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Abstract

We study a problem of classification of linear homogeneous second-order ODE's with polynomial coefficients based on qualitative properties of singularities. The corresponding combinatorial problem of counting the number of classes is then solved in terms of the initial number of singularities.

Un problème combinatoire en classification des EDO linéaires du second ordre

Résumé

Nous étudions un problème de classification d'équations linéaires homogènes du second ordre à coefficients polynomiaux fondée sur des propriétés qualitatives des singularités. Le problème combinatoire consistant à compter le nombre de classes dans la classification est ensuite résolu en fonction du nombre initial de singularités.

A Combinatorial Problem in the Classification of Second-Order Linear ODE's

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Abstract

We study a problem of classification of linear homogeneous second-order ODE's with polynomial coefficients based on qualitative properties of singularities. The corresponding combinatorial problem of counting the number of classes is then solved in terms of the initial number of singularities.

Key words: classification, second-order linear ODE, Pólya theory.

1 Introduction

Classes of special functions arising as solutions of ODE's with polynomial coefficients are subject to a comprehensive study in modern mathematics [3, 6]. But even the hypergeometric class, studied on a non-structured basis, looks like a zoo where many peculiar animals named after great mathematicians are parked at random. The first structural problem that arises is an enumeration one. How many types of specific equations can be distinguished in the hypergeometric class? How many distinguished equations lie in the next class in complexity—the Heun class? How many equations in the next, unnamed, class whose equations originate from the Riemann-Hilbert problem and isomorphic deformation approaches to the Painlevé class of nonlinear equations? How many distinguished equations arise as specialized and confluent equations from the third order hypergeometric equation?

In this note, we answer these questions in two steps. First we give a theoretical basis for a classification which, as is natural, involves the order of the equation and qualitative and quantitative characteristics of its singularities. Quantitative characteristics of types of singularities make it possible to distinguish between types of equations.

As a second step we solve the combinatorial problem of enumerating the number of types of equations per class under this classification. This enumeration is important because it will provide a basis for future structured studies of phenomena related to linear differential equations.

2 Classification

As a starting point we consider second-order Fuchsian equations with n singularities one of which is at infinity. The canonical form for these equations is represented by the symbol T^C—a polynomial in two variables z and D

$$T^C(z, D) := P_{n-1}(z)D^2 + P_{n-2}(z)D + P_{n-3}(z), \tag{1}$$

where D is the differentiation operator and z is the independent variable, P_k denotes a polynomial of degree k and furthermore, $P_{n-1}(z)$ has only simple roots.

Let $v(z)$ be a solution of the first-order associated Fuchsian equation related to the symbol

$$U(z, D) := A_{n-1}(z)D + A_{n-2}(z). \tag{2}$$

The function $v(z)$ has an explicit form in terms of a product of power-type monomials $(z - z_j)^{\mu_j}$. Changing the unknown function in (1) by multiplying it with $v(z)$ induces a s-homotopic transformation

$$S_U : T(z, D) \mapsto [v(z)]^{-1}T(z, D)v(z). \tag{3}$$

The following well-known lemma holds.

Lemma 1 *There exists a s-homotopic transformation transforming the canonical form T^C of the Fuchsian equation into its normal form with the symbol*

$$T^N(z, D) := S_U(T^C) = Q_{2n-2}(z)D^2 + Q_{2n-4}(z), \tag{4}$$

$Q_{2n-2}(z)$ having double zeros at the $n - 1$ regular singularities.

The singularities are located at infinity and at the roots of Q_{2n-2}. By changing values of parameters in the coefficients of the polynomials Q_{2n-2} and Q_{2n-4} we can move the singularities without changing their type. When two singularities coalesce we get a new equation with possibly new degrees of the polynomials in (4) and different from the original types of singularities, by so-called confluence.

Another type of transformation which can be performed on this equation consists in modifying its parameters so that the singularities do not move, but one of them degenerates (by cancelling the leading term of the local behaviour of the leading coefficient) resulting in a change in the type of the singularity. Both processes lead to an equation that in general can no more be regarded as

Fuchsian and should be called a confluent Fuchsian equation. Below we consider symbols that arise from a given equation (4) as a result of the processes discussed above.

The practical classification of types of singularities may be performed either on the basis of the notion of the s-rank of the singularity [7], or on the basis of a treatment of singular properties of the zeros of the symbolic indicial equation

$$T^N(z, D) = 0. \tag{5}$$

The difference between these approaches is not so crucial and is mainly due to historical reasons. The s-rank definition is based on the notion of elementary irregular points which, according to Ince [5], constitutes a special case of regular singular points. Both approaches give the same formulation of Theorem 1 below and here we focus on the second one.

Since equation (5) is of degree 2 in D, its solutions can be represented in the neighbourhood of singularities by Puiseux series of the following types

$$\begin{aligned} D_m(z_j) &= C_{mj}(z - z_j)^{-\mu_j} \sum_{k=0}^{\infty} h_k (z - z_j)^k, \quad C_{mj} \neq 0, \\ D_m(\infty) &= C_{m\infty} z^{\mu_\infty - 2} \sum_{k=0}^{\infty} h_k z^{-k}, \qquad C_{m\infty} \neq 0. \end{aligned}$$

with integer or half-integer μ_j satisfying

$$\frac{1}{2} \leq \mu_j \leq n - 1. \tag{6}$$

The first term of these expansions gives the behaviour of the logarithmic derivative of the solutions related to (4) in the neighbourhood of the corresponding singularity:

$$|\ln w(z)| \leq K|z - z_j|^{\mu_j} \qquad \text{for} \quad \mu_j \geq 1; \tag{7}$$

with an appropriate constant K determined by the coefficients of the polynomials in (4). In the special case $\mu_j = 1/2$ this inequality becomes

$$|\ln w(z)| \leq K|z - z_j|^{-1}, \tag{8}$$

where K depends only on z_j.

It is important to stress that for generic singularities $\mu_j = 1$, corresponding to regular singular points.

Proposition 1 *The μ_j's behave subadditively in case of confluence.*

Proof. Completely similar to the proof in [7] (see also [5]). □

When our μ_j's are integer, they differ by one from the Poincaré rank of the singularity. This difference is crucial in the above proposition.

We can now define the classification.
Definition. The set of values $\{\mu_1, \mu_2, \dots, \mu_\infty\}$ is called the *s-multisymbol* of the differential equation. Two equations have the same type if and only if they have the same *s-multisymbol*.

The Fuchsian equation with n singularities corresponds to a multisymbol with n times 1. By specialization of the coefficients any μ_j can be made equal to 1/2. In the case of confluence of two singularities, the s-multisymbol is changed by a decrease of one element and the corresponding two μ_j's give rise to a new one equal to the sum of the original ones. Now, starting from a Fuchsian equation with n singularities, the question is "How many distinct equations (w.r.t. this classification) can be obtained by specializations of the coefficients?".
Example. We start from the hypergeometric equation with three regular singularities at 0, 1 and infinity: it corresponds to the s-multisymbol $\{1, 1, 1\}$. Reductions to s-multisymbols $\{1/2, 1, 1\}$, $\{1/2, 1/2, 1\}$, $\{1/2, 1/2, 1/2\}$ can be regarded as subsequent specifications of parameters c, b, a for this equation. Under confluence of the points $z = 1$, $z = \infty$ one obtains the confluent hypergeometric equation with s-multisymbol $\{1, 2\}$. Reductions of this equation with s-multisymbols $\{1/2, 2\}, \{1, 3/2\}, \{1/2, 3/2\}$ can be regarded as confluent hypergeometric equations with fixed parameter c, Bessel equation, or Bessel equation with fixed index respectively. The equation resulting from the next confluence with s-multisymbol $\{2\}$ is the equation for the parabolic cylinder functions while its reduction to the s-multisymbol $\{5/2\}$ is the Airy equation. Thus from the Fuchsian equation with 3 regular singular points, we get 10 distinct equations. In our next section, we show how this number can be computed in an algorithmic way.

In terms of μ_j we can redefine our classification as follows. Suppose we have a set of n unities, corresponding to the regular singularities of the initial equation. Two operations are allowed: each of these unities can loose one-half and two unities can be added to each other. The question is: How many distinct sets can be obtained, under the assumption that that those sets which can be obtained from $n - 1$ unities do not count ?

It is possible to give a physical formulation of the problem. Suppose that there is a set of particles each of which can be characterized by a number taking two values 1 and 1/2. (The particles could be considered as bosons and fermions and the number is their spin). Suppose that these particles can be combined in clusters for which spins are summed so that clusters are distinguished only by their spin and not by the particles. By quantum ensemble we mean the union of single particles and clusters. How many ensembles is it possible to obtain starting with n particles?

The problem can be generalized to higher-order equations or systems of first-order equations. The crucial difference from what we have studied is that for the second-order equation the set of μ_j's is valid for both solutions, whereas for higher-order equations we need to characterize different solutions by different sets.

3 Enumeration

We shall prove the following.

Theorem 1 *The number of distinct equations obtained by specializing the Fuchsian second-order equation with n singularities for $n \geq 3$ under the classification of Section 1 is the nth Taylor coefficient at the origin of*

$$\exp \sum_{j>0} \frac{2z^j}{j(1-z^j)} = 1+2z+5z^2+10z^3+20z^4+36z^5+65z^6+110z^7+185z^8+O(z^9).$$

Proof. The proof is based on *generating function* techniques and we select to present it using the framework of [4]. The generating function of a class of combinatorial objects $\mathcal{C}$ is

$$C(z) = \sum_{n\geq 0} c_n z^n,$$

where c_n denotes the number of objects of size n in $\mathcal{C}$ (it is assumed here that all the c_n's are finite). Combinatorial constructs on classes of objects have direct translations into generating function equations. The simplest instance is provided by disjoint union: obviously the generating function of a class $\mathcal{A}$ whose elements are either elements of $\mathcal{B}$ or elements of $\mathcal{C}$, $\mathcal{B}$ and $\mathcal{C}$ having no common element is $B(z)+C(z)$. Similar reasoning shows that the generating function of the cartesian product $\mathcal{B}\times\mathcal{C}$ is $B(z)C(z)$. A classical lemma gives the generating function of multisets (sets where repetition is allowed) (see [2, 4]).

Lemma 2 *Let $\mathcal{A}$ be the class of multisets of elements of $\mathcal{B}$, $\mathcal{B}$ having no element of size 0. Then the generating functions $A(z)$ and $B(z)$ of $\mathcal{A}$ and $\mathcal{B}$ are related by*

$$A(z) = \exp\left[\sum_{j>0} \frac{B(z^j)}{j}\right].$$

From the combinatorial viewpoint, the classification of Section 1 translates directly into the following combinatorial specification, using a classical combinatorial language (see for instance [2, 4]):

$$\begin{aligned}
\text{Unity} &= \text{Atom}(1),\\
\text{minus_one_half} &= \text{Atom}(0),\\
\text{Integer} &= \text{Sequence}(\text{Unity}, \text{card} > 0),\\
\text{Integer_minus_one_half} &= \text{Prod}(\text{Integer}, \text{minus_one_half}),\\
A &= \text{MultiSet}(\text{Union}(\text{Integer}, \text{Integer_minus_one_half})).
\end{aligned}$$

This reads as follows: a "unity" is an atomic object of size 1, while "minus_one_half" is an atomic object of size 0. From these atoms, we build more complicated objects. The first one is an integer, which is simply a sequence of unities. Then comes an integer minus one half, obtained in the obvious way. Finally, we consider sets of integers of both types. The reason for using such a description is that it directly translates into generating function equations. We thus obtain (using the same symbol for the generating function and the class it enumerates):

$$\begin{aligned}
\mathrm{Unity}(z) &= z, \\
\mathrm{minus_one_half}(z) &= 1, \\
\mathrm{Integer}(z) &= z/(1-z) = \mathrm{Integer_minus_one_half}(z), \\
A(z) &= \exp\left[\sum_{i>0} \frac{2z^i}{i(1-z^i)}\right].
\end{aligned}$$

□

It is also possible, but slightly more complicated, to reach the same result using the physical interpretation given above. Note that under this latter description, the sequence can be recognised as the combinatorial object "partitions of n into parts of two kinds", which is the name under which it is listed in [8]. From partition theory, it is also easy to see that an alternative form for the generating function is

$$\prod_{n>0} \frac{1}{(1-z^n)^2}.$$

From this, one sees that the nth coefficient c_n in the series is related to the number p_n of partitions of the integer n by

$$c_n = \sum_{k=0}^{n} p_k p_{n-k}.$$

The Hardy-Ramanujan theorem on partitions [1] then leads to the conclusion that asymptotically

$$c_n \sim \frac{K}{n^2} e^{L\sqrt{n}}, \qquad K = \frac{1}{12}, \quad L = 2\pi/\sqrt{3}.$$

The reason why Theorem 1 is not valid for $n = 0, 1, 2$ lies in Liouville's theorem on the nonexistence of analytic functions with "extremely simple" behaviour.

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