Predictability of Oceanic Circulations

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Abstract: In this paper, we study the Navier-Stokes equation in which the unknown functions are the vorticity and the streamfunction. Our goal is to estimate the long-time influence of a small error in the initial data. In order to do this, we write down the symmetrized linearized Navier-Stokes equation and we estimate the eigenvalues of the corresponding operator. A comparison between these estimates and the numerical value of the eigenvalues is proposed.

Key-words: Navier-Stokes equation, Multilayer model, Oceanic circulation, Lyapunov exponents, Predictability.

(Résumé : tsvp)

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Prédicibilité des circulations océaniques

Résumé : Dans ce rapport, on considère l'équation de Navier-Stokes, avec pour fonctions inconnues la vorticité et la fonction de courant. Notre but est d'estimer l'influence à long terme d'une petite perturbation des conditions initiales. Pour cela, nous écrivons l'équation linéarisée et symmétrisée, et nous estimons les valeurs propres de l'opérateur correpondant. Une comparaison entre ces estimations et les valeurs numériques est proposée.

Mots-clé : Equation de Navier-Stokes, modèle multicouche, circulation océanique, exposants de Lyapunov, Prédicibilité.

1 Introduction.

Connected with the problem of data assimilation, the problem of predictability is now under investigation by many teams [1], [2], [6], [11], [15]. We are interested in the long-time influence of a small error in the initial data, so we linearize and symmetrize the Navier-Stokes equations and study the corresponding linear operator. The local behaviour of a small perturbation is described by the local Lyapounov exponents. They are the eigenvalues of the symmetrized operator. Our aim is to present the theoretical estimates of these eigenvalues as for one-layer modelisation and for the multi-layer one [7], [16], [4]. Relying on the well-known results of existence, uniqueness and existence of the attractor for the equation under consideration [4], [12], [17], [19], we construct and study its symmetrized linearized operator. We propose the proof of the continuity in time of its eigenvalues. Finally we establish a priori estimates for the eigenvalues using their variational characterization [14]. These a priori estimates of the eigenvalues take into account the inverse of the internal radii of deformation of each layer. A good correlation is observed between these estimates and numerical computations of the eigenvalues.

2 One-layer model.

2.1 Existence and Uniqueness Theorems.

Let Ω be a bounded open subset of $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. We consider the system :

$$\left\{\begin{array}{lll} \dfrac{\partial \xi}{\partial t} - \mu\Delta\xi + \dfrac{\partial \psi}{\partial x} + J(\psi,\xi) = v & \text{in} & \Omega\times]0,T[, \\ \xi = 0 & \text{on} & \partial\Omega\times]0,T[, \\ \Delta\psi = \xi & \text{in} & \Omega\times]0,T[, \\ \psi = 0 & \text{on} & \partial\Omega\times]0,T[, \\ \xi(0,x) = \xi_0(x), & & \end{array}\right. \tag{1}$$

where $J(f,g) = \dfrac{\partial f}{\partial x} \times \dfrac{\partial g}{\partial y} - \dfrac{\partial f}{\partial y} \times \dfrac{\partial g}{\partial x}$. We suppose v is given in $L^2(\Omega)$. We recall the following results [17], [19], [12], [4].

Theorem 2.1. *For ξ_0 in $H^{-1}(\Omega)$, system (1) admits a unique solution $\xi(t,x)$ in $\mathrm{C}([0,T],H^{-1}(\Omega)) \cap L^2(0,T,L^2(\Omega)) \cap L^2_{loc}(0,T,H^1_0(\Omega))$. The semi-group $G(t)$ from $H^{-1}(\Omega)$ in $H^{-1}(\Omega)$, $G(t)\xi_0(x) = \xi(t,x)$, associated to these equations, is such that there exists a maximal attractor A which is bounded in $L^2(\Omega)$, compact and connected in $H^{-1}(\Omega)$ and whose basin of attraction is the whole space $H^{-1}(\Omega)$. This attractor has finite Hausdorff and fractal dimensions. For ξ_0 in $H^1_0(\Omega)$, system (1) admits a unique solution $\xi(t,x)$ in $\mathrm{C}([0,T],H^1_0(\Omega)) \cap L^2(0,T,H^2(\Omega))$.*

We linearize the system (1) around a solution $\xi(t,x)$ to obtain

$$\begin{cases} \dfrac{\partial U}{\partial t} - \mu\Delta U + \dfrac{\partial V}{\partial x} + J(\psi, U) + J(V, \xi) = 0 & \text{in} \quad \Omega\times]0,T[, \\ U = 0 & \text{on} \quad \partial\Omega\times]0,T[, \\ \Delta V = U & \text{in} \quad \Omega\times]0,T[, \\ V = 0 & \text{on} \quad \partial\Omega\times]0,T[, \\ U(0,x) = U_0(x), \end{cases} \tag{2}$$

The existence and the uniqueness of the solution are stated next and follow from general results in [18]:

Theorem 2.2. *For U_0 in $H^{-1}(\Omega)$, system (2) admits a unique solution $U(t,x)$ in $\mathrm{C}([0,T], H^{-1}(\Omega)) \cap L^2(0,T,L^2(\Omega)) \cap L^2_{loc}(0,T,H^1_0(\Omega))$.*

The third proposition links the two systems (1) and (2) (see [19], [12], [4]).

Theorem 2.3. *For $t > 0$, the semi-group $G(t)$ is uniformly differentiable on A. Its differential in ξ_0 is the linear operator on $H^{-1}(\Omega)$ given by $U \to L(t_0,\xi_0)U_0 = U(t_0)$, where $U(t_0)$ is the value at time $t = t_0$ of the solution $U(t)$ of the linearized system (2). Moreover,*

$$\sup_{\xi_0 \in A} |L(t_0,\xi_0)|_{L(H^{-1})} < \infty.$$

From now on, we consider ξ_0 in $H^1_0(\Omega)$ and we denote by $B = B(t,\xi_0)$ the unbounded operator on $H^{-1}(\Omega)$ defined by

$$BU = -\mu\Delta U + \frac{\partial V}{\partial x} + J(\psi,U) + J(V,\xi), \text{ and } \Delta V = U$$

where $D(B) = H^1_0(\Omega)$. We denote by $S = S(t,\xi_0)$ the symmetric operator $S = \dfrac{B+B^*}{2}$. We identify $H^{-1}(\Omega)$ with its dual and we introduce on $H^{-1}(\Omega)$ the scalar product $((.,.))$ defined by

$$((u_1,u_2)) = - < u_1, v_2 >_{-1,1} = (\nabla v_1, \nabla v_2)$$

where $(.,.)$ is the usual scalar product in L^2 and where $< .,. >_{-1,1}$ denotes the usual duality pairing between $H^{-1}(\Omega)$ and $H^1(\Omega)$; v_i is defined by $\Delta v_i = u_i$, $v_i = 0$ on $\partial\Omega$. The above scalar product on $H^{-1}(\Omega)$ defines the usual norm $||.||$. We denote by $|.|$ the usual L^2-norm. Let us determine B^*. We recall that an integration by parts shows that $(J(f,g),h) = -(J(f,h),g)$ if one of the three functions f, g, h is equal to zero on the boundary [17], [19], [4]. We then obtain

$$\begin{aligned} ((Bu_1,u_2)) &= - < Bu_1, v_2 >_{-1,1} \\ &= < \mu\Delta u_1, v_2 >_{-1,1} - < \frac{\partial v_1}{\partial x}, v_2 >_{-1,1} - \\ &\quad - < J(\psi,u_1), v_2 >_{-1,1} - < J(v_1,\xi), v_2 >_{-1,1} \\ &= \mu\ (u_1,u_2) + (v_1, \frac{\partial v_2}{\partial x}) + \ (J(\psi,v_2),u_1) + \ (J(v_2,\xi),v_1) \\ &= \mu\ (u_2,u_1) + (\frac{\partial v_2}{\partial x}, v_1) + \ (\Delta J(\psi,v_2),v_1) + \ (J(v_2,\xi),v_1) \\ &= - < -\mu\Delta u_2 - \frac{\partial v_2}{\partial x} - \Delta J(\psi,v_2) - J(v_2,\xi),\ v_1 >_{-1,1} \end{aligned}$$

$$
\begin{aligned}
&= ((B^* u_2, u_1)) \\
&= ((u_1, B^* u_2))
\end{aligned}
$$

so that

$$B^* u_2 = -\mu \Delta u_2 - \frac{\partial v_2}{\partial x} - \Delta J(\psi, v_2) - J(v_2, \xi).$$

Consequently

$$Su = -\mu \Delta u + \frac{1}{2}\left[J(\psi, u) - \Delta J(\psi, v)\right].$$

2.2 The operator S.

Theorem 2.4. *The operator S is a self-adjoint and closed operator on $H^{-1}(\Omega)$ with compact resolvent. Its eigenvalues λ_i are real and bounded from below by*

$$
\begin{aligned}
\lambda_i &\geq \mu\mu_1 - ||\psi||_{H^2}\sqrt{\mu_1} \qquad \text{if} \qquad ||\psi||_{H^2} \leq 2\mu\sqrt{\mu_1}, \\
\lambda_i &\geq -\frac{||\psi||^2_{H^2}}{4\mu} \qquad \text{otherwise.}
\end{aligned}
\tag{3}
$$

where μ_1 is the first eigenvalue of the Laplacian operator with homogeneous Dirichlet boundary conditions.

Proof. Let us bound the norm of Su_1 from below. Using successively an integration by parts, the Sobolev's embedding theorem and the Gagliardo-Nirenberg's inequality, we obtain (see [4])

$$|(J(f, v_1), u_1)| \leq ||f||_{H^2(\Omega)} \times |\nabla v_1| \times |u_1|.$$

We now evaluate

$$
\begin{aligned}
((Su, u)) &= \mu(\Delta u, v) - \frac{1}{2}\left[(J(\psi, u), v) - (\Delta J(\psi, v), v)\right] \\
&\geq \mu(u, u) - ||\psi||_{H^2}\ |\nabla v|\ |u| \\
((Su, u)) &\geq \mu\ |u|^2 - ||\psi||_{H^2}||u||\ |u|.
\end{aligned}
$$

So let us consider the operator $T = S + hI$, where I is the identity operator and h a real number such that $h \geq \frac{||\psi||^2_{H^2}}{2\mu}$. We obtain

$$
\begin{aligned}
((Tu, u)) &= ((Su, u)) + h((u, u)) \\
((Tu, u)) &\geq \mu\ |u|^2 - ||\psi||_{H^2}||u||\ |u| + h||u||^2 \\
((Tu, u)) &\geq \left(\sqrt{\frac{\mu}{2}}|u| - \frac{||\psi||_{H^2}}{\sqrt{2\mu}}||u||\right)^2 + \frac{\mu}{2}|u|^2 \\
||Tu||\ ||u|| &\geq \frac{\mu}{2}|u|^2 \\
||Tu||\ ||u|| &\geq \frac{\mu\sqrt{\mu_1}}{2}|u|\ ||u|| \\
||Tu|| &\geq \frac{\mu\sqrt{\mu_1}}{2}|u|.
\end{aligned}
$$

Therefore the operator T has a bounded inverse. Then T is a closed operator and consequently so is S. Now, using the compact embedding from $L^2(\Omega)$ into $H^{-1}(\Omega)$, we prove that T is an operator with compact resolvent. We then conclude that the same property holds true for S. The operator S is a self-adjoint operator with compact resolvent. Its eigenvalues are real. We now bound them from below. We have already seen that

$$((Su,u)) + h((u,u)) \geq \mu \ |u|^2 - ||\psi||_{H^2}||u|| \ |u| + h||u||^2.$$

Let ϵ be a positive real number. We have for $h \geq \dfrac{||\psi||^2_{H^2}}{4\epsilon\mu}$ the estimate

$$((Su,u)) + h((u,u)) \geq \left(\sqrt{\epsilon\mu}|u| - \frac{||\psi||_{H^2}}{2\sqrt{\epsilon\mu}}||u||\right)^2 + (1-\epsilon)\mu|u|^2$$

so that if u is an eigenvector corresponding to the eigenvalue λ, we obtain

$$\begin{aligned} (\lambda+h)||u||^2 &\geq (1-\epsilon)\mu\mu_1||u||^2 \\ (\lambda+h) &\geq (1-\epsilon)\mu\mu_1 \\ \lambda &\geq (1-\epsilon)\mu\mu_1 - h. \end{aligned}$$

In particular, for $\epsilon = \frac{1}{2}$ we obtain :

$$\lambda \geq \frac{\mu\mu_1}{2} - \frac{||\psi||^2_{H^2}}{2\mu}. \tag{4}$$

Optimizing in ϵ, we get

$$\begin{array}{lll} \lambda_i \geq \mu\mu_1 - ||\psi||_{H^2}\sqrt{\mu_1} & if & ||\psi||_{H^2} \leq 2\mu\sqrt{\mu_1}, \\ \lambda_i \geq -\frac{||\psi||^2_{H^2}}{4\mu} & otherwise. & \end{array}$$

From now on, the eigenvalues are ordered in such a way that : $\lambda_1 \leq \lambda_2 \leq \ ...$

2.3 Eigenvalues of the operator S and predictability.

Let us consider m solutions of system (2), $U_1(t), U_2(t), .., U_m(t)$, corresponding to the m initial data $U_1^0, U_2^0, ..., U_m^0$. We denote by
$| \ U_1^0 \ \wedge \ U_2^0 \ \wedge \ ... \ \wedge \ U_m^0 \ |$ the m-volume of the parallelepiped with edges $U_1^0, U_2^0, ..., U_m^0$, and by $| \ U_1(t) \wedge U_2(t) \wedge ... \wedge U_m(t) \ |$ the m-volume of the parallelepiped with edges $U_1(t), U_2(t), .., U_m(t)$.
Define

$$\omega_m(t,\xi_0) = \sup_{\substack{U_i^0 \in H^{-1}(\Omega) \\ ||U_i^0|| \leq 1}} \frac{| \ U_1(t) \ \wedge \ U_2(t) \ \wedge \ ... \ \wedge \ U_m(t) \ |}{| \ U_1^0 \ \wedge \ U_2^0 \ \wedge \ ... \ \wedge \ U_m^0 \ |}.$$

The numbers $\omega_m(t,\xi_0)$ determine the largest distortion of an infinitesimal m-dimensional volume generated by $G(t)$ around the point $\xi_0 \in H^{-1}(\Omega)$.
One can prove that [19]

$$| \ U_1(t) \ \wedge \ ... \ \wedge \ U_m(t) \ | = | \ U_1^0 \ \wedge \ ... \ \wedge \ U_m^0 \ | \ exp\left(\int_0^t Tr(-B(s,\xi_0)oQ_m(s))ds\right)$$

where $Q_m(s) = Q_m(s, U_1^0, U_2^0, ..., U_m^0)$ is the projector from $H^{-1}(\Omega)$ onto the space spanned by $U_1(t), U_2(t), .., U_m(t)$, and where Tr denotes the trace operator. It follows that

$$\omega_m(t, \xi_0) = \sup_{\substack{U_i^0 \in H^{-1}(\Omega) \\ ||U_i^0|| \leq 1}} exp\left(\int_0^t Tr(-B(s, \xi_0)oQ_m(s))ds\right)$$

and so

$$\frac{1}{t}log\omega_m(t, \xi_0) = \sup_{\substack{U_i^0 \in H^{-1}(\Omega) \\ ||U_i^0|| \leq 1}} \left(\frac{1}{t}\int_0^t Tr(-B(s, \xi_0)oQ_m(s))ds\right) \tag{5}$$

But by definition, we have

$$Tr(-B(s, \xi_0)oQ_m(s)) = \sum_{i=1}^{i=m}((-B\phi_i(s), \phi_i(s)\ ))$$

where $\phi_i(s), i \in \mathbb{N}$, is an orthonormal basis of $H^{-1}(\Omega)$, $\phi_i(s) \in L^2(\Omega)$, with $\phi_1(s), ..., \phi_m(s)$ spanning $Q_m(s)H^{-1}(\Omega) = Span\,[U_1(t), U_2(t), .., U_m(t)]$. Noticing that

$$((-B\phi_i(s), \phi_i(s)\ )) = ((-S\phi_i(s), \phi_i(s)\ )),$$

we conclude that

$$Tr(-B(s, \xi_0)oQ_m(s)) = Tr(-S(s, \xi_0)oQ_m(s)),$$

and, since S is self-adjoint,

$$Tr(-S(s, \xi_0)oQ_m(s)) = \sum_{i=1}^{i=m}((-S\phi_i(s), \phi_i(s)\ )) \leq \sum_{i=1}^{i=m}\lambda_i(-S(s, \xi_0)).$$

From inequality (5), we obtain

$$\lim_{t\to 0} \frac{1}{t}\ log\omega_m(t, \xi_0)\ \leq\ \lim_{t\to 0}\left(\frac{1}{t}\int_0^t \sum_{i=1}^{i=m}\lambda_i(-S(s, \xi_0))\ ds\right) \tag{6}$$

We now prove :

Theorem 2.5. *The following equality holds :*

$$\lim_{t\to 0} \frac{1}{t}\ log\omega_m(t, \xi_0)\ =\ \sum_{i=1}^{i=m}\lambda_i(-S(0, \xi_0)) \tag{7}$$

We first prove the following.

Lemma. The eigenvalue $\lambda(S(t, \xi_0))$ converges to $\lambda(S(0, \xi_0))$ continuously when t tends to zero.
Remark. The continuity in time of the operator $Tr(-S(t, \xi_0)oQ_m(t))$ follows from this lemma.

We divide the proof into 3 steps.

First step. The operator S is continuous in time.

We prove the continuity of S in the H^{-1}-norm. We have

$$\begin{aligned}||S(t)U - S(t_0)U|| &= \frac{1}{2}||J(\psi(t),U) - \Delta J(\psi(t),V) - \\ &\quad -J(\psi(t_0),U) + \Delta J(\psi(t_0),V)|| \\ &= \frac{1}{2}||J(\psi(t)-\psi(t_0),U) - \Delta J(\psi(t)-\psi(t_0),V)|| \\ &= \frac{1}{2}||AU||\end{aligned}$$

where $AU = J(\psi(t)-\psi(t_0),U) - \Delta J(\psi(t)-\psi(t_0),V)$.
We introduce $\Psi = \psi(t)-\psi(t_0)$, $\Xi = \xi(t)-\xi(t_0)$. Let us evaluate $||AU||$. We have

$$\begin{aligned}< AU, v > &= < J(\Psi,U), v > - < \Delta J(\Psi,V), v >, \\ < AU, v > &= < J(\Psi,\Delta V), v > - < J(\Psi,V), u > .\end{aligned}$$

An integration by part of the first term leads to the estimates :

$$\begin{aligned}< AU, v > &\leq |< J(\Xi,V), v >| + 2\int |v_x J(\Psi_x,V) + v_y J(\Psi_y,V)|\, d\omega, \\ < AU, v > &\leq |\nabla\Xi| ||V||_\infty |\nabla v| + 2\left(|v_x||\nabla\Psi_x|_{L^4} + |v_y||\nabla\Psi_y|_{L^4}\right)|\nabla v|_{L^4}, \\ &\leq |\nabla\Xi||\nabla v||U| + c|\nabla v||\nabla\Xi||U|, \\ &\leq c'|\nabla\Xi||\nabla v||U|.\end{aligned}$$

So $||AU|| \leq c'|U||\nabla\Xi|$ and then

$$||S(t)U - S(t_0)U|| \leq c'|U|\ |\nabla\xi(t) - \nabla\xi(t_0)|.$$

Since $\nabla\xi(t)$ is continuous in time, we obtain the result.

Second step. The operator S is continuous in the "generalized sense" with respect to the viscosity μ.

Proof. We denote by S_n the closed operator in $H^{-1}(\Omega)$ defined by

$$S_n u_n = -\left(\mu + \frac{1}{n}\right)\Delta u_n + \frac{1}{2}\left(J(\psi,u_n) - \Delta J(\psi,v_n)\right)$$

and prove that S_n converges to S in the generalized sense defined by Kato [13]. Following [13], let us recall a useful criterion for convergence "in the generalized sense". Let T be a closed operator in H, with a non-empty resolvent set $\rho(T)$. We denote by $R(\zeta,T)$ the resolvent of T.
Lemma. In order that a sequence T_n of closed operators converges to T in the generalized sense, it is necessary that each $\zeta\in\rho(T)$ belong to $\rho(T_n)$ for sufficiently large n and that $||R(\zeta,T_n)-R(\zeta,T)|| \to 0$ when $n\to\infty$, while it is sufficient that this be true for some $\zeta\in\rho(T)$.

We know that $\lambda < \frac{\mu\mu_1}{2} - \frac{||\psi||^2_{H^2}}{2\mu}$ implies $\lambda\in\rho(T)$. So we now consider $\zeta \geq \frac{||\psi||^2_{H^2}}{2\mu}$ and estimate $||R(-\zeta,T_n) - R(-\zeta,T)||$. We introduce u_n and u the solutions of $(S_n+\zeta I)u_n = f$ and $(S+\zeta I)u = f$. Substracting these two equations, we obtain

$$S_n u_n - Su + \zeta(u_n - u) = 0.$$

Since $S_n u_n = Su_n - \frac{1}{n}\Delta u_n$, it follows that $S(u_n - u) + \zeta(u_n - u) = \frac{1}{n}\Delta u_n$.
We write $w = u_n - u$ and $z = v_n - v$ and we recall that $\Delta z = w$. We then rewrite the last equation as

$$-\mu\Delta w + \frac{1}{2}\left(J(\psi, w) - \Delta J(\psi, z)\right) + \zeta w = \frac{1}{n}\Delta u_n.$$

Multiplying by z and integrating on Ω, we obtain:

$$\begin{aligned}
\mu|w|^2 - (J(\psi, w), z) + \zeta|\nabla z|^2 &\leq \frac{1}{n}|u_n||w|, \\
\mu|w|^2 + \zeta|\nabla z|^2 &\leq \frac{1}{n}|u_n||w| + ||\psi||_{H^2}|\nabla z||w|, \\
&\leq \mu|w|^2 + \frac{1}{2n^2\mu}|u_n|^2 + \frac{1}{2\mu}||\psi||_{H^2}^2|\nabla z|^2
\end{aligned}$$

and so

$$|\nabla z|^2\left(\zeta - \frac{1}{2\mu}||\psi||_{H^2}^2\right) \leq \frac{1}{2n^2\mu}|u_n|^2. \tag{8}$$

We now estimate $|u_n|^2$. Multiplying the equation $(S_n + \zeta I)u_n = f$ by v_n and integrating on Ω, we have

$$\begin{aligned}
\left(\mu + \frac{1}{n}\right)|u_n|^2 + \zeta|\nabla v_n|^2 &\leq ||f||\ |\nabla v_n| + |(J(\psi, u_n), v_n)| \\
&\leq ||f||\ |\nabla v_n| + ||\psi||_{H^2}\ |\nabla v_n|\ |u_n| \\
&\leq ||f||\ |\nabla v_n| + \frac{\mu}{2}|u_n|^2 + \frac{||\psi||_{H^2}^2}{2\mu}|\nabla v_n|^2.
\end{aligned}$$

Consequently, we get

$$\begin{aligned}
\frac{\mu}{2}|u_n|^2 + \left(\zeta - \frac{||\psi||_{H^2}^2}{2\mu}\right)|\nabla v_n|^2 &\leq ||f||\ |\nabla v_n| \\
\frac{\mu}{2}|u_n|^2 + \left(\zeta - \frac{||\psi||_{H^2}^2}{2\mu}\right)|\nabla v_n|^2 &\leq ||f||\ \sqrt{\lambda_1}|u_n| \\
\frac{\mu}{2}|u_n| &\leq ||f||\ \sqrt{\lambda_1} \\
|u_n| &\leq \frac{2\ ||f||\ \sqrt{\lambda_1}}{\mu}.
\end{aligned}$$

Inserting the last estimate into (8), we obtain

$$\begin{aligned}
|\nabla z|^2\left(\zeta - \frac{1}{2\mu}||\psi||_{H^2}^2\right) &\leq \frac{2}{n^2\mu^3}||f||^2\ \lambda_1 \\
\frac{|\nabla z|}{||f||} &\leq \frac{1}{\mu n}\sqrt{\frac{4\lambda_1}{2\mu\zeta - ||\psi||_{H^2}^2}}.
\end{aligned}$$

Since

$$\begin{aligned}
||R(-\zeta, T_n) - R(-\zeta, T)|| &= \sup_{f\in H^{-1}} \frac{||u_n - u||}{||f||} \\
&= \sup_{f\in H^{-1}} \frac{|\nabla v_n - \nabla v|}{||f||} \\
&= \sup_{f\in H^{-1}} \frac{|\nabla z|}{||f||},
\end{aligned}$$

we get

$$||R(-\zeta,T_n)-R(-\zeta,T)|| \quad \leq \quad \frac{1}{\mu n}\sqrt{\frac{4\lambda_1}{2\mu\zeta-||\psi||^2_{H^2}}} \rightarrow 0$$

as n goes to infinity. This proves the convergence in the generalized sense of S_n to S.

Third step. We now prove the lemma.

We consider U such that $|\nabla V|=1$ and $E_t(U)=((S(t,\xi_0)U,U))$ and $E_{t,n}(U)=((S_n(t,\xi_0)U,U))$. We fix n. By the continuity of $|\nabla\xi|$, there exists a time T such that for all t $\leq$ T, $|\nabla\xi(t)-\nabla\xi(0)|\leq \frac{2}{n}$. By the continuity of S in time, we obtain

$$|E_t(U)-E_0(U)|\leq\frac{2}{n}|U|\leq\frac{1}{n}+\frac{|U|^2}{n}$$

$$E_0(U)-\frac{1}{n}-\frac{|U|^2}{n}\leq E_t(U)\leq E_0(U)+\frac{1}{n}+\frac{|U|^2}{n}$$

But $E_0(U)-\frac{|U|^2}{n}=E_{0,-n}(U)$ and $E_0(U)+\frac{|U|^2}{n}=E_{0,n}(U)$. Then, taking the inf-sup definition of the eigenvalues, it follows that

$$\lambda_{i,-n}(S(0,\xi_0))-\frac{1}{n}\leq\lambda_i(S(t,\xi_0))\leq\lambda_{i,n}(S(0,\xi_0))+\frac{1}{n}.$$

We now use a result of [13]. Let T be a closed operator in H.

Lemma. If a sequence T_n of closed operators converges to T in the genera-lized sense, then a finite system of eigenvalues of T_n converges continuously to the corresponding finite system of eigenvalues of T.

Since S_n converges to S in the generalized sense, the eigenvalues are continuous in n. So when t tends to 0, n tends to infinity and $\lambda_{i,n}(S(0,\xi_0))$ and $\lambda_{i,-n}(S(0,\xi_0))$ tend to $\lambda_i(S(0,\xi_0))$. Consequently $\lambda_i(S(t,\xi_0))$ tends to $\lambda_i(S(0,\xi_0))$.

Proof of theorem 2.5.

Since the operator $Tr(-S(s,\xi_0)oQ_m(s))$ is continuous in s, we have

$$Tr(-S(0,\xi_0)oQ_m(0))=\lim_{t\to 0}\frac{1}{t}\int_0^t Tr(-S(s,\xi_0)oQ_m(s))ds.$$

And so by (5), it follows that

$$Tr(-S(0,\xi_0)oQ_m(0))\leq\lim_{t\to 0}\frac{1}{t}log\omega_m(t,\xi_0).$$

Taking the supremum on the left side, we get

$$\sup_{\substack{U_i^0\in H^{-1}(\Omega)\\ ||U_i^0||\leq 1}} Tr(-S(0,\xi_0)oQ_m(0))\leq\lim_{t\to 0}\frac{1}{t}log\omega_m(t,\xi_0) \tag{9}$$

We now consider an orthonormal basis $\phi_1, ..., \phi_m$ consisting of the first m eigenvectors of $S(0, \xi_0)$ and we denote by Q_o the projector in $H^{-1}(\Omega)$ onto $V_o = Span\,[\phi_1, ..., \phi_m]$. So we have

$$Tr(-S(0,\xi_0)oQ_o) = \sum_{i=1,\ldots,m} ((-S\phi_i, \phi_i)) = \sum_{i=1,\ldots,m} \lambda_i(-S(0,\xi_0)).$$

But we also have

$$Tr(-S(0,\xi_0)oQ_o) \leq \sup_{\substack{U_i^0 \in H^{-1}(\Omega) \\ ||U_i^0|| \leq 1}} Tr(-S(0,\xi_0)oQ_m(0))$$

and consequently

$$\sum_{i=1,\ldots,m} \lambda_i(-S(0,\xi_0))) \leq \sup_{\substack{U_i^0 \in H^{-1}(\Omega) \\ ||U_i^0|| \leq 1}} Tr(-S(0,\xi_0)oQ_m(0)).$$

This result, together with estimate (9), gives us

$$\sum_{i=1,\ldots,m} \lambda_i(-S(0,\xi_0))) \leq \lim_{t\to 0} \frac{1}{t} log\omega_m(t,\xi_0).$$

To obtain the reverse inequality, we recall that (see (6) above)

$$\lim_{t\to 0} \frac{1}{t}\; log\omega_m(t,\xi_0) \;\leq\; \lim_{t\to 0} \left(\frac{1}{t} \int_0^t \sum_{i=1}^{i=m} \lambda_i(-S(s,\xi_0))\; ds \right).$$

Using now the continuity of the eigenvalues of S, the result follows.

2.4 Estimates of the eigenvalues of S.

We estimate the eigenvalues of S using their variational characterization [14]. Let

$$E(U) = ((SU, U)) = \mu |U|^2 + (J(\psi, V), U).$$

We are looking for the eigenvalues of E under the constraint K :

$$K = \left\{ U \in L^2(\Omega), ||U|| = (|\nabla V| =)1 \right\}.$$

As before , we bound the Jacobian term from above as

$$|(J(\psi, V), U)| \leq ||\psi||_{H^2}\; |\nabla V|\; |U|.$$

Since $|\nabla V| = 1$, we use Young's inequaliy to get

$$|(J(\psi, V), U)| \leq \mu\epsilon |U| + \frac{1}{4\epsilon\mu} ||\psi||^2_{H^2}.$$

So we obtain

$$\mu(1-\epsilon)|U|^2 - \frac{1}{4\epsilon\mu}||\psi||^2_{H^2} \leq E(U) \leq \mu(1+\epsilon)|U|^2 + \frac{1}{4\epsilon\mu}||\psi||^2_{H^2}\;. \tag{10}$$

We now consider the functional $F(U) = |U|^2$. The critical points μ_k of F under the constraint K are the eigenvalues of Laplace's operator with Dirichlet homogeneous boundary conditions. They are characterized by

$$\mu_k = \inf_{\substack{A \in L^2(\Omega) \\ \dim A = k}} \quad \sup_{\substack{U \in A \\ ||U|| = 1}} \quad F(U).$$

In the same way, the eigenvalues λ_k of S are given by

$$\lambda_k = \inf_{\substack{A \in L^2(\Omega) \\ \dim A = k}} \quad \sup_{\substack{U \in A \\ ||U|| = 1}} \quad E(U).$$

So, taking the supremum over $U \in A$ in (10), and then the infimum over all of the subspaces A of dimension k, we obtain for $\epsilon \in]0,1]$:

$$\mu(1-\epsilon)\mu_k - \frac{1}{4\epsilon\mu}||\psi||^2_{H^2} \leq \lambda_k \leq \mu(1+\epsilon)\mu_k + \frac{1}{4\epsilon\mu}||\psi||^2_{H^2}.$$

We optimize on ϵ to obtain

Theorem 2.6. *The eigenvalues λ_k of S satisfy*

$$\begin{array}{lllll} \lambda_k & \geq & \mu\mu_k - ||\psi||_{H^2}\sqrt{\mu_k} & \text{if} & ||\psi||_{H^2} \leq 2\mu\sqrt{\mu_k}, \\ \lambda_k & \geq & -\dfrac{||\psi||^2_{H^2}}{4\mu} & \text{otherwise.} & \end{array} \tag{11}$$

where μ_k is the k^{th} eigenvalue of Laplace's operator with Dirichlet homogeneous boundary conditions.

3 The multilayer problem.

3.1 Existence and Uniqueness Theorems.

Let Ω be a bounded open subset of $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. Let us introduce $\psi = (\psi_1, ..., \psi_N)$, $\theta = (\theta_1, ..., \theta_N)$. We consider the N layers system [7], [16], [3] :

$$\left\{ \begin{array}{lll} \dfrac{\partial\theta_k}{\partial t} - \mu\Delta^2\psi_k + J(\psi_k, \theta_k) + \beta\dfrac{\partial\psi_k}{\partial x} = S_k & \text{in} & \Omega\times]0,T[, \text{k} = 1,...,\text{N} \\ \Delta\psi_k(x,y,t) = 0 & \text{on} & \partial\Omega\times]0,T[, \text{k} = 1,...,\text{N} \\ \Delta\psi - W\psi = \theta & \text{in} & \Omega\times]0,T[, \\ \psi_k(x,y,t) = C_k(t) & \text{on} & \partial\Omega\times]0,T[, \text{k} = 1,...,\text{N} \\ \int_\Omega \psi_1 dx = \int_\Omega \psi_2 dx = \ldots = \int_\Omega \psi_N dx, & & \\ \psi_1^m = 0 & \text{on} & \partial\Omega\times]0,T[, \\ \theta(0,x,y) = \theta_0(x,y) & \text{in} & \Omega. \end{array} \right. \tag{12}$$

The k^{th} layer is characterized by its thickness H_k, its reduced gravity g_k and its streamfunction ψ_k. The forcing S infers only on the upper layer : $S_k = 0$ for k $\neq 1$ and $S_1 = \frac{v}{H_1}$ where v is the curl of the wind stress on the surface. We denote by θ the vector with components θ_k, k=1,...,N, ψ the vector with components ψ_k, k=1,...,N. The parameter of Coriolis in the middle of the bassin is f_0 and β is a physical constant given by the linear approximation of the parameter of Coriolis $f = f_0 + \beta y$. The matrix W is the $N \times N$ matrix defined by

$$\left\{\begin{array}{l} W_{k,j} = 0 \text{ if } |k-j| > 1, \\ W_{k,k} = \dfrac{f_0^2}{H_k}\left(\dfrac{1}{g_k} + \dfrac{1}{g_{k-1}}\right), \\ \\ W_{k,k+1} = \dfrac{-f_0^2}{H_k g_k}, \\ \\ W_{k,k-1} = \dfrac{-f_0^2}{H_k g_{k-1}}, \\ \end{array}\right. \tag{13}$$

where $\dfrac{1}{g_0} = 0$. We remark that the matrix W is positive defined. So we introduce Λ the diagonal matrix of the eigenvalues of W, and define P and P^{-1} the two matrices such that $\Lambda = P^{-1}\ W\ P$. We denote by ψ^m the vector $P^{-1}\ \psi$, $C^m = P^{-1}\ C$, etc.Notice that the boundary conditions on ψ^m are the following :
$\psi_1^m = 0$ on $\partial\Omega$, $\psi_k^m = C_k^m$ on $\partial\Omega$ for $k \neq 1$, $\int_\Omega \psi_k^m = 0$.

We denote by H^{-1} the space $(H^{-1}(\Omega))^N$, by L^2 the space $(L^2(\Omega))^N$, etc
We recall the following results [17], [19], [12], [4].

Theorem 3.1. *For θ_0 in H^{-1}, system (12) admits a unique solution $\theta(t,x)$ in $\mathrm{C}([0,T],H^{-1}) \cap L^2(0,T,L^2) \cap L^2_{loc}(0,T,H_0^1)$. The semi-group $G(t)$ from H^{-1} in H^{-1}, $G(t)\theta_0(x) = \theta(t,x)$, associated to these equations, is such that there exists a maximal attractor A which is bounded in L^2, compact and connected in H^{-1} and whose basin of attraction is the whole space H^{-1}. This attractor has finite Hausdorff and fractal dimensions. For θ_0 in H^1, system (12) admits a unique solution $\theta(t,x)$ in $\mathrm{C}([0,T],H^1) \cap L^2(0,T,H^2)$.*

This theorem is proved in [4].

We now consider the system linearized around a solution on the attractor :

$$\left\{\begin{array}{lll} \dfrac{\partial U_k}{\partial t} - \mu\Delta^2 V_k + J(\psi_k, U_k) + J(V_k, \theta_k) + \beta\dfrac{\partial V_k}{\partial x} = 0 & \text{in} & \Omega\times]0,T[, \\ \Delta V - WV = U & \text{in} & \Omega\times]0,T[, \\ \Delta V = 0 & \text{on} & \partial\Omega\times]0,T[, \\ V_k(x,y,t) = C_k(t) & \text{on} & \partial\Omega\times]0,T[, \\ \int_\Omega V_1 dx = \int_\Omega V_k dx = \ldots = \int_\Omega V_N dx, & & \\ V_1^m = 0 & \text{on} & \partial\Omega\times]0,T[, \\ U(0,x,y) = U_0(x,y) & \text{in} & \Omega. \end{array}\right. \tag{14}$$

From now , for every U, we define V by

$$\begin{cases} \Delta V - WV = U & \text{in} \quad \Omega\times]0,T[, \\ \Delta V = 0 & \text{on} \quad \partial\Omega\times]0,T[, \\ V_k(x,y,t) = C_k(t) & \text{on} \quad \partial\Omega\times]0,T[, \\ \int_\Omega V_1 dx = \int_\Omega V_k dx = \ldots = \int_\Omega V_N dx, & \\ V_1^m = 0 & \text{on} \quad \partial\Omega\times]0,T[, \end{cases} \tag{15}$$

and $\bar{V}$ by $V = \bar{V} + C$, $\bar{V} \in H_0^1(\Omega)$.

We introduce on H^{-1} the norm defined by the scalar product [4] :

$$((U,u)) =< U, H\bar{V} >= -\sum_{i=1}^{i=N} H_i \ < U_i, \bar{v}_i >_{-1,1}, \tag{16}$$

where $(H\bar{V})_k = H_k \bar{V}_k$ and $< .,. >=< .,. >_{-1,1}$ denotes the duality product $H^{-1} \times H_0^1$.

Introducing $p_i = \dfrac{f_0^2}{g_i}, p_0 = 0$, we obtain that

$$||U||_{-1}^2 = \sum_{i=1}^{i=N} H_i \ |\nabla V_i|^2 + p_i \ |V_{i+1} - V_i|^2.$$

This norm is equivalent to the usual one.
On L^2, we introduce the norm :

$$||U||_2^2 = \sum_{i=1}^{i=N} H_i \ |\Delta V_i|^2 + p_i \ |\nabla V_{i+1} - \nabla V_i|^2. \tag{17}$$

The three following norms $||.||_2$, $\left(\sum_{i=1}^{i=N} H_i \ |\Delta V_i|^2\right)^{1/2}$ and $|U|^2 = \left(\sum_{i=1}^{i=N} H_i |U_i|^2\right)^{1/2}$ where $|.|$ denotes the usual scalar product on $L^2(\Omega)$ are equivalent (see appendix A).

Theorem 3.2. *For $U(0,x,y) = U_0(x,y)$ in H^{-1}, system (14) admits a unique solution $U(t,x,y)$ in $C([0,T],H^{-1}) \cap L^2(0,T,L^2) \cap L^2_{loc}(0,T,H^1)$.*

Proof of theorem 3.2. We consider the operator B from H^{-1} to H^{-1}, with domain of definition $D(B) = H_0^1$ defined by

$$(BU)_k = -\mu\Delta^2 V_k + J(\psi_k, U_k) + J(V_k, \theta_k) + \beta\frac{\partial V_k}{\partial x}.$$

We introduce B_{λ_0} by $B_{\lambda_0} = B + \lambda_0 I$ where I is the operator identity. Let us consider the three spaces $L^2 \subset H^{-1} \subset H^{-2}$, each embedding is compact and each space is dense in the following. We introduce the bilinear form $b_{\lambda_0}(t, U^1, U^2) = - < B_{\lambda_0} U^1, H\bar{V}^2 >_{-1,1}$. We first prove that b_{λ_0} is continuous from L^2 to L^2.

$$\begin{aligned} b_{\lambda_0}(t,U^1,U^2) \quad &= \quad -\sum_{k=1}^{k=N} H_k \left(-\mu < \Delta^2 V_k^1, \bar{V}_k^2 > + < J(\psi_k, U_k^1), \bar{V}_k^2 > + \right. \\ &\left. + < J(V_k^1, \theta_k), \bar{V}_k^2 > +\beta < \frac{\partial V_k^1}{\partial x}, \bar{V}_k^2 > +\lambda_0 < U_k^1, \bar{V}_k^2 > \right). \end{aligned}$$

Thanks to the Green's inequality, and the usual estimate on the Jacobian term, we obtain

$$
\begin{aligned}
|b_{\lambda_0}(t, U^1, U^2)| \quad &\leq \quad \sum_{k=1}^{k=N} H_k \left(\mu |\Delta V_k^1||\Delta V_k^2| + |\nabla \psi_k|_{L^4} |\bar{V}_k^2|_{L^4} |U_k^1| + \right. \\
&\left. + |V_k^1|_{L^4} |\bar{V}_k^2|_{L^4} |\theta_k| + \beta |\nabla V_k^1||\bar{V}_k^2| + \lambda_0 |U_k^1||\bar{V}_k^2| \right).
\end{aligned}
$$

Sobolev embeddings give us, with c denoting any positive constant,

$$
\begin{aligned}
|b_{\lambda_0}(t, U^1, U^2)| \quad &\leq \quad \sum_{k=1}^{k=N} H_k \left(\mu |\Delta V_k^1||\Delta V_k^2| + c||\psi_k||_{H^2} |\nabla \bar{V}_k^2||U_k^1| + \right. \\
&\left. + c|\nabla V_k^1||\nabla \bar{V}_k^2||\theta_k| + \beta |\nabla V_k^1||\bar{V}_k^2| + \lambda_0 |U_k^1||\bar{V}_k^2| \right).
\end{aligned}
$$

From the Poincaré's inequality, and denoting by μ_1 the first eigenvalue of the Laplacian operator with Dirichlet boundary condition, we deduce

$$
\begin{aligned}
|b_{\lambda_0}(t, U^1, U^2)| \quad &\leq \quad \sum_{k=1}^{k=N} H_k \left(\mu + c|\theta_k|\mu_1 + \beta \mu_1 \sqrt{\mu_1} \right) |\Delta V_k^1||\Delta V_k^2| + \\
&+ \sum_{k=1}^{k=N} H_k \left(c||\psi_k||_{H^2} \sqrt{\mu_1} + \lambda_0 \mu_1 \right) |\Delta V_k^2||U_k^1|.
\end{aligned}
$$

But, for a solution on the attractor, there exists a constant r such that $c|\theta_k| < r$ and $c||\psi_k||_{H^2} < r$ [4]. Then

$$
\begin{aligned}
|b_{\lambda_0}(t, U^1, U^2)| \quad &\leq \quad \left(\mu + r\mu_1 + \beta \mu_1 \sqrt{\mu_1} \right) \sum_{k=1}^{k=N} H_k |\Delta V_k^1||\Delta V_k^2| + \\
&+ \left(r\sqrt{\mu_1} + \lambda_0 \mu_1 \right) \sum_{k=1}^{k=N} H_k |\Delta V_k^2||U_k^1|.
\end{aligned}
$$

and so, introducing $c_1 = \mu + r\mu_1 + \beta\mu_1\sqrt{\mu_1}$ and $c_2 = r\sqrt{\mu_1} + \lambda_0\mu_1$, we have

$$
\begin{aligned}
|b_{\lambda_0}(t, U^1, U^2)| \quad &\leq \quad c_1 \left(\sum_{k=1}^{k=N} H_k |\Delta V_k^1|^2 \right)^{1/2} \left(\sum_{k=1}^{k=N} H_k |\Delta V_k^2|^2 \right)^{1/2} + \\
&c_2 \left(\sum_{k=1}^{k=N} H_k |\Delta V_k^2|^2 \right)^{1/2} \left(\sum_{k=1}^{k=N} H_k |U_k^1|^2 \right)^{1/2}.
\end{aligned}
$$

By the equivalence of the norm $\left(\sum_{k=1}^{k=N} H_k |\Delta V_k|^2 \right)^{1/2}$ and $|U|$, we obtain the continuity of b

$$
|b_{\lambda_0}(t, U^1, U^2)| \quad \leq \quad c|U^1||U^2|.
$$

We now prove the coercivity of b in L^2.

$$
\begin{aligned}
|b_{\lambda_0}(t, U, U)| \quad &= \quad - \sum_{k=1}^{k=N} H_k \left(-\mu < \Delta^2 V_k, \bar{V}_k > + < J(\psi_k, U_k), \bar{V}_k > + \right. \\
&\left. + < J(V_k, \theta_k), \bar{V}_k > + \beta < \frac{\partial V_k}{\partial x}, \bar{V}_k > + \lambda_0 < U_k, \bar{V}_k > \right).
\end{aligned}
$$

Using Green's inequality, intregation by parts, and $< J(f+c,g), f >= 0$ for f or $g \in H_0^1(\Omega)$, $c \in \mathbb{R}$, we obtain

$$|b_{\lambda_0}(t,U,U)| = \sum_{k=1}^{k=N} H_k \mu |\Delta V_k|^2 - \sum_{k=1}^{k=N} H_k < J(\psi_k, U_k), \bar{V}_k > + \lambda_0 ||U||_{-1}^2. \tag{18}$$

Let us bound now the term $|\sum_{k=1}^{k=N} H_k < J(\psi_k, U_k), \bar{V}_k >|$ which we denote by $J(U)$.

$$\begin{aligned} J(U) \quad &\leq \quad |\sum_{k=1}^{k=N} H_k < J(\psi_k, \Delta V_k), \bar{V}_k > + \\ &\quad + \sum_{k=1}^{k=N} H_k < J(\psi_k, p_{k-1}(V_k - V_{k-1}) + p_k(V_{k+1} - V_k)), \bar{V}_k >|. \end{aligned}$$

And then

$$\begin{aligned} J(U) \quad &\leq \quad \sum_{k=1}^{k=N} H_k | < J(\psi_k, \Delta V_k), \bar{V}_k > | + \\ &\quad + \sum_{k=1}^{k=N} H_k p_{k-1} | < J(\psi_k, V_k), (V_k - V_{k-1}) > | + \\ &\quad + \sum_{k=1}^{k=N} H_k p_k | < J(\psi_k, V_k), (V_{k+1} - V_k) > |. \end{aligned}$$

Using the usual estimate of the Jacobian term, we deduce

$$\begin{aligned} J(U) \quad &\leq \quad \sum_{k=1}^{k=N} H_k ||\psi_k||_{H^2} |\nabla V_k| |\Delta V_k| + \\ &\quad + \sum_{k=1}^{k=N} H_k |\nabla \psi_k|_{L^4} |\nabla V_k|_{L^4} p_{k-1} |V_k - V_{k-1}| + \\ &\quad + \sum_{k=1}^{k=N} H_k |\nabla \psi_k|_{L^4} |\nabla V_k|_{L^4} p_k |V_{k+1} - V_k|. \end{aligned}$$

As before, the Sobolev embeddings give us

$$\begin{aligned} J(U) \quad &\leq \quad \sum_{k=1}^{k=N} H_k ||\psi_k||_{H^2} |\nabla V_k| |\Delta V_k| + \\ &\quad + \sum_{k=1}^{k=N} H_k ||\psi_k||_{H^2} |\Delta V_k| p_{k-1} |V_k - V_{k-1}| + \\ &\quad + \sum_{k=1}^{k=N} H_k ||\psi_k||_{H^2} |\Delta V_k| p_k |V_{k+1} - V_k|. \end{aligned}$$

We obtain, for all $\epsilon > 0$,

$$J(U) \quad \leq \quad \frac{\mu\epsilon}{3} \sum_{k=1}^{k=N} H_k |\Delta V_k|^2 + \frac{3}{4\mu\epsilon} \sum_{k=1}^{k=N} H_k ||\psi_k||_{H^2}^2 |\nabla V_k|^2 +$$

$$+\frac{\mu\epsilon}{3}\sum_{k=1}^{k=N} H_k|\Delta V_k|^2 + \frac{3}{4\mu\epsilon}\sum_{k=1}^{k=N}\frac{p_{k-1}^2}{H_k}||\psi_k||_{H^2}^2|V_k - V_{k-1}|^2 +$$
$$+\frac{\mu\epsilon}{3}\sum_{k=1}^{k=N} H_k|\Delta V_k|^2 + \frac{3}{4\mu\epsilon}\sum_{k=1}^{k=N}\frac{p_k^2}{H_k}||\psi_k||_{H^2}^2|V_{k+1} - V_k|^2.$$

And so,

$$\begin{aligned}
J(U) &\leq \mu\epsilon\sum_{k=1}^{k=N} H_k|\Delta V_k|^2 + \\
&\frac{3}{4\mu\epsilon}||\psi||_{H^2}^2\sum_{k=1}^{k=N}\left(H_k|\nabla V_k|^2 + \frac{p_{k-1}^2}{H_k}|V_k - V_{k-1}|^2 + \frac{p_k^2}{H_k}|V_{k+1} - V_k|^2\right), \\
J(U) &\leq \mu\epsilon\sum_{k=1}^{k=N} H_k|\Delta V_k|^2 + \\
&\frac{3}{4\mu\epsilon}||\psi||_{H^2}^2\left(\sum_{k=1}^{k=N} H_k|\nabla V_k|^2 + \sum_{k=1}^{k=N}\left(\frac{p_k}{H_k} + \frac{p_k}{H_{k+1}}\right)p_k|V_{k+1} - V_k|^2\right), \\
J(U) &\leq \mu\epsilon\sum_{k=1}^{k=N} H_k|\Delta V_k|^2 + \\
&\frac{3||\psi||_{H^2}^2}{4\mu\epsilon}\left(\sum_{k=1}^{k=N} H_k|\nabla V_k|^2 + \sum_{k=1}^{k=N}\left(\frac{p_k}{H_k} + \frac{p_k}{H_{k+1}}\right)p_k|V_{k+1} - V_k|^2\right).
\end{aligned}$$

Then, for all $\epsilon > 0$, we have

$$J(U) \leq \mu\epsilon\sum_{k=1}^{k=N} H_k|\Delta V_k|^2 + \frac{3||\psi||_{H^2}^2}{4\mu\epsilon}\max_k\left(1, p_k\left(\frac{1}{H_k} + \frac{1}{H_{k+1}}\right)\right)||U||_{-1}^2. \tag{19}$$

Taking $\epsilon = \frac{1}{2}$, we obtain

$$J(U) \leq \frac{\mu}{2}\sum_{k=1}^{k=N} H_k|\Delta V_k|^2 + \frac{3||\psi||_{H^2}^2}{2\mu}\max_k\left(1, \left(\frac{p_k}{H_k} + \frac{p_k}{H_{k+1}}\right)\right)||U||_{-1}^2.$$

Inserting this inequality in (18), we deduce

$$\begin{aligned}
b_{\lambda_0}(t, U, U) &\geq \mu\sum_{k=1}^{k=N} H_k|\Delta V_k|^2 + \lambda_0||U||_{-1}^2 - \\
&-\frac{\mu}{2}\sum_{k=1}^{k=N} H_k|\Delta V_k|^2 - \frac{3||\psi||_{H^2}^2}{2\mu}\max_k\left(1, \left(\frac{p_k}{H_k} + \frac{p_k}{H_{k+1}}\right)\right)||U||_{-1}^2.
\end{aligned}$$

Consequently,

$$\begin{aligned}
b_{\lambda_0}(t, U, U) &\geq \frac{\mu}{2}\sum_{k=1}^{k=N} H_k|\Delta V_k|^2 + \\
&+\left(\lambda_0 - \frac{3||\psi||_{H^2}^2}{2\mu}\max_k\left(1, \left(\frac{p_k}{H_k} + \frac{p_k}{H_{k+1}}\right)\right)\right)||U||_{-1}^2.
\end{aligned}$$

On the attractor, $||\psi||_{H^2} < r$, and then,

$$\begin{aligned} b_{\lambda_0}(t,U,U) &\geq \frac{\mu}{2}c|U|^2 + \\ &+ \left(\lambda_0 - \frac{3r^2}{2\mu}\max_k\left(1, \left(\frac{p_k}{H_k} + \frac{p_k}{H_{k+1}}\right)\right)\right)||U||_{-1}^2. \end{aligned}$$

We conclude that, for

$$\lambda_0 > \frac{3r^2}{2\mu}\max_k\left(1, \left(\frac{p_k}{H_k} + \frac{p_k}{H_{k+1}}\right)\right)$$

we have the result. Using now a classical theorem (see, for example [18], p 257, theorem 4.1), we conclude that there exists a unique solution U_{λ_0} for the system (14) with b_{λ_0}. Let us now introduce $U(t,x,y) = e^{\lambda_0 t}U_{\lambda_0}(t,x,y)$. It is easy to see that U satisfies the system (14).
Let us now prove briefly that $\Delta V \in L^2_{loc}(0,T,H^1_0)$.
Using estimates, we obtain that

$$\begin{aligned} \frac{1}{2}\frac{\partial ||U||_2^2}{\partial t} + \mu\sum_k H_k|\nabla\Delta V_k|^2 &\leq c\sum_{k,j}|\nabla\theta_k||\nabla\Delta V_k||\Delta V_j|, \\ &\leq \frac{\mu}{2}\sum_k H_k|\nabla\Delta V_k|^2 + c\sum_{k,j}|\nabla\theta_k|^2|\Delta V_j|^2, \\ \frac{\partial ||U||_2^2}{\partial t} + \mu\sum_k H_k|\nabla\Delta V_k|^2 &\leq c\sum_{k,j}|\nabla\theta_k|^2|\Delta V_j|^2. \end{aligned}$$

Since we are on the attractor, we have that $|\nabla\theta_k|^2 \in C([0,T])$, and then $|\nabla\theta_k|^2 \leq M\forall t \in [0,T]$. So

$$\frac{\partial ||U||_2^2}{\partial t} + \mu\sum_k H_k|\nabla\Delta V_k|^2 \leq cM\sum_j|\Delta V_j|^2.$$

We multiply this inequality by t

$$\begin{aligned} t\frac{\partial ||U||_2^2}{\partial t} + t\mu\sum_k H_k|\nabla\Delta V_k|^2 &\leq cMT\sum_j|\Delta V_j|^2, \\ \frac{\partial (t||U||_2^2)}{\partial t} + \mu t\sum_k H_k|\nabla\Delta V_k|^2 &\leq cMT\sum_j|\Delta V_j|^2 + ||U||_2^2. \end{aligned}$$

Integrating in time between 0 and t, we obtain

$$t||U(t)||_2^2 + \mu\int_0^t t\sum_k H_k|\nabla\Delta V_k|^2 \leq cMT\int_0^t\sum_j|\Delta V_j|^2 + \int_0^t ||U||_2^2 \leq M'$$

since $\Delta V \in L^2(0,T,L^2)$ and $U \in L^2(0,T,L^2)$. The proof is completed.

The third theorem links the two systems (12) and (14) (see [4]).

Theorem 3.3. *For $t > 0$, the semi-group $G(t)$ is uniformly differentiable on A. Its differential in θ_0 is the linear operator on $H^{-1}(\Omega)$ given by $U \to L(t_0, \theta_0)U_0 = U(t_0)$, where $U(t_0)$ is the value at time $t = t_0$ of the solution $U(t)$ of the linearized system (14). Moreover,*

$$\sup_{\theta_0 \in A} |L(t_0, \theta_0)|_{L(H^{-1})} < \infty.$$

Proof of theorem 3.3. Let us consider two solutions θ^1 and θ^2 of the system (12). We denote by θ the difference $\theta^1 - \theta^2$, and by ψ the difference $\psi^1 - \psi^2$. So, θ is solution of the system :

$$\frac{\partial \theta_k}{\partial t} - \mu \Delta^2 \psi_k + J(\psi_k^1, \theta_k^1) - J(\psi_k^2, \theta_k^2) + \beta \frac{\partial \psi_k}{\partial x} = 0 \text{ in } \Omega \times]0, T[, \tag{20}$$

with the boundary and initial conditions :

$$\begin{cases} \Delta \psi_k(x, y, t) = 0 & \text{on} \quad \partial\Omega \times]0, T[, \\ \Delta \psi - W\psi = \theta & \text{in} \quad \Omega \times]0, T[, \\ \psi_k(x, y, t) = C_k(t) & \text{on} \quad \partial\Omega \times]0, T[, \\ \int_\Omega \psi_1 dx = \int_\Omega \psi_k dx = \ldots = \int_\Omega \psi_N dx, & \\ \psi_1^m = 0 & \text{on} \quad \partial\Omega \times]0, T[, \\ \theta(0, x, y) = \theta_0^1(x, y) - \theta_0^2(x, y) & \text{in} \quad \Omega. \end{cases} \tag{21}$$

We rewrite the difference of the nonlinear term in the following form :

$$J(\psi_k^1, \theta_k^1) - J(\psi_k^2, \theta_k^2) = J(\psi_k, \theta_k^1) + J(\psi_k^2, \theta_k).$$

And then, (20) becomes :

$$\frac{\partial \theta_k}{\partial t} - \mu \Delta^2 \psi_k + J(\psi_k, \theta_k^1) + J(\psi_k^2, \theta_k) + \beta \frac{\partial \psi_k}{\partial x} = 0 \text{ in } \Omega \times]0, T[. \tag{22}$$

Let us now introduce U the solution of the system (14) linearized around the solution θ^2. Then, U verifies

$$\frac{\partial U_k}{\partial t} - \mu \Delta^2 V_k + J(\psi_k^2, U_k) + J(V_k, \theta_k^2) + \beta \frac{\partial V_k}{\partial x} = 0 \quad \text{in} \quad \Omega \times]0, T[,$$

with the boundary and initial conditions :

$$\begin{cases} \Delta V - WV = U & \text{in} \quad \Omega \times]0, T[, \\ \Delta V = 0 & \text{on} \quad \partial\Omega \times]0, T[, \\ V_k(x, y, t) = C_k(t) & \text{on} \quad \partial\Omega \times]0, T[, \\ \int_\Omega V_1 dx = \int_\Omega V_k dx = \ldots = \int_\Omega V_N dx, & \\ V_1^m = 0 & \text{on} \quad \partial\Omega \times]0, T[, \\ U(0, x, y) = \theta_0^1(x, y) - \theta_0^2(x, y) & \text{in} \quad \Omega. \end{cases} \tag{23}$$

We introduce $Y = \theta - U$, and $z = \psi - V$. Then Y verifies :

$$\begin{aligned}\frac{\partial Y_k}{\partial t} - \mu\Delta^2 z_k + J(\psi_k, \theta_k^1) + J(\psi_k^2, \theta_k) - \\ -J(\psi_k^2, U_k) - J(V_k, \theta_k^2) + \beta\frac{\partial z_k}{\partial x} = 0 \quad \text{ in } \quad \Omega\times]0, T[,\end{aligned}$$

with the same boundary conditions as U or θ, and with the initial condition

$$Y(0, x, y) = 0.$$

Rewriting the Jacobian terms, we obtain :

$$\frac{\partial Y_k}{\partial t} - \mu\Delta^2 z_k + J(\psi_k, \theta_k) + J(\psi_k^2, Y_k) + J(z_k, \theta_k^2) + \beta\frac{\partial z_k}{\partial x} = 0. \tag{24}$$

Let us multiply this equation by $H_k\bar{z}_k$, integrate on Ω and sum on k. We obtain

$$\begin{aligned}\frac{1}{2}\frac{\partial ||Y||_{-1}^2}{\partial t} + \mu\sum H_k|\Delta z_k|^2 &= \sum H_k(J(\psi_k, \theta_k), \bar{z}_k) + \\ &+ \sum H_k(J(\psi_k^2, Y_k), \bar{z}_k),\end{aligned}$$

$$\begin{aligned}&\frac{1}{2}\frac{\partial ||Y||_{-1}^2}{\partial t} + \mu\sum H_k|\Delta z_k|^2 \le \sum H_k|(J(\psi_k, \theta_k), \bar{z}_k)| + \\ &+ \sum H_k|(J(\psi_k^2, \Delta z_k), \bar{z}_k)| + \sum p_k|(J(\psi_k^2, z_{k+1} - z_k), \bar{z}_k)| + \\ &+ \sum p_{k-1}|(J(\psi_k^2, z_k - z_{k-1}), \bar{z}_k)|.\end{aligned} \tag{25}$$

We look at each term :

$$\begin{aligned}|(J(\psi_k^2, \Delta z_k), \bar{z}_k)| &\le |\nabla z_k|_{L^4}^2||\psi_k^2||_{H^2} \\ &\le c|\nabla z_k||\Delta z_k|||\psi_k^2||_{H^2} \\ &\le \frac{\mu}{4}|\Delta z_k|^2 + \frac{c}{\mu}|\nabla z_k|^2||\psi_k^2||_{H^2}^2.\end{aligned}$$

On the attractor, $||\psi^2||_{H^2} \le r$, and then

$$|(J(\psi_k^2, \Delta z_k), \bar{z}_k)| \le \frac{\mu}{4}|\Delta z_k|^2 + \frac{cr^2}{\mu}|\nabla z_k|^2. \tag{26}$$

For the next term, we obtain :

$$\begin{aligned}p_k|(J(\psi_k^2, z_{k+1} - z_k), \bar{z}_k)| &= p_k|(J(\psi_k^2, z_{k+1}), \bar{z}_k)| \\ &\le p_k|\nabla\psi_k^2|_{L^4}|\nabla z_{k+1}||\bar{z}_k|_{L^4} \\ &\le cp_k||\psi_k^2||_{H^2}|\nabla z_{k+1}||\nabla\bar{z}_k| \\ &\le cp_k r|\nabla z_{k+1}||\nabla z_k| \\ &\le \frac{1}{2}cp_k r|\nabla z_{k+1}|^2 + \frac{1}{2}cp_k r|\nabla z_k|^2 \\ &\le c_1|\nabla z_{k+1}|^2 + c_2|\nabla z_k|^2.\end{aligned}$$

By the same way, we have :

$$p_{k-1}|(J(\psi_k^2, z_k - z_{k-1}), \bar{z}_k)| \leq c_3|\nabla z_k|^2 + c_4|\nabla z_{k-1}|^2. \tag{27}$$

Let us now consider the last term $|(J(\psi_k, \theta_k), \bar{z}_k)|$. We first develop θ_k :

$$\begin{aligned} H_k|(J(\psi_k, \theta_k), \bar{z}_k)| \quad &\leq \quad H_k|(J(\psi_k, \Delta\psi_k), \bar{z}_k)| + p_k|(J(\psi_k, \psi_{k+1}), \bar{z}_k)| + \\ &\quad +p_{k-1}|(J(\psi_k, \psi_{k-1}), \bar{z}_k)|. \end{aligned}$$

Let us consider the first term :

$$\begin{aligned} |(J(\psi_k, \Delta\psi_k), \bar{z}_k)| \quad &\leq \quad c|\nabla z_k|_{L^4}|\nabla\psi_k|_{L^4}|\Delta\psi_k|, \\ &\leq \quad c|\Delta z_k|(|\Delta\psi_k|^{1-\sigma}|\nabla\psi_k|^{\sigma})|\Delta\psi_k|, \\ &\leq \quad c|\Delta z_k||\Delta\psi_k|^{2-\sigma}|\nabla\psi_k|^{\sigma}, \\ &\leq \quad \frac{\mu}{4}|\Delta z_k|^2 + c|\Delta\psi_k|^{4-2\sigma}|\nabla\psi_k|^{2\sigma}. \end{aligned}$$

Since ψ^1 and ψ^2 are on the attractor, then $|\Delta\psi_k| \leq r$, and finally we obtain that for all $\sigma \in]0, 1[$

$$|(J(\psi_k, \Delta\psi_k), \bar{z}_k)| \quad \leq \quad \frac{\mu}{4}|\Delta z_k|^2 + cr^{2-2\sigma}|\Delta\psi_k|^2|\nabla\psi_k|^{2\sigma}. \tag{28}$$

We now consider the second term :

$$\begin{aligned} p_k|(J(\psi_k, \psi_{k+1}), \bar{z}_k)| \quad &= \quad p_k|(J(\psi_k, \psi_{k+1} - \psi_k), \bar{z}_k)|, \\ &\leq \quad p_k|\nabla z_k|_{L^4}|\nabla\psi_k|_{L^4}|\psi_{k+1} - \psi_k|, \\ &\leq \quad cp_k|\Delta z_k||\Delta\psi_k||\psi_{k+1} - \psi_k|, \\ &\leq \quad \frac{\mu H_k}{4}|\Delta z_k|^2 + cp_k|\Delta\psi_k|^2|\psi_{k+1} - \psi_k|^2, \\ &\leq \quad \frac{\mu H_k}{4}|\Delta z_k|^2 + \\ &\quad +cp_k|\Delta\psi_k|^2|\psi_{k+1} - \psi_k|^{2-2\sigma}|\psi_{k+1} - \psi_k|^{2\sigma}. \end{aligned}$$

Since we are on the attractor, the norm of ψ in H^2 is bounded and then

$$p_k|(J(\psi_k, \psi_{k+1}), \bar{z}_k)| \quad \leq \quad \frac{\mu H_k}{4}|\Delta z_k|^2 + c|\Delta\psi_k|^2|\psi_{k+1} - \psi_k|^{2\sigma}. \tag{29}$$

And so, inserting (26), (27), (28), (29) in (25), we deduce

$$\begin{aligned} \frac{1}{2}\frac{\partial||Y||_{-1}^2}{\partial t} \quad &+ \quad \mu\sum H_k|\Delta z_k|^2 \leq \sum H_k\mu|\Delta z_k|^2 + c\sum H_k|\nabla z_k|^2 + \\ &+ \quad c\sum H_k|\Delta\psi_k|^2(|\nabla\psi_k|^{2\sigma} + |\psi_{k+1} - \psi_k|^{2\sigma} + |\psi_k - \psi_{k-1}|^{2\sigma}) \end{aligned}$$

$$\frac{1}{2}\frac{\partial||Y||_{-1}^2}{\partial t} \leq c||Y||_{-1}^2 + c||\theta||_{-1}^{2\sigma}\mu\sum H_k|\Delta\psi_k|^2. \tag{30}$$

We are now interested in $||\theta||_{-1}$. We multiply (22) by $H_k\bar{\psi}_k$, integrate over Ω, and sum on k. We have

$$\frac{1}{2}\frac{\partial||\theta||_{-1}^2}{\partial t} + \mu\sum H_k|\Delta\psi_k|^2 \quad \leq \quad \sum H_k|(J(\psi_k^2, \theta_k), \bar{\psi}_k)|,$$

$$
\begin{aligned}
&\leq \sum H_k|(J(\psi_k^2,\Delta\psi_k),\bar{\psi}_k)| + \\
\sum p_k|(J(\psi_k^2,\psi_{k+1}-\psi_k),\bar{\psi}_k)| \quad &+ \quad \sum p_{k-1}|(J(\psi_k^2,\psi_k-\psi_{k-1}),\bar{\psi}_k)|.
\end{aligned}
$$

Using (26) and (27), we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{\partial||\theta||_{-1}^2}{\partial t} + \mu\sum H_k|\Delta\psi_k|^2 &\leq \frac{\mu}{4}\sum H_k|\Delta\psi_k|^2 + c\sum H_k|\nabla\psi_k|^2, \\
&\leq \frac{\mu}{2}\sum H_k|\Delta\psi_k|^2 + c||\theta||_{-1}^2, \\
\frac{\partial||\theta||_{-1}^2}{\partial t} + \mu\sum H_k|\Delta\psi_k|^2 &\leq c||\theta||_{-1}^2.
\end{aligned}
$$

Integrating in time between 0 and t, we have :

$$
||\theta(t)||_{-1}^2 + \mu\int_0^t \sum H_k|\Delta\psi_k|^2 \leq ||\theta(0)||_{-1}^2 + c\int_0^t ||\theta(s)||_{-1}^2 ds.
$$

The Gronwall lemma implies:
$||\theta(0)||_{-1}^2 + c\int_0^t ||\theta(s)||_{-1}^2 ds \leq ||\theta(0)||_{-1}^2 exp(cT)$, for all $t \in [0,T]$.
And then

$$
||\theta(t)||_{-1}^{2\sigma} \leq ||\theta(0)||_{-1}^{2\sigma} exp(\sigma cT), \tag{31}
$$

$$
\mu\int_0^t \sum H_k|\Delta\psi_k|^2 \leq ||\theta(0)||_{-1}^2 exp(cT). \tag{32}
$$

Inserting (31) in (30), we obtain that for all $t \in [0,T]$

$$
\frac{1}{2}\frac{\partial||Y||_{-1}^2}{\partial t} \leq c||Y||_{-1}^2 + c||\theta(0)||_{-1}^{2\sigma} exp(\sigma cT)\mu\sum H_k|\Delta\psi_k|^2.
$$

Using one more time the Gronwall lemma, we deduce :

$$
||Y(t)||_{-1}^2 \leq \left(||Y(0)||_{-1}^2 + c\int_0^t ||\theta(0)||_{-1}^{2\sigma} exp(\sigma cT)\mu\sum H_k|\Delta\psi_k(s)|^2 ds\right) exp(ct),
$$

and then

$$
||Y(t)||_{-1}^2 \leq c||\theta(0)||_{-1}^{2\sigma} exp(\sigma cT)\left(\mu\int_0^t \sum H_k|\Delta\psi_k(s)|^2 ds\right) exp(ct).
$$

Remembering now (32), we have

$$
||Y(t)||_{-1}^2 \leq c||\theta(0)||_{-1}^{2\sigma} exp(\sigma cT)||\theta(0)||_{-1}^2 exp(cT) exp(cT),
$$

then

$$
\frac{||Y(t)||_{-1}^2}{||\theta(0)||_{-1}^2} \leq K||\theta(0)||_{-1}^{2\sigma}
$$

which tends to 0 with $||\theta(0)||_{-1}^2$.

Let us prove now the last part of the theorem :

$$
\sup_{\theta_0\in A} |L(t_0,\theta_0)|_{L(H^{-1})} < \infty.
$$

We multiply the equation(23) by $-H_k\bar{V}_k$, integrate on Ω and sum on k to obtain :

$$\frac{1}{2}\frac{\partial||U||_{-1}^2}{\partial t}+\mu\sum H_k|\Delta Y_k|^2=\sum H_k(J(\psi_k^2,U_k),V_k),$$

$$\frac{1}{2}\frac{\partial||U||_{-1}^2}{\partial t}+\mu\sum H_k|\Delta Y_k|^2\leq\frac{\mu}{2}\sum H_k|\Delta Y_k|^2+c\sum|\nabla Y_k|^2$$

thanks to (26) and (27). Then

$$\frac{\partial||U||_{-1}^2}{\partial t}+\mu\sum H_k|\Delta Y_k|^2\leq c||U||_{-1}^2,$$

and the Gronwall lemma gives

$$\frac{||U(t_0)||_{-1}^2}{||U_0||_{-1}^2}\leq exp(ct_0)$$

which concludes the proof.

In all that follows, we consider $\theta_0\in H_0^1$ and denote by $B=B(t,\theta_0)$ the unbounded operator on H^{-1} with $D(B)=H_0^1$, defined by

$$(BU)_k=-\mu\Delta^2V_k+J(\psi_k,U_k)+J(V_k,\theta_k)+\beta\frac{\partial V_k}{\partial x}.$$

As before, we want to introduce $S=\dfrac{B+B^*}{2}$. Let us first determine B^*.

$$\begin{aligned}
E &= ((BU^1,U^2))_{-1}=-<BU^1,H\bar{V}^2>,\\
&= -\sum\Big(-\mu<\Delta^2V_k^1,H_k\bar{V}_k^2>+<J(\psi_k,U_k^1),H_k\bar{V}_k^2>+\\
&\quad+<J(V_k^1,\theta_k),H_k\bar{V}_k^2>+\beta<\frac{\partial V_k^1}{\partial x},H_k\bar{V}_k^2>\Big),\\
&= -\sum\Big(-\mu H_k(\Delta V_k^1,\Delta V_k^2)-H_k<J(\psi_k,V_k^2),\bar{U}_k^1>-\\
&\quad-H_k<J(V_k^2,\theta_k),\bar{V}_k^1>-\beta H_k<\frac{\partial V_k^2}{\partial x},\bar{V}_k^1>\Big),\\
&= -\sum\Big(-\mu H_k<\Delta^2V_k^2,\bar{V}_k^1>-H_k<J(\psi_k,V_k^2),\Delta V_k^1>-\\
&\quad-p_k<J(\psi_k,V_k^2),\bar{V}_{k+1}^1-\bar{V}_k^1>+\\
&\quad+p_{k-1}<J(\psi_k,V_k^2),\bar{V}_k^1-\bar{V}_{k-1}^1>-\\
&\quad-H_k<J(V_k^2,\theta_k),\bar{V}_k^1>-\beta H_k<\frac{\partial V_k^2}{\partial x},\bar{V}_k^1>\Big),\\
&= -\sum\Big(-\mu H_k<\Delta^2V_k^2,\bar{V}_k^1>-H_k<\Delta J(\psi_k,V_k^2),\bar{V}_k^1>-\\
&\quad-p_k<J(\psi_k,V_k^2),\bar{V}_{k+1}^1>+p_k<J(\psi_k,V_k^2),\bar{V}_k^1>+\\
&\quad+p_{k-1}<J(\psi_k,V_k^2),\bar{V}_k^1>-p_{k-1}<J(\psi_k,V_k^2),\bar{V}_{k-1}^1>-\\
&\quad-H_k<J(V_k^2,\theta_k),\bar{V}_k^1>-\beta H_k<\frac{\partial V_k^2}{\partial x},\bar{V}_k^1>\Big),\\
&= -\sum\Big(-\mu H_k<\Delta^2V_k^2,\bar{V}_k^1>-H_k<\Delta J(\psi_k,V_k^2),\bar{V}_k^1>-\\
&\quad-p_{k-1}<J(\psi_{k-1},V_{k-1}^2),\bar{V}_k^1>+p_k<J(\psi_k,V_k^2),\bar{V}_k^1>+\\
&\quad+p_{k-1}<J(\psi_k,V_k^2),\bar{V}_k^1>-p_k<J(\psi_{k+1},V_{k+1}^2),\bar{V}_k^1>-
\end{aligned}$$

$$-H_k < J(V_k^2,\theta_k),\bar{V}_k^1 > -\beta H_k < \frac{\partial V_k^2}{\partial x},\bar{V}_k^1 > \Big),$$

and then

$$\begin{aligned}(B^*U^2)_k &= -\mu\Delta^2 V_k^2 - \Delta J(\psi_k, V_k^2) - \\ &- \frac{p_{k-1}}{H_k}J(\psi_{k-1},V_{k-1}^2) + \frac{p_k}{H_k}J(\psi_k,V_k^2) + \\ &+ \frac{p_{k-1}}{H_k}J(\psi_k,V_k^2) - \frac{p_k}{H_k}J(\psi_{k+1},V_{k+1}^2) - \\ &- J(V_k^2,\theta_k) - \beta\frac{\partial V_k^2}{\partial x}.\end{aligned}$$

So

$$\begin{aligned}(SU)_k &= -\mu\Delta^2 V_k + \frac{1}{2}\left(J(\psi_k,U_k) - \Delta J(\psi_k,V_k)\right) - \\ &-\frac{p_{k-1}}{2H_k}\left(J(\psi_{k-1},V_{k-1}) - J(\psi_k,V_k)\right) + \\ &+\frac{p_k}{2H_k}\left(J(\psi_k,V_k) - J(\psi_{k+1},V_{k+1})\right).\end{aligned}$$

or

$$\begin{aligned}(SU)_k &= -\mu\Delta^2 V_k + \frac{1}{2}\left(J(\psi_k,\Delta V_k) - \Delta J(\psi_k,V_k)\right) + \\ &+\frac{p_{k-1}}{2H_k}J(\psi_k-\psi_{k-1},V_{k-1}) - \frac{p_k}{2H_k}J(\psi_{k+1}-\psi_k,V_{k+1}).\end{aligned}$$

3.2 The operator S.

Theorem 3.4. *The operator S is a self-adjoint closed operator on H^{-1} with compact resolvent. Its eigenvalues λ_i are real and bounded from below by*

$$\begin{aligned}\lambda_i &\geq \mu\nu_1 - ||\psi||_{H^2}\sqrt{3M\nu_1} \qquad \textit{if} \qquad ||\psi||_{H^2} \leq 2\mu\sqrt{\frac{\nu_1}{3M}}, \\ \lambda_i &\geq -||\psi||_{H^2}^2\frac{3M}{4\mu} \qquad \textit{otherwise.}\end{aligned}$$

where

$$M = \max_k\left(1, p_k\left[\frac{1}{H_k} + \frac{1}{H_{k+1}}\right]\right).$$

and ν_1 is the largest positive constant such that

$$\sum_{k=1}^{k=N} H_k|\Delta V_k|^2 \geq \nu_1||U||_{-1}^2.$$

Proof of the theorem. The operator S is self-adjoint by construction. Let us prove that S is closed. We look at the scalar product $((SU,U))$.

$$((SU,U)) = -< SU, H\bar{V} > .$$

$$\begin{aligned}
= & -\sum\left(-\mu < \Delta^2 V_k, H_k\bar{V}_k > +\right.\\
& +\frac{1}{2}\left(< J(\psi_k,U_k), H_k\bar{V}_k > - < \Delta J(\psi_k,V_k), H_k\bar{V}_k >\right) -\\
& -\frac{1}{2}\left(p_{k-1} < J(\psi_{k-1},V_{k-1}), \bar{V}_k > -p_{k-1} < J(\psi_k,V_k), \bar{V}_k >\right) +\\
& \left.+\frac{1}{2}\left(p_k < J(\psi_k,V_k), \bar{V}_k > -p_k < J(\psi_{k+1},V_{k+1}), \bar{V}_k >\right)\right),\\
= & -\sum\left(-\mu H_k|\Delta V_k|^2 +\right.\\
& +\frac{1}{2}\left(< J(\psi_k,U_k), H_k\bar{V}_k > - < J(\psi_k,V_k), H_k\Delta\bar{V}_k >\right) -\\
& -\frac{1}{2}\left(p_k < J(\psi_k,V_k), \bar{V}_{k+1} > -p_{k-1} < J(\psi_k,V_k), \bar{V}_k >\right) +\\
& \left.+\frac{1}{2}\left(p_k < J(\psi_k,V_k), \bar{V}_k > -p_{k-1} < J(\psi_k,V_k), \bar{V}_{k-1} >\right)\right).\\
= & -\sum\left(-\mu H_k|\Delta V_k|^2 +\right.\\
& +\frac{1}{2}\left(- < J(\psi_k,V_k), H_k\bar{U}_k > - < J(\psi_k,V_k), H_k\Delta\bar{V}_k >\right) -\\
& -\frac{1}{2}\left(p_k < J(\psi_k,V_k), \bar{V}_{k+1} > -p_k < J(\psi_k,V_k), \bar{V}_k >\right) +\\
& \left.+\frac{1}{2}\left(p_{k-1} < J(\psi_k,V_k), \bar{V}_k > -p_{k-1} < J(\psi_k,V_k), \bar{V}_{k-1} >\right)\right).\\
= & \sum\left(\mu H_k|\Delta V_k|^2 -\right.\\
& \left.-\frac{1}{2}\left(- < J(\psi_k,V_k), H_k\bar{U}_k > - < J(\psi_k,V_k), H_k\bar{U}_k >\right)\right).
\end{aligned}$$

So

$$((SU,U)) = \sum \mu H_k|\Delta V_k|^2 + \sum < J(\psi_k,V_k), H_k\bar{U}_k > . \tag{33}$$

From (19) with $\epsilon = \frac{1}{2}$, we have

$$\begin{aligned}
\left|\sum < J(\psi_k,V_k), H_k\bar{U}_k >\right| \leq & \frac{\mu}{2}\sum_{k=1}^{k=N} H_k|\Delta V_k|^2 +\\
& +\frac{3||\psi||^2_{H^2}}{2\mu}\max_k\left(1, p_k\left(\frac{1}{H_k}+\frac{1}{H_{k+1}}\right)\right)||U||^2_{-1}.
\end{aligned}$$

So we obtain

$$((SU,U)) \geq \frac{\mu}{2}\sum_{k=1}^{k=N} H_k|\Delta V_k|^2 - \frac{3||\psi||^2_{H^2}}{2\mu}\max_k\left(1, p_k\left(\frac{1}{H_k}+\frac{1}{H_{k+1}}\right)\right)||U||^2_{-1}.$$

Let us consider the operator $T = S + hId$ where Id is the operator Identity on H^{-1} and h a non negative real. We have

$$((TU,U)) = ((SU,U)) + h||U||^2_{-1} > \frac{\mu}{2}\sum_{k=1}^{k=N} H_k|\Delta V_k|^2 \geq 0 \tag{34}$$

for any real h such that

$$h > \frac{3||\psi||_{H^2}^2}{2\mu} \max_k \left(1, p_k \left(\frac{1}{H_k} + \frac{1}{H_{k+1}}\right)\right).$$

On the other hand, the Hölder's inequality gives us that

$$((TU, U)) \leq ||TU||_{-1}||U||_{-1}.$$

From (34), we deduce

$$\frac{\mu}{2} \sum_{k=1}^{k=N} H_k |\Delta V_k|^2 \leq ||TU||_{-1}||U||_{-1}. \tag{35}$$

The Poincaré's inequality implies

$$\frac{\mu}{2}\sqrt{\mu_1} \sum_{k=1}^{k=N} H_k |\nabla V_k||\Delta V_k| \leq ||TU||_{-1}||U||_{-1},$$

and by the Hölder's inequality and the equivalence of the following norms

$$\sum_{k=1}^{k=N} H_k |\nabla V_k|^2 \geq c||U||_{-1}^2,$$

$$\sum_{k=1}^{k=N} H_k |\Delta V_k|^2 \geq c||U||_2^2,$$

we have

$$c||U||_{-1}||U||_2 \leq ||TU||_{-1}||U||_{-1},$$

and then

$$c||U||_2 \leq ||TU||_{-1}.$$

And since $||U||_{-1} \leq ||U||_2$, we deduce that the operator T^{-1} is bounded. So the operator T is closed and so is S. From the compactness of the embedding from L^2 to H^{-1}, we conclude that T^{-1} is a compact operator. Then S is an operator with compact resolvent. We denote by λ_i the eigenvalues of S. If U is the eigenvector associated to λ_k, then, from (19) it follows that (U, λ_k) verifies

$$((SU, U)) \geq \mu \sum_{k=1}^{k=N} H_k |\Delta V_k|^2 - \sum_{k=1}^{k=N} H_k |(J(\psi_k, V_k), \bar{U}_k)|.$$

We recall (19)

$$\begin{aligned}\sum_{k=1}^{k=N} H_k |(J(\psi_k, V_k), \bar{U}_k)| &\leq \mu\epsilon \sum_{k=1}^{k=N} H_k |\Delta V_k|^2 + \\ &+ \frac{3||\psi||_{H^2}^2}{4\mu\epsilon} \max_k \left(1, p_k \left(\frac{1}{H_k} + \frac{1}{H_{k+1}}\right)\right) ||U||_{-1}^2.\end{aligned}$$

Inserting this inequality in the last estimate, we obtain

$$\begin{aligned}\lambda_i ||U||_{-1}^2 &\geq \mu(1-\epsilon) \sum_{k=1}^{k=N} H_k |\Delta V_k|^2 - \\ &- \frac{3||\psi||_{H^2}^2}{4\mu\epsilon} \max_k \left(1, p_k \left(\frac{1}{H_k} + \frac{1}{H_{k+1}}\right)\right) ||U||_{-1}^2.\end{aligned}$$

We denote by ν_1 the largest positive constant such that

$$\sum_{k=1}^{k=N} H_k |\Delta V_k|^2 \geq \nu_1 ||U||_{-1}^2.$$

We then obtain that for all $\epsilon \geq 0$,

$$\begin{aligned}
\lambda_i ||U||_{-1}^2 &\geq \mu(1-\epsilon)\nu_1 ||U||_{-1}^2 - \\
&- \frac{3||\psi||_{H^2}^2}{4\mu\epsilon} \max_k \left(1, p_k \left(\frac{1}{H_k} + \frac{1}{H_{k+1}}\right)\right) ||U||_{-1}^2, \\
\lambda_i &\geq (1-\epsilon)\mu\nu_1 - \\
&- \frac{3||\psi||_{H^2}^2}{4\mu\epsilon} \max_k \left(1, p_k \left(\frac{1}{H_k} + \frac{1}{H_{k+1}}\right)\right).
\end{aligned}$$

Let us now optimize in ϵ. We introduce a function $f(\epsilon) = (1-\epsilon)\mu\nu_1 - \frac{R}{\epsilon}$ where

$$R = \frac{3||\psi||_{H^2}^2}{4\mu} \max_k \left(1, p_k \left(\frac{1}{H_k} + \frac{1}{H_{k+1}}\right)\right).$$

The derivative of f is equal to

$$f'(\epsilon) = -\mu\nu_1 + \frac{R}{\epsilon^2}$$

and the optimum of f will be in $\hat{\epsilon} = \sqrt{\frac{R}{\mu\nu_1}}$. Then

$$\begin{aligned}
\lambda_i &\geq f(\hat{\epsilon}), \\
\lambda_i &\geq \mu\nu_1 - ||\psi||_{H^2} \sqrt{3\nu_1 \max_k \left(1, p_k \left(\frac{1}{H_k} + \frac{1}{H_{k+1}}\right)\right)}
\end{aligned}$$

which ended the proof of theorem 3.4.

We now consider the eigenvalues of the operator S and prove the theorem 3.5.

Theorem 3.5. *The following equality holds :*

$$\lim_{t \to 0} \frac{1}{t} log \omega_m(t, \theta_0) = \sum_{i=1}^{i=m} \lambda_i(-S(0, \theta_0)).$$

We first prove the lemma :

Lemma. The eigenvalue $\lambda_i(S(t, \theta_0))$ converges to $\lambda_i(S(t_0, \theta_0))$ continuously when t tends to t_0.

As before, the proof is divided in three steps.
First step : The operator S verify for all U in L^2:
for all $\varepsilon > 0$, there exists T such that for all t with $|t - t_0| \leq T$,

$$||S(t)U - S(t_0)U|| \leq \varepsilon \sqrt{\sum_k H_k |\Delta V_k|^2}.$$

Let us estimate $E = < S(t)U - S(t_0)U, Hv >$.

$$\begin{aligned}
E &= \sum_k \frac{H_k}{2} < J(\psi_k(t), U_k) - \Delta J(\psi_k(t), V_k), v_k > - \\
&- \sum_k \frac{H_k}{2} < J(\psi_k(t_0), U_k) - \Delta J(\psi_k(t_0), V_k), v_k > + \\
&+ \sum_k \frac{p_{k-1}}{2} < J(\psi_k(t), V_k) - J(\psi_{k-1}(t), V_{k-1}), v_k > - \\
&- \sum_k \frac{p_{k-1}}{2} < J(\psi_k(t_0), V_k) - J(\psi_{k-1}(t_0), V_{k-1}), v_k > - \\
&- \sum_k \frac{p_k}{2} < J(\psi_{k+1}(t), V_{k+1}) - J(\psi_k(t), V_k), v_k > + \\
&+ \sum_k \frac{p_k}{2} < J(\psi_{k+1}(t_0), V_{k+1}) - J(\psi_k(t_0), V_k), v_k > .
\end{aligned}$$

We introduce $\Psi_k = \psi_k(t) - \psi_k(t_0)$. We obtain

$$\begin{aligned}
E &= \sum_k \frac{H_k}{2} < J(\Psi_k, U_k) - \Delta J(\Psi_k, V_k), v_k > + \\
&+ \sum_k \frac{p_{k-1}}{2} < J(\Psi_k, V_k) - J(\Psi_{k-1}, V_{k-1}), v_k > - \\
&- \sum_k \frac{p_k}{2} < J(\Psi_{k+1}, V_{k+1}) - J(\Psi_k, V_k), v_k > .
\end{aligned}$$

Since $H_k U_k = H_k \Delta V_k + p_k(V_{k+1} - V_k) - p_{k-1}(V_k - V_{k-1})$, we have

$$\begin{aligned}
E &= \sum_k \frac{H_k}{2} < J(\Psi_k, \Delta V_k) - \Delta J(\Psi_k, V_k), v_k > + \\
&+ \sum_k \frac{p_k}{2} < J(\Psi_k, V_{k+1} - V_k), v_k > - \\
&- \sum_k \frac{p_{k-1}}{2} < J(\Psi_k, V_k - V_{k-1}), v_k > + \\
&+ \sum_k \frac{p_{k-1}}{2} < J(\Psi_k, V_k) - J(\Psi_{k-1}, V_{k-1}), v_k > - \\
&- \sum_k \frac{p_k}{2} < J(\Psi_{k+1}, V_{k+1}) - J(\Psi_k, V_k), v_k > .
\end{aligned}$$

Using the same estimates as before, we deduce

$$\begin{aligned}
| < J(\Psi_k, \Delta V_k) - \Delta J(\Psi_k, V_k), v_k > | &\leq c|\nabla \Delta \Psi_k||\nabla v_k||\Delta V_k|, \\
| < J(\Psi_k, V_{k+1}), v_k > | &\leq c|\Delta \Psi_k||\nabla V_{k+1}||\nabla v_k|, \\
&\leq c'|\Delta \Psi_k||\Delta V_{k+1}||\nabla v_k|
\end{aligned}$$

thanks to the Poincaré's inequality.

And since $|\nabla\Delta\psi_k(t)|$ and $|\Delta\psi_k(t)|$ are continuous in time, for all $\varepsilon > 0$, there exists T such that for all t with $|t-t_0| \le T$, $|\nabla\Delta\Psi_k| \le \varepsilon, |\Delta\Psi_k| \le \varepsilon \forall k \in 1,\dots,N$. So

$$\begin{aligned}\sum_k \frac{H_k}{2} | < J(\Psi_k, \Delta V_k) - \Delta J(\Psi_k, V_k), v_k > | &\le c\varepsilon \sum_k H_k |\nabla v_k||\Delta V_k|,\\ &\le c\varepsilon \sqrt{\sum_k H_k |\nabla v_k|^2}\sqrt{\sum_k H_k|\Delta V_k|^2},\end{aligned}$$

$$\begin{aligned}\sum_k \frac{p_k}{2} | < J(\Psi_k, V_{k+1}), v_k > | &\le c\varepsilon \sum_k |\Delta V_{k+1}||\nabla v_k|,\\ &\le c\varepsilon \sqrt{\sum_k H_{k+1}|\Delta V_{k+1}|^2}\sqrt{\sum_k H_k|\nabla v_k|^2},\end{aligned}$$

and then

$$< S(t)U - S(t_0)U, Hv > \le c\varepsilon \sqrt{\sum_k H_k|\nabla v_k|^2}\sqrt{\sum_k H_k|\Delta V_k|^2}.$$

We conclude that

$$||S(t)U - S(t_0)U||_{-1} \le c\varepsilon \sqrt{\sum_k H_k|\Delta V_k|^2}.$$

Second step: The operator S is continuous in μ in the generalized sense defined by Kato in [13].

Let us introduce S_n the operator defined by $S_nU = SU + \frac{1}{n}\Delta^2 V$ and u and u^n the two solutions of the following equations : $Su + \zeta u = f$, $S_n u^n + \zeta u^n = f$. Replacing S_n by its definition , we obtain: $Su^n + \frac{1}{n}\Delta^2 v^n + \zeta u^n = f$. Introducing $z = v^n - v$ and $w = u^n - u$, we have :

$$Sw + \zeta w = \frac{1}{n}\Delta^2 v^n.$$

We multiply each equation by $H_k\bar{z}_k$, integrate on Ω and sum over k to obtain

$$\begin{aligned}\sum_k \left(-\mu H_k(\Delta^2 z_k, \bar{z}_k) + \frac{H_k}{2}(< J(\psi_k, \Delta z_k) - \Delta J(\psi_k, z_k), \bar{z}_k >)\right) +\\ + \sum_k \frac{p_{k-1}}{2} < J(\psi_k - \psi_{k-1}, z_{k-1}), \bar{z}_k > -\\ - \sum_k \frac{p_k}{2} < J(\psi_{k+1} - \psi_k, z_{k+1}), \bar{z}_k > - \zeta||w||^2_{-1} = \frac{1}{n}\sum_k H_k(\Delta^2 v_k^n, \bar{z}_k).\end{aligned}$$

So,

$$\begin{aligned}\mu \sum_k H_k|\Delta z_k|^2 + \zeta||w||^2_{-1} &\le \sum_k H_k||\psi_k||_{H^2}|\Delta z_k||\nabla z_k| +\\ &+ \sum_k p_k| < J(\psi_{k+1} - \psi_k, z_{k+1}), \bar{z}_k > | +\\ &+ \frac{1}{n}\sum_k H_k(\Delta^2 v_k^n, \bar{z}_k).\end{aligned}$$

And then

$$\begin{aligned}\mu \sum_k H_k |\Delta z_k|^2 + \zeta ||w||_{-1}^2 &\leq \sum_k H_k ||\psi_k||_{H^2} |\Delta z_k| |\nabla z_k| + \\ &+ \sum_k p_k |\nabla z_{k+1}| |\bar{z}_k|_{L^4} |\nabla \psi_{k+1} - \nabla \psi_k|_{L^4} + \\ &+ \frac{1}{n} \sum_k H_k |\Delta v_k^n| |\Delta z_k|.\end{aligned}$$

The Sobolev embeddings give us

$$\begin{aligned}\mu \sum_k H_k |\Delta z_k|^2 + \zeta ||w||_{-1}^2 &\leq \sum_k H_k ||\psi_k||_{H^2} |\Delta z_k| |\nabla z_k| + \\ &+ c \sum_k p_k |\nabla z_{k+1}| |\nabla z_k| |\Delta \psi_{k+1} - \Delta \psi_k| + \\ &+ \frac{1}{n} \sum_k H_k |\Delta v_k^n| |\Delta z_k|.\end{aligned}$$

Therefore

$$\begin{aligned}\mu \sum_k H_k |\Delta z_k|^2 + \zeta ||w||_{-1}^2 &\leq \frac{1}{2\mu} \sum_k H_k ||\psi_k||_{H^2}^2 |\nabla z_k|^2 + \frac{\mu}{2} \sum_k H_k |\Delta z_k|^2 + \\ &+ c\frac{1}{2} \sum_k p_k |\nabla z_{k+1}|^2 |\Delta \psi_{k+1} - \Delta \psi_k| + \\ &+ c\frac{1}{2} \sum_k p_k |\Delta \psi_{k+1} - \Delta \psi_k| |\nabla z_k|^2 + \\ &+ \frac{1}{2n^2\mu} \sum_k H_k |\Delta v_k^n|^2 + \frac{\mu}{2} \sum_k H_k |\Delta z_k|^2.\end{aligned}$$

And we obtain

$$\begin{aligned}\zeta ||w||_{-1}^2 &\leq \frac{1}{2\mu} \sum_k H_k ||\psi_k||_{H^2}^2 |\nabla z_k|^2 + \\ &+ c\frac{1}{2} \sum_k p_k |\nabla z_{k+1}|^2 |\Delta \psi_{k+1} - \Delta \psi_k| + \\ &+ c\frac{1}{2} \sum_k p_k |\Delta \psi_{k+1} - \Delta \psi_k| |\nabla z_k|^2 + \\ &+ \frac{1}{2n^2\mu} \sum_k H_k |\Delta v_k^n|^2.\end{aligned}$$

Let us introduce

$$C = \max_k \left(\frac{1}{2\mu} ||\psi_k||_{H^2}^2 + \frac{cp_k}{2H_k} |\Delta \psi_{k+1} - \Delta \psi_k| + \frac{cp_{k-1}}{2H_k} |\Delta \psi_k - \Delta \psi_{k-1}| \right),$$

we have

$$\zeta ||w||_{-1}^2 \leq C \sum_k H_k |\nabla z_k|^2 + \frac{1}{2n^2\mu} \sum_k H_k |\Delta v_k^n|^2.$$

Since $\sum_k H_k|\nabla z_k|^2 \leq ||w||_{-1}^2$, we conclude

$$(\zeta - C)||w||_{-1}^2 \leq \frac{1}{2n^2\mu}\sum_k H_k|\Delta v_k^n|^2. \tag{36}$$

Let us now estimate the sum $E = \sum_k H_k|\Delta v_k^n|^2$. We consider the equation$(S_n + \zeta Id)u^n = f$. We multiply this equation by $H_k v_k^n$, integrate on Ω and sum on k.We obtain

$$-(\mu + \frac{1}{n})\sum_k H_k(\Delta^2 v_k^n, v_k^n) + \sum_k H_k < J(\psi_k, \Delta v_k^n), v_k^n > +$$
$$+\sum_k p_k < J(\psi_{k+1} - \psi_k, v_{k+1}^n), v_k^n > -\zeta||w||_{-1}^2 = \sum_k H_k(f_k, v_k^n).$$

Using the usual inequalities, we have

$$\begin{aligned}(\mu + \frac{1}{n})E + \zeta||w||_{-1}^2 \quad &\leq \quad \sum_k H_k||\psi_k||_{H^2}|\nabla v_k^n||\Delta v_k^n| + \\ &+\sum_k cp_k|\Delta\psi_{k+1} - \Delta\psi_k||\nabla v_{k+1}^n||\nabla v_k^n| + \\ &+\sum_k H_k||f_k||_{-1}|\nabla v_k^n|.\end{aligned}$$

So

$$\begin{aligned}(\mu + \frac{1}{n})E + \zeta||w||_{-1}^2 \quad &\leq \quad \sum_k \frac{H_k}{2\mu}||\psi_k||_{H^2}^2|\nabla v_k^n|^2 + \\ &+\frac{\mu}{2}\sum_k H_k|\Delta v_k^n|^2 + \\ &+\sum_k \frac{cp_k}{2}|\Delta\psi_{k+1} - \Delta\psi_k||\nabla v_{k+1}^n|^2 + \\ &+\sum_k \frac{cp_k}{2}|\Delta\psi_{k+1} - \Delta\psi_k||\nabla v_k^n|^2 + \\ &+\sqrt{\sum_k H_k||f_k||_{-1}^2}\sqrt{\sum_k H_k|\nabla v_k^n|^2}.\end{aligned}$$

And then

$$\begin{aligned}(\frac{\mu}{2} + \frac{1}{n})E + \zeta||w||_{-1}^2 \quad &\leq \quad C\sum_k H_k|\nabla v_k^n|^2 + \\ &+\sqrt{\sum_k H_k||f_k||_{-1}^2}\sqrt{\sum_k H_k|\nabla v_k^n|^2},\end{aligned}$$

$$\begin{aligned}(\frac{\mu}{2} + \frac{1}{n})E + \zeta||w||_{-1}^2 \quad &\leq \quad C||w||_{-1}^2 + \\ &+\sqrt{\sum_k H_k||f_k||_{-1}^2}\sqrt{\sum_k H_k\frac{1}{\mu_1}|\Delta v_k^n|^2}.\end{aligned}$$

For $\zeta \leq C$, we have

$$(\frac{\mu}{2} + \frac{1}{n})\sqrt{E} \quad \leq \quad \frac{1}{\sqrt{\mu_1}}\sqrt{\sum_k H_k ||f_k||_{-1}^2}.$$

So

$$\frac{\mu\sqrt{\mu_1}}{2}\sqrt{E} \quad \leq \quad \sqrt{\sum_k H_k ||f_k||_{-1}^2}.$$

We conclude that

$$E \quad \leq \quad \frac{4}{\mu^2\mu_1}||f||_{-1}^2.$$

Replacing in (36), we obtain

$$(\zeta - C)||w||_{-1}^2 \leq \frac{1}{2n^2\mu}\frac{4}{\mu^2\mu_1}||f||_{-1}^2.$$

So

$$\frac{||w||_{-1}^2}{||f||_{-1}^2} \leq \frac{2}{n^2\mu^3\mu_1(\zeta - C)},$$

and

$$\frac{||w||_{-1}}{||f||_{-1}} \leq \frac{1}{n\mu}\sqrt{\frac{2}{\mu_1(\zeta - C)}}.$$

Since

$$||R(-\zeta, S_n) - R(-\zeta, S)|| = \sup_{f \in H^{-1}} \frac{||w||_{-1}}{||f||_{-1}},$$

we have

$$||R(-\zeta, S_n) - R(-\zeta, S)|| \leq \frac{1}{n\mu}\sqrt{\frac{2}{\mu_1(\zeta - C)}},$$

which tends to zero when n tends to infinity. So, S_n converges to S in the generalized sense.

Third step : Proof of the lemma.
We consider U such that $||U||_{-1} = 1$ and introduce the functionals $E_t(U) = ((S(t)U, U))$,$E_0(U) = ((S(t_0)U, U))$ and $E_t^{1/n}(U) = ((S_n(t)U, U))$,$E_0^{1/n}(U) = ((S_n(t_0)U, U))$. Notice that $E_t^{1/n}(U) = E_t(U) + \frac{1}{n}\sum_k H_k|\Delta V_k|^2$. We denote by $\lambda_i(S(t))$ the eigenvalues of $S(t)$ and by $\lambda_i(S_n(t))$ the eigenvalues of $S_n(t)$. By definition, we have :

$$|E_t(U) - E_0(U)| = |((S(t)U, U)) - ((S(t_0)U, U))| \leq ||S(t)U - S(t_0)U||_{-1}||U||_{-1}.$$

Let ε fixed. By Kato [13], since S_n converges to S in the generalized sense, then $\lambda_i(S_{-n}(t_0))$ converges to $\lambda_i(S(t_0))$. We choose n such that

$$\left\{ \begin{array}{lll} |\lambda_i(S_n(t_0)) - \lambda_i(S(t_0))| & \leq & \varepsilon, \\ |\lambda_i(S_{-n}(t_0)) - \lambda_i(S(t_0))| & \leq & \varepsilon, \\ \frac{1}{n} & \leq & \varepsilon. \end{array} \right.$$

The first step and $||U||_{-1} = 1$ give us that for this n, there exists T such that for all t with $|t - t_0| \leq T$

$$|E_t(U) - E_0(U)| \leq \frac{2}{n}\sqrt{\sum_k H_k|\Delta V_k|^2} \leq \frac{1}{n} + \frac{1}{n}\sum_k H_k|\Delta V_k|^2.$$

and then

$$\begin{array}{rll} E_0(U) - \frac{1}{n}\sum_k H_k|\Delta V_k|^2 - \frac{1}{n} & \leq E_t(U) \leq & E_0(U) + \frac{1}{n}\sum_k H_k|\Delta V_k|^2 + \frac{1}{n}, \\ E_0^{-1/n}(U) - \frac{1}{n} & \leq E_t(U) \leq & E_0^{1/n}(U) + \frac{1}{n}. \end{array}$$

By the inf-sup definition of the eigenvalues, we have

$$\begin{array}{rll} \lambda_i(S_{-n}(t_0)) - \frac{1}{n} & \leq \lambda_i(S(t)) \leq & \lambda_i(S_n(t_0)) + \frac{1}{n}, \\ \lambda_i(S(t_0)) - 2\varepsilon & \leq \lambda_i(S(t)) \leq & \lambda_i(S(t_0)) + 2\varepsilon. \end{array}$$

We conclude that $\lambda_i(S(t))$ converges to $\lambda_i(S(t_0))$ when t tends to t_0.

The proof of theorem 3.5 is now the same as the proof of theorem 2.5.

3.3 The eigenvalues of the operator S.

As before, we estimate the eigenvalues of S using some ideas of the theory of critical points [14]. We are looking for the critical points of $E_0(U)$ under the constraint K :

$$K = \left\{U \in L^2, \Delta V = 0 \text{ on } \partial\Omega, ||U||_{-1} = 1\right\}.$$

We recall that

$$E_0(U) = \mu\sum_k H_k|\Delta V_k|^2 + J(U)$$

where

$$\begin{array}{rcl} J(U) & = & -\sum_k H_k < J(\psi_k, V_k), \Delta V_k > + \\ & & +\sum_k p_k < J(\psi_k, V_k), V_{k+1} - V_k > - \\ & & -\sum_k p_{k-1} < J(\psi_k, V_k), V_k - V_{k-1} > . \end{array}$$

Let us remember that (19) gives us

$$\begin{array}{rcl} |J(U)| & \leq & \mu\epsilon\sum_{k=1}^{k=N} H_k|\Delta V_k|^2 + \\ & + & \frac{3||\psi||_{H^2}^2}{4\mu\epsilon}\max_k\left(1, p_k\left(\frac{1}{H_k} + \frac{1}{H_{k+1}}\right)\right)||U||_{-1}^2. \end{array}$$

Since by the constraint K, $||U||_{-1}^2 = 1$, we obtain

$$E_0(U) \geq \mu(1-\varepsilon)\sum_k H_k|\Delta V_k|^2 - \frac{3M}{4\mu\varepsilon}\sum_k ||\psi_k||_{H^2}^2.$$

where

$$M = \max_k\left(1, p_k\left[\frac{1}{H_k} + \frac{1}{H_{k+1}}\right]\right).$$

As before, we denote by ν_i the critical points of the functional

$$G(U) = \sum_k H_k |\Delta V_k|^2$$

under the constraint K. Taking the inf-sup definition of the eigenvalues, we obtain

$$\lambda_i(S(t_0)) \quad \geq \quad \mu(1-\varepsilon)\nu_i - \frac{3M}{4\mu\varepsilon}||\psi||^2_{H^2}.$$

We now optimize in $\varepsilon \in [0,1]$. We obtain :

Theorem 3.6 : *The eigenvalues of S satisfie*

$$\begin{array}{lll} \lambda_i & \geq & \mu\nu_i - ||\psi||_{H^2}\sqrt{3M\nu_i} \qquad \textit{if} \qquad ||\psi||_{H^2} \leq 2\mu\sqrt{\frac{\nu_i}{3M}}, \\ \lambda_i & \geq & -||\psi||^2_{H^2}\frac{3M}{4\mu} \qquad \textit{otherwise.} \end{array}$$

where

$$M = \max_k \left(1, p_k\left[\frac{1}{H_k} + \frac{1}{H_{k+1}}\right]\right).$$

and ν_i are the critical points of the functional $G(U) = \sum_k H_k|\Delta V_k|^2$.

Let us determine these critical points ν_i of the functional $G(U)$ under the constraint

$$K = \left\{U \in L^2, \Delta V = 0 \text{ on } \partial\Omega, ||U||_{-1} = 1\right\}.$$

These points are the eigenvalues of the problem :

$$H_k\Delta^2 V_k \quad = \quad \nu\left(-H_k\Delta V_k - p_k(V_{k+1} - V_k) + p_{k-1}(V_k - V_{k-1})\right)$$

i.e.

$$\Delta^2 V \quad = \quad \nu\left(\Delta V - WV\right)$$

Let us work in the basis where W is a diagonal matrix Λ with coefficients Λ_k (remember that $\Lambda_1 = 0$). We recall that P and P^{-1} are the two matrices such that $\Lambda = P^{-1}WP$, and $V^m = P^{-1}V$, $U^m = P^{-1}U$, ...etc. In this basis, our problem is :

$$\Delta^2 V^m \quad = \quad \nu\left(\Delta V^m - \Lambda V^m\right),$$

with the following boundary conditions (CL1) :

$$\begin{cases} V^m = C^m & \text{on} \quad \partial\Omega, \\ \int_\Omega V^m = 0. \end{cases}$$

As before, we introduce $\bar{V}^m$ such that

$$V^m = \bar{V}^m + C^m = \bar{V}^m - \frac{1}{|\Omega|}\int_\Omega \bar{V}^m,$$

where $|\Omega| = mes(\Omega)$, and $\bar{V}^m \in H^1_0$, and rewrite the problem on the following form:

$$\Delta^2\bar{V}^m \quad = \quad \nu\left(\Delta\bar{V}^m - \Lambda\bar{V}^m + \frac{\Lambda_k}{|\Omega|}\int_\Omega \bar{V}^m\right),$$

with the following boundary conditions :

$$\bar{V}^m = 0 \quad \text{on} \quad \partial\Omega,$$

and so, ν is a critical point of the functional $G^m(U^m) = \sum_k |\Delta \bar{V}_k^m|^2$ under the constraint

$$K^m = \left\{ U^m \in L^2, \Delta \bar{V}_k^m = 0 \text{ on } \partial\Omega, \sum_k |\nabla \bar{V}_k^m|^2 + \Lambda_k |\bar{V}_k^m|^2 - \frac{\Lambda_k}{|\Omega|} \left(\int_\Omega \bar{V}^m \right)^2 = 1 \right\}.$$

Remember that the Schwartz inequality gives us

$$\Lambda_k |\bar{V}_k^m|^2 - \frac{\Lambda_k}{|\Omega|} \left(\int_\Omega \bar{V}^m \right)^2 \geq 0.$$

Taking the inf-sup definition, we obtain :

$$\nu_i \quad = \quad \inf_{A \in L^2, dim A = i,} \; \sup_{U^m \in A \bigcap K^m} G^m(U^m),$$

or

$$\nu_i \quad = \quad \inf_{A \in L^2, dim A = i} \; \sup_{U^m \in A, \Delta \bar{V}^m = 0 \text{ on } \partial\Omega} \frac{\sum_k |\Delta \bar{V}_k^m|^2}{\sum_k |\nabla \bar{V}_k^m|^2 + \Lambda_k |\bar{V}_k^m|^2 - \frac{\Lambda_k}{|\Omega|} \left(\int_\Omega \bar{V}^m \right)^2},$$

which we denote by

$$\nu_i \quad = \quad \inf_{A \in L^2, dim A = i} \; \sup_{U^m \in A, \Delta \bar{V}^m = 0 \text{ on } \partial\Omega} \frac{G^m(U^m)}{||U^m||^2}.$$

But since

$$\sum_k |\nabla \bar{V}_k^m|^2 \leq \sum_k |\nabla \bar{V}_k^m|^2 + \Lambda_k |\bar{V}_k^m|^2 - \frac{\Lambda_k}{|\Omega|} \left(\int_\Omega \bar{V}^m \right)^2 \leq \sum_k |\nabla \bar{V}_k^m|^2 + \Lambda_k |\bar{V}_k^m|^2,$$

we obtain

$$\frac{\sum_k |\Delta \bar{V}_k^m|^2}{\sum_k |\bar{\nabla} V_k^m|^2} \geq \frac{G^m(U^m)}{||U^m||^2} \geq \frac{\sum_k |\Delta \bar{V}_k^m|^2}{\sum_k |\nabla \bar{V}_k^m|^2 + \Lambda_k |\bar{V}_k^m|^2}.$$

As before, we now take the supremum over $U \in A$ with $\Delta V_k^m = 0$ on $\partial\Omega$, and then the infimum over all subspaces A of $L^2(\Omega)$ with dimension i. We denote by μ_i the i-th eigenvalues of the Laplacian operator with homogeneous Dirichlet boundary condition. We recognize on the left hand side μ_i. Let us determine the right hand side. We are looking for the critical points of the functional $G^m(U^m)$ under the constraint

$$\bar{K} = \left\{ U^m \in L^2, \Delta \bar{V}_k^m = 0 \text{ on } \partial\Omega, \sum_k |\nabla \bar{V}_k^m|^2 + \Lambda_k |\bar{V}_k^m|^2 = 1 \right\}.$$

These are the eigenvalues η_i of the following problem :

$$\Delta^2 \bar{V}_k^m = \eta(-\Delta \bar{V}_k^m + \Lambda_k \bar{V}_k^m)$$

It is easy to see that

$$\{\eta_i, i = 1, \ldots, \infty\} = \left\{ \frac{\mu_i^2}{\mu_i + \Lambda_k}, k = 1, \ldots, N; i = 1, \ldots, \infty \right\}.$$

So we obtain

$$\mu_i \geq \nu_i \geq \eta_i.$$

Notice that for homogeneous boundary conditions on V, we have $\nu_i = \eta_i$.

We remark that the estimates take into account the inverse of radii of deformation of each layer.

4 Numerical results.

Numerical experiments were provided by Dr. Kazantsev for a one-layer model on the sphere . At each time of integration, we compute instantaneous Lyapunov exponents, which are defined as the eigenvalues of the discrete operator [9], [10] and the estimate of theorem 2.6.. We renorm these results and compare (see picture). On the following table, we write the coefficient of correlation between these two sequences of numbers. We compute these coefficients for different lengh of sequences. The time integration is 300 days, with a result storage twice a day.

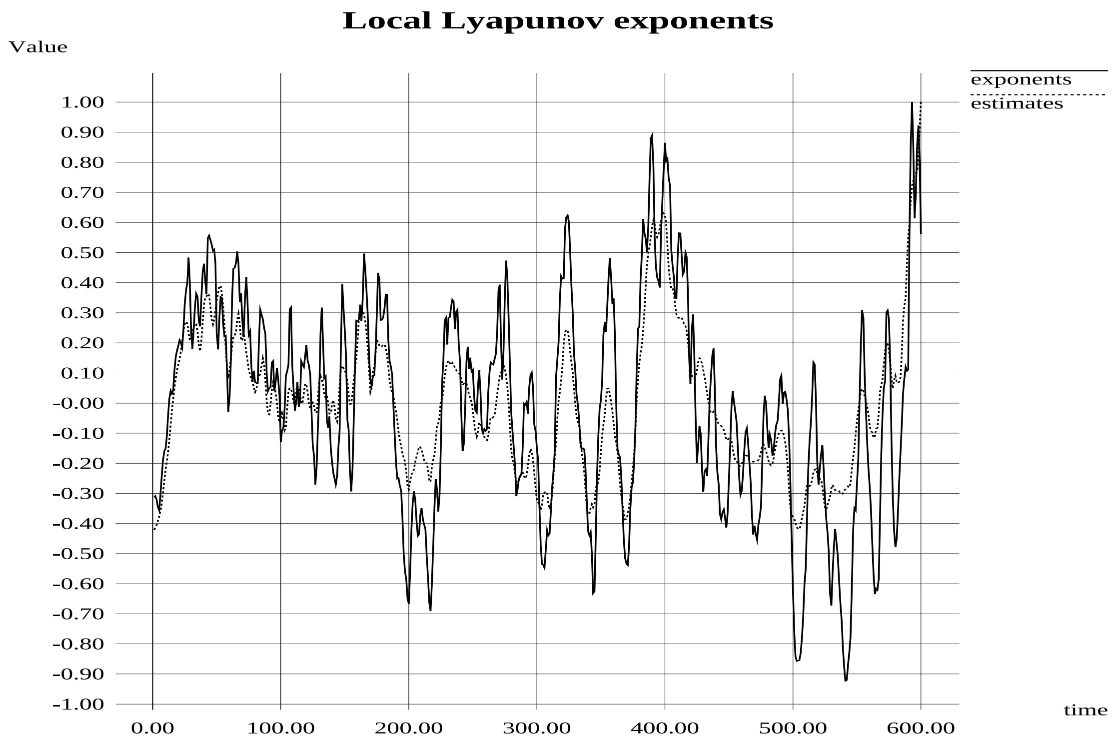

Table 1: Coefficients of correlation.

Vector lengh (in days)	30	50	100	150	300
Number of vectors	10	6	3	2	1
Average correlation	0.825000	0.855829	0.848636	0.833358	0.835897
Maximum of correlation	0.969472	0.915011	0.870233	0.836611	0.835897
Minimum of correlation	0.352854	0.758368	0.819056	0.830105	0.835897

5 Conclusion.

The a priori estimates that we have obtained give us not only a bound from below for each eigenvalue, but also a bound from above for the number of negative eigenvalues. These estimates are still valid for a very small coefficient of viscosity. We obtain a good coefficient of correlation between our estimates and the numerical value (between 0.80 and 0.85). Our estimates are easy to compute and give a good indice of predictability of the system.

6 Appendix A : equivalence of norms.

In this section, we will prove the equivalence between the three following norms in L^2:

$$\begin{aligned}
||U||^2 &= \sum_k H_k|\Delta V_k|^2 + p_k|\nabla V_{k+1} - \nabla V_k|^2, \\
|U|^2 &= \sum_k H_k|U_k|^2, \\
G(U) &= \sum_k H_k|\Delta V_k|^2.
\end{aligned}$$

First, we prove the equivalence between $||U||^2$ and $G(U)$.
By definition of these norms, we have $G(U) \leq ||U||^2$. We are interested by the reverse inequality. Let us estimate :

$$\begin{aligned}
||U||^2 &= \sum_k H_k|\Delta V_k|^2 + p_k|\nabla V_{k+1} - \nabla V_k|^2, \\
&\leq \sum_k H_k|\Delta V_k|^2 + 2p_k|\nabla V_{k+1}|^2 + 2p_k|\nabla V_k|^2, \\
&\leq \sum_k H_k|\Delta V_k|^2 + 2(p_{k-1} + p_k)|\nabla V_k|^2, \\
&\leq \sum_k H_k|\Delta V_k|^2 + 2(p_{k-1} + p_k)\mu_1|\Delta V_k|^2, \\
&\leq \sum_k H_k|\Delta V_k|^2(1 + 2\mu_1\frac{p_{k-1} + p_k}{H_k}), \\
&\leq \max_k(1 + 2\mu_1\frac{p_{k-1} + p_k}{H_k})\sum_k H_k|\Delta V_k|^2, \\
&\leq \max_k(1 + 2\mu_1\frac{p_{k-1} + p_k}{H_k})G(U).
\end{aligned}$$

We prove now the equivalence between $|U|^2$ and $G(U)$.

Let us work first in the basis where the matrix W is the diagonal matrix Λ. By definition, we have

$$U_k^m = \Delta V_k^m - \Lambda_k V_k^m,$$

and then

$$|U_k^m|^2 = |\Delta V_k^m|^2 + \Lambda_k^2 |V_k^m|^2 - 2\Lambda_k \int_\Omega \Delta V_k^m V_k^m \, dx. \tag{37}$$

We remark that

$$\begin{aligned}
|V_k^m|^2 &= \int_\Omega V_k^m \bar{V}_k^m \, dx + \int_\Omega V_k^m C_k^m \, dx, \\
&= \int_\Omega V_k^m \bar{V}_k^m \, dx + C_k^m \int_\Omega V_k^m, \\
&= \int_\Omega V_k^m \bar{V}_k^m \, dx, \\
&\leq |V_k^m||\bar{V}_k^m|,
\end{aligned}$$

and then

$$|V_k^m| \leq |\bar{V}_k^m|.$$

Plugging this result into (37), we obtain

$$\begin{aligned}
|U_k^m|^2 &\leq |\Delta V_k^m|^2 + \Lambda_k^2 |V_k^m|^2 + 2\Lambda_k |\Delta V_k^m||V_k^m|, \\
&\leq |\Delta V_k^m|^2 + \Lambda_k^2 |\bar{V}_k^m|^2 + 2\Lambda_k |\Delta V_k^m||\bar{V}_k^m|,
\end{aligned}$$

and the Poincaré's inequality gives us

$$\begin{aligned}
|U_k^m|^2 &\leq |\Delta V_k^m|^2 + \Lambda_k^2 \frac{1}{\mu_1^2} |\Delta V_k^m|^2 + 2\Lambda_k |\Delta V_k^m| \frac{1}{\mu_1} |\Delta V_k^m| \\
&\leq \left\{ 1 + \frac{\Lambda_k}{\mu_1} \right\}^2 |\Delta V_k^m|^2, \\
|U_k^m| &\leq (1 + \frac{\Lambda_k}{\mu_1}) |\Delta V_k^m|.
\end{aligned}$$

Let us now look for the reverse inequality. We begin with the definition of U_k^m, multiply by ΔV_k^m and integrate over Ω. We obtain

$$\begin{aligned}
|\Delta V_k^m|^2 - \Lambda_k \int_\Omega \Delta V_k^m V_k^m \, dx &= \int_\Omega \Delta V_k^m U_k^m \, dx, \\
|\Delta V_k^m|^2 + \Lambda_k |\nabla V_k^m|^2 + \Lambda_k \int_{\partial\Omega} \frac{\partial V_k^m}{\partial n} C_k^m \, d\sigma &\leq |\Delta V_k^m||U_k^m|, \\
|\Delta V_k^m|^2 + \Lambda_k |\nabla V_k^m|^2 - \Lambda_k |C_k^m| |\int_{\partial\Omega} \frac{\partial V_k^m}{\partial n} d\sigma| &\leq |\Delta V_k^m||U_k^m|.
\end{aligned}$$

But since

$$\int_{\partial\Omega} \frac{\partial V_k^m}{\partial n} = -\int_\Omega \Delta V_k^m \, dx$$

and

$$\int_\Omega \Delta V_k^m \, dx = \int_\Omega U_k^m + \Lambda_k V_k^m \, dx = \int_\Omega U_k^m,$$

we have

$$|\int_{\partial\Omega} \frac{\partial V_k^m}{\partial n}| \le \int_\Omega |U_k^m| \le |U_k^m|\sqrt{|\Omega|}.$$

On the other hand, we have

$$|C_k^m| = \frac{1}{|\Omega|}\int_\Omega \bar{V}_k^m \, dx \le \frac{1}{\sqrt{|\Omega|}}|\bar{V}_k^m|$$

thanks to the Schwartz inequality. Now, by the Poincaré's inequality, we obtain

$$|C_k^m| \le \frac{1}{\mu_1\sqrt{|\Omega|}}|\Delta V_k^m|.$$

We then conclude that

$$\begin{aligned}
|\Delta V_k^m|^2 + \Lambda_k|\nabla V_k^m|^2 &\le |\Delta V_k^m||U_k^m| + \frac{\Lambda_k}{\mu_1}|\Delta V_k^m||U_k^m|, \\
|\Delta V_k^m| &\le (1 + \frac{\Lambda_k}{\mu_1})|U_k^m|.
\end{aligned}$$

We now use the relation $V^m = P^{-1}V$ to conclude. Let us introduce

$$\begin{aligned}
p^{-1} &= \max_{j,k} |p_{j,k}^{-1}|, \\
p &= \max_{j,k} |p_{j,k}|,
\end{aligned}$$

where $p_{j,k}$ and $p_{j,k}^{-1}$ denote the entries of the matrices P and P^{-1} respectively, and

$$\begin{aligned}
H' &= \min_j H_j, \\
\bar{H} &= \max_j H_j.
\end{aligned}$$

We have

$$\begin{aligned}
|\Delta V_k|^2 &\le |\sum_j p_{k,j}^{-1}\Delta V_j^m|^2, \\
&\le N(p^{-1})^2 \sum_j |\Delta V_j^m|^2,
\end{aligned}$$

so

$$\sum_k H_k|\Delta V_k|^2 \le N^2(p^{-1})^2\bar{H}\sum_j |\Delta V_j^m|^2,$$

$$\leq \quad N^2(p^{-1})^2\bar{H}\max_j(1+\frac{\Lambda_j}{\mu_1})\sum_j |U_j^m|,$$

$$\leq \quad N^2(p^{-1})^2\bar{H}\max_j(1+\frac{\Lambda_j}{\mu_1})N^2p^2\frac{1}{H'}\sum_k H_k|U_k|^2,$$

$$\leq \quad N^4(p^{-1})^2p^2\frac{\bar{H}}{H'}\max_j(1+\frac{\Lambda_j}{\mu_1})\sum_k H_k|U_k|^2,$$

and with the same tools, we have

$$\sum_k H_k|U_k|^2 \quad \leq \quad N^4(p^{-1})^2p^2\frac{\bar{H}}{H'}\max_j(1+\frac{\Lambda_j}{\mu_1})\sum_k H_k|\Delta V_k|^2,$$

which concludes the proof.

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