Large deviation probability and local density of sets

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Abstract: Let $X_1, X_2, \ldots, X_n$ be n independent identically distributed real random variables and $S_n := \sum_{i=1}^{n} X_i$. We obtain precise asymptotics for $P(S_n \in n\,A)$ for rather arbitrary Borel sets A, in terms of the density of the dominating points in A.

Our result extends classical theorems in the field of large deviations for independent samples. We also obtain asymptotics for $P(S_n \in \gamma_n A)$, with $\gamma_n / n \to \infty$.

Key-words: large deviation, fractal, local density.

AMS (1991) Classification : 60F10 ; 28A80.

(Résumé : tsvp)

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INRIA ROCQUENCOURT

Probabilité de grande déviation et dimension locale des ensembles

Résumé : Soient $X_1, X_2, \ldots, X_n$, n variables aléatoires réelles indépendantes identiquement distribuées et $S_n := \sum_{i=1}^{n} X_i$. Nous obtenons des évaluations asymptotiques de $P(S_n \in n\ A)$ pour des ensembles Boréliens A assez généraux, en termes de la densité locale des points dominants dans A. Ces résultats généralisent des résultats classiques dans la théorie des grandes déviations pour des suites de variables indépendantes. Nous obtenons également des équivalents asymptotiques de $P(S_n \in \gamma_n A)$, lorsque $\gamma_n/n \to \infty$.

Mots-clé : grande déviation, fractale, densité locale.

AMS (1991) Classification : 60F10 ; 28A80.

1. Introduction

Let $X_1, X_2, \ldots, X_n$ be n independent and identically distributed (i.i.d) real valued random variables (r.v'.s) and let $S_n := \sum_{1 \le i \le n} X_i$. The main goal of this paper is to obtain precise asymptotics for $P(S_n \in n\ A)$ for rather arbitrary Borel sets A. This problem is strongly connected to various topics studied in the literature about large deviations and about fractal geometry.

Assume that

(H1) X_1 is non degenerate, i.e. $P(X_1 = c) < 1$ for all $c \in IR$,

(H2) $\phi(t) = E(e^{tX_1}) < \infty$ on some non-void set $\mathcal{D}$ with $0 \in Int\mathcal{D}$.

Set

$$t_0 := \sup\{t : \phi(t) < \infty\}.$$

Define the Chernoff transform

$$\Lambda(x) := Sup\{tx - \log \phi(t)\,;\, t \in IR\}$$

and set

$$\Lambda(A) := \inf\{\Lambda(x)\,;\, x \in A\}.$$

Chernoff's (1952) theorem asserts that, for any Borel set A,

$$\begin{aligned} -\Lambda(Int\ A) &\le \liminf_{n\to\infty}\ n^{-1} \log P(S_n \in n\ A) \\ &\le \limsup_{n\to\infty}\ n^{-1} \log P(S_n \in n\ A) \le -\Lambda(cl\ \ A). \end{aligned} \tag{1.1}$$

Thus, if $\Lambda(Int\ A) = \Lambda(cl\ \ A)$,

$$\lim_{n\to\infty} n^{-1} \log P(S_n \in n\ A) = -\Lambda(A). \tag{1.2}$$

How good is (1.1)? For instance, if $A = \{1\} \cup [2, \infty)$ and $E(X_1) = 0$, then $\Lambda(A) = \Lambda(1)$, while (1.2) holds with $\Lambda(2) = \Lambda(Int\ A)$. Of course this example just stresses the rather well known fact that one should take an essential infimum instead of an infimum in the definition of $\Lambda(A)$.

If we rewrite (1.2) as

$$P(S_n \in n\ A) = \exp(-n\Lambda(A))(1 + o(1)) \text{ as } n \to \infty, \tag{1.3}$$

it is natural to look for an equivalent of the probability itself instead of its logarithm. Further, we wish such a result to reflect some information about the fine structure of A.

Bahadur and Ranga-Rao (1960), and Petrov (1965) solve partly this problem. Consider the following hypothesis, stronger than H1,

$(H3)$ X_1 has a continuous distribution.

Set $m(t) := (\log \phi)'(t)$ and $s^2(t) := (\log \phi)''(t)$.

Petrov's (1965) theorem asserts that, if $A = [a, \infty), a > E(X_1)$, then

$$P(S_n \in n\ A) = P(S_n \geq n\ a) = \frac{e^{-n\Lambda(a)}}{\sqrt{n}\Psi(a)}(1 + o(1)), \text{ as } n \to \infty \tag{1.4}$$

where $\Psi(a) := t^* s(t^*)\sqrt{2\pi}$, and t^* is the unique root of the equation $a = m(t)$ in $\mathcal{D}$. For sets A more general than $[a, \infty)$, (1.4) suggests that all what really matters is the essential infimum of A. This is clearly true if the essential infimum a of A is the end point of an interval contained in A. For the multidimensional case, see Ney (1983).

What about a very discontinuous set A? Recall that the density of a in A is defined, whenever it exists, by

$$d(a) := \lim_{\varepsilon \to 0} \frac{|A \cap [a - \varepsilon, a + \varepsilon]|}{2\varepsilon}.$$

Is (1.4) still true if $d(a) = 0$? We shall investigate the link between (1.4) and the structure of A near its essential infimum; see Falconer (1990) for the properties of d and related concepts.

We also mention other works related to (1.4). Blackwell and Hodges (1959) proved a version of (1.4) for lattice distributions. Petrov and Širikova (1973) showed that if $\phi(t) = \infty$ for any $t > 0$, then $\limsup_{n \to \infty} \rho^{-n}\ P(S_n \geq na) = 0$ for any $\rho \in I\!R$ (see also Steinebach (1980) and references therein). Finally, Jensen (1988) obtained uniform expansions for

$P(S_n \geq na)$ under stronger assumptions than those given above; for a survey about statistical applications of these results, see Fields and Ronchetti (1990).

Our investigation of $P(S_n \in n\ A)$ for rather arbitrary Borel sets A (not only closed or open) is also related to the recent interest in large deviations without topology (see Ben Arous and Ledoux (1993) for Schilder's theorem and Deheuvels and Lifshits (1993) for related investigations on Strassen's (1964) law of the iterated logarithm). The study of $P(S_n \in n\ A)$ is also related to the Erdös-Rényi (1970) law of large numbers (see e.g. Deheuvels, Devroye and Lynch (1986)).

We shall also study asymptotics for $P(S_n \in \gamma_n A)$ for general sequences satisfying $\gamma_n/n \to \infty$. The case $A = [a, +\infty)$ has been partly explored by Broniatowski and Mason (1994).

Section 2 presents our results, along with some remarks about a new definition of the local density of a in A. Section 3 contains the proofs.

2. Results

2.0.0

2.1 Large deviation probabilities

In the sequel we assume that $(H1), (H2)$ and $(H3)$ hold.

Denote α the essential infimum of A with respect to the Lebesgue measure,

$$\alpha := \ essinf A := \inf\{x : \text{ for all } \epsilon > 0, |[x, x+\epsilon] \cap A| > 0\},$$

with $\inf \emptyset = -\infty$.

Assume

$(H4)$ $\qquad \alpha > -\infty.$

$(H4)$ means that we consider rather thick sets A; for example we do not consider Cantor type sets.

The density of α in A will not be measured in the ordinary way, but will be related to the more appropriate quantity

$$M(t) := t\int I_{A-\alpha}(y)e^{-ty}dy, \qquad t>0. \tag{2.1.1}$$

For any set A, $0 \le M(t) \le 1$. Notice that if $A=[\alpha,\infty)$, then $M(t)=1$ for any $t>0$. If there exists an interval $[\alpha,\alpha+\varepsilon] \subset A$, then $\lim_{t\to\infty} M(t)=1$.

Here is an example for a self-similar set. Let $p>2$ and $I_p := \left[\frac{p-1}{p},1\right]$. Set $A := A_p := \bigcup_{n\in\mathbb{Z}} p^n I_p$.Then $0 = essinf A_p$ and $pA_p = A_p$. Consequently, for any $t \ge 0, M(tp)=M(t)$. Thus

$$\inf_{1\le u\le p} M(u) = \liminf_{t\to\infty} M(t) \le \limsup_{t\to\infty} M(t) = \sup_{1\le u\le p} M(u).$$

It is convenient to define

$$M_n(t) := M(nt)/t = \int I_{A-\alpha}(y)e^{-nty}dy \tag{2.1.2}$$

and

$$\Psi_n(t) := n\log\phi(t) + \log M_n(t) - n\ \alpha\ t, \tag{2.1.3}$$

for all $t>0$ such that $\phi(t)<\infty$.

It is well known that when $(H1)$ and $(H2)$ hold, $\log\phi$ is a strictly convex function. Furthermore, $m(0)=E(X_1)$ and $m(t)$ is a strictly increasing function, which we assume to satisfy:

$(H5)$ $\qquad \lim_{t\to t_0} m(t) = +\infty$.

See Deheuvels, Devroye and Lynch (1986) for a discussion on $(H5)$ in connection with Petrov's (1965) theorem.

We quote some properties on the function M defined in (2.1.1). Set $\mu_n(t) := (1/n)\log M_n(t)$. For any $n\ge 1, \mu_n$ is a decreasing function on $[0,\infty)$, negative for n large enough. Furthermore, $\mu_n'(t) = \mu_1'(nt)$, and μ_1' is non decreasing on $[0,\infty)$.

Let $\overline{\mu} := \lim_{t\to\infty}\ \mu_1'(t)$ and $\underline{\mu} := \lim_{t\to 0}\mu_1'(t)$.

Lemma 2.1.1. *Assume that* $(H1)$, $(H2)$ *and* $(H5)$ *hold. Then the equation* $\Psi_n'(t) = 0$ *has a unique solution* t_n *in* $(0, t_0)$ *for* $\alpha \in (E(X_1) + \overline{\mu}, \infty)$.

Furthermore

(1) *If* $\alpha \leq E(X_1) + \underline{\mu}$, *then* $\lim_{n\to\infty} t_n = 0$.

(2) *If* $\alpha > E(X_1) + \underline{\mu}$, *then there exists a compact subset* $K \subset (0, t_0)$ *such that* $t_n \in K$ *for all* $n \geq 1$.

We postpone the proof to Section 3.

From now on, we suppose that $\alpha > E(X_1) + \underline{\mu}$.

Set $\psi_n(t) := \Psi_n''(t)$ and suppose that for any $\lambda > 0$,

$$(H6) \qquad \lim_{n\to\infty} \sup_{|u|<\lambda} \frac{\psi_n\left(t_n + u/\sqrt{\psi_n(t_n)}\right)}{\psi_n(t_n)} = 1$$

where t_n is the solution of

$$\Psi_n'(t) = 0 \tag{2.1.4}$$

in the range $(0, t_0)$.

The following result shows that $(H6)$ is not very restrictive.

Recall that a function l is slowly varying at $+\infty$ if for any $\lambda > 0$, $\lim_{x\to+\infty} l(\lambda x)/l(x) = 1$. In this case we note $l \in \mathcal{R}_0(+\infty)$. We say that l is slowly varying at 0 if $l(1/.) \in \mathcal{R}_0(+\infty)$; we denote $l \in \mathcal{R}_0(0)$. A function g is regularly varying as 0 or $+\infty$ if $g(x) = x^\rho l(x)$ for some $l \in \mathcal{R}_0(0)$ or $l \in \mathcal{R}_0(+\infty)$. We denote $g \in \mathcal{R}_\rho(0)$ or $g \in \mathcal{R}_\rho(+\infty)$; see Bingham, Goldie and Teugels (1987) for the theory of regular variation.

Lemma 2.1.2. *Assume that* $(H1)$, $(H2)$ *and* $(H5)$ *hold.*

A sufficient condition for $(H6)$ *is*

$$\log(M(t)/t) \in \mathcal{R}_\rho(\infty) \text{ for some } \rho \in [0, 1].$$

We also need the following condition:

$$(H7) \qquad \limsup_{t\to\infty} t\,(\log M(t))'' < \infty.$$

A sufficient condition for $(H7)$ is $\log(M(t)/t) \in \mathcal{R}_\rho$, for $0 \leq \rho < 1$.

We now state our main result.

Theorem. ***2.1.1*** *Assume that (H2) - (H7) hold. Then, for* $\alpha = essinf > E(X_1) + \underline{\mu}$,

$$P(S_n \in n\ A) = \frac{\phi^n(t_n).M_n(t_n).e^{-nt_n\alpha}}{\psi_n(t_n)\sqrt{2\pi}}(1+o(1))\ \textit{as}\ n \to \infty, \qquad (2.1.5)$$

with t_n *satisfying* (2.1.4)*, provided that the function* $x \mapsto P(S_n \in nA + x)$ *is nonincreasing for* n *large enough. In particular, this last condition holds if*

(i) *(Petrov):* $A = (\alpha, \infty)$ *or* $A = [\alpha, \infty)$ *(in this case* $(H7)$ *is satisfied and* $M_n(t) = 1/t$*), or*

(ii) X_1 *has a symetric unimodal distribution or*

(iii) X_1 *has a strongly unimodal distribution .*

The shape of A near α is reflected in the behavior of the function $M(t)$ for large values of t. By (2.1.2) and Lemma 2.1.1, the more n is large, the more the shape of A near α is relevant in (2.1.5).

One should notice that the formula given in Theorem is the exact analogue of (1.4). Indeed, using Hölder Inequality, one easily checks that $\Psi_n(t)$ is convex.

By Lemma 2.1.1, t_n is unique in K, and t_n achieves $\sup\{n\alpha t - n\log\phi(t) + \log M_n(t)\}$; compare with the definition of $\Lambda(n\alpha)$. However, recall that the function M_n depends upon the regularity of the set A near α, while the function Λ only depends upon the distribution of X_1.

Notice that $M_n(t)e^{-nt\alpha} = \int I_A(y)e^{-nty}dy$, from which we see that α plays no role in (2.1.5). Therefore we can substitute α by any real number γ for which $\int I_{A-\gamma}(y)e^{-ty}dy$ converges. Further, t_n defined in (2.1.4) is independent upon α. The so-called dominating point α in A can therefore be defined by $\alpha := \lim_{t\to\infty} t^{-1}\log\int I_A(y)e^{-ty}dy$.

2.2 Very large deviation probabilities

We consider asymptotics for

$$P(S_n \in \gamma_n A),\ \text{with}\ \gamma_n/n \to +\infty.$$

For all $\alpha \in (E(X_1) + \underline{\mu}, \infty)$, let $t_n = t_n(\alpha)$ be the unique solution of the equation

$$\Psi_n(t) = 0$$

in the range $(0, t_0)$, where now

$$\Psi_n(t) = n \log \phi(t) + \log M_n(t) - \gamma_n t\alpha$$

and

$$M_n(t) = M(\gamma_n t)/t.$$

It is easy to check that $\lim_{n\to\infty} t_n = t_0$.

The following result holds:

Theorem. ***2.2.1*** *Assume that X_1 has a strongly unimodal distribution and (H2), (H4) - (H7) hold. With the above notation, (*2.1.5*) holds.*

Whenever $t_0 = +\infty$, $s^2(t)$ and $(\log M(t))''$ are regularly varying functions at $+\infty$, it is easy to characterize sequences γ_n such that $(H6)$ holds.

2.3 Density of α in A and the function $M(t)$

In order to investigate further the role played in (2.1.5) by the regularity of the set A near its essential infimum α, we define

$$G(\epsilon) := |A \cap [\alpha, \alpha + \epsilon]|, \qquad \varepsilon \geq 0.$$

We interpret G as defining a measure on IR^+. Shifting slightly the terminology used in the study of multifractal measures (see e.g. Falconer (1990)), we define the pointwise Hölder dimension of A at α as

$$\delta(\alpha) := \lim_{\epsilon\to 0} \frac{\log G(\epsilon)}{-\log \epsilon},$$

if the limit exists.

For instance, for the set A_p defined above, $G(\varepsilon) = 1/p^k(p-1)$ if $p^{-k-1} \leq \varepsilon < p^{-k}$, so that $\varepsilon/(p-1) \leq G(\varepsilon) \leq p\varepsilon/(p-1)$ for any $\varepsilon > 0$, and thus $\delta(0) = 1$. So, A_p is reasonably big around 0.

There is a strong connection between the density, the pointwise Hölder dimension and the behavior of the function $M(t)$.

Proposition 2.3.1. *i) If $M(t)/t \in \mathcal{R}_{-\rho}(\infty)$, then $0 \le \rho \le 1$. Moreover, $M(t) \sim c\ t^{-\rho+1} l(t)$ as $t \to \infty$, for some $l \in \mathcal{R}_0(\infty)$ iff $G(\varepsilon) \sim c\ \varepsilon^{\rho} l(1/\varepsilon)/\Gamma(1+c)$, as $\varepsilon \to 0$.*

ii) If $\log(M(t)/t) \in \mathcal{R}_{1/(1+\rho)}(\infty)$, then $\rho \ge 0$. Moreover $-\log G(\varepsilon) \sim c/\phi^{\leftarrow}(1/\varepsilon)$ as $\varepsilon \to 0$ and for $\phi \in \mathcal{R}_{-\rho}(0^+)$ and $c > 0$ iff $-\log(M(t)/t) \sim (1+\rho)\left(\frac{c}{\rho}\right)^{\rho/(\rho+1)} \dfrac{1}{\Psi^{\leftarrow}(t)}$ as $t \to \infty$ where here $\Psi(\varepsilon) = \phi(\varepsilon)/\varepsilon \in \mathcal{R}_{-\rho-1}(0^+)$, and $\phi^{\leftarrow}$ (resp. $\Psi^{\leftarrow}$) denotes the generalized inverse of ϕ (resp. Ψ).

This proposition follows readily from classical Abel–Tauber theorems (see Bingham, Goldie and Teugels (1987), Ch. 2 and 4). For instance, we deduce from Proposition 2.3.1 that $G(\varepsilon) \sim \varepsilon^{\delta(\alpha)}$ (as $\varepsilon \to 0$) iff $M(t) \sim c\ t^{-\delta(\alpha)+1}\Gamma(1+\delta(\alpha))$ (as $t \to \infty$). Consequently, if $M_n(t) \to 1$ as $t \to \infty$ then $M(t) \sim t$ as $t \to \infty$ and $G(\varepsilon) \sim \varepsilon$ as $\varepsilon \to 0$.

3. Proofs

3.1 Proof of Lemma 2.1.1.

By the properties of the function $\mu_1, -\infty < \underline{\mu} \le 0$. The mapping $\alpha \to t_n$ is one to one from $(E(X_1) + \overline{\mu}, \infty)$ onto $(0, t_0)$. Note that the solution t_n of (2.1.4) may not be unique on $\mathcal{D}$ ($f.i$, if X_1 is normally distributed and $A = [\alpha, \infty)$).

i) Assume that $\liminf_{n\to\infty} t_n > t > 0$. For some sequence $\{k\} \subset \{n\}$, $\lim_{k\to\infty} m(t_k) + \mu_1'(k\ t_k) = \alpha > m(t) + \underline{\mu} > E(X_1) + \underline{\mu}$, a contradiction.

ii) If $\limsup_{n\to\infty} t_n = t_0$, then, for some $\{k\} \subset \{n\}$, $\lim\limits_{k\to\infty} m(t_k) + \mu_1'(k\ t_k) = \alpha$, which implies $\lim\limits_{k\to\infty} \mu_1'(k\ t_k) = -\infty$, a contradiction.

If $\liminf_{n\to\infty} t_n = 0$, then $\lim_{k\to\infty} m(t_k) + \mu_1'(k\ t_k) = \alpha < E(X_1)$, a contradiction. ■

3.2 Proof of Lemma 2.1.2.

The expression in (H6) is equal to

$$\frac{\mu_1''\left(n\left(t_n + u/\sqrt{\psi_n(t_n)}\right)\right)}{\mu_1''(nt_n)}(1+o(1)) \text{ as } n \to \infty.$$

Since t_n belongs to some compact set K (see Lemma 2.1.1) and $\lim_{n\to\infty}\psi_n(t_n) = +\infty$, the result follows from known facts on regular variation. ■

3.3 Proof of Theorem 2.1.1

The proof will be covered by several lemmas.

Lemma 3.3.1. *For any $t \in (0, t_0)$, if (H4) holds,*

$$\int e^{tx} P(S_n \in n\ A + x)dx = \phi(t)^n M_{n,\gamma}(t) e^{-nt\gamma} \tag{3.3.1}$$

for all $n \geq 1$ and $\gamma \in IR$, where $M_{n,\gamma}(t) = (1/t)\int I_{A-\gamma}(y)e^{-nty}dy$.

Proof of Lemma 3.3.1.

Since all the functions that we integrate are nonnegative, we can use Tonelli's theorem and obtain

$$\begin{aligned}
\int e^{tx} P(S_n \in n\ A + x)\ dx &= \int\int e^{tx} I_{\ x+nA}(u)\ dx\ P(S_n = du) \\
&= \int\int e^{tx} I_{\ u-nA}(x)\ dx\ P(S_n = du) = \int\int e^{t(u-ny)} I_{\ A}(y)\ n\ dy\ P(S_n = du) \\
&= n\int e^{tu} P(S_n = du) \int I_{\ A}(y) e^{-nty} dy = n\phi(t)^n e^{-tn\gamma} \int I_{\ A-\gamma}(y) e^{-tny} dy. \quad ■
\end{aligned}$$

Choosing $\gamma = \alpha$, we obtain

$$\int e^{tx} P(S_n \in n\ A + x)\ dx = \phi(t)^n M_n(t) e^{-tn\alpha}. \tag{3.3.2}$$

Notice that Lemma 3.3.2 also holds in IR^d:

$$\int e^{<t,x>} P(S_n \in n\ A + x)\ dx = \phi(t)^n e^{-n<t,\gamma>} M_{n,\gamma}(t),$$

with

$$M_n(t) := n^d \int I_{A-\gamma}(y) e^{-n<t,y>} dy$$

and assuming that $\int e^{<t,x>} P(S_n \in n\ A + x)\ dx < +\infty$.

We now define a family of densities indexed by $t \in (0, t_0)$ namely, by (3.3.2),

$$g_n(x,t) := \frac{e^{tx} P(S_n \in n\ A + x)}{\phi(t)^n M_n(t) e^{-tn\alpha}}.$$

The main line of the proof consists in showing that $g_n(.,t)$, suitably rescaled, converges to the Gaussian density, and then take $g_n(0,t_n)$, with t_n defined as in Lemma 2.1.1.

Notice that the Laplace transform of $g_n(.,t)$ is

$$L_n(\lambda,t) := \int e^{\lambda x} g_n(x,t)\ dx = \frac{\phi^n(t+\lambda)}{\phi^n(t)} \frac{M_n(t+\lambda)}{M_n(t)} e^{-n\lambda\alpha}. \tag{3.3.3}$$

We now calculate the expectation and the variance of $g_n(.,t)$.

Lemma 3.3.2. *We have* $c_n(t) = \int x\ g_n(x,t)\ dx = n\ m(t) + (\log M_n)'(t)$ *and* $v_n^2(t) := \int (x - \mu_n(t))^2 g_n(x,t)\ dx = n\ s^2(t) + (\log M_n)''(t)$.

Proof of Lemma 3.3.2. Take the derivative at $\lambda = 0$ in (3.3.3) in order to calculate the cumulants of the density $g_n(.,t_n)$ ■.

It is convenient to introduce a family of r.v.'s, say Y_n, with density $g_n(.,t_n)$. Recall that we have assumed $\alpha > E(X_1) + \underline{\mu}$. Also, by Lemma 3.3.2, $E(Y_n) = c_n(t_n) = 0$.

Our next Lemma is a central limit theorem for Y_n.

Lemma 3.3.3. *Whenever (H6) holds,* $Y_n/v_n(t) \xrightarrow{d} \mathcal{N}(0,1)$ *as* $n \to \infty$.

Proof of Lemma 3.3.3. Taylor's expansion at $\lambda = 0$, Lemma 3.3.2, the choice of t_n and the definition of Ψ_n yields, for some $\theta = \theta_n(\lambda) \in (0,1)$,

$$\begin{aligned}
\log E(e^{\lambda Y_n / v_n(t_n)}) &= \log L_n(\lambda/v_n(t_n), t_n) \\
&= \frac{\lambda^2}{2 v_n(t_n)^2} (\log L_n(.,t_n))''(\theta\lambda/v_n(t_n)) \\
&= \frac{\lambda^2}{2} \frac{\psi_n(t_n + \theta\lambda/\sqrt{\psi_n(t_n)})}{\psi_n(t_n)}.
\end{aligned}$$

Assuming $(H6)$ the proof follows from classical convergence criteria (see e.g. Billingsley (1978), p. 345). ■

Having a central limit theorem for $Y_n/v_n(t_n)$, we turn it into a local central limit theorem on the density of $Y_n/v_n(t_n)$.

Let

$$h_n(x) := v_n(t_n)g_n(xv_n(t_n), t_n)$$

and its characteristic function (c.f.)

$$\hat{h}_n(u) := \int e^{iux} h_n(x)dx.$$

Lemma 3.3.4. *Assume that* $(H6)$ *holds. Then there exists two positive constant* C *and* δ *such that*

$$|\hat{h}_n(\omega)| \leq e^{-C\ \omega^2} \quad \textit{for} \quad |\omega| < \delta v_n(t_n).$$

Proof of Lemma 3.3.4. Since

$$\left| e^{ix} - \left(1 + ix - \frac{x^2}{2}\right)\right| \leq \frac{|x|^3}{6},$$

we have

$$\frac{|\phi(t_n + iu)|}{\phi(t_n)} \leq |1 - (a - ib)| + c$$

with

$$a = \frac{u^2}{2}\left(\frac{\phi''}{\phi}\right)(t_n), \qquad b = u\left(\frac{\phi'}{\phi}\right)(t_n), \qquad c = \frac{|u|^3}{6}\left(\frac{\phi^{(3)}}{\phi}\right)(t_n)$$

and

$$\phi^{(3)}(t) := \int |x|^3 e^{t\ x} dF(x)$$

where F is the d.f. of X_1.

Let

$$A = 2a - b^2 - a^2 - c^2 - 2c\sqrt{1 + a^2 + b^2 - 2a} = u^2 s^2(t_n) - a^2 - c^2 - 2c\sqrt{1 + a^2 + b^2 - 2a}.$$

Since t_n is in a compact subset of $(0, 1)$, (Lemma 2.1.1), we have $|a^2 + c^2 + 2c\sqrt{1 + a^2 + b^2 - 2a}| \leq c_1 u^3$ for some $c_1 > 0$ and provided $|u|$ is small enough. Therefore, in taking some $\delta > 0$ small enough, for $|u| \leq \delta$ we have

$$u^2 s^2(t_n)/2 \leq A \leq 2u^2 s^2(t_n) \leq 1/2.$$

Hence, for $|u| \leq \delta$,

$$\frac{|\phi(t_n + iu)|}{\phi(t_n)} \leq |1 - A|^{1/2} \leq 1 - A/2 \leq 1 - u^2 s^2(t_n).$$

Consequently, if $|\omega| \leq \delta v_n(t_n)$,

$$\frac{|\phi(t_n + i\omega/v_n(t_n))|^n}{\phi(t_n)^n} \leq \left(1 - \omega^2 \frac{s^2(t_n)}{v_n^2(t_n)}\right)^n.$$

If δ is small enough, the $\limsup_{n\to\infty} \delta s_n^2(t_n)/v_n^2(t_n) < 1$ and therefore, $\omega^2 s^2(t_n)/v_n(t_n) < 1$ if $|\omega| < \delta v_n(t_n)$ and so, for n large enough and $|\omega| < \delta v_n(t_n)$,

$$\frac{|\phi(t_n + i\omega/v_n(t_n))|^n}{\phi(t_n)^n} \leq \exp\left(-\omega^2 \frac{s^2(t_n)}{v_n^2(t_n)}\right). \quad \blacksquare$$

We state now the local central limit theorem.

Theorem. ***3.3.1*** *If $(H1) - (H7)$ hold, then, for any $a < b$ and $x \in IR$,*

$$\lim_{n\to\infty} v_n(t_n) P(Y_n/v_n(t_n) \in (x + a/v_n(t_n), x + b/v_n(t_n))) = (b - a)\varphi(x),$$

where $\varphi(x) = (1/\sqrt{2\pi})e^{-x^2/2}$.

Proof of Theorem 3.3.1. Making use of the proof of Theorem 5.4 in Durrett (1991), it is enough to show that, for all θ and $x \in IR$,

$$v_n(t_n) E(l_\theta(Y_n - x\ v_n(t_n))) \to \varphi(x) \int l_\theta(y)dy \text{ as } n \to \infty, \tag{3.3.4}$$

where

$$l_\theta(y) = e^{i\theta\ y} l_0(y)$$

and

$$l_0(y) = \frac{1}{\pi}\frac{1 - \cos y}{y^2}$$

is the Pólya density function.

Denote $\hat{l}_\theta$ and $\hat{l}_0$ the Fourier transforms of l_θ and l_0. Denote F_n the d.f. of the r.v. $Y_n - x\ v_n(t_n)$.

We have

$$\begin{aligned} v_n(t_n)E(l_\theta(Y_n - xv_n(t_n))) &= v_n(t_n)\int l_\theta(y)dF_n(y) \\ &= \frac{v_n(t_n)}{2\pi}\int\int e^{-iuy}\hat{l}_\theta(u)dudF_n(y) \\ &= \frac{v_n(t_n)}{2\pi}\int e^{-iuy}dF_n(y)\int \hat{h}_\theta(u)du, \end{aligned}$$

where we have used the inversion formula for the Fourier transform and Fubini's theorem.

Now,

$$\begin{aligned} \int e^{-iuy}dF_n(y) &= E\left(e^{iu(Y_n - xv_n(t_n))}\right) \\ &= E\left(e^{-iuY_n}\right)e^{-iuxv_n(t_n)}. \end{aligned}$$

Thus

$$\int e^{-iuy}dF_n(y) = e^{-iuxv_n(t_n)}\hat{h}_n(-uv_n(t_n)),$$

which yields

$$v_n(t_n)E(l_\theta(Y_n - xv_n(t_n))) = \frac{v_n(t_n)}{2\pi}\int \hat{h}_n(-uv_n(t_n))e^{iuxv_n(t_n)}\hat{h}_\theta(u)du. \tag{3.3.5}$$

Let $M > 0$ such that $l_\theta(u) = 0$ for $u \in [-M, M]^{-c}$. Let $\delta > 0$ be as in Lemma 3.3.4, $I = [-\delta, \delta]$ and $J = [-M, M] \setminus I$. We split the integral in (3.3.5) on I and J.

$$\begin{aligned} &\left|\frac{v_n(t_n)}{2\pi}\int_J \hat{h}_n(-uv_n(t_n))e^{iuxv_n(t_n)}\hat{l}_\theta(u)du\right| \\ \leq\ & \frac{v_n(t_n)}{2\pi}\int_J \left|\hat{h}_n(-uv_n(t_n))\right| du \\ \leq\ & \frac{M}{\pi}v_n(t_n)\sup_{u\in J}\left|\hat{h}_n(-uv_n(t_n))\right|. \end{aligned}$$

Moreover

$$\left|\hat{h}_n(-uv_n(t_n))\right| \leq \frac{|\phi(t_n - iu)|^n}{\phi(t)^n}.$$

Since F is continuous, there exists $\eta, 0 < \eta < 1$, such that

$$\left|\frac{v_n(t_n)}{2\pi}\int_J \hat{h}_n(-uv_n(t_n))du\right| \leq v_n(t_n)\eta^n M/\pi, \tag{3.3.6}$$

which tends to 0 as $n \to \infty$.

We consider now

$$\frac{v_n(t_n)}{2\pi} \int_I \hat{h}_n(-uv_n(t_n)) e^{iuv_n(t_n)x} \hat{h}_\theta(u) du.$$

By Lemma 3.3.3 and 3.3.4 and the dominated convergence Theorem, we see that this expression tends to

$$\begin{aligned} \int e^{-\omega^2/2} e^{i\omega x} \hat{h}_\theta(0) d\omega &= \varphi(x) \hat{l}_\theta(0) \\ &= \varphi(x) \int l_\theta(y) dy. \end{aligned}$$

This completes the proof of Theorem 3.3.1. ■

We now complete the proof of Theorem 2.1.1.

By Theorem 3.3.1, we have, for all $\delta > 0$:

$$\frac{v_n(t_n)}{\delta} \int_0^{\delta/v_n(t_n)} h_n(x) dx = \rho(0) + \varepsilon(n, \delta), \tag{3.3.7}$$

with $\lim_{n\to\infty} \varepsilon(n, \delta) = 0$. Theorem 3.3.1 implies:

$$\frac{v_n(t_n) P(S_n \in n\ A)}{M_n(t_n) \phi(t_n)^n e^{-nt_n\alpha}} \frac{e^{t_n\delta}}{t_n\delta} \geq \varphi(0) + \varepsilon(n, \delta)/\delta$$

which yields

$$\frac{v_n(t_n) P(S_n \in nA)}{M_n(t_n) \phi(t_n)^n e^{-nt_n\alpha}} \geq \left(\varphi(0) + \frac{\varepsilon(n, \delta)}{\delta}\right) (1 - t_n\delta). \tag{3.3.8}$$

By Lemma 2.1.1, t_n belongs to some compact $K \subset (0, t_0)$. Taking the $\liminf$ as $n \to \infty$ on both sides of (3.3.8) yields:

$$\liminf_{n\to\infty} \frac{v_n(t_n) P(S_n \in nA)}{M_n(t_n) \phi(t_n)^n e^{-nt_n\alpha}} \geq \varphi(0).$$

The lower bound for the $\limsup$ is handled in the same way, using Theorem 3.3.1 with $x = 0, a = -\delta/v_n(t_n)$ and $b = 0$. ■

3.4 Proof of Theorem 2.2.1

Arguing as previously, we define

$$g_n(x,t) := \frac{e^{tx}P(S_n \in \gamma_n A + x)}{\phi^n(t)M_n(t)e^{-\gamma_n t\alpha}}.$$

The central limit theorem (Lemma 3.3.3) still holds whenever (H6) holds (see Lemma 2.1.2). We cannot any longer prove the local central limit theorem 3.3.1. Instead, we prove that

Lemma 3.4.1. *For any $n \geq 1, g_n(.,t)$ is a* log*-concave function.*

Proof of Lemma 3.4.1. It is enough to prove that $x \longmapsto \log P(S_n \in \gamma_n A + x)$ is concave on $I\!R$.

The argument will go throught discrete approximations. Consider first an integer-valued r.v. X with log-concave distribution, meaning that

$$P(X=m)^2 \geq P(X=m-1)P(X=m+1), m \in \mathbb{Z}.$$

It is known that $S_n = \sum_{i=1}^n X_i$ also possesses a log-concave distribution, where the X_i's are n independent copies of X. Set $p_m = P(S_n = m), n \in I\!N$. Consider first the case where $A = \{0,1,2,\ldots,K\}$.

Define $\tilde{p} := (\ldots, p_0, p_1, p_2, \ldots)$ and $\tilde{1}_A := (\ldots, 0, 1, 1, 1, \ldots)$, where the first 1 in $\tilde{1}_A$ is $\tilde{1}_A(-K) = 1$. Clearly $\tilde{p} * \tilde{1}_A = \left(\sum_{i=m}^{m+K} p_i\right)_{m \in \mathbb{Z}}$. Since $\tilde{1}_A$ is log-concave, the same is $\tilde{p} * \tilde{1}_A$. Now, $(\tilde{p} * \tilde{1}_A)(m) = P(S_n \in A + m)$, which therefore is a log-concave function of m.

Consider now $C = A \cup B$ where A and B are two disjoint intervals of $I\!N$, say $A = \{0,1,\ldots,K\}$ and $B = \{\delta, \delta+1, \ldots, \delta+L\}, \delta > K$. Let $q_m := P(S_n \in A \cup B + m) = \sum_{i=m}^{m+K} p_i + \sum_{j=m+\delta}^{m+\delta+L} p_j$.

Direct calculation shows that

$$q_m^2 \geq q_{m-1}q_{m+1}$$

is equivalent to

$$\begin{aligned} & p_{m-1}p_{m+\delta+L}\left(\frac{p_m}{p_{m-1}}-\frac{p_{m+1+\delta+L}}{p_{m+\delta+L}}\right) \\ \geq \quad & p_{m-1+\delta}p_{m+K}\left(\frac{p_{m+1+K}}{p_{m+K}}-\frac{p_{m+\delta}}{p_{m-1+\delta}}\right). \end{aligned} \tag{3.4.1}$$

Since (p_m) is log-concave,

$$\frac{p_{j+1}}{p_j} \leq \frac{p_l}{p_{l-1}} \text{ for } j > l,$$

by which (3.4.1) holds. Thus (q_m) is a log-concave sequence.

For the absolutely continuous case, it is enough to approximate the function $x \longmapsto P(S_n \in \gamma_n A + x)$ by a sequence of lattice-valued log-concave functions with a decreasing grid-width, since log-concavity is preserved under simple limits. We omit the details. ■

We now complete the proof of Theorem 2.1.1.

Using Lemma 2 in Feigin and Yashchin (1983), Lemmas 3.3.3 and 3.4.1 we obtain

$$\lim_{n\to\infty} g_n(0, t_n) = \varphi(0)$$

as sought. ■

4. *

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