The Euler Scheme for Lévy driven Stochastic Differential Equations

Philip Protter, Denis Talay

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

The Euler Scheme for Lévy driven Stochastic Differential Equations

Philip PROTTER , Denis TALAY

N° 2621

Juillet 1995

PROGRAMME 6

Rapport de recherche

ISSN 0249-6399

The Euler Scheme for Lévy driven Stochastic Differential Equations

Philip PROTTER* , Denis TALAY**

Programme 6 — Calcul scientifique, modélisation et logiciel numérique

Projet Omega

Rapport de recherche n˚ 2621 — Juillet 1995 — 31 pages

Abstract: In relation with Monte-Carlo methods to solve some integro-differential equations, we study the approximation problem of $\mathbb{E}g(X_T)$ by $\mathbb{E}g(\bar{X}_T^n)$, where $(X_t, 0 \leq t \leq T)$ is the solution of a stochastic differential equation governed by a Lévy process (Z_t), $(\bar{X}_t^n)$ is defined by the Euler discretization scheme with step $\frac{T}{n}$. With appropriate assumptions we show that the error $\mathbb{E}g(X_T) - \mathbb{E}g(\bar{X}_T^n)$ can be expanded in powers of $\frac{1}{n}$ if the Lévy measure of Z has finite moments of order high enough. Otherwise the rate of convergence is slower and its speed depends on the behavior of the tails of the Lévy measure.

The simulation of the increments of (Z_t) is also discussed.

AMS Classifications. Primary:60H10, 65U05; Secondary: 65C05, 60J30, 60E07, 65R20.

Key-words: Stochastic Differential Equations, approximation, Lévy processes.

(Résumé : tsvp)

*Supported in part by NSF♯ INT-9401109 and NSA♯ MDA-904-94-H-2049. Departments of Mathematics and Statistics, Purdue University, West Lafayette, IN 47907-1395, USA.
E-mail : protter@math.purdue.edu

**INRIA Sophia-Antipolis, 2004 route des Lucioles, B.P. 109, 06561 Valbonne Cedex, France.
E-mail : talay@sophia.inria.fr

Le schéma d'Euler pour les équations différentielles stochastiques conduites par un processus de Lévy

Résumé : En relation avec des méthodes de Monte Carlo pour résoudre certaines équations intégro-différentielles, nous étudions la vitesse de convergence de $\mathbb{E}g(\bar{X}_T^n)$ vers $\mathbb{E}g(X_T)$, où $(X_t, 0 \leq t \leq T)$ est la solution d'une équation différentielle stochastique gouvernée par un processus de Lévy (Z_t), $(\bar{X}_t^n)$ est défini par le schéma d'Euler de pas $\frac{T}{n}$. Sous des hypothèses appropriées nous montrons que l'erreur d'approximation $\mathbb{E}g(X_T) - \mathbb{E}g(\bar{X}_T^n)$ peut être développée en puissances de $\frac{1}{n}$ si la mesure de Lévy de Z a des moments d'ordre assez élevé. Dans le cas contraire, la vitesse de convergence est plus faible et dépend de la queue de la mesure de Lévy.

Nous abordons également la question de la simulation des incréments de (Z_t).

AMS Classifications. Primary:60H10, 65U05; Secondary: 65C05, 60J30, 60E07, 65R20.

Mots-clé : Equations différentielles stochastiques, approximation, processus de Lévy.

1 Introduction

We consider the following stochastic differential equation:

$$X_t = X_0 + \int_0^t f(X_{s-})dZ_s \ , \tag{1}$$

where X_0 is an $\mathbb{R}^d$-valued random variable, $f(\cdot)$ is a $d \times r$-matrix valued function of $\mathbb{R}^d$, and (Z_t) is an r-dimensional Lévy process, null at time 0. For background on Lévy processes and stochastic differential equations governed by general semimartingales, we refer to Protter [14]. In this paper, we consider the problem of computing $\mathbb{E}g(X_T)$ for a given function $g(\cdot)$ and a fixed non random time T.

We have two main motivations. The first one is the numerical solution by Monte-Carlo methods of integro-differential equations of the type

$$\frac{\partial u}{\partial t}(t,x) = \mathcal{A}u(t,x) + \int_{\mathbb{R}^d} \{u(t,x+z) - u(t,x) - < z, \nabla u(t,x) > \ \mathbb{1}_{[\|z\| \leq 1]}\} M(x,dy) \tag{2}$$

where $\mathcal{A}$ is an elliptic operator with Lipschitz coefficients and the measure $M(x,\cdot)$ is defined as follows: let ν be a measure on $\mathbb{R}^d - \{0\}$ such that

$$\int_{\mathbb{R}^d} (\| x \|^2 \wedge 1) \nu(dx) < \infty$$

and let $f(\cdot)$ be a $d \times r$-matrix valued Lipschitz function defined in $\mathbb{R}^d$; then, for any Borel set $B \subset \mathbb{R}^d$ whose closure does not contain 0, set

$$M(x,B) := \nu\{z \ ; \ < f(x), z > \in B\} \ .$$

Our second motivation is the computation of the expectation of functionals of solutions of SDE's arising from probabilistic models, for example the calculation of the energy of the response of a stochastic dynamical system: in the latter case, obviously the Markovian structure of (X_t) is important to develop simple algorithms of simulation; a result due to Jacod and Protter [9] states that, under an appropriate condition on $f(\cdot)$, the solution of a stochastic differential equation of type (1) is a strong Markov process if and only if the driving noise (Z_t) is a Lévy process; this explains our focus on this case.

When Z is a Brownian motion Talay and Tubaro [20] have shown that when $f(\cdot)$ is smooth and if $(\bar{X}_t^n)$ is the process corresponding to the Euler scheme with step T/n (see below for a definition), then for a smooth function $g(\cdot)$ with polynomial growth, the error $\mathbb{E}g(X_T) - \mathbb{E}g(\bar{X}_T^n)$ can be expanded with respect to n:

$$\mathbb{E}g(X_T) - \mathbb{E}g(\bar{X}_T^n) = \frac{C}{n} + \mathcal{O}\left(\frac{1}{n^2}\right) \ .$$

Using the techniques of stochastic calculus of variation, Bally and Talay [1] have shown that the result also holds for any measurable and bounded function $g(\cdot)$ when the infinitesimal generator of (X_t) satisfies a "uniform hypoellipticity" condition.

Here we follow the strategy of [20]: we suppose that $g(\cdot)$ has derivatives up to order 4 but we make no assumption on the generator of (X_t). The proof used for the Brownian case does not carry through and needs to be adapted. The changes in approach are commented on in detail in Subsection 4.3. The nature of the results moreover is different. When the jumps of Z are bounded the order of convergence $\mathcal{O}\left(\frac{1}{n}\right)$ is preserved. When the jumps are unbounded the order of convergence depends on the tail of the Lévy measure of Z. However if the jumps are well behaved, as reflected by the Lévy measure having its first several moments finite, we still have a rate $1/n$ of convergence.

The discretization of Brownian driven SDE's has been now analysed in many papers for various convergence criteria: see Talay [19] or Kloeden and Platen [11] for reviews. The case of SDE's driven by discontinuous semimartingales has barely been investigated. Kurtz and Protter [13] have studied the convergence in law of the normalized error for the path by path Euler scheme, and L^p estimates of the Euler scheme error are given by Kohatsu–Higa and Protter [12].

An important point is the numerical efficiency of the Euler scheme compared to other approximation methods of (X_t). In particular the Euler scheme supposes that one can simulate the increments of the Lévy process Z. Actually, in practical situations the law of $Z_t - Z_s$ may be explicitly known: for example, Stuck and Kleiner [17] have proposed a model for telephone noise that could be interpreted as a symmetric stable Lévy process of index α (they found $\alpha \simeq 1.95$). Section 3 presents algorithmic procedures for the simulation of the increments of a class of Lévy processes which are likely to include useful models arising from engineering applications.

In a forthcoming paper we will discuss three important problems related to the present article. First, for more complex situations than those investigated here, it is sometimes possible to approximate the law of $Z_t - Z_s$ itself, which is desirable in view of simulation problems; we describe the effect of this additional approximation on the convergence rate of the Euler scheme. Second, we will study the convergence rate of another approximation method of (X_t), based upon the approximation of Z by a compound Poisson process: this approach allows the consideration of all the cases where one is given the Lévy measure of Z, which probably is more common than those for which one is given the law of the increments of Z (which generally cannot be easily derived from the Lévy measure). We also compare the numerical efficiency of this procedure to the Euler scheme when both can be used. It is worthwhile nevertheless to announce here that frequently the Euler scheme is the more efficient algorithm (in terms of the number of computations to run to ensure a given accuracy). Finally, we will extend the latter numerical procedure and its error analysis to the case of SDE's driven by diffusions and Poisson random measures, which thus includes Lévy processes.

We make a rather detailed presentation of results which are well known by specialists of Lévy processes but are perhaps nevertheless not well known in general.

Notation.

We denote by ΔZ_s the jump of (Z_t) at time s: $\Delta Z_s = Z_s - Z_{s-}$.

The Lévy decomposition of Z is:

$$Z_t = \sigma W_t + \beta t + \int_{\|x\|<1} x(N_t(\omega, dx) - t\nu(dx)) + \sum_{0<s\leq t} \Delta Z_s \, \mathbb{1}_{[\|\Delta Z_s\|\geq 1]} \, . \tag{3}$$

For a function ψ defined on $[0,T] \times \mathbb{R}^d$, $\partial_0 \psi$ will denote the derivative with respect to the time variable, and $\partial_i \psi$ will denote the derivative with respect to the i^{th} space coordinate. In the same way, $\partial_{00}\psi$ will denote the second derivative of ψ with respect to the time variable, and for a multiindex I $\partial_I \psi$ denotes the derivative with respect to space coordinates.

2 Rate of convergence of the Euler scheme

Let X be the solution of (1) for a given and fixed Lévy process Z.

In general, the law of the random variable X_T is unknown. We propose to discretize (1) in time. Let $\frac{T}{n}$ be the discretization step of the time interval $[0,T]$ and let $(\bar{X}_t^n)$ be the piecewise constant process defined by $\bar{X}_0^n = X_0$ and

$$\bar{X}^n_{(p+1)T/n} = \bar{X}^n_{pT/n} + f(\bar{X}^n_{pT/n})(Z_{(p+1)T/n} - Z_{pT/n}) \, . \tag{4}$$

From a pratical point of view, this scheme requires that the law of the stationary and independent increments $Z_{(p+1)T/n} - Z_{pT/n}$ can be simulated on a computer. For considerations on this point, see Section 3.

We now state our rate of convergence results. The case where Z has bounded jumps, or even simply where the Lévy measure has all its moments up to k for some k large enough, allows us to relax the assumptions on $f(\cdot)$ and $g(\cdot)$, and we obtain a faster rate.

For $K > 0$, $m > 0$ and $p \in \mathbb{N} - \{0\}$, set

$$\begin{aligned} \rho_p(m) \;:=\; & 1 + \|\beta\|^2 + \|\sigma\|^2 + \int_{-m}^{m} \|z\|^2 \nu(dz) \\ & + \|\beta\|^p + \|\sigma\|^p + \left(\int_{-m}^{m} \|z\|^2 \nu(dz)\right)^{p/2} + \int_{-m}^{m} \|z\|^p \nu(dz) \end{aligned} \tag{5}$$

where ν is the Lévy measure as in (3), and

$$\eta_{K,p}(m) := \exp\left(K\rho_p(m)\right) \, . \tag{6}$$

For $m > 0$ we define

$$h(m) := \nu(\{x; \|x\| \geq m\}) \, . \tag{7}$$

Theorem 2.1 *Suppose:*

(H1) *the function $f(\cdot)$ is of class $\mathcal{C}^4$; $f(\cdot)$ and all derivatives up to order 4 are bounded;*

(H2) *the function $g(\cdot)$ is of class $\mathcal{C}^4$; $g(\cdot)$ and all derivatives up to order 4 are bounded;*

(H3) $X_0 \in L^4(\Omega)$.

Then there exists a strictly increasing function $K(\cdot)$ depending only on d, r and the L^∞-norm of the partial derivatives of $f(\cdot)$ and $g(\cdot)$ up to order 4 such that, for any discretization step of type $\frac{T}{n}$, for any integer m,

$$|\mathbb{E}g(X_T) - \mathbb{E}g(\bar{X}_T^n)| \leq 4\|g\|_{L^\infty(\mathbb{R}^d)}(1 - \exp(-h(m)T)) + \frac{\eta_{K(T),8}(m)}{n} \ . \tag{8}$$

Thus, the convergence rate is governed by the rate of increase to infinity of the functions $h(\cdot)$ and $\eta_{K(T),8}(\cdot)$. The proof is given in Section 4.

Theorem 2.1 is probably far from being optimal. We include it in order to provide at least some rate estimates for all Lévy processes. Our main result is Theorem 2.2.

Theorem 2.2 *Suppose:*

(H1') *the function $f(\cdot)$ is of class $\mathcal{C}^4$; all derivatives up to order 4 of $f(\cdot)$ are bounded;*

(H2') *the function $g(\cdot)$ is of class $\mathcal{C}^4$ and moreover $|\partial_I g(x)| = \mathcal{O}(\|x\|^{M'})$ for $|I| = 4$ and some $M' \geq 2$;*

(H3') $\int_{\|x\|\geq 1} \|x\|^\gamma \nu(dx) < \infty$ *for* $2 \leq \gamma \leq M'^* := \max(2M', 8)$ *and* $X_0 \in L^{M'^*}(\Omega)$.

Then there exists an increasing function $K(\cdot)$ such that, for all $n \in \mathbb{N} - \{0\}$,

$$|\mathbb{E}g(X_T) - \mathbb{E}g(\bar{X}_T^n)| \leq \frac{\eta_{K(T),M'^*}(\infty)}{n} \ . \tag{9}$$

Suppose now:

(H1") *the function $f(\cdot)$ is of class $\mathcal{C}^8$; all derivatives up to order 8 of $f(\cdot)$ are bounded;*

(H2") *the function $g(\cdot)$ is of class $\mathcal{C}^8$ and moreover $|\partial_I g(x)| = \mathcal{O}(\|x\|^{M''})$ for $|I| = 8$ and some $M'' \geq 2$;*

(H3") $\int_{\|x\|\geq 1} \|x\|^\gamma \nu(dx) < \infty$ *for* $2 \leq \gamma \leq M''^* := 2\max(2M'', 16)$ *and* $X_0 \in L^{M''^*}(\Omega)$.

Then there exists a function $C(\cdot)$ and an increasing function $K(\cdot)$ such that, for any discretization step of type $\frac{T}{n}$, one has

$$\mathbb{E}g(X_T) - \mathbb{E}g(\bar{X}^n_T) = \frac{C(T)}{n} + R^n_T \tag{10}$$

and $\sup_n n^2|R^n_T| \leq \eta_{K(T),M''*}(\infty)$.

The proofs are given in Section 5.

The functions $C(\cdot)$ and $K(\cdot)$ depend on $g(\cdot)$, $f(\cdot)$ and moments of X_0. They can be described (we do this in the proofs of the theorems in Section 5), in terms of the solution of a Cauchy problem related to the infinitesimal generator of (X_t) and the derivatives of this solution.

We remark that if the first 4 (resp. 8) derivatives of $g(\cdot)$ are bounded, then $M' = M'' = 0$. Also, if the Lévy process Z has bounded jumps and X_0 is (for example) constant then (H3') and (H3") are automatically satisfied.

The main interest of establishing the expansion in the second half of Theorem 2.2 (compared with just an upper bound for the error) is to be able to apply the Romberg extrapolation technique:

Corollary 2.3 *Suppose (H1"), (H2") and (H3"). Let $X^{n/2}$ be the Euler scheme with step size $n/2$. Then*

$$|\mathbb{E}g(X_T) - \{2\mathbb{E}g(\bar{X}^{n/2}_T) - \mathbb{E}g(\bar{X}^n_T)\}| \leq \frac{K(T)}{n^2}\ .$$

The result is an immediate consequence of (10). The numerical cost of the Romberg procedure is much smaller than the cost corresponding to schemes of order n^{-2}. See [20] for a discussion and illustrative numerical examples for the case Z is a Brownian motion.

If $f(\cdot)$ and $g(\cdot)$ are smooth enough and ν has moments of all orders larger than 2, the arguments used in the proof can also be used to show that, for any integer $k > 0$, there exists constants $C_1, \ldots, C_{k+1}$ such that

$$\mathbb{E}g(X_T) - \mathbb{E}g(\bar{X}^n_T) = \frac{C_1}{n} + \frac{C_2}{n^2} + \ldots + \frac{C_k}{n^k} + R^n_T$$

and $\sup_n n^{k+1}|R^n_T| \leq C_{k+1}$.

Finally, we underline that no ellipticity condition is required on the infinitesimal generator of X.

Remark 2.4 *Theorems 2.1 and 2.2 are stated for a vector $Z = (Z^1, \ldots, Z^r)$ of driving semimartingales where Z is a Lévy process; however they also remain true if the driving semimartingales are strong Markov processes of a certain type. Indeed, Çinlar and Jacod [6] have shown that up to a random time change every semimartingale Hunt process* *can be represented as the solution of a stochastic differential equation driven by a Wiener process, Lebesgue measure, and a compensated Poisson random measure (see Theorem 3.35, p. 207). Our situation is more restrictive since we use Lévy processes, themselves semimartingales, rather than random measures. The difference is essentially this: the coefficient for the random measure term is of the form $k(x,z)$; if $k(x,z)=f(x)h(z)$ (i.e., if it factors), then the random measure term becomes equivalent to considering Lévy process differentials. We conclude then that a large class of semimartingale Hunt processes (essentially quasi left continuous strong Markov processes with technical regularity conditions) can be represented as solutions of SDE's driven by Lévy processes. Hence if Z is such a Hunt process we can write*

$$Z_t = Z_0 + \int_0^t g(Z_{s-})dY_s$$

where Y is a (vector) Lévy process and equation (1) can be rewritten

$$X_t = X_0 + \int_0^t f(X_{s-})g(Z_{s-})dY_s$$

and by passing to a larger system we obtain

$$X_t = X_0 + \int_0^t \hat{f}(X_{s-})dY_s$$

with a new coefficient $\hat{f}(\cdot)$.

Example 2.5 *Let $\tilde{Z}$ be a real valued Lévy process with no Brownian part such that its Lévy measure ν has a finite second moment. Then $\mathbb{E}\tilde{Z}_t$ and $\mathbb{E}(\tilde{Z}_t)^2$ are finite. Set $Z_t := \tilde{Z}_t - \mathbb{E}\tilde{Z}_t$, $f(x)=x$ and $g(x)=x^2$. An easy calculation shows that*

$$\mathbb{E}(X_t)^2 = \int x^2\nu(dx)\mathbb{E}\int_0^t \mathbb{E}(X_s)^2 ds \ , \ 0 \le t \le T \ ,$$

so that

$$\mathbb{E}(X_T)^2 = \exp\left(\int x^2\nu(dx)T\right) \ .$$

Similarly, one has

$$\mathbb{E}(\bar{X}_T^n)^2 = \left(1 + \frac{T}{n}\int x^2\nu(dx)\right)^n \ .$$

Thus, the rate of convergence is $\frac{1}{n}$. We conclude that Theorem 2.2 is optimal with respect to the rate of convergence, even with no Brownian component. One cannot a priori *hope this example is typical with Lévy processes with finite second moments, since it is the linear (or exponential) case, and thus the derivatives of $\mathbb{E}_x g(X_t)$ are zero for order three or higher: indeed, in the proof of Theorem 2.2 one can use this fact to eliminate several terms that effectively slow the rate.*

Example 2.6 *Let Z be a Lévy process which is a compound Poisson process with Lévy measure*

$$\nu(dx) = \mathbb{1}_{\mathbb{R}_+}(x)\frac{1}{1+x^9}dx \ .$$

(Thus ν does not have a finite 8^{th} moment and one cannot apply Theorem 2.2). Theorem 2.1 can still be used however and we have $\rho_8(m)$ is of order $\log(m)$ as m tends to infinity. Also $h(m)$ is of order $\frac{1}{m^8}$. Therefore Theorem 2.1 gives us a rate of convergence

$$\frac{m^{K(T)}}{n} + \frac{1}{m^8} \ .$$

We are free to choose m as a function of n, so let $m = n^\gamma$. The optimal choice of γ is $\frac{1}{8+K(T)}$ and we obtain a rate of convergence of $n^{-8/(8+K(T))}$, which may be only slightly worse than $\frac{1}{n}$. Note however that if ν were of the form

$$\nu(dx) = \mathbb{1}_{\mathbb{R}_+}(x)\frac{1}{1+x^8}dx \ ,$$

which of course is farther away from having 8 moments, analogous calculations yield a rate of convergence $\frac{1}{\log(n)^\gamma}$ for some $\gamma > 0$.

3 A Discussion on Simulation

If one considers a stochastic differential equation of the type

$$X_t = X_0 + \int_o^t \sigma(X_s)dW_s + \int_0^t b(X_s)ds$$

where (W_t) is a standard Wiener process, then to implement methods of the type considered here (using the Euler scheme) one needs to be able to simulate the increments of the Wiener process $W_{(k+1)T/n} - W_{kT/n}$. Since the Wiener process has independent increments, this amounts to having to simulate a (finite) i.i.d. sequence of normal random variables, for which efficient methods are well known.

In contrast, simulation problems for equations of type (1) can be formidable. It is perhaps first appropriate to discuss a little what a Lévy process is. By the independence and stationarity of the increments, we can write

$$Z_1 = \sum_{k=1}^{n}(Z_{(k+1)/n} - Z_{k/n}) \ ,$$

and thus Z_1 is the sum of n i.i.d. random variables for any n. Hence Z_1 is infinitely divisible (indeed, Z_t is infinitely divisible for all $t > 0$). Thus "knowing" Lévy processes can be equated with "knowing" infinitely divisible distributions. Many familiar classsical distributions are infinitely divisible such as the Normal, Gamma, Chi-squared, Cauchy, Laplace, Negative Binomial, Pareto, Logarithmic, Logistic, Compound Geometric, Student, Fisher, and Log-normal (that the last three are infinitely divisible is non trivial; see e.g. Steutel [16]). Goldie's theorem [8] allows one to generate such at will: the product UV of random variables is infinitely divisible if U is arbitrary but nonnegative, V is exponential, and U and V are independent.

From our standpoint, however, it is perhaps more appropriate to deal with Fourier transforms. Indeed, using the Lévy-Khintchine formula (see, e.g., Protter [14]), one can imagine a description of the process (Z_t) being given in applications by a description of the diffusive constant σ, a description of the drift constant β and a description of the behavior of the jumps (remember (3)). Since the Brownian component (W_t) and the jumps of the Lévy process Z are independent, we will treat here only the simulation of the jumps. Mathematically speaking, being given a description of the jumps is tantamount to being given the Lévy measure.

3.1 A finite Lévy measure ν.

The following is well known and elementary but we include a proof for the sake of completeness.

Theorem 3.1 *Assume (Z_t) is a Lévy process with no Brownian term and no drift term and a finite Lévy measure ν. Let $\lambda := \nu(\mathbb{R}^r)$. Then, (Z_t) is a compound Poisson process with jump arrival rate λ and its jumps have distribution $\frac{1}{\lambda}\nu$.*

Proof. Due to the independence and stationarity of the increments, the Lévy-Khintchine formula uniquely determines the distribution of the entire process (Z_t). We have

$$\mathbb{E}\left[e^{i<u,Z_t>}\right] = e^{-t\phi(u)} \ ,$$

where, for some $a \in \mathbb{R}^r$,

$$\phi(u) := \int_{\|x\|\geq 1}(1-e^{i<u,x>})\nu(dx) + \int_{\|x\|<1}(1-e^{i<u,x>} + i<u,x>)\nu(dx) + i<a,u> \ .$$

Let (N_t) be Poisson with arrival rate λ, and let T_j ($j \in \mathbb{N}$) be its arrival times. Let U_j be an i.i.d. sequence with $\mathcal{L}(U_j) = \mu(dx) = \frac{1}{\lambda}\nu(dx)$, and let

$$M_t^\lambda := \sum_{j=1}^{\infty} U_j \ \mathbb{1}_{[t\geq T_j]} \ .$$

Then

$$\begin{aligned}
\mathbb{E}\left[e^{iuM_t^\lambda}\right] &= \sum_k \mathbb{E}\left[exp(i<u,Z_t>)|N_t=k\right]\mathbb{P}(N_t=k) \\
&= \sum_{k=1}^{\infty}\mathbb{E}\left[\exp\left(i\sum_{j=1}^{k}<u,U_j>\right)\right]\mathbb{P}(N_t=k) \\
&= \exp\left(-t\int(1-e^{i<u,x>})\nu(dx)\right) \ ,
\end{aligned}$$

and the result follows. ∎

Thus if ν is a finite measure, we need only to simulate the increments of compound Poisson processes, and this too is well understood: the problem is reduced to the simulation of random variables having law $\frac{1}{\lambda}\nu$: for example, one can use a rejection method, see Bouleau and Lépingle [5] or Devroye [7].

3.2 A Lévy measure with a countable number of point masses.

Here we assume the Lévy measure is of the form

$$\nu(dx) = \tau(dx) + \sum_{k=1}^{\infty} \alpha_k \epsilon_{\beta_k}(dx) \ , \tag{11}$$

where $\epsilon_{\beta_k}(dx)$ denotes the point mass at $\beta_k \in \mathbb{R}$ of size 1; $\tau(dx)$ is a finite measure on $\mathbb{R}$ not including any point masses at the $\{\beta_k\}_{k\leq 1}$, and also we assume

$$\sum_{k=1}^{\infty} \beta_k^2 \alpha_k < \infty \ . \tag{12}$$

Note that without loss of generality we can assume $\beta_k \in [-\delta, \delta]$, all k, for some $\delta > 0$, since otherwise we can put the jumps into $\tau(dx)$. With this assumption the hypothesis (12) is automatically satisfied (and hence redundant) since all Lévy measures ν satisfy

$$\int_{\mathbb{R}} (x^2 \wedge 1)\nu(dx) < \infty \ .$$

Theorem 3.2 *Suppose (11)and (12) with $\tau = 0$. Let (N_t^k) be independent Poisson processes with parameters α_k. Then*

$$M_t := \sum_{k=1}^{\infty} \beta_k (N_t^k - \alpha_k t)$$

is a Lévy process with Lévy measure ν.

Proof. Let

$$M_t^n := \sum_{k=1}^{n} \beta_k (N_t^k - \alpha_k t) \ .$$

Then (M_t^n) is a square integrable martingale, and

$$\mathbb{E}[(M_t^n)^2] = \sum_{k=1}^{n} \beta_k^2 \alpha_k t \ .$$

Then $M := \lim_n M^n$ exists as a limit in $L^2(\Omega)$, and by Doob's martingale quadratic inequality $\lim_n M^n = M$ in $L^2(\Omega)$, uniformly in t on compacts; moreover M is also a martingale and a Lévy process. Finally note that

$$\begin{aligned}
\mathbb{E}\left[e^{iuM_t}\right] &= \lim_n \mathbb{E}\left[e^{iuM_t^n}\right] \\
&= \lim_n \mathbb{E}\left[e^{iu\sum_{k=1}^n \beta_k(N_t^k-\alpha_k t)}\right] \\
&= \lim_n \prod_{k=1}^n \mathbb{E}\left[e^{iu\beta_k(N_t^k-\alpha_k t)}\right] \\
&= \lim_n \prod_{k=1}^n e^{-t\phi_k(u)}
\end{aligned}$$

where

$$\phi_k(u) := \int (e^{iux} - 1 - iux)\alpha_k \epsilon_{\beta_k}(dx) \; . \; \blacksquare$$

Corollary 3.3 *Suppose (11) and (12) and set*

$$\lambda := \int \tau(dx) \; .$$

Then the process (Z_t) *has the form*

$$Z_t = H_t + J_t \; ,$$

where (H_t) *is a compound Poisson process with jumps having law* $\frac{1}{\lambda}\tau(dx)$ *and arrival intensity* λ*, and where* (J_t) *is independent of* (H_t) *and is of the form*

$$J_t := \sum_{k=1}^{\infty} \beta_k (N_t^k - \alpha_k t)$$

for (N_t^k) *independent Poisson processes of intensities* α_k*.*

Proof. This is simply a combination of Theorems 3.1 and 3.2. $\blacksquare$

The simulation problems here begin to get a little complicated. Clearly one will have to truncate the infinite series expression for J_t. We hope to address these issues in future work.

3.3 Symmetric Stable Processes.

Recall that a real valued Lévy process (Z_t) is called *stable* if for every $c > 0$ there exists $a > 0$ and $b \in \mathbb{R}$ such that the process (cZ_t) has the same law as the process $(Z_{at} + bt)$. If one takes $b = 0$ then (Z_t) is *strictly stable*. It follows from the Lévy-Khintchine formula that if (Z_t) is stable then $a = c^\alpha$, for some α, $0 < \alpha < 2$. The constant α thus determines the process and it is called the order of the process. In this case the Lévy measure takes the form

$$\nu(dx) = (m_1 \, \mathbb{1}_{x<0} + m_2 \, \mathbb{1}_{x>0})|x|^{-(1+\alpha)} dx$$

for $0 < \alpha < 2$, $m_1 \geq 0$, $m_2 \geq 0$. If $m_1 = m_2$ then (Z_t) is called a *symmetric stable process*.

If $0 < \alpha < 1$, then the densities of some stable random variables are known "explicitly". Indeed, let $p(\cdot, \alpha)$ denote the density on $[0, +\infty)$ of a stable random variable with Laplace transform $\exp(-s^\alpha)$, for $s > 0$. The corresponding Lévy processes are known as *stable subordinators*, and they have non-decreasing sample paths. Note that if $U_1, \ldots, U_n$ are i.i.d. random variables with density $p(\cdot, \alpha)$ having Laplace transform $\exp(-s^\alpha)$, then $n^{-1/\alpha} \sum_{j=1}^n U_j$ also has density $p(\cdot, \alpha)$, whence $p(\cdot, \alpha)$ is the density of a stable law of index α (cf, e.g., p.110 in Revuz & Yor [15]). In this case for $x \geq 0$, $p(x, \alpha)$ is given by (see Kanter [10]):

$$p(x, \alpha) = \frac{1}{\pi} \left(\frac{\alpha}{1-\alpha} \right) \left(\frac{1}{x} \right)^{1/(1-\alpha)} \int_0^\pi a(z) \exp\left(- \left(\frac{1}{x} \right)^{\frac{\alpha}{1-\alpha}} a(z) \right) dz \tag{13}$$

where

$$a(z) := \left(\frac{sin(\alpha z)}{sin(z)} \right)^{1/(1-\alpha)} \left(\frac{sin((1-\alpha)z)}{sin(\alpha z)} \right) . \tag{14}$$

Theorem 3.4 *Let (Z_t) be a vector valued symmetric strictly stable process of index α, $0 < \alpha < 2$, and let Σ be a symmetric positive matrix such that*

$$\mathbb{E} \left[e^{i<u, Z_t>} \right] = e^{-t<\Sigma u, u>^\alpha} .$$

Then

$$\mathcal{L}aw(Z_t - Z_s) = \mathcal{L}aw \left((t-s)^{1/(2\alpha)} V^{1/2} G \right)$$

where $\mathcal{L}aw(G) = \mathcal{N}(0, \Sigma)$, V is independent of G and

$$V = \left(\frac{a(U)}{L} \right)^{\frac{1-\alpha}{\alpha}} ,$$

where U is uniform on $[0, \pi]$; L is exponential of parameter 1; U and L are independent; the function $a(\cdot)$ is given in (14).

Proof. It is well known that (Z_t) has the representation

$$(Z_t) = (W_{Y_t}) ,$$

where Y is a stable subordinator of index $\frac{\alpha}{2}$, and (W_t) is an independent standard Wiener process (see, e.g., page 111 in [15]). As Herman Rubin observed (see Corollary 4.1 of [10], p.703), the function $p(\cdot,\alpha)$ of (13) is the density of $(a(U)/L)^{1-\alpha/\alpha}$ where $a(\cdot)$ is given in (14); U is uniform on $[0,\pi]$; L is exponential of parameter 1; U and L are independent. Therefore

$$\mathcal{L}aw(Y_1) = \mathcal{L}aw\left((a(U)/L)^{(1-\alpha/\alpha)}\right) ,$$

and by scaling we have

$$\mathcal{L}aw(Y_1) = \mathcal{L}aw\left((t-s)^{-1/\alpha}Y_{t-s}\right) = \mathcal{L}aw\left((t-s)^{-1/\alpha}(Y_t - Y_s)\right) .$$

Since

$$\begin{aligned}
\mathcal{L}aw(Z_t - Z_s) &= \mathcal{L}aw\left(W_{Y_t} - W_{Y_s}\right) \\
&= \mathcal{L}aw\left(W_{Y_t - Y_s}\right) \\
&= \mathcal{L}aw\left(\sqrt{Y_t - Y_s}G\right) \\
&= \mathcal{L}aw\left(\sqrt{(t-s)^{1/\alpha}V}G\right) ,
\end{aligned}$$

we are done. ∎

Note that Theorem 3.4 implies that in order to simulate the increments of a strictly stable symmetric process of index α, it is enough to simulate three independent random variables: a Gaussian, an exponential and a uniform.

3.4 The case $\nu(\mathbb{R}) = \infty$.

We have already treated two cases where $\nu(\mathbb{R}) = \infty$: first, the case where the infinite mass comes only from the contribution of point masses (subsection 3.2); and second, the case of symmetric stable processes. In certain cases one knows what process corresponds to an infinite Lévy measure, and also one knows how to simulate the increments of such a process. Such examples are rare! The most well known is the *Gamma process*: a Lévy process (Z_t) is called a Gamma process if

$$\mathcal{L}aw\left(Z_t\right) = \Gamma(1,t) , \ \forall t > 0 .$$

That is, the law of Z_t has density

$$p(x) = \frac{x^{1-t}e^{-x}}{\Gamma(t)} \, \mathbb{1}_{x>0} .$$

Its characteristic function is

$$\mathbb{E}\left[e^{iuZ_t}\right] = \frac{1}{(1-iu)^t}$$

which is clearly infinitely divisible since

$$\frac{1}{(1-iu)^t} = \left(\frac{1}{(1-iu)^{t/n}}\right)^n , \ \forall n \geq 1 .$$

One can then calculate the Lévy measure to be

$$\nu(dx) = \frac{1}{x} e^{-x} \, \mathbb{1}_{x>0} dx \ .$$

Thus reasoning backwards, if one knows

$$\nu(dx) = \frac{1}{x} e^{-x} \, \mathbb{1}_{x>0} dx \ ,$$

one can simulate the increments of (Z_t) by simulating gamma random variables. For such random variables many techniques are known. See, for example, p. 379 in Bouleau [4].

If one is not so lucky as to be given ν corresponding to a known (and nice) process, various other techniques are possible. We plan to present these in subsequent work.

4 Proof of Theorem 2.1

4.1 Preliminary remarks.

In order to avoid having to treat the case where Z reduces to being continuous (which was the case studied in [20]), from now on we suppose:

(H0) the discontinuous part of Z is not the null process.

A naive copy of the arguments in [20] would involve estimates on the moments of the increments of Z which were they to hold, would imply by Kolmogorov's lemma that Z had continuous paths. Since we are assuming Z has jumps, such estimates do not exist.

We introduce an intermediate process Z^m defined by

$$Z_t^m := Z_t - \sum_{0<s\leq t} \Delta Z_s \, \mathbb{1}_{\|\Delta Z_s\|>m} \ .$$

Note that Z^m is a Lévy process (see Theorem 36 of Chapter 1 in Protter [14] e.g.), therefore (see Chapter 6 in [14] e.g.) the process (X_t^m) which is a solution to

$$dX_t^m = f(X_{t-}^m) dZ_t^m$$

is also a Markov process. Applying the Euler scheme to (X_t^m), we define a discrete time process $(X_t^{m,n})$.

Decompose the global discretization error into three terms:

$$\begin{aligned} |\mathbb{E}g(X_T) - \mathbb{E}g(X_T^n)| &\leq |\mathbb{E}g(X_T) - \mathbb{E}g(X_T^m)| \\ &\quad + |\mathbb{E}g(X_T^m) - \mathbb{E}g(X_T^{m,n})| + |\mathbb{E}g(X_T^{m,n}) - \mathbb{E}g(X_T^n)| \\ &=: A_1 + A_2 + A_3 \ . \end{aligned} \tag{15}$$

Before bounding from above the A_i's, we need some intermediate results.

We start by a technical lemma. It appears in a more general setting in Bichteler and Jacod [3] with a proof for $Q = 2^i$, i an integer, and a slightly different result is proven in Bichteler [2] (p. 536). We give a detailed proof here for the sake of completeness.

Lemma 4.1 *Let Q be a real number with $Q \geq 2$. Let $\mathcal{L}(Q)$ be the class of Lévy processes L such that $L_0 = 0$ and the Lévy measures ν_L have moments of order q with $2 \leq q \leq Q$. Let $\mathcal{H}(Q)$ be the class of predictable processes H such that*

$$\mathbb{E}\left[\int_0^T \|H_s\|^Q ds\right] < \infty \ . \tag{16}$$

For $L \in \mathcal{L}(Q)$ we rewrite (3) as follows:

$$L_t = \sigma_L W_t + b_L t + \int_{\|x\|<1} x(N_t(\omega, dx) - t\nu_L(dx)) + \sum_{0<s\leq t} \Delta L_s \ \mathbb{1}_{[\|\Delta L_s\|\geq 1]} \ . \tag{17}$$

There exists an increasing function $K_Q(\cdot)$ depending on the dimension of L such that, for any $L \in \mathcal{L}(Q)$, for any $H \in \mathcal{H}(Q)$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t H_s dL_s\right|\right]^Q \\
&\leq \ K_Q(T)\left[\| \ b_L \ \|^Q + \| \ \sigma_L \ \|^Q + \left(\int \|z\|^2\nu_L(dz)\right)^{Q/2} + \int \|z\|^Q\nu_L(dz)\right]\int_0^T \mathbb{E}\|H\|_s^Q ds
\end{aligned} \tag{18}$$

Proof. We give the case for L one dimensional.

It is clear that without loss of generality we can suppose $\sigma = 0$ (for Brownian stochastic integrals the inequality (18) is classical). Since ν_L has a second moment we know that $\mathbb{E}|L_t|^2 < \infty$. Let β_L be such that $\mathbb{E}L_t := \beta_L t$. Then $(L_t - \beta_L t)$ is a martingale. For $L_t = \beta_L t$ the inequality (18) obviously holds. Thus we consider the case $\beta_L = 0$, that is, L is a martingale.

In the computations below, the constants C_p and the functions $K_p(\cdot)$ vary from line to line.

Choose the rational number k such that $2^k \leq p < 2^{k+1}$. Applying Burkholder's inequality for $p \geq 2$ we have

$$\mathbb{E}\left|\int_0^t H_s dL_s\right|^p \leq (4p)^p \mathbb{E}\left|\int_0^t |H_s|^2 d[L,L]_s\right|^{p/2} \ . \tag{19}$$

Set

$$\alpha_L := \mathbb{E}[L,L]_1 = \mathbb{E}\left\{\sum_{s\leq 1}(\Delta L_s)^2\right\} = \int |x|^2\nu_L(dx) < \infty \ .$$

Since $[L,L]$ is also a Lévy process, we have that $[L,L]_t - \alpha_L t$ is also a martingale. Therefore (19) becomes:

$$\mathbb{E}\left|\int_0^t H_s dL_s\right|^p \leq C_p \mathbb{E}\left|\int_0^t |H_s|^2 d([L,L]_s - \alpha_L s)\right|^{p/2} + K_p(t)\alpha_L^{p/2}\mathbb{E}\left|\int_0^t |H_s|^2 ds\right|^{p/2} . \quad (20)$$

We apply Burkholder's inequality again to the first term on the right side of (20) to obtain:

$$\mathbb{E}\left|\int_0^t H_s dL_s\right|^p \leq C_p \mathbb{E}\left\{\sum_{s\leq t}|H_s \Delta L_s|^4\right\}^{p/4} + K_p(t)\left(\int |x|^2 \nu_L(dx)\right)^{p/2} \mathbb{E}\int_0^t |H_s|^p ds .$$

We continue recursively to get

$$\begin{aligned}\mathbb{E}\left|\int_0^t H_s dL_s\right|^p &\leq C_p \mathbb{E}\left\{\sum_{s\leq t}|H_s\Delta L_s|^{2^{k+1}}\right\}^{p/2^{k+1}} \\ &\quad + K_p(t)\left(\sum_{i=1}^k \left[\int |x|^{2^i}\nu_L(dx)\right]^{p2^{-i}}\right)\int_0^t |H_s|^p ds . \qquad (21)\end{aligned}$$

Next we use the fact that, for any sequence a such that $\|a\|_{l^q}$ is finite, $\|a\|_{l^2} \leq \|a\|_{l^q}$ for $1 \leq q \leq 2$. As $1 \leq \frac{p}{2^k} < 2$ we get:

$$\begin{aligned}\left\{\sum_{s\leq t}|H_s\Delta L_s|^{2^{k+1}}\right\}^{p/2^{k+1}} &= \left\{\sum_{s\leq t}\left(|H_s\Delta L_s|^{2^k}\right)^2\right\}^{\frac{1}{2}\frac{p}{2^k}} \\ &\leq \sum_{s\leq t}|H_s\Delta L_s|^p\end{aligned}$$

whence

$$\mathbb{E}\left\{\sum_{s\leq t}|H_s\Delta L_s|^{2^{k+1}}\right\}^{p/2^{k+1}} \leq \mathbb{E}\sum_{s\leq t}|H_s\Delta L_s|^p .$$

Note that $\sum_{s\leq t}|\Delta L_s|^p$ is an increasing, adapted, càdlàg process, and its compensator is $t\int |x|^p \nu_L(dx)$, which is finite by hypothesis. Since $|H|^p$ is a predictable process,

$$\left(\int_0^t |H_s|^p d\left(\sum_{r\leq s}|\Delta L_r|^p - s\int |x|^p \nu_L(dx)\right)\right)$$

is a martingale with zero expectation. Therefore (21) yields:

$$\mathbb{E}\left|\int_0^t H_s dL_s\right|^p \leq \left[C_p\int |x|^p\nu_L(dx) + K_p(t)\sum_{i=1}^k\left(\int |x|^{2^i}\nu_L(dx)\right)^{p2^{-i}}\right]\mathbb{E}\int_0^t |H_s|^p ds .$$

It remains to show that, for any $1 \leq i \leq k$,

$$\left(\int |x|^{2^i}\nu_L(dx)\right)^{p2^{-i}} \leq \left(\int |x|^2\nu_L(dx)\right)^{p/2} + \int |x|^p\nu_L(dx)\ .$$

Let $\lambda_L := \int |x|^2\nu_L(dx)$, so that

$$\mu_L(dx) := \frac{1}{\lambda_L}|x|^2\nu_L(dx)$$

is a probability measure. Denote 2^i by q. One has to show:

$$\lambda_L^{p/q}\left(\int |x|^{q-2}\mu_L(dx)\right)^{p/q} \leq \lambda_L^{p/2} + \lambda_L\int |x|^{p-2}\mu_L(dx)\ . \tag{22}$$

If

$$\left(\int |x|^{q-2}\mu_L(dx)\right)^{p/q} \leq \lambda_L^{p/2-p/q}$$

the inequality (22) is obvious. On the other hand, if

$$\lambda_L \leq \left(\int |x|^{q-2}\mu_L(dx)\right)^{2/(q-2)}$$

then it is sufficient to prove that

$$\lambda_L^{p/q-1}\left(\int |x|^{q-2}\mu_L(dx)\right)^{p/q} \leq \int |x|^{p-2}\mu_L(dx)\ .$$

But the bound on λ_L and Jensen's inequality give the result. ∎

The preceding lemma leads to bounds for the derivatives of the flows $x \to X^m(x,t,\omega)$.

Lemma 4.2 *We assume (H1).*

For any multiple index I denote by $\partial_I X_t^m(\cdot,\omega)$ the derivative of order I of the flow $x \to X_t^m(x,\omega)$. Then, for any integer p, there exists a strictly increasing function $K_p(\cdot)$ such that for any multi-index I with length $|I| \leq 4$,

$$I\!\!E|\partial_I X^m(x,t,\omega)|^{2p} \leq \eta_{K_p(T),2p|I|}(m)\ . \tag{23}$$

Proof. Let ν^m be the Lévy measure of the process Z^m.

Let DX_t^m denote the Jacobian matrix of the stochastic flow $X_t^m(\cdot,\omega)$. It solves (see Theorems 39 and 40 in Chapter 5 of Protter [14], e.g.):

$$DX_t^m = Id + \sum_{\alpha=1}^r \int_0^t \nabla f_\alpha(X_{s-}^m)DX_{s-}^m d(Z_s^m)^\alpha\ .$$

Lemma 4.1 shows that there exists an increasing function $K_p(\cdot)$ depending only on d, r, p and the L^∞-norm of the first derivatives of $f(\cdot)$ such that

$$\begin{aligned} \mathbb{E}|(DX_t^m)_k^i|^{2p} \;\; &\leq \;\; 1 + K_p(T) \Big[\|\beta\|^{2p} + \|\sigma\|^{2p} \\ &\qquad + \left(\int \|z\|^2 d\nu^m(z) \right)^p + \int \|z\|^{2p} d\nu^m(z) \Big] \int_0^t \mathbb{E}|(DX_s^m)_k^i|^{2p} ds \ . \end{aligned}$$

Gronwall's lemma leads to

$$\mathbb{E}\left[\sup_{0 \leq s \leq t} |(DX_s^m)_k^i|^{2p} \right] \leq \eta_{K_p(T),2p}(m)$$

(with a possible change of the function $K_p(\cdot)$).

We then write the stochastic differential system satisfied by the flow $X_t^m(\cdot,\omega)$ and its derivatives up to order 2. The preceding estimate and a new application of Gronwall's lemma provide the estimate for $|I| = 2$.

The conclusion is obtained by successive differentiations of the flow. ∎

Corollary 4.3 *Assume (H1) and (H2).*

Set

$$v^m(t,x) := \mathbb{E}_x g(X_{T-t}^m) \ . \tag{24}$$

Then, there exists an increasing function $K(\cdot)$ such that for any multi-index I with $|I| \leq 4$,

$$|\partial_I v^m(t,x)| \leq \eta_{K(T),8}(m) \ . \tag{25}$$

Proof. For $I = i \in \{1, \ldots, d\}$ one has

$$\partial_i v^m(t,x) = \mathbb{E}_x[DX_{T-t}^m \partial g(X_{T-t}^m)] \tag{26}$$

from which

$$|\partial_i v^m(t,x)| \leq C\mathbb{E}_x \|DX_{T-t}^m\| \leq C\sqrt{\mathbb{E}_x \|DX_{T-t}^m\|^2}$$

where $\|\cdot\|$ stands for any of the equivalent norms on the space of $d \times d$ matrices. Thus, Lemma 4.2 induces (25) for $|I| = 1$.

The conclusion is then obtained by successive differentiations from (26). ∎

4.2 An upper bound for $A_1+A_3 = |\mathbb{E}g(X_T)-\mathbb{E}g(X_T^m)|+|\mathbb{E}g(X_T^n)-\mathbb{E}g(X_T^{m,n})|$.

The objective of this subsection is to prove:

Proposition 4.4 *Suppose (H1) and (H2). Then*

$$A_1 + A_3 \leq 4\|g\|_{L^\infty(\mathbb{R}^d)}(1-\exp(-h(m)T)) \ , \tag{27}$$

where the function $h(\cdot)$ is as in (7).

Proof. For $m>0$ define

$$T^m := \inf\{t>0 \ : \ \|\Delta Z_t\| > m\} \ . \tag{28}$$

One has, since $X_t^m = X_t$ for $t \leq T_m$,

$$\begin{aligned}
A_1 &= \left| \mathbb{E}\left[(g(X_T) - g(X_T^m)) \ \mathbb{1}_{[T^m \leq T]} \right] \right| \\
&\leq 2\|g\|_{L^\infty(\mathbb{R}^d)} \mathbb{P}(T^m \leq T) \\
A_3 &= \left| \mathbb{E}\left[(g(X_T^{m,n}) - \mathbb{E}g(X_T^n)) \ \mathbb{1}_{[T^m \leq T]} \right] \right| \\
&\leq 2\|g\|_{L^\infty(\mathbb{R}^d)} \mathbb{P}(T^m \leq T) \ .
\end{aligned}$$

The conclusion follows from the next Proposition. ∎

Proposition 4.5 *Let L be a Lévy process with Lévy measure ν_L. Set*

$$T^m = \inf\{t>0 \ : \ \|\Delta L_t\| > m\} \ .$$

For all $m>0$, it holds that

$$\mathbb{P}(T^m > T) = \exp\left(-T\nu_L\{x; \|x\| \geq m\}\right) \ . \tag{29}$$

Proof. We recall that T is a fixed non-random time denoting the endpoint of our time interval.

We truncate the jumps of L from below. For $m>0$ and $0<\delta<1$ we define

$$\hat{L}_t^{\delta m} := \sum_{0 \leq s < t} \Delta L_s \ \mathbb{1}_{[\|\Delta L_s\| > \delta m]} \ .$$

Set

$$\hat{T}^{\delta m} := \inf\{t>0; \|\Delta \hat{L}_t^{\delta m}\| > m\} \ .$$

Then,

$$\mathbb{P}[T^m > T] = \mathbb{P}[\hat{T}^{\delta m} > T] \ .$$

Theorem 3.1 implies that $\hat{L}^{\delta m}$ is a compound Poisson process with jump arrival rate

$$\lambda^{\delta m} := \nu_L\{x; \|x\| \geq \delta m\} \ .$$

We set

$$\hat{L}_t^{\delta m} := \sum_{i=1}^{\infty} U_i^{\delta m} \ \mathbb{1}_{[T_i^{\delta m} \leq t]}$$

and

$$N_t^{\delta m} := \sum_{i=1}^{\infty} \ \mathbb{1}_{[T_i^{\delta m} \leq t]} \ .$$

Thus $N^{\delta m}$ is a standard Poisson process with arrival rate $\lambda^{\delta m}$. Set

$$\alpha^{\delta m} := \mathbb{P}[\|U_1^{\delta m}\| \leq m] = \frac{1}{\lambda^{\delta m}} \nu_L\{x; \delta m \leq \|x\| \leq m\} \ .$$

Thus,

$$\begin{aligned}
\mathbb{P}[T^m > T] &= \sum_k \mathbb{P}\left[\cap_{i=1}^k \|U_i^{\delta m}\| \leq m | N_T^{\delta m} = k\right] \mathbb{P}[N_T^{\delta m} = k] \\
&= \sum_k \mathbb{P}\left[\cap_{i=1}^k \|U_i^{\delta m}\| \leq m\right] \exp(-\lambda^{\delta m} T) \frac{(\lambda^{\delta m} T)^k}{k!} \\
&= \exp(-\lambda^{\delta m} T) \sum_k \frac{(\alpha^{\delta m} \lambda^{\delta m} T)^k}{k!} \\
&= \exp(-\lambda^{\delta m} T(1 - \alpha^{\delta m})) \\
&= \exp\left(-T \nu_L\{x; \|x\| \geq m\}\right) \ ,
\end{aligned}$$

which is independent of the choice of δ. ∎

Note that in this subsection the boundedness of the function $g(\cdot)$ was essential. This is not surprising: except when the jumps of Z are bounded or have finiteness properties reflected by ν having finite moments, in general the law of X_T has no moments. A contrario we will not use the boundedness of $g(\cdot)$ to bound A_2 from above.

4.3 An upper bound for $A_2 = |\mathbb{E} g(X_T^m) - \mathbb{E} g(X_T^{m,n})|$.

The objective of this subsection is to prove the following

Proposition 4.6 *Assume (H1), (H2) and (H3) hold.*

Let $m \in \mathbb{N}$, $m \geq 1$, and $p \in \mathbb{N}$. Then for some increasing function $K(\cdot)$ depending only on X_0, the dimensions d, r and on the L^∞-norm of the partial derivatives of $f(\cdot)$ and $g(\cdot)$ up to order 4, one has

$$\forall (m,n) \in \mathbb{N} - \{0\} \times \mathbb{N} - \{0\} \ , \ A_2 = |\mathbb{E}g(X_T^m) - \mathbb{E}g(X_T^{m,n})| \leq \frac{\eta_{K(T),8}(m)}{n} \ , \qquad (30)$$

where the function $\eta_{K(T),8}(\cdot)$ is as in (6).

Proof. It is useful (see [18], [20]) to modify the original approximation problem in the estimation of the difference $\mathbb{E}v^m(T, X_T^m) - \mathbb{E}v^m(T, \bar{X}_T^{m,n})$ in terms of

$$\mathbb{E}v^m(T - T/n, X_{T-T/n}^m) - \mathbb{E}v^m(T - T/n, \bar{X}_{T-T/n}^{m,n}) \ .$$

It can be checked using the Meyer-Itô formula that the function $v^m(t, \cdot)$ defined in (24) solves

$$\begin{cases} (\partial_0 v^m + A^m) v^m(t,x) = 0 \ , \ 0 \leq t < T \ , \\ \\ v^m(T, \cdot) = g(\cdot) \ , \end{cases} \qquad (31)$$

where A^m is the infinitesimal generator of the process (X_t^m): A^m is of the type (2) with ν^m instead of ν.

In view of (31), $\partial_{00} v^m(t,x) = -A^m(A^m v^m(t,x))$, so that, by (25),

$$\|\partial_{00} v^m(t,x)\|_{L^\infty([0,T]\times \mathbb{R}^d)} \leq \eta_{K(T),8}(m) \ .$$

Therefore, one has

$$\begin{aligned} \mathbb{E}v^m(T, \bar{X}_T^{m,n}) &= \mathbb{E}v^m(T - T/n, \bar{X}_T^{m,n}) + \frac{T}{n} \mathbb{E}\partial_0 v^m(T - T/n, \bar{X}_T^{m,n}) + R_{T-T/n}^{m,n} \\ &= \mathbb{E}v^m(T - T/n, \bar{X}_T^{m,n}) - \frac{T}{n} \mathbb{E}A^m v^m(T - T/n, \bar{X}_T^{m,n}) + R_{T-T/n}^{m,n} (32) \end{aligned}$$

with

$$|R_{T-T/n}^{m,n}| \leq \frac{\eta_{K(T),8}(m)}{n^2} \ .$$

We now are going to expand the right side of (32) around $\bar{X}_{T-T/n}^{m,n}$ in order to prove:

$$\mathbb{E}v^m(T, \bar{X}_T^{m,n}) = \mathbb{E}v^m(T - T/n, \bar{X}_{T-T/n}^{m,n}) + S_{T-T/n}^{m,n}$$

with

$$|S_{T-T/n}^{m,n}| \leq \frac{\eta_{K(T),8}(m)}{n^2} \ .$$

If Z^m were a Brownian motion, this could be done by simply making a Taylor expansion using the fact that, for $p > 1$, $\mathbb{E}|W_T - W_{T-T/n}|^{2p}$ is smaller than n^{-2}. In the general case, this does not apply: any moment of $Z_T^m - Z_{T-T/n}^m$ is of order $1/n$ (otherwise Z would of necessity have continuous paths by Kolmogorov's lemma). We proceed in a different way, using the Markov property of Z^m.

Let $\tilde{Z}^m$ denote the Lévy process $(Z^m_{s+T-T/n} - Z^m_{T-T/n}\ ,\ 0 \leq s \leq T/n)$ and let $\tilde{G}^m$ denote its infinitesimal generator. For any function $\psi(\cdot)$ of class $\mathcal{C}^2_b(\mathbb{R}^d)$, Dynkin's formula holds:

$$\begin{aligned}
\mathbb{E}\psi(\tilde{Z}^m_{T/n}) &= \psi(0) + \int_0^{\frac{T}{n}} \mathbb{E}\tilde{G}^m(\tilde{Z}^m_s)ds \\
&= \psi(0) + \sum_i \beta_i \int_0^{\frac{T}{n}} \mathbb{E}\partial_i\psi(\tilde{Z}^m_s)ds + \frac{1}{2}\sum_{i,j}(\sigma\sigma^*)^i_j \int_0^{\frac{T}{n}} \mathbb{E}\partial_{ij}\psi(\tilde{Z}^m_s)ds \\
&\quad + \mathbb{E}\int_0^{\frac{T}{n}}\int_{\mathbb{R}^r}\left\{\psi(\tilde{Z}^m_s + y) - \psi(\tilde{Z}^m_s) - \sum_j \partial_j\psi(\tilde{Z}^m_s)y_j\ \mathbb{1}_{[|y|\leq 1]}\right\}\nu^m(dy)ds
\end{aligned} \tag{33}$$

Now, each subexpression of the right side of the above equality is considered as a function of $\tilde{Z}^m_s$ and, supposing that $\psi(\cdot)$ is of class $\mathcal{C}^4_b(\mathbb{R}^d)$, we make a first-order Taylor expansion around 0; remembering the definition (5), we observe that

$$\|\mathbb{E}\tilde{Z}^m_s\| \leq s\left(\|\beta\| + \int_{[1\leq\|z\|\leq m]}\|z\|\nu(dz)\right) \leq \rho_2(m)s \tag{34}$$

and that

$$\mathbb{E}\|\tilde{Z}^m_s\|^2 \leq \rho_2(m)(s+s^2)\ . \tag{35}$$

We thus obtain

$$\mathbb{E}\psi(\tilde{Z}^m_{T/n}) = \psi(0) + \frac{T}{n}\tilde{G}^m\psi(0) + \tilde{R}^{m,n}\ , \tag{36}$$

with

$$\mathbb{E}|\tilde{R}^{m,n}| \leq \frac{\eta_{K(T),2}(m)}{n^2}\sum_{1\leq|I|\leq 4}\|\partial_I\psi\|_{L^\infty(\mathbb{R}^d)} \tag{37}$$

for some increasing function $K(\cdot)$ uniform with respect to $\psi(\cdot)$, β, σ, ν and n.

Choose

$$\psi^m(z) := v^m\left(T - T/n, \bar{X}^{m,n}_{T-T/n} + f\left(\bar{X}^{m,n}_{T-T/n}\right)z\right)\ .$$

This function $\psi^m(\cdot)$ of course is of class $\mathcal{C}^4_b(\mathbb{R}^d)$ as a consequence of the hypotheses, and (36) can be used. We get:

$$\mathbb{E}v^m\left(T - T/n, \bar{X}^{m,n}_T\right) = \mathbb{E}v^m\left(T - T/n, \bar{X}^{m,n}_{T-T/n}\right) + \frac{T}{n}\mathbb{E}A^m v^m\left(T - T/n, \bar{X}^{m,n}_{T-T/n}\right) + \bar{R}^{m,n}_{T-T/n} \tag{38}$$

with (we use (25))

$$\mathbb{E}|\bar{R}^{m,n}_{T-T/n}| \leq \frac{\eta_{K(T),2}(m)}{n^2}\sum_{1\leq|I|\leq 4}\|\partial_I v^m(T-T/n,\cdot)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{\eta_{K(T),8}(m)}{n^2}\ .$$

We now come back to (32), use (38), make a first-order Taylor expansion around 0 of

$$z \rightarrow A^m v^m \left(T - T/n, \bar{X}^{m,n}_{T-T/n} + f\left(\bar{X}^{m,n}_{T-T/n}\right) z\right)$$

and use (34), (35). We obtain:

$$\mathbb{E} v^m(T, \bar{X}^{m,n}_T) = \mathbb{E} v^m(T - T/n, \bar{X}^{m,n}_{T-T/n}) + S^{m,n}_{T-T/n}$$

with

$$|S^{m,n}_{T-T/n}| \leq \frac{\eta_{K(T),8}(m)}{n^2} .$$

Proceeding in the same way to expand $\mathbb{E} v^m(T - T/n, \bar{X}^{m,n}_{T-T/n})$ around $\mathbb{E} v^m(T - 2T/n, \bar{X}^{m,n}_{T-2T/n})$, and so on, one finally gets

$$\begin{aligned}
\mathbb{E} g(\bar{X}^{m,n}_T) &= \mathbb{E} v^m(T, \bar{X}^{m,n}_T) = \mathbb{E} v^m(0, X^{m,n}_0) + \sum_{k=0}^{n-1} S^{m,n}_{kT/n} \\
&= \mathbb{E} v^m(0, X^m_0) + \sum_{k=0}^{n-1} S^{m,n}_{kT/n} \\
&= \mathbb{E} v^m(T, X^m_T) + \sum_{k=0}^{n-1} S^{m,n}_{kT/n} \\
&= \mathbb{E} g(X^m_T) + \sum_{k=0}^{n-1} S^{m,n}_{kT/n} , \qquad (39)
\end{aligned}$$

with

$$|S^{m,n}_{kT/n}| \leq \frac{\eta_{K(T),8}(m)}{n^2} .$$

Thus, one has

$$|\mathbb{E} g(X^m_T) - \mathbb{E} g(\bar{X}^{m,n}_T)| \leq \frac{\eta_{K(T),8}(m)}{n} .$$

■

5 Proof of Theorem 2.2

5.1 Preliminary remarks.

We start by two lemmas.

The following lemma is given in Bichteler and Jacod [3] in a more general context. Due to its importance for our results, we include it here.

Lemma 5.1 *Let $p \in \mathbb{N}$, $p \geq 2$. Suppose that $\int \|z\|^p \nu(dz) < \infty$ and that $f(\cdot)$ is Lipschitz. Then the solution X of (1) is in $L^p(\Omega)$ and*

$$\mathbb{E}[\sup_{0<s\leq t} \|X_s\|^p] \leq \eta_{K(T),p}(\infty)(1+\mathbb{E}\|X_0\|^p) \ . \tag{40}$$

Proof. We know by the general theory (see, e.g., [14]) that equation (1) has a solution and it is unique. Let X denote the solution with the convention $X_{0-}=0$ and define

$$T^k := \inf\{t>0 \ ; \ \|X_t\| > k\} \ .$$

Let

$$X_t^{T^k-} := X_t \ 1\!\!1_{t<T^k} + X_{T^k-} \ 1\!\!1_{t\geq T^k} \ .$$

Then $X^{T^k-} = X$ on $[0,T^k) \cap [\|X_0\| \leq k]$ and moreover the T^k's are increasing with $\lim_{k\to\infty} T^k = \infty$ a.s.

The hypothesis on ν allows us to apply Lemma 4.1 to deduce

$$\begin{aligned}
\mathbb{E}[\sup_{0<s\leq t} \|X_s^{T^k-}\|^p] &= C_p \left(\mathbb{E}\left[\sup_{0<s\leq t} \left\| \int_0^s f(X_{\theta-}^{T^k-}) dZ_\theta \right\|^p \right] + \mathbb{E}\|X_0\|^p \right) \\
&\leq \rho_p(\infty) \int_0^T \mathbb{E}\|f(X_{\theta-}^{T^k-})\|^p d\theta + C_p \mathbb{E}\|X_0\|^p
\end{aligned}$$

where the right side is finite, because $\|X^{T^k-}\| \leq k$, and $f(\cdot)$ is continuous. Since $f(\cdot)$ is Lipschitz,

$$\|f(X_{\theta-}^{T^k-})\| \leq C(f) \left(\|f(0)\| + \|X_{\theta-}^{T^k-}\| \right)$$

and applying Gronwall's lemma we have

$$\mathbb{E}\left[\sup_{0<s\leq t} \|X_s^{T^k-}\|^p \right] \leq \eta_{K(T),p}(\infty)(1+\mathbb{E}\|X_0\|^p) \ .$$

The right side is independent of k, so Fatou's lemma gives the result. ∎

In view of the preceding lemma, our proof of Corollary 4.3 can be rewritten to get:

Corollary 5.2 *Assume (H1'), (H2') and (H3') (resp. (H2") and (H3")).*

Set

$$v(t,x) := \mathbb{E}_x g(X_{T-t}) \ . \tag{41}$$

Then, there exists an increasing function $K(\cdot)$ such that for any multi-index I with $|I| \leq 4$ (resp. 8),

$$|\partial_I v(t,x)| \leq \eta_{K(T),M^\sharp}(1+\|x\|^{M^\sharp}) \tag{42}$$

with $M^\sharp = \max(2M', 2|I|)$ (resp. $\max(2M'', 2|I|)$).

Lemma 5.3 *Assume that $\int \|z\|^{2p}\nu(dz) < \infty$ for some integer $p \geq 1$ and that $f(\cdot)$ is Lipschitz. Then there exists an increasing function $K(\cdot)$ such that, uniformly in n one has*

$$\max_{0\leq k\leq n} \mathbb{E}\|\bar{X}^n_{kT/n}\|^{2p} \leq \eta_{K(T),2p}(\infty)(1+\mathbb{E}\|X_0\|^{2p}) \ . \tag{43}$$

Proof. For $p=1$, one has

$$\mathbb{E}\|\bar{X}^n_{(k+1)T/n}\|^2 \leq \mathbb{E}\|\bar{X}^n_{kT/n}\|^2 + \mathbb{E}\|f(\bar{X}^n_{kT/n})(Z_{(k+1)T/n} - Z_{kT/n})\|^2 \ .$$

The Lévy-Khintchine formula provides an analytical expression for the characteristic function of $Z_{T/n}$; since $Z_{T/n}$ has moments of orders up to $2p$, differentiation under the integral sign of $\int (1 - exp(i < u, x >) + i < u, x >) \ \mathbb{1}_{[\|x\|\leq 1]}\nu(dx)$ permits the computation of these moments. Under (H1), one can then check that

$$\mathbb{E}\|\bar{X}^n_{(k+1)T/n}\|^2 \leq \mathbb{E}\|\bar{X}^n_{kT/n}\|^2 + \frac{C\rho_{2p}(\infty)T^2}{n}$$

for some constant C depending only on $f(\cdot)$. One then sums over k to obtain the result for $p=1$. One then proceeds by induction. ■

We are now in a position to prove (9).

5.2 Proof of (9)

In this subsection we suppose (H1'), (H2'), (H3'). We follow the guidelines of Subsection 4.3.

Let $\tilde{Z}$ denote the Lévy process $(Z_{s+T-T/n} - Z_{T-T/n} \ , \ 0 \leq s \leq T/n)$ and let $\tilde{G}$ denote its infinitesimal generator. Consider functions ψ in $\mathcal{C}^4(\mathbb{R}^d)$ such that

$$\sum_{1\leq|I|\leq 4} |\partial_I\psi(z)| \leq C_\psi(1+\|z\|^{M_\psi}) \tag{44}$$

for some positive real number C_ψ and some integer $M_\psi \geq 2$. Consider Dynkin's formula (33) with $\tilde{Z}$ instead of $\tilde{Z}^m$ and ν instead of ν^m. Make a Taylor expansion to get the approximate Dynkin formula, similar to (36):

$$\mathbb{E}\psi(\tilde{Z}_{T/n}) = \psi(0) + \frac{T}{n}\tilde{G}\psi(0) + \tilde{R}^n \ , \tag{45}$$

with

$$\mathbb{E}|\tilde{R}^n| \leq \frac{\eta_{K(T),M_\psi}(\infty)}{n^2}$$

and furthermore the increasing function $K(\cdot)$ is uniform with respect to β, σ, ν and n, and depends on $\psi(\cdot)$ only through the constants C_ψ and M_ψ appearing in (44).

Choose

$$\psi(z) := v\left(T - T/n, \bar{X}^n_{T-T/n} + f\left(\bar{X}^n_{T-T/n}\right) z\right) \ .$$

This function $\psi(\cdot)$ is of class $\mathcal{C}^4(\mathbb{R}^d)$ and satisfies (44) with $M_\psi = M^\sharp = M'^* = \max(2M', 8)$ (remember that M'^* appears in (H3') and use (42)). Thus (36) can be used. We get:

$$\mathbb{E}v\left(T - T/n, \bar{X}^n_T\right) = \mathbb{E}v\left(T - T/n, \bar{X}^n_{T-T/n}\right) + \frac{T}{n}\mathbb{E}Av\left(T - T/n, \bar{X}^n_{T-T/n}\right) + \bar{R}^n_{T-T/n} \tag{46}$$

with (we use (42) and (43))

$$\mathbb{E}|\bar{R}^n_{T-T/n}| \leq \frac{\eta_{K(T),M'^*}(\infty)}{n^2}$$

Proceeding as in (39) with X^n instead of $X^{m,n}$ and $v(\cdot,\cdot)$ instead of $v^m(\cdot,\cdot)$, we deduce:

$$|\mathbb{E}g(X_T) - \mathbb{E}g(X^n_T)| \leq \frac{\eta_{K(T),M'^*}}{n}$$

for any function $g(\cdot)$ satisfying the hypothesis (H2').

5.3 Proof of (10).

To obtain the expansion of the Euler scheme error (10), we must now refine the strategy. From now on, we suppose (H1"), (H2"), (H3").

It can be checked using the Meyer-Itô formula that the function $v(t,\cdot)$ defined in (41) solves

$$\begin{cases} (\partial_0 v + A)v(t,x) = 0 \ , \ 0 \leq t < T \ , \\ v(T,\cdot) = g(\cdot) \ , \end{cases} \tag{47}$$

where A is the infinitesimal generator of the process (X_t) (see (2)).

In view of (47), $\partial_{000}v(t,x) = -A \circ A \circ Av(t,x)$. The estimate (42) shows that, for an increasing function $K(\cdot)$,

$$|\partial_{000}v(t,x)| \leq \eta_{K(T),M^\sharp}(\infty)\left(1 + \|x\|^{M^\sharp}\right)$$

where $M^\sharp = 2M + 12$.

Instead of (32), we now write:

$$\begin{aligned} \mathbb{E}v(T, \bar{X}^n_T) &= \mathbb{E}v(T - T/n, \bar{X}^n_T) + \frac{T}{n}\mathbb{E}\partial_0 v(T - T/n, \bar{X}^n_T) \\ &\quad + \frac{T^2}{2n^2}\mathbb{E}\partial_{00}v(T - T/n, \bar{X}^n_T) + R^n_T \\ &= \mathbb{E}v(T - T/n, \bar{X}^n_T) - \frac{T}{n}\mathbb{E}Av(T - T/n, \bar{X}^n_T) \\ &\quad + \frac{T^2}{2n^2}\mathbb{E}A(Av)(T - T/n, \bar{X}^n_T) + R^n_T \end{aligned} \tag{48}$$

with (we use (43))

$$|R_T^n| \leq \frac{\eta_{K(T),M''*}(\infty)}{n^3} . \tag{49}$$

In order to expand the right side of (32) around $\bar{X}^n_{T-T/n}$, we need an "approximate Dynkin formula" more precise than (36).

Suppose that $\psi(\cdot)$ is of class $\mathcal{C}^6(\mathbb{R}^d)$ and that

$$\sum_{1\leq|I|\leq 6} |\partial_I(z)| \leq C_\psi(1+\|z\|^{M_\psi}) \tag{50}$$

for some positive real number C_ψ and some integer M_ψ. Apply Dynkin's formula twice:

$$\mathbb{E}\psi(\tilde{Z}_{T/n}) = \psi(0) + \frac{T}{n}\tilde{G}\psi(0) + \int_0^{T/n}\int_0^s \mathbb{E}\tilde{G}\circ\tilde{G}\psi(\tilde{Z}_\theta)d\theta ds .$$

We make a Taylor expansion of Z_θ around 0; we obtain:

$$\mathbb{E}\psi(\tilde{Z}_{T/n}) = \psi(0) + \frac{T}{n}\tilde{G}\psi(0) + \frac{T^2}{2n^2}\tilde{G}\circ\tilde{G}\psi(0) + \tilde{R}^n , \tag{51}$$

with

$$\mathbb{E}|\tilde{R}^n| \leq \frac{\rho_{K(T),M_\psi}(\infty)}{n^3}$$

and furthermore the increasing function $K(\cdot)$ is uniform with respect to β, σ, ν and n, and depends on $\psi(\cdot)$ only through the constants C_ψ and M_ψ appearing in (50).

Choose

$$\psi(z) \;:=\; v\left(T-T/n, \bar{X}^n_{T-T/n} + f\left(\bar{X}^n_{T-T/n}\right)z\right) .$$

This function $\psi(\cdot)$ is of class $\mathcal{C}^8(\mathbb{R}^d)$ and satisfies (50) with $M_\psi = M^\sharp = \max(2M'', 12)$ (use (42) again). Thus, we can apply (51).

Then apply (36) to

$$\psi(z) := Av\left(T-T/n, \bar{X}^n_{T-T/n} + f\left(\bar{X}^n_{T-T/n}\right)z\right) ,$$

and finally make a Taylor expansion around 0 for

$$z \to A\circ Av\left(T-T/n, \bar{X}^n_{T-T/n} + f\left(\bar{X}^n_{T-T/n}\right)z\right) .$$

As in the preceding subsection, easy computations lead to:

$$\mathbb{E}v(T,\bar{X}^n_T) = \mathbb{E}v(T-T/n,\bar{X}^n_{T-T/n}) + \frac{T^2}{n^2}\mathbb{E}\phi(T-T/n,\bar{X}^n_{T-T/n}) + S^n_{T-T/n}$$

where

$$|S^n_{T-T/n}| \leq \frac{\eta_{K(T),\max(2M,12)}(\infty)}{n^3}$$

and where the function $\phi(\cdot,\cdot)$ is defined as follows:

$$\phi(t,x) := \frac{1}{2}A^2 v(t,x) + \frac{1}{2}\tilde{G}\circ\tilde{G}\circ v^{t,x}(0) + \tilde{G}\circ A v^{t,x}(0)$$

where

$$v^{t,x}(z) := v(t, x + f(x)z) \ .$$

We conclude as in [20]: consider now $\phi(t,\cdot)$, $0 \le t < T$, instead of $g(\cdot)$ in (9); $\phi(t,\cdot)$ satisfies (H2') with $M'^* = \max(2M'', 16)$, so that

$$\mathbb{E}v(T, \bar{X}^n_T) = \mathbb{E}v(T-h, \bar{X}^n_{T-T/n}) + \frac{T^2}{n^2}\mathbb{E}\phi(T - T/n, X_{T-T/n}) + U^n_{T-T/n} \ ,$$

with

$$|U^n_{T-T/n}| \le \frac{\eta_{K(T),M''^*}(\infty)}{n^3} \ .$$

Proceeding as in (39), we obtain:

$$\mathbb{E}v(T, \bar{X}^n_T) = \mathbb{E}v(T, X_T) + \frac{T^2}{n^2}\sum_{k=0}^{n-1}\mathbb{E}\phi(kT/n, X_{kT/n}) + \sum_{k=0}^{n-1} U^n_{kT/n} \ .$$

Finally, we observe that

$$\frac{T^2}{n^2}\sum_{k=0}^{n-1}\mathbb{E}\phi(kT/n, X_{kT/n}) = \frac{T}{n}\int_0^T \mathbb{E}\phi(s, X_s)ds + r^n$$

with

$$|r^n| \le \frac{\eta_{K(T),M''^*}(\infty)}{n^2} \ .$$

Thus,

$$\left|\mathbb{E}g(X_T) - \mathbb{E}g(\bar{X}^n_T) + \frac{T}{n}\int_0^T \mathbb{E}\phi(s, X_s)ds\right| \le \frac{\eta_{K(T),M''^*}(\infty)}{n^2} \ .$$

■

References

[1] V. BALLY and D. TALAY. The law of the Euler scheme for stochastic differential equations (I) : convergence rate of the distribution function. *Probability Theory and Related Fields*, To appear.

[2] K. BICHTELER. Stochastic integrators with independent increments. *Zeit. für Wahr.*, 58:529–548, 1981.

[3] K. BICHTELER and J. JACOD. Calcul de Malliavin pour les diffusions avec sauts : existence d'une densité dans le cas unidimensionnel. In *Séminaire de Probabilités XVII*, number 986 in Lecture Notes in Mathematics, pages 132–157. Springer-Verlag, 1983.

[4] N. BOULEAU. *Probabilités de l'Ingénieur, Variables Aléatoires et Simulation.* Hermann, 1986.

[5] N. BOULEAU and D. LEPINGLE. *Numerical Methods for Stochastic Processes.* J. Wiley, 1993.

[6] E. CINLAR and J. JACOD. Representation of semimartingale Markov processes in terms of Wiener processes and Poisson random measures. In *Seminar on Stochastic Processes*, pages 159–242. Birkhauser, Boston, 1981.

[7] L. DEVROYE. *Non Uniform Random Variate Generation.* Springer, 1985.

[8] C. GOLDIE. A class of infinitely divisible random variables. *Proc. Camb. Phil. Soc.*, 63:1141–1143, 1967.

[9] J. JACOD and P. PROTTER. Une remarque sur les équations différentielles stochastiques à solutions Markoviennes. In *Séminaire de Probabilités XXV*, volume 1485 of *Lecture Notes in Mathematics*, pages 138–139. Springer-Verlag, 1991.

[10] G.M. KANTER. Stable densities with change of scale and total variation inequalities. *Ann. Prob.*, 3:697–707, 1975.

[11] P.E. KLOEDEN and E.PLATEN. *Numerical Solution of Stochastic Differential Equations.* Springer–Verlag, 1992.

[12] A. KOHATSU-HIGA and P. PROTTER. The Euler scheme for SDEs driven by semimartingales. In H. Kunita and H.H. Kuo, editors, *Stochastic Analysis on Infinite Dimensional Spaces*, pages 141–151. Pitman, 1994.

[13] T. KURTZ and P. PROTTER. Wong–Zakai corrections, random evolutions and numerical schemes for S.D.E.'s. In *Stochastic Analysis: Liber Amicorum for Moshe Zakai*, pages 331–346, 1991.

[14] P. PROTTER. *Stochastic Integration and Differential Equations.* Springer–Verlag, Berlin, 1990.

[15] D. REVUZ and M. YOR. *Continuous Martingales and Brownian Motion.* Springer–Verlag, Berlin, 1991.

[16] F.W. STEUTEL. Infinite divisibility in theory and in practice. *Scand. J. Stat.*, 6:57–64, 1979.

[17] B.W. STUCK and B. KLEINER. A statistical analysis of telephone noise. *The Bell System Technical Journal*, 53(7):1263–1320, 1974.

[18] D. TALAY. Discrétisation d'une E.D.S. et calcul approché d' espérances de fonctionnelles de la solution. *Mathematical Modelling and Numerical Analysis*, 20(1), 1986.

[19] D. TALAY. Simulation and numerical analysis of stochastic differential systems: a review. In P. Krée and W.Wedig, editors, *Probabilistic Methods in Applied Physics*, volume 451 of *Lecture Notes in Physics*, chapter 3, pages 54–96. Springer-Verlag, 1995.

[20] D. TALAY and L. TUBARO. Expansion of the global error for numerical schemes solving stochastic differential equations. *Stochastic Analysis and Applications*, 8(4):94–120, 1990.

Unité de recherche INRIA Lorraine, Technopôle de Nancy-Brabois, Campus scientifique,
615 rue du Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY
Unité de recherche INRIA Rennes, Irisa, Campus universitaire de Beaulieu, 35042 RENNES Cedex
Unité de recherche INRIA Rhône-Alpes, 46 avenue Félix Viallet, 38031 GRENOBLE Cedex 1
Unité de recherche INRIA Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex
Unité de recherche INRIA Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex

Éditeur
INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
ISSN 0249-6399