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Abstract: *S-functions are a generalisation of Berry's stable functions, for being intensionally stable, so that they need not domains to be distributif.*
In this paper, we show that together with S-domains, they constitute an order-enriched Λ -category which is a retract of the category of domains and continuous functions, and we provide a sequential algorithm for computing all the finite elements of S-domains.

Key-words: *S-functions, S-domains, oder-enriched* Λ *-category*

(Résumé : tsvp)

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UNE THEORIE DE FONCTIONS SEQUENTIELLES

Résumé : Nous présentons ici, une généralisation des fonctions stables (définies par Berry): les S-fonctions qui ne requièrent pas que leurs domaines soient distributifs. Nous montrons, qu'avec les S-domaines, qui sont une généralisation des domaines concrèts de Kahn et Plotkin, elles constituent une catégorie cartésienne close qui est "order-enrichie " par l'ordre point-a-point. Pour finir, nous montrons que cette catégorie est un rétract de la catégorie des domaines et des fonctions continues et nous fournissons un algorithme séquentiel pour calculer les éléments finis des S-domaines.

Mots-clé : S-fonctions, S-domains, order-enrichie Λ-categorie

1 Introduction

The purpose of this paper is twofold: first it aims at generalising the notion of stable functions introduced by Berry in [Ber79], in order to construct a Λ-category whose homsets are extensionally ordered (ie: using a pointwise order) like in [Gam92]; and secondly at obtaining models for languages like PCF (a typed λcalculus together with arithmetic and boolean operators, augmented with a fixpoint operator) whose terms are sequentialy computable, so as to construct fully-abstract models.

It is known that the sequentiality problem is highly related to "the fully-abstraction question" for λcalculus based languages. In [Mil77] Milner posed the problem in the following form : *" One would like to find a concept of sequential continuous functions and show that the model of sequential functions exists and yields a fully-abstract intrepretation ".* Actually the sequentiality problem was originally raised in its most typical form and shown difficult by Plotkin in [Plo77]; there the model derived from Scott's of continuous functions was shown to be non fully-abstract for the language PCF. Indeed Plotkin exhibited two PCF terms which are operationally equivalent but which are denoted by to functions differing on an argument which is undefinable in the language. Such an argument is typically the "parallel or" function, a binary boolean function which yields value true, as soon as one of its arguments is true. The reason of this failure of full-abstraction is essentially an inadequate treatment of sequentiality: it is well known that PCF-like languages can be evaluated sequentialy; therefore in order to provide fully-abstract models for such languages, one should restrict to sequential continuous functions (since it is known that the model of continuous functions is complete). On this way, Milner and Vuillemin proposed two different definitions of sequentiality (see[Mil77] and [Vui74]); unfortunately their definitions rely on the product structure of the input space of functions and are adequate only for first order functions (not for functionnals).

In order to solve the problem, Kahn and Plotkin(see [KaP78]) proposed a more general definition which abstracts from the product structure of the input domains. Their proposition relies on an axiomatisation of the notion of the place of an argument in a particular class of domains called *concrete domains;* nevertheless it cannot be used to provide the fully-abstract model of PCF since the category of concrete domains and sequential functions (as defined by Kahn and Plotkin) is not cartesian closed. Notice however that subsituting the notion of sequential algorithm for that of function and relaxing the extensional order, allow to provide PCF with a fully-abstract but non order-extensional model(see [Cur86] and [Ber79]).

Attacks on the problem led Berry to define the notion of *stable function* (see [Ber79]). Intuitively, stable functions are continuous functions together with a particular minimality property. Formally, a function f is stable if for any element x and for any approximant b of $f(x)$, there exists an approximant $m(f,x,b)$ of x s.t. : for any approximant y of x, b

approximates $f(y)$ if and only if $m(f,x,b)$ approximates y. A typical example of a non stable function is the parallel-or function since $m(por,(tt,tt),tt)$ does not exist.
These functions together with distributive and finitary domains, provide a cartesian closed category which cannot be order-enriched by the pointwise-order and thus cannot provide the fully-abstract model of PCF. Indeed Milner(see [Mil77]) showed that *the function-based fully-abstract model of PCF is unique(up to isomorphim) and is ordered extensionally.*
Actually the stability notion was introduced as an approximation of that of sequentiality; but in sight of Milner's results it is clear that the fully-abstract functions-based model cannot have distributive domains; one can immediately convince oneself of the non distributivity of domains of sequential functions (as far as the extensional order is used): for example, considering the domain $[T^2 \longrightarrow T]$, where T is the domain of thruth values, let:
$f_1 = ((\perp, tt) \Longrightarrow tt)$, $f_2 = ((tt, \perp) \Longrightarrow tt)$ & $g = ((ff, ff) \Longrightarrow tt)$;
then $f_1 \vee f_2 = \lambda x.tt$ *and* $g \wedge (f_1 \vee f_2) = g$ *but* $(g \wedge f_1) \vee (g \wedge f_2) = \lambda x.\perp$.
In the meantime it is clear that functions f_1 , f_2 and g are sequential and PCF definable. More crucially, Berry left the problem open by exhibiting a function which is stable but non sequential; such a function is the least continuous 3-ary boolean function f such that $f(tt, ff, \perp) = f(ff, \perp, tt) = f(\perp, tt, ff) = tt$.
Indeed this function has no sequentility index at $(\perp, \perp, \perp)$.

Following this way, Bucciarelli and Ehrhard(see[BuE93] and [Ehr93]) proposed the notion of *strong stability,* but their proposition also relies on the distributivity of domains.
Finally, Abramsky, Jagadeesan and Malacaria(see [AJM93]) proposed strategies in a game model but their strategies are not a functional model and the description of the fully-abstract model still leaves to be desired.
To sum up, what we want is to relax the distributivity of domains and to use the pointwise order, in order to semantically reach Milner's model.
After some notations, given in section II, we introduce in section III *S-domains,* which are structures provided with a partial order relation together with a preorder relation. The two relations are intended to "compare" the behaviour of individuals both extensionally and intensionally. The intensional preorder does not impose any significant structure on domains, it only allows to generalise Kahn and Plotkin's notion of the place of an argument in a wider class of domains(not only the concrete ones). In section IV, we use the intensional preorder and define the *S-functions* which are continuous functions together with some intensional conditions. In section V, we show that *S-domains together with S-functions constitute a cartesian close order-enriched pointwisely category.* Finally, we compare our model to the Scott's model of continuous functions, we show that our model is a *retract* of Scott's, and we define *sequential algorithms for computing the finite elements of S-domains.*

2 NOTATION

In this section we introduce the stuctures used through out the paper and we precise our terminology and notations:
Let $\mathbf{D}=< D, \leq >$ be a partial order (po); $\mathbf{D}$ is a *complete lattice* if any every subset X of D has a least upper bound denoded $\bigvee X$and a greatest lower bounded denoded $\bigwedge X$. A subset $X \subseteq D$ is *directed* if it is non empty and is s.t.: $\forall \alpha, \beta \in X, \exists \gamma \in X \ : \alpha \leq \gamma \geq \beta$.
An element x of D is *finite(or isolated)* if for any directed subset X of D,
$(x \leq \bigvee X) \Longrightarrow (\exists \alpha \in X : x \leq \alpha)$.
We denote $\mathcal{B}(D)$ the set of finite elements of a complete lattice $\mathbf{D}$ and we call it the *base* of $\mathbf{D}$; $\mathcal{B}(x)$ represents the set of finite elements dominated by x. If for all finite element $x, \mathcal{B}(x)$ is finite, then $\mathbf{D}$ is said to be *finitary*.
A complete lattice $\mathbf{D}$ is *algebraic* if for any element x , $\mathcal{B}(x)$ is directed and has lub x ; if $\mathcal{B}(D)$ is denumerable, then it is *ω-algebraic*. Notice that a complete lattice has a least element (we shall denote it $\perp$) and top element (denoted $\top$).
Henceforth, we refer to finitary ω-algebraic complete lattice as *domains*.

We now introduce S-domains :

3 S-DOMAINS

3.1 <u>definition</u> : *S-domains* are triples $\mathbf{D} = < D, \leq, \preceq >$ s.t.:$< D, \leq >$ is a domain and the relation " $\preceq$ " is a preoder on D. ∎

The partial order "$\leq$ " is called *extensional.* The *intensional* preorder (denoted "$\preceq$") does not impose any significant structures on S-domains, it is only used to define their morphisms.

The whole class of S-domains will be inductively constructed from *basic* (flat) domains, using the product and exponentiation operation.
For this class, the intensional preorder is defined as the following:

3.2 <u>definition</u> : let $\mathbf{D}$ be an S-domain; then
$x \preceq y$ iff $\forall \alpha \in \mathcal{B}(D), (\alpha \not\leq y \ \& \ \alpha \Uparrow y) \Rightarrow (\alpha \not\leq x \ \& \ \alpha \Uparrow x)$; where
$x \Uparrow y$ iff $\exists z \in \mathcal{B}(D) / \exists ! z' \longrightarrow\!\!< z \ : \ x \leq z \geq y$; where
$z' \ \longrightarrow\!\!< z$ iff $z' < z \ \& \ \forall x \ : \ z' \leq x \leq z \Longrightarrow x = z'$ <u>*or*</u> $x = z$. ∎

3.3 <u>proposition</u> : relation " $\preceq$ " is a preoder on $\mathbf{D}$.
<u>proof</u> : reflexivity: $x \preceq x : \forall \alpha \in \mathcal{B}(D)$, $(\alpha \not\leq x \ \& \alpha \Uparrow x) \Rightarrow (\alpha \not\leq x \ \& \ \alpha \Uparrow x)$
transitivity: $x \preceq y \preceq z \Rightarrow x \preceq z$:
$(\forall \alpha \in \mathcal{B}(D)$, $(\alpha \not\leq y \ \& \ \alpha \Uparrow y) \Rightarrow (\alpha \not\leq x \ \& \ \alpha \Uparrow x)$ &
$\forall \alpha \in \mathcal{B}(D)$, $(\alpha \not\leq z \ \& \ \alpha \Uparrow z) \Rightarrow (\alpha \not\leq y \ \& \ \alpha \Uparrow y)) \Rightarrow$

$\forall\alpha \in \mathcal{B}(D)\ , (\alpha \not\leq z\ \&\ \alpha \Uparrow z) \Rightarrow\ (\alpha \not\leq y\ \&\ \alpha \Uparrow y) \Rightarrow\ (\alpha \not\leq x\ \&\ \alpha \Uparrow x)$ ∎

3.4 proposition : let **D** be an S-domain then:
$\forall x, y \in D, (x \preceq y\ \&\ x \Uparrow y) \Rightarrow x \leq y$.
proof : suppose $x, y \in$ D and let $x \preceq y$ but $x \not\leq y$;
then $\exists\alpha \in \mathcal{B}(D) : \alpha \leq x\ \&\ \alpha \not\leq y$ in this case either :

- $\alpha \Uparrow y$ which is excluded since $\alpha \not\leq y\ \&\ \alpha \Uparrow y$ but $\alpha \leq x$ or
- $\neg(\alpha \Uparrow y)$ which is excluded since $\alpha \leq x\ \&\ x \Uparrow y$. ∎

3.5 collary : let **D** be an S-domain then:
$\forall x, y \in D, (x \preceq y \preceq x\ \&\ x \Uparrow y) \Rightarrow x = y$.
From now on we denote " $\approx$ " the equivalence induced by the preorder " $\preceq$ " . ∎

The proposition (resp. the collary) above shows that the domain partial order (resp. the equivalence) has been broken into two part: first the intensional preorder (rep. the intensional equivalence) and secondly the relation "$\Uparrow$" on elements.
Intuitively, $x \approx y$ can be interpreted as follows: considering any piece of information, if it can be used to significantly increase the information contained in x then it can be made so with y and vise versa. The attempt here, is to generalise the notion of "the place of an argument" of a function so as to leave away any reference to the product structure of the input domain.
Let us go further and illustrate on examples the relation between the notion of place and our definition. We want to show here, that our intensional equivalence allows to verify whether or not two sequences of arguments are defined on the same components. That is to say that we are able to deal with arguments' places without counting them.

Define domain $\mathbf{D} = \mathbf{N}^k$ where $k \in \mathbf{N}$ and finite and where $\mathbf{N}$ is the flat S-domain of integers: the extensional relation is the discrete one and the intensional relation is defined as indicated above. The elements of **D** are sequences of integers with lengh k, and they are ordered componentwisely, using the product order. Is is immediate that **D** is an S-domain. There are particular finite elements (the completely prime elements) which are sequences which contain only element $\perp$ except for a unique component. Such completely primes can thus be represented by pairs (p, v) , where $1 \leq p \leq k$ and $v \in \mathbf{N}$.
Now we claim: if $a = (p_a, v_a)$ and $b = (p_b, v_b)$ are two completely prime and intensionally equivalent elements, then $p_a = p_b$.
Indeed, letting $c = (p, v)$, be a complete prime element, it is easily seen that
$(a \Uparrow c\ \&\ c \not\leq a)$ iff $((p_a = p) \Rightarrow (v_a = v))\ \&\ ((p_a \neq p\ \underline{or}\ v_a \neq v))$;
ie: $a \approx b \Rightarrow\ \forall p, p \neq p_a$ iff $p \neq p_b$ and thus $p_a = p_b$.
We can therefore talk about arguments' positions in a list of arguments.
Now we can get confidence that our definition introduces a much more general notion of place in domains which are abstract than concrete domains. Before we proceed, let us illus-

trate it on some more examples:

example 1 : Let **D** be the integer domain **N** ordered s.t.:
$n \leq m$ iff $n = m$ or $m = \top$ or $n = \bot$ then : $\forall x, y \neq \top$, $(x \neq \bot \neq y) \Rightarrow x \approx y$.

example 2 : Set **D** $=\mathbf{T}^2$ ordered with the product order, **T** is the flat domain of boolean values; then :

1. $(\bot, \bot) \preceq (tt, \bot) \approx (ff, \bot)$
2. $(tt, \bot) \npreceq (\bot, tt)$

since : $(tt, \bot) \neq (\bot, ff) \neq (\bot, tt)$ & $(\bot, ff) \Uparrow (tt, \bot)$
but $\neg((\bot, ff) \Uparrow (\bot, tt))$.

4 S-FUNCTIONS

First of all, let us recall that a function f from domain **D** to domain **E** is said to be continuous iff : $\forall x \in D, \forall \beta \in E : \beta$ isolated & $\beta \leq f(x), \exists \alpha \in D$ s.t. : α isolated &
$\alpha \leq x$ & $\beta \leq f(\alpha)$.
Equivalently f is continuous iff it preserves lubs of directed subsets. Thus in domains, a continuous function is completely characterized by its values on the finite elements. Therefore, while considering continuous functions, we shall rectrict to finite elements only, without any lost of generality. We only consider as programs, those continuous functions having as image, the flat domains.

Now, given two S-domains **D** and **E**, we are interested only in those continuous functions from D to E which satisfy the following condition (S):

$$\forall x, z \in D : x \leq z, \forall \beta \in E, (f(z) \geq \beta \nleq f(x)) \Rightarrow \exists \mu(f, x, \beta) \in D :$$
$$z \geq \mu(f, x, \beta) \nleq x \quad \&$$
$$\forall y \in D, \ (x \leq y \ \ \& \ \ y \leq z \ \ \& \ \ \beta \preceq f(y)) \Rightarrow \mu(f, x, \beta) \preceq y \ .$$

Intuitively, a function verifies condition (S) iff any increase of information in its result necessitates the increase of an intensionaly predefined information in its argument. Indeed this idea is a generalization of Kahn and Plotkin's definition of sequential functions. By the way, $\mu(f, x, \beta)$ can be viewed as *the index of f in x, for value β*; one thus see, as already noticed by Vuillemin, that this index is not necessarily unique. Henceforth, we write $[D \xrightarrow[S]{} E]$ for the set of continuous functions from D to E which satisfy the (S)-condition and we call them S-functions.
Before pursuing, let us verify that S-functions exist and that together with S -domains, they form a category.

4.1 proposition : *the identity function* $1_D \in [D \longrightarrow D]$ *s.t. :*
$\forall x \in D,\ \ 1_D(x) = x,$ *is an S-function.*
proof : One immediately verifies that we can set $\mu(1_D, x, \beta) = \beta$ ∎

4.2 proposition : *given two S-functions:* $f \in [\mathrm{D} \underset{s}{\longrightarrow} \mathrm{E}]$ and $g \in [\mathrm{E} \underset{s}{\longrightarrow} \mathrm{F}]$, *their composition* $g \circ f \in [D \longrightarrow F]$, *is an S-function.*
proof : let $x, z \in D :\ x \leq z,\ \beta \in F\ \ \&\ \ g \circ f(z) \geq \beta \not\leq g \circ f(x) :$ then
$\exists\ \mu_1 = \mu(g, f(x), \beta) \in E : f(z) \geq \mu_1 \not\leq f(x)\ \ \&$
$\forall y \in E,\ \ (f(x) \leq y\ \ \&\ \ y \leq f(z)\ \ \&\ \ \beta \preceq g(y)) \Rightarrow \mu_1 \preceq y;$
and thus $\exists \alpha \in E : \alpha \leq \mu_1\ \ \&$
$f(z) \geq \alpha\ \not\leq f(x)$ hence $\exists\ \mu_2 \leq \mu_1(f, x, \alpha) \in D :\ \ x \not\geq \mu_2 \leq z\ \ \&$
$\forall y \in D, (x\ \leq y\ \ \&\ \ y \leq z\ \ \&\ \ \alpha \preceq f(y)) \Rightarrow \mu_2 \preceq y.$
Now, letting $\mu(g \circ f, x, \beta) = \mu_2 = \mu,$ we get:

1. $x \not\geq \mu \leq z$
2. $\forall y \in D, (x \leq y\ \ \&\ \ y \leq z\ \ \&\ \ \beta \preceq g \circ f(y)) \Rightarrow \mu \preceq y.$ ∎

The above results allow to speak of the category whose objects are S-domains and whose arrows are S-functions. We shall call it **SD**.
Actually, our intent is to show that **SD** is a Λ -category. Before, let's examine more closely S-functions and some of their properties.

Fact 1 : *S-functions, which are different from the identity, do exist:*
an example of S-function is the least function f s.t.:
$f : \mathbf{T} \longrightarrow \mathbf{T}$ s.t.: $f(\perp) = \perp,\ \ f(tt) = f(ff) = tt.$
Indeed, it is easily verified that we can set $\mu(f, \perp, tt) = tt$.
Another example of the existence of S-functions is given by the following:

4.3 proposition :*Let d and e be two finite elements of S-domains* **D** and **E** *then the function f denoted* $(d \Longrightarrow e)$ *and s.t.:* $f(x) =$ if $x \geq d$ then e else $\perp$, *is an S-function from* **D** to **E**.
proof : function f is continuous since d and e are finite elements.
Now $\mu(f, x, \beta)$ is definable only if $\beta \preceq e$ and if $(x \not\geq d)\ \&\ (\exists\ z > x$ s.t.: $z \geq d);$
but in that case one immediately verifies that $\mu(f, x, \beta)$ can be set to d. ∎

Corollary *any constant function is an S-function.*

Fact 2 : *S-functions are not divided in strict versus constant continuous functions*: for example, function $f : \mathbf{T}^2 \longrightarrow \mathbf{T}^2$ s.t.:

$$f(x) = \text{ if } x \geq (\perp, tt) \text{ then } (tt, tt) \text{ else } (tt, \perp)\ ,$$

is an S-function which is neither strict nor constant.

Fact 3 : *S-functions are strictly included in continuous functions* :
1. The paradigmatic "parallel or" function is not an S-function
since $\mu(por,(\bot,\bot),tt)$ does not exist; indeed $(\bot,tt)\not\leq(tt,\bot)\not\leq(\bot,tt)$.
2. Berry's function is not an S-function : It is the least continuous function
$f:\ \mathbf{T}^3\longrightarrow\mathbf{T}$ s.t.: $f(tt,\mathrm{ff},\bot)=f(\mathrm{ff},\bot,tt)=f(\bot,tt,\mathrm{ff})=tt$.
We can now remark that $\not\exists\,\mu(f,(\bot,\bot,\bot),tt)$.
3. The Curien's examples in proposition 4.4.2 [Cur86] are not S-functions; let us look for example the function A^1 :

A^1 is characterised by :
$A^1(a_1^1)=tt,\ A^1(a_2^1)=\mathrm{ff}$, where a_1^1,a_2^1 are themself characterized by:
$a_1^1(tt,\bot)=tt,\ a_1^1(\mathrm{ff},tt)=tt,\ a_1^1(\mathrm{ff},\mathrm{ff})=\mathrm{ff}$
$a_2^1(\bot,tt)=tt,\ a_2^1(tt,\mathrm{ff})=tt,\ a_2^1(\mathrm{ff},\mathrm{ff})=\mathrm{ff}$

Since a_3^1 which s.t.: $a_3^1(\bot,\bot)=tt$, and $a_3^1(\mathrm{ff},\mathrm{ff})=\top$ which is greater than
a_4^1 : $a_4^1(\mathrm{ff},\mathrm{ff})=\mathrm{ff}$ and $a_4^1(\mathrm{ff},tt)=tt$ and $a_4^1(tt,\mathrm{ff})=tt$ and
$A^1(a_4^1)=\bot$ and $A^1(a_3^1)=\top$ but there those not exist an index for the value a_4^1 at value tt.

Fact 4 : *absence of indexes are not nessarily detected in* $\bot$:
define f to be the least continuous function $\in[\mathbf{T}^3\longrightarrow\mathbf{T}]$ s.t.:
$f(x)=tt$ iff $x\geq(tt,\mathrm{ff},\bot)$ or $x\geq(tt,\bot,\mathrm{ff})$; then
f is not an S-function since $\not\exists\,\mu(f,(tt,\bot,\bot),tt)$ thought $\exists\,\mu(f,(\bot,\bot,\bot),tt)$.

Now we relate S-functions to stable functions.
In fact S-functions are stable functions; the difference between them is that S-functions are also *"intensionally stable"* in particular in domains, where elements could be intensionally preordered without being extensionally ordered.
For example, considering the function $f\in[\,\mathbf{T}^3\longrightarrow\mathbf{T}]$ s.t.:
$f(x)=tt$ iff $x\geq(tt,tt,\bot)$ or $x\geq(\mathrm{ff},\bot,tt)$ then f is an S-function, though it cannot be exhibited an element ϵ which could be put for $\mu(f,(\bot\bot\bot),tt)$ and s.t.: $tt\leq f(\epsilon)$.

A first result of section is the following:

4.4 theorem : *the function set of S-functions between two S-domains is an S-domain, where greatest lower bounds are taken pointwisely. Its finite elements are those lubs of finite set of elements of the form* $\{(d_i\Longrightarrow e_i)/i\in I\}$ *(where d and e are finite elements of the two S-domains) which verify condition (S).*
We use the following lemma:
lemma: let be two S-domains $\mathbf{D}$ and $\mathbf{E}$; then their set of S-functions is a lower-complete lattice where glb are taken pointwisely:
proof: the constant function $\lambda x.\bot\in[\mathrm{D}\underset{s}{\longrightarrow}\mathrm{E}]$ is trivialy the bottom
in $[\mathrm{D}\underset{s}{\longrightarrow}\mathrm{E}]$. Now let $\mathcal{S}=\{f_i/i\in I\}$; then $\exists f\in[D\longrightarrow E]$ s.t.: $f=\bigwedge\mathcal{S}$.
We now proof that f is $\in[\mathrm{D}\underset{s}{\longrightarrow}\mathrm{E}]$.

Let $x, z \in D : x \leq z$ and $\beta \in E$ s.t.:
$f(x) \not\geq \beta \leq f(z)$ ie: $\{\bigwedge f_i(x)/i \in I\} \not\geq \beta \leq \{\bigwedge f_i(z)/i \in I\}$ and thus
$\exists f_{i_0} \in \mathcal{S}$ s.t.: $f_{i_0}(x) \not\geq \beta \leq f_{i_0}(z)$ then $\exists\ \mu = \mu(f_{i_0}, x, \beta)$ s.t.:

$$x \not\geq \mu \leq z \quad \&$$

$\forall y \in D, (y \geq x\ \&\ y \leq z\ \&\ \beta \preceq f(y)) \Rightarrow \mu \preceq y$.

the proof of the theorem:
let be two S-domains **D** and **E** ; then of S-functions ($[D \underset{s}{\longrightarrow} E]$) is a lattice, since its is a lower complete lattice and the constant $\lambda x.\top$ is trivially its greatest element.
Function: $f = \bigvee\{(d_i \Longrightarrow e_i)/i \in I\}$ with I finite, and $d_i \in \mathcal{B}(D)$ and $e_i \in \mathcal{B}(E)$, are finite elements in $[D \longrightarrow E]$; *so*, as soon as they verify condition (S), they are also in $[D \underset{s}{\longrightarrow} E]$.
It is trivial that $[D \underset{s}{\longrightarrow} E]$ is finitary. ∎

Let us now examine the category **SD**.

5 THE Λ-CATEGORY SD

First of all, recall that a Λ-category is a cartesian closed category which is order-enriched, together with some additional continuity properties.

5.1 theorem : **SD** *is cartesian closed.*
proof : we verify that **SD** has products and exponentials.
1) *product* :
i) The cartesian product of a denumerable family of domains is a domain. This is a well known fact. It's partial order is the product of the orderd of the domains. Now to get an S-domain, we define the intensional preorder to be also the product of the intensional preorders of the domains.
ii) The projection functions $\pi_i : [(D_1 \times D_2) \longrightarrow D_i]$ *for* $i = 1, 2$ are S-functions:
It is easy to verify that for any $\beta \in D_i,\ i = 1, 2$ and for
$x =< x_1, x_2 >, z =< z_1, z_2 >\in D_1 \times D_2\ : x \leq z$ one can set:
$\mu(\pi_1, x, \beta) =< \beta, \perp >$ and $\mu(\pi_2, x, \beta) =< \perp, \beta >$.
iii) If $f \in$ [F $\underset{s}{\longrightarrow}$ D] & $g \in$ [F $\underset{s}{\longrightarrow}$ E] then $< f, g >\in$ [F $\underset{s}{\longrightarrow}$ $D \times E$] :
one can easely verify that $\mu(< f, g >, x, \beta)$ can be set to $\mu(f, x, \pi_1(\beta))$ or to $\mu(g, x, \pi_2(\beta))$.

2) exponential :

i) *function app :* $[< [D \underset{s}{\longrightarrow} E] \times D > \longrightarrow E]$ *is an S-function:*
Let $< f, x >, < h, z >\in<$ [D $\underset{s}{\longrightarrow}$ E] $\times D >$ s.t.: $< f, x > \leq < h, z >$,
and let $\beta \in E : h(z) \geq \beta \not\leq f(x)$; we have to exhibit $\mu(app, < f, x >, \beta) =$

$\mu \in < [D \underset{s}{\longrightarrow} E] \times D >$ s.t.: $< h, z > \ \geq \ \mu \ \not\leq \ < f, x >$ &
$\forall < g, y > \in < [D \underset{s}{\longrightarrow} E] \times D >, (< f, x > \leq < g, y > \leq < h, z > \ \& \ \beta \preceq g(y)) \Rightarrow$
$\mu \preceq < g, y >$ various cases:

$-\exists z \geq y \not\leq x : \beta \leq f(y)$ in this case there exist $\mu(f, x, \beta)$ s.t.:
$z \geq \mu(f, x, \beta) \not\leq x$ &
$\forall y : (y \geq x \ \ \& \ z \geq y \ \ \& \ \beta \preceq f(y)) \Rightarrow \mu(f, x, \beta) \preceq y$.
In this case $< f, \mu(f, x, \beta) >$ is a possible value for μ .

$-\forall y \geq x, \beta \not\leq f(y)$
In this case, $< (z \Longrightarrow \beta), x >$ can be put for μ .

ii) *function curry is an S-isomorphism of S-domains:*
A) consider an S-function $f \in [\ D \times E \underset{s}{\longrightarrow} F]$: we want to show that function $f_a \in [E \longrightarrow F]$ s.t.: $\forall b \in E, f_a(b) = f(a, b)$, verifies (S).
Let $x, z \in E : z \geq x$ and let $\beta \in F$ and suppose $f_a(x) \not\geq \beta \leq f_a(z)$ then we have: $(a, x), (a, z) \in D \times E : (a, x) \leq (a, z) \ \ \& \ \ \beta \in F \ \ \& \ \ f(a, x) \not\geq \beta \leq f(a, z)$;
thus $\exists \ \mu(f, (a, x), \beta) = \mu$ s.t.:$(a, x) \not\geq \mu \leq (a, z) \ \& \ \forall (a, y) \in D \times E$:
$((a, x) \leq (a, y) \ \& \ (a, y) \leq (a, z) \ \& \ \beta \preceq f(a, y)) \Rightarrow \mu \preceq (a, y)$
We can set $\mu(f_a, x, \beta) = \pi_2(\mu)$.
B) considering the function curry itself, we have to show:
$\forall f, g \in [\ D \times E \underset{s}{\longrightarrow} F] : g \geq f, \quad \forall \phi \in [D \underset{s}{\longrightarrow} [E \underset{s}{\longrightarrow} F]\ ]$,
$curry(f) \not\geq \phi \leq curry(g) \ \exists \mu(curry, f, \phi) = \mu$ s.t.: $f \not\geq \mu \leq g$ &
$\forall h, (f \leq h \leq g \ \ \& \ \phi \preceq curry(h)) \Rightarrow \mu \preceq h$.
It trivial that a suitable value for μ is the function $\phi' \in [\ D \times E \underset{s}{\longrightarrow} F]$ s.t.:
$\forall < x, y > \in D \times E, \phi'(< x, y >) = \phi(x)(y)$.
To end the proof, notice that curry is an isomorphism of domains and thus it is an isomorphism of S-domains by its definition and the above. ∎

A significant result is the following:

5.2 <u>theorem</u> : **SD** *is an order-enriched* Λ*-category whose objects are S-domains in which every finite element is computable in a sequentialy way* .
<u>proof</u> : the fact **SD** is an order-enriched Λ-category, comes from the above theorem; notice that we do not need to worry about continuity, since we are using the pointwise partial order.
The finite elements of an S-domain are lub of finite subset $F = \{(d_i \Longrightarrow e_i)/i \in I\}$ which verify axiom (S), s.t.: given $x \in D$, its image is
$f(x) = \bigvee \{e_i/(d_i \Longrightarrow e_i) \in F \ \& \ d_i \leq x\}$. A sequential algorithm for computing this value is then the following:
Suppose x s.t.: $f(x) \neq \perp$. Assume that f is not a constant. Then must exist an element $\mu(f, \perp, f(x))$. Either $f(\mu(f, \perp, f(x))) = f(x)$ and the process stops, <u>or</u>
we choose $\mu(f, \perp, f(x)) \leq x$ (which must exist!) and we itterate the process: $\mu(f, \mu(\ldots), f(x))$;

since our S-domains are finitary, the sequence must stop, yielding the value $f(x)$. ∎

Now, considering the case of PCF (which we modify by adding a constant representing $\top^\sigma$ at all types σ), we can claim, using the algebraicity lemma (see[Mil77]) that the model is fully-abstract. And considering Milner's results, it is unique (up to isomorphism).

For finishing, we relate S-functions to the continuous ones

5.3 theorem : *Let* $< \mathbf{D}, \leq >$ *where* $\mathbf{D} = [D_1 \longrightarrow D_2]$ *be the set of continuous functions from* D_1 *to* D_2*; and suppose their S-functions* $< \mathbf{D}_s, \leq >$ *where* $\mathbf{D}_s = [D_1 \underset{s}{\longrightarrow} D_2]$ *we want to show that they constitute a retraction.*
proof : we want to show that the mappings $s : \mathbf{D} \longrightarrow \mathbf{D}_s$ defined by
$s(f) = \bigwedge\{g \in \mathbf{D}_s / f \leq g\}$ is a projection, and the mapping $i : \mathbf{D}_s \longrightarrow \mathbf{D}$ defined by $i(g) = g$, (which is an injection) constitutes a retraction.
It is immediate to see that s is a projection ($f_1 \leq f_2 \Longrightarrow s(f_1) \leq s(f_2)$:
$f_1 \leq f_2 \Longrightarrow \forall g \in \mathbf{D}_s, g \geq f_2 \Longrightarrow g \in \mathbf{D}_s, g \geq f_1 \Longrightarrow s(f_1) \leq s(f_2)$.
$s(f) \geq f$ is immediate .
$s \circ s(f) = f : f_1 = s(f) = \bigwedge\{g \in \mathbf{D}_s / f \leq g\}$; and thus
$s \circ s(f) = s(f_1) = f_1$ since f_1 is in $\mathbf{D}_s$.)
That $i \circ s(f) \geq f$ is trivial since $s(f) \geq f$.
To see $s \circ i(g) = g$ is immediate, since $s \circ i(g) = s(g) = \bigwedge\{g' \in \mathbf{D}_s / g \leq g'\} = g$. ∎

6 CONCLUDING REMARKS

In this paper, we generalize the notion of stable function, getting intensionally stable functions, which can be combined to S-domains, to get an order-enriched Λcategory. We provide sequential algorithm to evaluate the finite elements of S-domains. Thus we can imagine that our model is the same as Milner's model (which is fully-abstract and unique).

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