

$$\sum_{g \in \mathbf{G}} z_g g^s = \beta_s, \quad \forall s \in T.$$

When  $p = 2$  we obtain a system of *diagonal equations* over the finite field of order  $2^m$ ; the problem consists in *finding the number of solutions*

$$(X_1, \dots, X_\lambda), \quad X_i \in GF(2^m), \quad X_i \neq X_j \quad \forall i, j$$

*satisfying*

$$\sum_{i=0}^{\lambda} X_i^s = \beta_s, \quad \forall s \in T. \quad (6)$$



## 2.2 Cosets of codes of $\mathcal{A}$

Let  $C$  be an  $[N, k]$ -code of  $\mathcal{A}$ . We first recall the MAC-WILLIAMS transform, which determines uniquely the weight polynomial of the dual of  $C$  from the weight polynomial of the code  $C$  itself.

**Theorem 1** [15, p.146] *Let  $C$  be a linear  $[N, k]$ -code. Define its weight polynomial:*

$$W_C(X, Y) = \sum_{i=0}^N A_i X^{N-i} Y^i \quad , \quad A_i = \text{card} \{ c \in C \mid \text{wt}(c) = i \} \quad ,$$

where  $\text{wt}(c)$  is the weight of  $c$ . Then the weight polynomial of the dual code  $C^\perp$  is

$$W_{C^\perp}(X, Y) = \frac{1}{p^k} W_C(X + (p-1)Y, X - Y) \quad . \quad (7)$$

Denote by  $D_x$  the  $[N, k+1]$ -code generated by  $C$  and a coset  $C_x = x + C$  of  $C$ :

$$D_x = \bigcup_{a \in K} (ax + C) \quad .$$

As  $C \subset D_x$ ,  $D_x^\perp \subset C^\perp$ ; then every weight of  $C^\perp$  could be a weight of  $D_x^\perp$ . Moreover the weight distribution of  $x + C$  can be deduced from the weight distribution of  $C^\perp$  and of  $D_x^\perp$ . Indeed suppose that the weight distribution of  $C^\perp$  is known and that we can compute the weight polynomial of  $D_x^\perp$ . The dimension of  $D_x$  equals  $k+1$  so that the dimension of  $D_x^\perp$  equals  $N - (k+1)$ . Applying (7) we obtain the weight polynomial of  $D_x$ :

$$W_{D_x}(X, Y) = \frac{1}{p^{N-(k+1)}} W_{D_x^\perp}(X + (p-1)Y, X - Y) \quad .$$

But, by definition,  $W_{D_x}(X, Y) = (p-1)Q_x(X, Y) + W_C(X, Y)$ , where  $Q_x(X, Y)$  is the weight polynomial of the coset  $x + C$ . Then the expression of  $Q_x(X, Y)$  is as follows:

$$\begin{aligned} Q_x(X, Y) &= \frac{1}{p-1} \left( \frac{1}{p^{N-(k+1)}} W_{D_x^\perp}(X + (p-1)Y, X - Y) \right. \\ &\quad \left. - \frac{1}{p^{N-k}} W_{C^\perp}(X + (p-1)Y, X - Y) \right) \\ &= \frac{1}{(p-1)p^{N-k}} \left( p W_{D_x^\perp}(X + (p-1)Y, X - Y) \right. \\ &\quad \left. - W_{C^\perp}(X + (p-1)Y, X - Y) \right) \quad . \quad (8) \end{aligned}$$

In the following we always will assume that the polynomial  $W_{C^\perp}(X, Y)$  is known and we want to have results on the polynomial  $Q_x(X, Y)$ , for every  $x$ . Then the problem consists in the determination of the polynomials  $W_{D_x^\perp}(X, Y)$ .

Let  $\lambda$  be a non zero weight of  $C^\perp$  and  $A_\lambda(x)$  be the number of codewords of weight  $\lambda$  in  $D_x^\perp$ . So  $A_\lambda(x)$  is the number of elements of  $C^\perp$  of weight  $\lambda$  which are orthogonal to  $x$ :

$$A_\lambda(x) = \text{card} \{ y \in C^\perp \mid \text{wt}(y) = \lambda \text{ and } \langle y, x \rangle = 0 \} \quad . \quad (9)$$

Indeed  $y \in D_x^\perp$  if and only if  $\langle y, z \rangle = 0$  for all  $z$  in  $D_x$ . But  $z = ax + c$ ,  $a \in K$  and  $c \in C$ ; as  $y \in C^\perp$ ,  $\langle y, c \rangle = 0$ . Hence  $y \in D_x^\perp$  is equivalent to  $\langle y, x \rangle = 0$ ,

### 3 Weight polynomials of cosets

#### 3.1 Translations on codewords

Let  $h$  be any non zero element of  $\mathbf{G}$ . We denote by  $\tau_h$  the permutation on  $\mathbf{G}$  which acts as follows on the elements of  $\mathcal{A}$ :

$$\tau_h \left( \sum_{g \in \mathbf{G}} x_g X^g \right) = \sum_{g \in \mathbf{G}} x_g X^{g+h} = \sum_{g \in \mathbf{G}} x_{g-h} X^g .$$

We will say that  $\tau_h$  is an  $h$ -translation on the codewords. An ideal of  $\mathcal{A}$  is a code invariant under every  $\tau_h$ .

Let  $x$  and  $y$  in  $\mathcal{A}$ . It follows easily from the definition of the multiplication in  $\mathcal{A}$  that

$$xy = \sum_{h \in \mathbf{G}} \left( \sum_{g \in \mathbf{G}} x_g y_{h-g} \right) X^h = \sum_{h \in \mathbf{G}} \langle x, X^h y \rangle X^h = \sum_{h \in \mathbf{G}} \langle x, \tau_h(y) \rangle X^h .$$

Therefore we obtain a relation which links up the weight of  $xy$  with the set of codewords orthogonal to  $x$ :

$$wt(xy) = \text{card} \{ h \in \mathbf{G} \mid \langle x, \tau_h(y) \rangle \neq 0 \} \quad (\mathbf{I}) .$$

Suppose that the code  $C$  is an ideal of  $\mathcal{A}$ . Then the dual of  $C$  is also an ideal. Let  $y \in C^\perp$ ; so the set of the  $\tau_h(y)$ ,  $h \in \mathbf{G}$ , is a subset of codewords of  $C^\perp$  of same weight. Hence Formula (I) means that the weight of  $xy$  equals the number of elements of that subset which are orthogonal to  $x$ . In the applications we study later, we always consider the ideals  $C$  of  $\mathcal{A}$  which are between two consecutive powers of the radical. We explain now this definition.

**Proposition 1** *Let  $y$  in  $\mathcal{A}$  such that  $\text{depth}(y) = j$  with  $j \in [0, m(p-1) - 1]$ . Then*

$$\tau_h(y + \mathcal{P}^{j+1}) = y + \mathcal{P}^{j+1} , \quad \forall h \in \mathbf{G} .$$

Therefore let  $C$  be a code of  $\mathcal{A}$  of depth  $j$  satisfying

$$\mathcal{P}^{j+1} \subset C \subset \mathcal{P}^j , \quad \text{i.e.} \quad \mathcal{P}^{m(p-1)-j+1} \subset C^\perp \subset \mathcal{P}^{m(p-1)-j} .$$

Then  $C$  is an ideal of  $\mathcal{A}$ , so that  $C^\perp$  is also an ideal.

*Proof:* Let  $h \in \mathbf{G}$ . Note that  $X^h - 1$  is an element of  $\mathcal{P}$ . Then

$$X^h y = (X^h - 1)y + y \quad \text{where} \quad (X^h - 1)y \in \mathcal{P}^{j+1} .$$

Thus the coset  $y + \mathcal{P}^{j+1}$  is invariant under any  $h$ -translation. The code  $C$  can be considered as a union of such cosets. Hence it is an ideal of  $\mathcal{A}$ .

□

**Remark:** Assume that  $p = 2$ . It follows immediately from the proposition above that any coset of depth  $j$  of the RM-code  $\mathcal{P}^{j+1}$  is an *orphan* (see the terminology in [5]). Indeed the minimum weight codewords of these cosets cover all coordinate positions.

### 3.2 Application of Formula (I)

In the application we present now, Formula (I) is most interesting, because  $xy$  has few possible weights, since the depth of  $y$  is  $m-2$ ; moreover the  $\tau_h(y)$  form the set of codewords of a given weight in the coset  $y + \mathcal{P}^{m-1}$  (see Proposition 2). From now on, in this Section,  $p = 2$ ,  $K$  and  $\mathbf{G}$  will be respectively the finite field of order 2 and  $2^m$ . We will consider linear codes  $C$  of depth 2 such that:

$$\mathcal{P}^3 \subset C \subset \mathcal{P}^2 \quad (\text{i.e. } \mathcal{P}^{m-1} \subset C^\perp \subset \mathcal{P}^{m-2}).$$

From Proposition 1, the code  $C$  is an ideal of the algebra  $\mathcal{A}$ . The code  $C^\perp$  is an union of cosets of the Reed-Muller code of order 1, which are contained in the Reed-Muller code of order 2. These cosets are precisely described in [15, Chapter 15]; the reader can also refer to [8]. We only recall the results we need.

Such a coset  $y + \mathcal{P}^{m-1}$  is uniquely defined by the symplectic form associated to  $y$ . Since  $y$  is in  $\mathcal{P}^{m-2}$ , then  $y$  can be identified to a quadratic boolean function  $f_y$  :

$$y = \sum_{g \in \mathbf{G}} f_y(g) X^g \quad - \text{ i.e. } y_g = f_y(g).$$

The associated symplectic form of  $f_y$  is

$$\Psi_y : (u, v) \in \mathbf{G}^2 \quad \mapsto \quad \Psi_y(u, v) = f_y(0) + f_y(u) + f_y(v) + f_y(u+v) \in K.$$

The kernel of  $\Psi_y$  is as follows defined:

$$\mathcal{E}_y = \{ u \in \mathbf{G} \mid \forall v \in \mathbf{G} : \Psi_y(u, v) = 0 \}.$$

The set  $\mathcal{E}_y$  is a  $\mathbf{K}$ -subspace of  $\mathbf{G}$  of dimension  $\kappa = m - 2h$ , where  $2h$  is the rank of  $\Psi_y$ . Let  $E_y = y + \mathcal{P}^{m-1}$ ; then  $\Psi_y = \Psi_b$  for all  $b \in E_y$ . Moreover the weight distribution of  $E_y$  only depends on  $h$ ; that is (cf. [15, p. 441]):

weights	$2^{m-1} - 2^{m-h-1}$	$2^{m-1}$	$2^{m-1} + 2^{m-h-1}$
number	$2^{2h}$	$2^{m+1} - 2^{2h+1}$	$2^{2h}$

where  $h \in [0, \lfloor m/2 \rfloor]$ . Note that such a coset has exactly three weights unless  $m$  is even and  $h = m/2$ .

**Definition 4** Let  $y \in \mathcal{P}^{m-2}$ . Let  $2h$  be the rank of the symplectic form associated to  $y$ . We will say that the coset  $E_y = y + \mathcal{P}^{m-1}$  is of type  $(h)$ .

In this section we will always consider  $y \in \mathcal{P}^{m-2} \setminus \mathcal{P}^{m-1}$  and  $E_y = y + \mathcal{P}^{m-1}$ . If  $C_x = x + C$  is any coset of  $C$ , then we will denote by  $D_x$  the code  $C_x \cup C$  (see Section 2.2). We suppose that the weight polynomial of  $C^\perp$  is known and we want to determine the weight polynomial of  $D_x^\perp$ . In accordance with Formula (8), the weight distribution of the coset  $x + C$  is:

$$Q_x(X, Y) = \frac{1}{2^{2^m-k}} \left( 2W_{D_x^\perp}(X+Y, X-Y) - W_{C^\perp}(X+Y, X-Y) \right) \quad (10)$$

where  $k$  is the dimension of  $C$ .

**Proposition 2** Let  $y \in \mathcal{P}^{m-2}$ . Set  $E_y = y + \mathcal{P}^{m-1}$  and assume that  $E_y$  is of type (h). Let  $\lambda$  be a weight of  $E_y$  such that  $\lambda \neq 2^{m-1}$  and suppose that  $wt(y) = \lambda$ . We denote by  $\kappa$  the dimension of the kernel of the symplectic form  $\Psi_y$ . Then we have:

$$\{ X^g y \mid g \in \mathbf{G} \} = \{ a \in E_y \mid wt(a) = \lambda \},$$

and the cardinality of the set above is  $2^{m-\kappa}$ . So each codeword  $a$  of  $E_y$ , of weight different from  $2^{m-1}$  is invariant under  $2^\kappa$  translations

*Proof:* Set  $\kappa = m - 2h$ . According to the table above, we have

$$\text{card} \{ a \in E_y \mid wt(a) = \lambda \} = 2^{2h},$$

where  $\lambda = 2^{m-1} \pm 2^{m-h-1}$ . Let  $u$  and  $v$  in  $\mathbf{G}$ . By definition,  $\Psi_y(u, v) = y_0 + y_u + y_v + y_{u+v}$ . Furthermore for any  $u \neq 0$

$$(X^u + 1)y = \sum_{v \in \mathbf{G}} y_v X^{u+v} + \sum_{v \in \mathbf{G}} y_v X^v = \sum_{v \in \mathbf{G}} (y_{u+v} + y_v) X^v,$$

and we have obviously

$$\text{card} \{ X^g y \mid g \in \mathbf{G} \} = \frac{\text{card } \mathbf{G}}{\text{card} \{ g \in \mathbf{G} \mid y = X^g y \}}.$$

An element  $u$  is in  $\mathcal{E}_y$  if and only if  $\Psi_y(u, v) = 0$ , for all  $v$ . Therefore  $u$  is an element of  $\mathcal{E}_y$  if and only if

$$(X^u + 1)y = \sum_{v \in \mathbf{G}} (y_0 + y_u) X^v = (y_0 + y_u) \mathbf{I}.$$

That means:  $(X^u + 1)y$  is either the all-one vector or the null vector. Assume that  $u$  is in  $\mathcal{E}_y$ . Since  $wt((X^u + 1)y) \leq 2wt(y) < 2^m$  then  $(X^u + 1)y = 0$  (equivalently  $y_0 = y_u$ ) in the equality above when  $wt(y) < 2^{m-1}$ . This result holds for  $wt(y) > 2^{m-1}$  because  $\mathcal{P}^{m-1}$  contains the all-one vector  $\mathbf{I}$ , which means that there exists  $b \in E_y$  such that  $y = b + \mathbf{I}$  with  $wt(b) < 2^{m-1}$ . Obviously  $(X^u + 1)y = (X^u + 1)b$ .

Conversely  $X^u y = y$  implies  $y_{u+v} = y_v$  for all  $v$ , which means  $\Psi_y(u, v) = 0$ , for all  $v$ . So we have proved that  $X^u y = y$  if and only if  $u \in \mathcal{E}_y$ . Hence

$$\text{card} \{ g \in \mathbf{G} \mid y = X^g y \} = \text{card } \mathcal{E}_y = 2^\kappa,$$

which yields  $\text{card} \{ X^g y \mid g \in \mathbf{G} \} = 2^{m-\kappa} = 2^{2h}$ , completing the proof.

□

**Lemme 1** Let  $x \in \mathcal{A} \setminus \mathcal{P}^2$ ,  $y \in C^\perp$ ,  $E_y = y + \mathcal{P}^{m-1}$ ;  $D_x$  is the code  $C_x \cup C$ , where  $C_x$  is the coset  $x + C$ . Then  $D_x^\perp$  contains half of elements of  $E_y$ :

$$\text{card} \{ a \in E_y \mid \langle x, a \rangle = 0 \} = \frac{\text{card } E_y}{2} = 2^m.$$

*Proof:* Recall that the dimension of  $\mathcal{P}^{m-1}$  equals  $m+1$  (see (4)). Then  $E_y$  has  $2^{m+1}$  elements. The function  $f : u \in \mathcal{P}^{m-1} \mapsto \langle x, u \rangle \in K$  is linear. By hypothesis  $x \notin (\mathcal{P}^{m-1})^\perp$ . Then  $\dim(\ker f) = \dim(\mathcal{P}^{m-1}) - 1 = m$  which yields that the number of  $u \in \mathcal{P}^{m-1}$  satisfying  $\langle x, u \rangle = 0$  equals  $2^m$ . The equality  $\langle x, a \rangle = \langle x, y \rangle + \langle x, u \rangle$ ,  $a \in E_y$  and  $u \in \mathcal{P}^{m-1}$ , completes the proof.

□

**Proposition 3** *Let  $x \in \mathcal{A} \setminus \mathcal{P}^2$ . Let  $\lambda$  be a weight of  $E_y$  and suppose that  $wt(y) = \lambda$ . We denote by  $N_\lambda$  the number of codewords of  $E_y$  of weight  $\lambda$ . Set*

$$\widehat{N}_\lambda = \text{card} \{ a \in E_y \mid wt(a) = \lambda \text{ and } \langle x, a \rangle = 0 \}.$$

Then

$$\begin{aligned} x \in \mathcal{P} \setminus \mathcal{P}^2 &\implies \widehat{N}_\lambda = \widehat{N}_{2^m - \lambda} \\ x \in \mathcal{A} \setminus \mathcal{P} &\implies \widehat{N}_\lambda = N_\lambda - \widehat{N}_{2^m - \lambda}. \end{aligned}$$

Moreover, if  $\lambda \neq 2^{m-1}$  we have:

$$\widehat{N}_\lambda = 2^{-\kappa} (2^m - wt(xy)). \quad (11)$$

*Proof:* If  $x \in \mathcal{P} \setminus \mathcal{P}^2$  then the weight of  $x$  is even; thus  $\langle x, \mathbf{1} \rangle = 0$  (where  $\mathbf{1}$  is the all-one vector). That implies that for all  $a \in E_y$ ,  $\langle x, a \rangle = \langle x, a + \mathbf{1} \rangle$ . We know that  $N_\lambda = N_{2^m - \lambda}$  and that  $\mathbf{1} \in \mathcal{P}^{m-1}$  (i.e. if  $E_y$  contains  $a$  it contains  $a + \mathbf{1}$ ). Hence  $\widehat{N}_\lambda = \widehat{N}_{2^m - \lambda}$ .

Assume now that  $x \in \mathcal{A} \setminus \mathcal{P}$ . Then  $wt(x)$  is odd and  $\langle x, \mathbf{1} \rangle = 1$ . So we have  $\langle x, a \rangle + 1 = \langle x, a + \mathbf{1} \rangle$ , for all  $a \in E_y$ . Hence

$$\widehat{N}_\lambda = \text{card} \{ a \in E_y \mid wt(a) = 2^m - \lambda \text{ and } \langle y, a \rangle = 1 \} = N_{2^m - \lambda} - \widehat{N}_{2^m - \lambda},$$

$N_{2^m - \lambda} = N_\lambda$  completing the proof.

We suppose now that  $\lambda \neq 2^{m-1}$ . Recall that  $\kappa$  is the dimension of the kernel of the symplectic form associated to  $y$ . From Proposition 2, the set of the  $X^g y$  equals the set of the codewords of  $E_y$  of weight  $\lambda$  and there are  $2^{m-\kappa}$  such codewords ( $N_\lambda = 2^{m-\kappa}$ ). We have:

$$\begin{aligned} \widehat{N}_\lambda &= 2^{-\kappa} \text{card} \{ g \in \mathbf{G} \mid \langle x, X^g y \rangle = 0 \} \\ &= 2^{-\kappa} (2^m - wt(xy)) \quad (\text{from (I)}). \end{aligned}$$

□

By hypothesis the code  $C^\perp$  is an ideal of  $\mathcal{A}$ , since it is invariant under any translation. Thus the cosets of  $C$  of minimum weight 1 have the same weight distribution. By applying Formula (11), we can compute this weight distribution.

**Corollary 1** Suppose that the dual of the code  $C$  consists of the code  $\mathcal{P}^{m-1}$  itself and  $L$  sets of cosets of same type; that is :  $L_i$  cosets of type  $(h_i)$ ,  $i \in [1, L]$ . Let  $x = X^g$ ,  $g \in G$  and  $D_x = (x + C) \cup C$ . Set

$$A_\ell = \text{card} \{ c \in D_x^\perp \mid \text{wt}(c) = \ell \} \quad \text{and} \quad \lambda_i = 2^{m-1} - 2^{m-h_i-1}, \quad i \in [1, L].$$

So the weights of  $D_x^\perp$  are elements of the set  $\{ 0, 2^{m-1}, \lambda_i, 2^m - \lambda_i \mid (i \in [1, r]) \}$ . Then the coefficients of the weight polynomial of  $D_x^\perp$  are, for each  $i$  in  $[1, L]$ :

$$A_{\lambda_i} = L_i(2^{2h_i-1} + 2^{h_i-1}), \quad A_{2^m - \lambda_i} = L_i(2^{2h_i-1} - 2^{h_i-1}),$$

$A_0 = 1$  and  $A_{2^m-1} = \beta/2$  where  $\beta$  is the number of codewords of weight  $2^{m-1}$  in  $C^\perp$ . The weight distribution of the coset  $x + C$  can be calculated from the Formula (10).

*Proof:* We will apply the Proposition 3 to each type of cosets. Let  $E_y = y + \mathcal{P}^{m-1}$  be a coset of type  $(h_i)$ ; recall that the dimension of the kernel of the symplectic form associated to  $y$  is  $\kappa = m - 2h_i$ . Assume that  $\text{wt}(y) = \lambda_i$ . Since  $\text{wt}(x) = 1$ ,  $\text{wt}(xy) = \text{wt}(y)$ . Thus, according to (11), we have:

$$\begin{aligned} \widehat{N}_{\lambda_i} &= 2^{2h_i-m}(2^m - \lambda_i) = 2^{2h_i-m}(2^m - 2^{m-1} + 2^{m-h_i-1}) \\ &= 2^{2h_i} - 2^{2h_i-1} + 2^{h_i-1} = 2^{2h_i-1} + 2^{h_i-1}, \end{aligned}$$

and  $A_{\lambda_i} = L_i \widehat{N}_{\lambda_i}$ . Since  $\widehat{N}_{2^m - \lambda_i} = N_{\lambda_i} - \widehat{N}_{\lambda_i}$ , then

$$\widehat{N}_{2^m - \lambda_i} = 2^{2h_i} - 2^{2h_i-1} - 2^{h_i-1} = 2^{2h_i-1} - 2^{h_i-1}.$$

The null vector belongs to  $D_x^\perp$  while the all-one vector does not. Since  $2^{m-1} = 2^m - 2^{m-1}$ , we obtain for any coset of any type  $2\widehat{N}_{2^m-1} = N_{2^m-1}$ . That means that  $D_x^\perp$  contains half of codewords of  $C^\perp$  of weight  $2^{m-1}$ .

□

**Example 1:** Cosets of minimum weight 1 of the extended triple-error-correcting BCH-codes of length  $2^m$ ,  $m$  even. We denote by  $B$  any such extended BCH-code; the dual code  $B^\perp$  is a subcode of  $\mathcal{P}^{m-2}$  with parameters

$$[ N = 2^m, 3m + 1, 2^{m-1} - 2^{(m+2)/2} ], \quad \text{for even } m \geq 6.$$

The weight distribution of  $B^\perp$  has been found by BERLEKAMP; we will use the table given in [15, p. 669]. The code  $B^\perp$  consists of  $2^{2m}$  cosets of  $\mathcal{P}^{m-1}$ , each coset containing  $2^{m+1}$  codewords. Using the general weight distribution of such cosets (see at the beginning of Section 3.2), and the values of the weights of  $B^\perp$ , we can deduce that  $B^\perp$  consists of the code  $\mathcal{P}^{m-1}$  itself and

- $L_1$  cosets of type  $(h_1)$ ,  $L_1 = (2^m - 1)(2^m - 4)/60$  and  $h_1 = m/2 - 2$ .
- $L_2$  cosets of type  $(h_2)$ ,  $L_2 = 7 \cdot 2^m(2^m - 1)/12$  and  $h_2 = m/2 - 1$ .
- $L_3$  cosets of type  $(h_3)$ ,  $L_3 = 2(2^m - 1)(3 \cdot 2^m + 8)/15$  and  $h_3 = m/2$ .

By applying Corollary 1, we obtain the weight distribution of the code  $D_x^\perp$  for any  $x$  such that  $wt(x) = 1$ :

$\lambda$	$A_\lambda$	$\lambda$	$A_\lambda$
$2^{m-1} - 2^{(m+2)/2}$	$L_1(2^{m-5} + 2^{m/2-3})$	$2^{m-1} - 2^{(m-2)/2}$	$L_3(2^{m-1} + 2^{m/2-1})$
$2^{m-1} + 2^{(m+2)/2}$	$L_1(2^{m-5} - 2^{m/2-3})$	$2^{m-1} + 2^{(m-2)/2}$	$L_3(2^{m-1} - 2^{m/2-1})$
$2^{m-1} - 2^{m/2}$	$L_2(2^{m-3} + 2^{m/2-2})$	$2^{m-1}$	$(2^m - 1)(29 \cdot 2^{2m-6} - 2^{m-4} + 1)$
$2^{m-1} + 2^{m/2}$	$L_2(2^{m-3} - 2^{m/2-2})$	$0$	$1$

(with notation of Corollary 1). Now the weight distribution of the coset  $x + B$  can be calculated from the weight enumerators of  $B^\perp$  and of  $D_x^\perp$ , by using the Formula (10).

### 3.3 Shifts of codewords

From now on we will treat only extended cyclic codes as they are defined in Section 2.1. Such codes are invariant under the *shifts*. Recall that we consider codes of length  $N = p^m$  over the finite field  $K$  of order  $p$ . A shift of a given codeword  $x$  is as follows:

$$\sigma_j : \sum_{g \in \mathbf{G}} x_g X^g \mapsto \sum_{g \in \mathbf{G}} x_g X^{\alpha^j g}, \quad j \in [0, p^m - 2].$$

**Proposition 4** Set  $S = [0, n]$ , where  $n = p^m - 1$ . Any  $s \in S$  is identified with its  $p$ -ary expansion  $(s_0, \dots, s_{m-1})$ . Then we define a partial order on  $S$ :

$$\text{for all } s, t \text{ in } S : s \prec t \iff \forall i \in [0, m-1] : s_i \leq t_i$$

Let  $x \in \mathcal{A}$  and  $y \in \mathcal{A}$ . Then

$$\forall s \in S : \phi_s(xy) = \sum_{i \prec s} \binom{s}{i} \phi_i(x) \phi_{s-i}(y). \quad (12)$$

*Proof:*

$$\begin{aligned} \phi_s(xy) &= \sum_{g \in \mathbf{G}} x_g \sum_{h \in \mathbf{G}} y_h (g+h)^s = \sum_{g \in \mathbf{G}} x_g \sum_{h \in \mathbf{G}} y_h \sum_{i=0}^s \binom{s}{i} g^i h^{s-i} \\ &= \sum_{i=0}^s \binom{s}{i} \sum_{g \in \mathbf{G}} x_g g^i \sum_{h \in \mathbf{G}} y_h h^{s-i} = \sum_{i \prec s} \binom{s}{i} \phi_i(x) \phi_{s-i}(y), \end{aligned}$$

since, by LUCAS's Theorem,  $\binom{s}{i} \not\equiv 0 \pmod{p}$  is equivalent to  $i \prec s$ .

□

**Theorem 2** Let  $x \in \mathcal{P}^r$  and  $y \in \mathcal{P}^{m(p-1)-r}$ . Then the product  $xy$  is zero if and only if  $\phi_n(xy) = 0$ . Moreover if  $xy = 0$  then  $x(y + \mathcal{P}^{m(p-1)-r+1}) = \{0\}$ . Define the polynomial of  $\mathbf{G}[Z]$ :

$$R(Z) = \sum_{\omega_p(i)=r} \phi_i(x) \phi_{n-i}(y) Z^{n-i}.$$

Then we have for all  $j \in [0, n-1]$  :

$$\phi_n(x\sigma_j(y)) = (-1)^r R(\alpha^j) \quad (\text{II}).$$

The number of shifts of  $y$  (i.e. the  $\sigma_j(y)$ ) which are orthogonal to  $x$  equals the number of zeros of the polynomial  $R(Z)$ .

*Proof:* The product  $xy$  is in  $\mathcal{P}^r \mathcal{P}^{m(p-1)-r} = \mathcal{P}^{m(p-1)}$ . A codeword  $z$  of  $\mathcal{P}^{m(p-1)}$  is defined by  $\phi_s(z) = 0$  unless  $s = n$ . If  $\phi_n(z) = 0$ ,  $z$  is the null vector; otherwise  $z = a\mathbf{1}$ ,  $a \in K$  and  $\mathbf{1}$  is the all-one vector (see Section 2.1). If  $xy = 0$  then  $xb = 0$  for every codeword  $b$  of the coset  $y + \mathcal{P}^{m(p-1)-r+1}$ , since  $x\mathcal{P}^{m(p-1)-r+1}$  is in  $\mathcal{P}^{m(p-1)+1}$  which is the null space.

From the definition of the functions  $\phi_s$  and of  $\sigma_j$ , we have:

$$\phi_s(\sigma_j(y)) = \sum_{g \in \mathbf{G}} y_g (\alpha^j g)^s = \alpha^{js} \phi_s(y).$$

Thus (noting that every  $i$  in  $S$  satisfies  $i \prec n$ ):

$$\phi_n(x\sigma_j(y)) = \sum_{i=0}^n \binom{n}{i} \phi_i(x) \phi_{n-i}(\sigma_j(y)) = \sum_{i=0}^n \binom{n}{i} \phi_i(x) \phi_{n-i}(y) \alpha^{-ji}$$

Since  $x \in \mathcal{P}^r$ ,  $\phi_i(x) = 0$  for all  $i$  such that  $\omega_p(i) < r$  (see Definition 3). When  $\omega_p(i) > r$ ,  $\omega_p(n-i) < m(p-1) - r$ ; in this case  $\phi_{n-i}(y) = 0$ , since  $y \in \mathcal{P}^{m(p-1)-r}$ . Then the sum above is a sum on the  $i$  such that  $\omega_p(i) = r$ .

Now we consider such an  $i$  and its  $p$ -ary expansion  $(i_0, \dots, i_{m-1})$  and we want to compute the coefficient  $\binom{n}{i}$  modulo  $p$ :

$$\binom{n}{i} = \prod_{t=0}^{m-1} \binom{p-1}{i_t} = \prod_{t=0}^{m-1} (-1)^{i_t} = (-1)^{\omega_p(i)} = (-1)^r,$$

since  $\binom{p-1}{i_t} = (p-1) \dots (p-i_t) / (i_t!) = (-1)^{i_t}$  (modulo  $p$ ). Therefore

$$\phi_n(x\sigma_j(y)) = \sum_{\omega_p(i)=r} (-1)^r \phi_i(x) \phi_{n-i}(y) \alpha^{-ji} = (-1)^r R(\alpha^j).$$

□

**Remarks:** 1) The set of the  $i$  whose  $p$ -weight equals  $r$  is invariant under the multiplication by  $p$  (modulo  $n$ ); moreover  $\phi_{sp}(z) = \sum_{g \in \mathbf{G}} z_g g^{ps} = (\phi_s(z))^p$ , for any  $z$ . The polynomial  $R(Z)$  is in fact a sum of trace-functions from  $\mathbf{G}$  to  $K$ .

2) Suppose that a code  $C$  is such that its dual is a union of  $L$  cosets  $y + \mathcal{P}^{m(p-1)-r+1}$ ,  $y \in \mathcal{P}^{m(p-1)-r}$ , where all cosets have the same weight enumerator. Then there is only one weight distribution for the cosets  $x + C$ ,  $x \in \mathcal{P}^r$  and this result holds for non extended cyclic codes  $C$ . It becomes from the fact that  $L/2$  cosets composing  $C^\perp$  are orthogonal to  $x$ , for any  $x$ . We will treat later a special application of this fact (see Corollary 3).



### 3.4 Application of Formula (II)

In this section we will consider extended cyclic codes  $C$  of depth  $r$ , for some  $r$ , such that:

$$\mathcal{P}^{r+1} \subset C \subset \mathcal{P}^r \quad (\text{i.e. } \mathcal{P}^{m(p-1)-r+1} \subset C^\perp \subset \mathcal{P}^{m(p-1)-r}).$$

Let  $T^\perp$  be the defining set of the dual code  $C^\perp$ . In accordance with Definition 3 and with the definition of  $C$ ,  $T^\perp$  has the following form:

$$T^\perp = I_p(m(p-1) - r + 1, m) \setminus \{s_1, \dots, s_\ell\}, \quad (13)$$

where the  $s_i$  are some elements of  $S$  of  $p$ -weight  $m(p-1) - r$ . Note that the defining set of  $C$  is  $T = I_p(r, m) \cup \{n - s_1, \dots, n - s_\ell\}$ . Set  $U = \{s_1, \dots, s_\ell\}$  and let  $E$  be the cyclic code (in the algebra  $\mathcal{R}$ ) whose nonzeros are the  $\alpha^t$ ,  $t \in U$ . The code  $E$  can be viewed in the algebra  $\mathcal{A}$ , by adding an all-zero column to the generator matrix. Clearly  $\text{depth}(E)$  equals  $\text{depth}(C^\perp)$  equals  $m(p-1) - r$ . Then, since the dimension of  $E$  equals the number of elements of  $U$ , which is the dimension of the quotient space  $C^\perp / \mathcal{P}^{m(p-1)-r+1}$ , we have clearly

$$C^\perp = \bigcup_{y \in E} (y + \mathcal{P}^{m(p-1)-r+1}). \quad (14)$$

The definitions of  $C^\perp$ ,  $U$  and  $E$  hold in all this section. Recall that  $D_x$  is the linear code generated by  $C$  and the coset of  $C$  containing  $x$ ; we want to determine the weight distribution of the dual of  $D_x$ .

**Proposition 5** *Let a coset of  $C$ :  $C_x = x + C$ ,  $x \in \mathcal{P}^r \setminus C$ .*

*Let  $y$  in  $E$ . Then, for every  $j \in [0, n-1]$ :*

$$\phi_n(x \sigma_j(y)) = 0 \quad \text{if and only if} \quad \sum_{t=1}^{\ell} \phi_{n-s_t}(x) \phi_{s_t}(y) \alpha^{s_t j} = 0.$$

*That means that the number of shifts of  $y$  orthogonal to  $x$  is related with the weight of the codeword  $y'$  of  $E$  defined by*

$$\phi_{s_t}(y') = \phi_{n-s_t}(x) \phi_{s_t}(y), \quad t \in [1, \ell]. \quad (15)$$

*Let  $n(y)$  be the number of elements of the set  $\{\sigma_j(y) \mid j \in [0, n-1]\}$ . Then we have*

$$L(y) = \text{card} \{ \sigma_j(y) \mid \langle x, \sigma_j(y) \rangle = 0 \} = \frac{n(y)}{n} (n - wt(y'));$$

*the code  $D_x^\perp$  contains  $L(y)$  cosets  $\sigma_j(y) + \mathcal{P}^{m(p-1)-r+1}$ .*

*Proof:* In accordance with Theorem 2,  $\phi_n(x \sigma_j(y)) = 0$  if and only if  $R(\alpha^j) = 0$ . Moreover, by hypothesis,  $\phi_{n-i}(y) = 0$  for every  $i$  such that  $\omega_p(i) = r$  unless  $n - i \in U$ . Since  $U = \{s_1, \dots, s_\ell\}$  we have :

$$R(\alpha^j) = \sum_{n-i \in U} \phi_i(x) \phi_{n-i}(y) \alpha^{-ij} = \sum_{r=1}^{\ell} \phi_{n-s_r}(x) \phi_{s_r}(y) \alpha^{s_r j} = 0.$$

Let  $z \in \mathcal{A}$ . Recall that the Mattson-Solomon polynomial of  $z$  is

$$MS_z(Z) = \sum_{i=0}^{n-1} \phi_i(z) Z^{n-i}.$$

It is well-known that  $MS_z(\alpha^j) = z_{\alpha^j}$  [15, p.239]. Then

$$wt(z) = n - (\text{card } \{j \in [0, n-1] \mid MS_z(\alpha^j) = 0\}) + \epsilon.$$

where  $\epsilon$  equals 0 if  $z_0 = 0$  and 1 otherwise.

Let  $y'$  be the codeword of  $E$  whose coefficients  $\phi_{s_t}$  are given by Formula (15) (the other coefficients  $\phi_i(y')$  are zero, by definition of the cyclic code  $E$ ). For every  $j$  in  $[0, n-1]$ , we have obviously

$$MS_{y'}(\alpha^j) = \sum_{t=1}^{\ell} \phi_{s_t}(y')(\alpha^j)^{n-s_t} = \sum_{t=1}^{\ell} \phi_{n-s_t}(x)\phi_{s_t}(y)(\alpha)^{-js_t} = R(\alpha^j).$$

Hence  $R(\alpha^j) = 0$  if and only if  $y'_{\alpha^j} = 0$ . Let  $n(y)$  be the number of distinct shifts of  $y$  and let  $n = n(y)n_1$ . Then

$$\text{card } \{ \sigma_j(y) \mid \langle x, \sigma_j(y) \rangle = 0 \} = \frac{1}{n_1} \text{card } \{ j \mid \langle x, \sigma_j(y) \rangle = 0 \} = \frac{1}{n_1} (n - wt(y')).$$

Note that  $y'_0 = 0$ , by definition.

□

Now we want to show how it is possible to achieve the computation of the weight distribution of the code  $D_x^\perp$  - then of the coset  $x + C$  of depth  $r$ . We will examine this problem when  $U$  contains only one cyclotomic class of  $p$  modulo  $n$ . From Proposition 5, we have immediate corollaries.

**Corollary 2** Let  $U = \{ s, ps, \dots \}$ ,  $x \in \mathcal{P}^r \setminus C$ ; let  $y$  be any element in  $E$ . Set  $\beta = \phi_{n-s}(x)\phi_s(y)$ . Let  $t = (n, s)$  and  $n = tn_1$ . Then, for every  $j \in [0, n-1]$ :

$$\phi_n(x \sigma_j(y)) = 0 \quad \text{if and only if} \quad \text{Tr}(\beta \alpha^{js}) = 0.$$

Let  $y'$  be the element of  $E$  such that  $\phi_s(y') = \beta$ . Then the number  $L(y)$  of cosets  $\sigma_j(y) + \mathcal{P}^{m(p-1)-r+1}$  contained in  $D_x^\perp$  equals  $(n - wt(y'))/t$ .

Note that in this case the code  $E$  is an irreducible cyclic code viewed as a cyclic code of length  $n$ , even when  $n$  and  $s$  are not relatively prime (by  $t$  repetitions of each symbol). When  $t = 1$ ,  $E$  is equivalent to the simplex code whose all codewords have a same weight  $\lambda$ . In this case the corollary above means that the weight polynomial of  $D_x^\perp$  does not depend on  $\beta$ , i.e. on  $x$ . In fact  $L(y) = n - \lambda$  and every element in  $E$  is a shift of  $y$ .

**Corollary 3** If all codewords of the code  $E$  have same weight, every coset of  $C$  of depth  $r$  (the same depth than  $C$ ) have the same weight enumerator.

We suppose now that  $E$  has more than one weight. The code  $C^\perp$  consists of  $L$  separate sets:

$$\bigcup_{j \in [0, n-1]} (\sigma_j(y_\ell) + \mathcal{P}^{m(p-1)-r+1}), \quad y_\ell \in E, \quad \ell \in [1, L].$$

Each set is a union of cosets of same weight polynomial, since the RM-codes are invariant under the shift. Assume that for each  $\ell$ , this weight polynomial, the weight of  $y_\ell$  and  $\phi_s(y_\ell)$  are known. Then, by applying Corollary 2, we can compute the values  $L(y_\ell)$  for every  $\ell$  and obtain the weight polynomial of  $D_x^\perp$ .

The most simple case appears when the code  $E$  is such that only two weights occur. For instance, using the tools we present here, we treat in [10] the cosets of depth 2 of the extended 2-error-correcting BCH codes. More generally consider  $E$  as a two-weights irreducible cyclic codes characterized in [3, Theorem 6]. When the cosets  $y + \mathcal{P}^{m(p-1)-r+1}$ ,  $y \in E$ , have two possible weight distributions, then we can assert that there are at most two distinct weight distributions for the cosets of  $C$  of depth  $r$ .

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