The Synthesis Problem for Elementary Net Systems is NP-Complete

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Abstract: The so-called synthesis problem consists in deciding for a class of nets whether a given graph is isomorphic to the case graph of some net and then constructing the net. This problem has been solved for various classes of nets, ranging from elementary nets to Petri nets. The general principle is to compute *regions* in the graph, i.e. subsets of nodes liable to represent extensions of places of an associated net. The naive method of synthesis which relies on this principle leads to exponential algorithms for an arbitrary class of nets. In an earlier study, we gave algorithms that solve the synthesis problem in polynomial time for the class of bounded Petri nets. We show here that in contrast the synthesis problem is indeed NP-complete for the class of elementary nets. This result is independent from the results of Kunihiko Hiraishi, showing that both problems of separation and inhibition by regions *at a given node of the graph* are NP-complete.

Key-words: Synthesis Problem for Nets, Elementary Nets Systems, Regions, NP-Completeness

(Résumé : tsvp)

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La synthèse de réseaux élémentaires est NP-complète

Résumé : Le problème de synthèse consiste à décider, pour une classe de réseaux, si un graphe donné est isomorphe au graphe des marquages d'un réseau, et si oui à construire ce réseau. Ce problème a été résolu pour diverses classes de réseaux, allant des réseaux élémentaires aux réseaux de Petri. Le principe général est de calculer les *régions* du graphe, ou sous-ensembles de noeuds susceptibles de représenter les extensions des places d'un réseau associé. La méthode de synthèse naïve qui s'appuie sur ce principe conduit à des algorithmes exponentiels pour une classe de réseaux arbitraire. Dans une étude antérieure, nous avons donné des algorithmes de synthèse en temps polynomial pour la classe des réseaux de Petri bornés. Nous montrons ici que le problème de synthèse est par contre effectivement NP-complet pour la classe des réseaux élémentaires. Ce résultat est indépendant des résultats de Kunihiko Hiraishi, montrant que les problèmes de séparation et d'inhibition par les régions *en un noeud donné* du graphe sont NP-complets.

Mots-clé : Synthèse de réseaux, réseaux élémentaires, régions, NP-complétude

1 Introduction / Preliminaries

A *net* is a quadruple $N = (P, E, {}^\bullet() \;,\; ()^\bullet\;)$, where the set of *places* P and the set of *events* E are disjoint, and the pair of mappings ${}^\bullet()\;,\;()^\bullet\; : E \rightarrow 2^P$ state preconditions and postconditions in P for each event $e \in E$. An *elementary net* is a net which is *pure*: ${}^\bullet e \;\cap\; e^\bullet \;= \emptyset$ for all $e \in E$ and *simple*: $({}^\bullet x \;= {}^\bullet y \;\wedge\; x^\bullet \;=\; y^\bullet\;) \Rightarrow x = y$ for all $x, y \in P \cup E$ (where $e \in {}^\bullet p \;\Leftrightarrow\; p \in\; e^\bullet$ and $e \in\; p^\bullet \;\Leftrightarrow\; p \in {}^\bullet e$), and which has no *isolated* event: ${}^\bullet e \;\neq \emptyset$ or $e^\bullet \;\neq \emptyset$ for all $e \in E$. An *elementary net system* is a quintuple $NS = (P, E, {}^\bullet()\;,\;()^\bullet\;, M_0)$, where $(P, E, {}^\bullet()\;,\;()^\bullet\;)$ is an elementary net and M_0 is a subset of P called the *initial marking*. The set of the *accessible* markings of NS is the inductive closure of the singleton set $\{M_0\}$ with respect to the relation over 2^P defined as $M \rightarrow M'$ if $M \setminus M' = {}^\bullet e$ and $M' \setminus M =\; e^\bullet$ for some event $e \in E$. The *state graph* of NS is the automaton $NS^* = (\mathcal{M}(NS), E, \mathcal{T}(NS), M_0)$, where $\mathcal{M}(NS)$ is the set of accessible markings of NS and $\mathcal{T}(NS)$ is the set of transitions $M \xrightarrow{e} M'$ such that $M, M' \in \mathcal{M}(NS)$, $e \in E$, and $M \setminus M' = {}^\bullet e$ and $M' \setminus M =\; e^\bullet$. An *elementary transition system* is an automaton $A = (S, E, T, s_0)$, where $T \subseteq S \times E \times S$ and $s_0 \in S$, which is isomorphic to the state graph of some elementary net system i.e. such that $A \cong NS^*$ for some elementary net system NS. The *synthesis problem* for elementary net systems consists in deciding uniformly in A and constructively in NS whether there exists NS such that $A \cong NS^*$. This problem has been solved by Ehrenfeucht and Rozenberg who gave in [ER90] a combinatorial method for the decision, based on the computation of *regions* in the underlying transition systems. It appears clearly from the work of these authors that the synthesis problem for elementary net systems is in NP when restricted to finite transition systems. The purpose of this note is to prove that this problem is NP-complete (whereas the synthesis problem for bounded Petri nets is in P, as we proved in [BBD94]).

According to Ehrenfeucht and Rozenberg, a *region* in a transition system $TS = (S, E, T)$ is a subset of states $R \subseteq S$ whose characteristic function $\sigma_R : S \rightarrow \{0, 1\}$ admits a (unique) companion map $\eta_R : E \rightarrow \{-1, 0, 1\}$ such that $\sigma_R(s') = \sigma_R(s) + \eta_R(e)$ for every transition $s \xrightarrow{e} s'$ in τ. A region $R \equiv (\sigma_R, \eta_R)$ induces an *atomic* net $N_R(TS) = (\{R\}, E, {}^\bullet()\;,\;()^\bullet\;)$ with flow relations set according to the mapping η_R, namely:

$$R \in {}^\bullet e \quad \textbf{iff} \quad \eta_R(e) = -1 \qquad \textit{and} \qquad R \in\; e^\bullet \quad \textbf{iff} \quad \eta_R(e) = 1$$

A subset of regions $\mathbf{R}$, together with an initial state $s_0 \in S$, induces similarly a net system $NS_{\mathbf{R}}(TS, s_0) = (\mathbf{R}, E, {}^\bullet()\;,\;()^\bullet\;, M_0)$, where $M_0 = \{R \in \mathbf{R} \mid s_0 \in R\}$. Given a finite automaton A, a natural candidate for the solution of the synthesis problem for A is the net system $A^* = NS_{Reg(TS)}(A)$, where $Reg(TS)$ is the set of all regions in its underlying transition system TS. Ehrenfeucht and Rozenberg proved in fact that A is an elementary transition system if and only if $A \cong A^{**}$. In order to give a precise account of their result, let us call *pre-elementary transition system* an automaton $A = (S, E, T, s_0)$ which is *reduced*: $\forall e \in E \;\; \exists (s \xrightarrow{e} s') \in T$, *accessible*: $\forall s \in S$ $s_0 \xrightarrow{*} s$ where $\rightarrow$ is the union of the labelled transition relations $\xrightarrow{e}$, and which presents neither *loops*: $s \xrightarrow{e} s' \Rightarrow s \neq s'$, nor *multiple transitions*: $(s \xrightarrow{e} s' \wedge s \xrightarrow{e'} s') \Rightarrow e = e'$. Ehrenfeucht and Rozenberg proved that an automaton $A = (S, E, T, s_0)$ is elementary if and only if it pre-elementary and $Reg(TS)$ contains enough regions for solving all instances of the following two *separation problems*, in which case $A \cong A^{**}$.

States separation problem (SSP):
Given $TS = (S, E, T)$ and a pair of distinct states $(s_1, s_2) \in S \times S$, find a region $R \equiv (\sigma_R, \eta_R)$ in $Reg(TS)$ *separating* s_1 from s_2 in the sense that $\sigma_R(s_1) = 0$ and $\sigma_R(s_2) = 1$ or vice-versa.

Event/State separation problem (ESSP):
Given $TS = (S, E, T)$ and a pair $(s, e) \in S \times E$ such that e is *not enabled* at s ($s \xrightarrow{e} s'$ in T for no $s' \in S$), find a region $R \equiv (\sigma_R, \eta_R)$ in $Reg(TS)$ *inhibiting* e at s in the sense that $\sigma_R(s) = 0$ and $\eta_R(e) = -1$.

Desel and Reisig proved in [DR92] that $A \cong (NS_{\mathbf{R}}(A))^*$ for a subset of regions $\mathbf{R} \subset Reg(TS)$ if and only if $A = (TS, s_0)$ is pre-elementary and $\mathbf{R}$ contains enough regions for solving all the instances of the separation problems. Since the number of these instances is quadratic in the size of TS, this makes clear that the synthesis problem for elementary net systems is in NP, because problems SSP and ESSP are in NP (one may certainly check in polynomial time whether a given region solves a fixed instance of the separation problems). To be more precise, Hiraishi proved in [Hir94] that both problems SSP and ESSP are NP-complete. However, as this author remarked, it does not follow from this fact that the synthesis problem for elementary net systems is NP-complete. We give in the remaining sections an independent proof of the intractability of the synthesis problem for elementary net systems. For that purpose, we construct a polynomial reduction of 3-SAT to the synthesis problem, showing that the latter problem is NP-hard and therefore NP-complete. This will be done in two stages. In section 2, we reduce 3-SAT to satisfiability of systems of additive or multiplicative clauses over the boolean ring, a structure which fits exactly the needs for expressing and solving separation problems in transition systems. In section 3, we encode systems of clauses over the boolean ring to transition systems, and we classify regions in the latter with respect to solutions of the former. We finally prove in section 4 that satisfiability of systems of clauses over the boolean ring reduces through this polynomial encoding to the solvability of all instances of the separation problems for transition systems, and hence to the synthesis problem for elementary net systems.

2 Reducing 3-SAT to satisfiability of systems of additive or multiplicative clauses over the boolean ring

Recall that 3-SAT is the problem whether, given a finite set V of boolean variables and a finite system C of (disjunctive) clauses over V, with exactly three literals per clause, there exists a truth assignment for V satisfying all clauses in C. Problem 3-SAT is known to be NP-complete, see e.g. [GJ79]. The purpose of the section is to reduce 3-SAT to an equivalent problem over the boolean ring, a structure which fits exactly the needs for expressing and solving separation problems in labelled transition systems. Recall that $\mathbf{2} = \{0,1\}$ may be equipped alternatively with the structure of a boolean algebra $\mathbf{2} = (\{0,1\}, \vee, \wedge, \neg)$ or with the structure of a boolean ring $\mathbf{2} = (\{0,1\}, +, \cdot)$, in which product is logical conjunction ($x \cdot y = x \wedge y$) and sum is symmetric difference ($x + y = (x \wedge \neg y) \vee (\neg x \wedge y)$). Consider now a transition system $TS = (S, E, T)$. Call *abstract regions* in TS the second projections $\eta_R : E \rightarrow \{-1, 0, 1\}$ of regions $R \equiv (\sigma_R, \eta_R)$ in $Reg(TS)$, and call *floating regions* in TS the mappings $|\eta_R| : E \rightarrow \{0,1\}$ which are derived from abstract regions η_R by setting $|\eta_R|(e) = |\eta_R(e)|$ for $e \in E$, where $|\cdot|$ maps integers to their absolute value. The set of the floating regions in (the underlying transition system of) a pre-elementary transition system $A = (TS, s_0)$ may be easily characterized by a finite system of equations over the boolean ring in the variables $|\eta_R(e)|$. The construction is as follows. Extract from TS a spanning tree T rooted at s_0, and form a basis of cycles for TS with one cycle for each transition in $TS \setminus T$ (see e.g. [GM85]). Each cycle in the basis is thus represented by a chain in TS, let $s_1 \stackrel{e_1}{\leftarrow} s_2 \stackrel{e_2}{\leftarrow} s_3 \ldots \stackrel{e_{n-1}}{\leftarrow} s_n \stackrel{e_n}{\rightarrow} s_{n+1} \stackrel{e_{n+1}}{\rightarrow} \ldots s_{n+m} \stackrel{e_{n+m}}{\rightarrow} s_{n+m+1} \stackrel{e_{n+m+1}}{\rightarrow} s_1$, formed by concatenating an elementary chain in T and a transition $s_{n+m+1} \stackrel{e_{n+m+1}}{\rightarrow} s_1$ in $TS \setminus T$. Set for each cycle in the basis a corresponding equation $|\eta_R(e_1)| + \ldots + |\eta_R(e_{n+m+1})| = 0$. Next, for each elementary chain in T, say $s_1 \stackrel{e_1}{\leftarrow} s_2 \stackrel{e_2}{\leftarrow} s_3 \ldots \stackrel{e_{n-1}}{\leftarrow} s_n \stackrel{e_n}{\rightarrow} s_{n+1} \stackrel{e_{n+1}}{\rightarrow} \ldots s_{n+m} \stackrel{e_{n+m}}{\rightarrow} s_{n+m+1}$, and for each event $e \in E$ which is enabled in TS at both ends of that chain, set the equation $|\eta_R(e)| \cdot (\, |\eta_R(e_1)| + \ldots + |\eta_R(e_{n+m})| \,) = 0$. We claim that the system of equations $\mathcal{E}$ assembled in this way characterizes exactly the set of the floating regions in TS (the proof for that claim is quite similar to the one given in [BBD94], where the ring of integers is used in place of the boolean ring). Relying on that claim, instances of problems SSP and ESSP may be set in equational form. For each pair of distinct states $(s', s'') \in S \times S$, let $s' = s_1 \stackrel{e_1}{\leftarrow} s_2 \stackrel{e_2}{\leftarrow} s_3 \ldots \stackrel{e_{n-1}}{\leftarrow} s_n \stackrel{e_n}{\rightarrow} s_{n+1} \stackrel{e_{n+1}}{\rightarrow} \ldots s_{n+m} \stackrel{e_{n+m}}{\rightarrow} s_{n+m+1} = s''$ be an elementary chain connecting s' and s'' in T, then the states separation problem SSP is solvable at (s', s'') if and only if the system of equations $\mathcal{E} \bigcup \; \{ \, |\eta_R(e_1)| + \ldots + |\eta_R(e_{n+m})| \;) = 1 \; \}$ has a solution. For each pair $(s', e) \in S \times E$ such that e is not enabled at s' in TS, let s'' be an arbitrary state at which e is enabled in TS and let $s' = s_1 \stackrel{e_1}{\leftarrow} s_2 \stackrel{e_2}{\leftarrow} s_3 \ldots \stackrel{e_{n-1}}{\leftarrow} s_n \stackrel{e_n}{\rightarrow} s_{n+1} \stackrel{e_{n+1}}{\rightarrow} \ldots s_{n+m} \stackrel{e_{n+m}}{\rightarrow} s_{n+m+1} = s''$ be an elementary chain connecting s' and s''' in T, then the event/state separation problem ESSP is solvable at (s', e) if and only if the system of equations $\mathcal{E} \bigcup \; \{ \, |\eta_R(e_1)| + \ldots + |\eta_R(e_{n+m})| \;) = 1, \; |\eta_R(e)| = 1\}$ has a solution On the intuitive ground given by the above claims (which we do not prove because they are not needed for the technical development), we focus our attention on *systems of additive or multiplicative clauses* over the boolean ring, defined as follows.

Definition 1 (Systems of clauses over the boolean ring) *Let $X = \{x_0, \ldots, x_n\}$ be a finite set of boolean variables, with a distinguished element x_0. A* system of clauses over the boolean ring *is a pair (Σ, Π) where Σ is a finite set of additive clauses σ_α $(\alpha \in A)$ and Π is a finite set of multiplicative clauses π_β $(\beta \in B)$ with respective forms $x_{\alpha_0} + x_{\alpha_1} + x_{\alpha_2}$ and $x_{\beta_1} \cdot x_{\beta_2}$, subject to the following restrictions: each additive clause has exactly three variables, two additive clauses have at most one common variable, each multiplicative clause has exactly two variables, and the distinguished variable x_0 does not occur in any multiplicative clause. The system (Σ, Π) is said to be* satisfiable *if there exists a truth assignment for X such that $x_0 = 1$, $\sigma_\alpha = 0$ for all $\alpha \in A$, and $\pi_\beta = 0$ for all $\beta \in B$. Such a truth assignment is called a solution of (Σ, Π).*

Observe that in the boolean ring, the equations $z_0 = z_1 + \ldots + z_n$ and $z_0 + z_1 + \ldots + z_n = 0$ are equivalent in view of the inversion law $z + z = 0$. Using this remark, one can show that each instance of problems SSP and ESSP reduces to the satisfiability of a corresponding system of clauses, with size polynomial in the size of the transition system.

Let CBR denote the satisfiability problem for systems of clauses over the boolean ring. It is patent that CBR reduces polynomially to 3-SAT. The converse is shown by the following.

Proposition 2 *3-SAT reduces polynomially to CBR.*

Proof: Let (V, C) be an instance of 3-SAT. Define $W = V \cup \neg V$ where $\neg V = \{\neg v \mid v \in V\}$, and equip this set with the involution $\neg(v) = \neg v$ and $\neg(\neg v) = v$ for $v \in V$. We construct from (V, C) a system of clauses (Σ, Π) over the boolean ring with variables in the set $X = \{x_0\} \cup W \cup (W \times W)$. The additive clauses in Σ are defined as follows: for each literal $c \in W$ set the clause $x_0 + c + \neg c$, and for each pair of literals $ab \in W \times W$ set the clause $a + ab + a\neg b$. The multiplicative clauses in Π are defined as follows: for each pair of literals $ab \in W \times W$ set the two clauses $\neg a \cdot ab$ and $\neg b \cdot ab$, and for each disjunctive clause $a \vee b \vee c$ in C set the clause $\neg a \cdot \neg b \neg c$. It is easily verified that every truth assignment for V which satisfies C extends to a truth assignment for X which satisfies (Σ, Π) by setting $[\![\neg v]\!] = \neg [\![v]\!]$ for $v \in V$ and $[\![ab]\!] = [\![a]\!] \wedge [\![b]\!]$ for $a, b \in W$. Similarly, every truth assignment for X which satisfies (Σ, Π) restricts to a truth assignment for V which satisfies C, and the proposition follows. ∎

3 Encoding systems of clauses over the boolean ring to labelled transition systems

The purpose of the section is to encode uniformly systems of clauses (Σ, Π) over the boolean ring to automata $A(\Sigma, \Pi)$ with size polynomial in the size of (Σ, Π), such that (Σ, Π) is satisfiable if and only if $A(\Sigma, \Pi)$ is an elementary transition system. The encoding will be done in two stages. In the first stage, we construct from (Σ, Π) a system of equations (Σ', Π') with a larger set of variables, and we state the relationships between the respective solutions of (Σ, Π) and (Σ', Π'). In the second stage, we construct from (Σ', Π') an automaton $A(\Sigma, \Pi)$ and we observe that (Σ, Π) is satisfiable if $A(\Sigma, \Pi)$ is elementary. The proof for the converse is left to a later section.

Now for (Σ', Π'). Let $X = \{x_0, \ldots, x_{n-1}\}$ be the set of variables of (Σ, Π), where the respective sets of clauses $\Sigma = \{\sigma_\alpha \mid \alpha \in A\}$ and $\Pi = \{\pi_\beta \mid \beta \in B\}$ have typical elements

$$(\sigma_\alpha): \quad x_{\alpha_0} + x_{\alpha_1} + x_{\alpha_2} \qquad (\pi_\beta): \quad x_{\beta_1} \cdot x_{\beta_2}$$

Let $X' = X \cup \{x^\alpha \mid \alpha \in A\} \cup \{x_n, \ldots, x_N\}$, where $N - n + 1 = 3 \times size(A)$. Define $\Sigma' = \{\sigma'_\alpha \mid \alpha \in A\} \cup \{\sigma''_\alpha \mid \alpha \in A\}$, where σ'_α and σ''_α are the equations:

$$(\sigma'_\alpha): \quad x_{\alpha_0} + x_{\alpha_1} + x_{\alpha_2} + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_5} = 0$$

$$(\sigma''_\alpha): \quad x^\alpha + x_{\alpha_3} + x_{\alpha_4} + x_{\alpha_5} = 0$$

where $\{x_n, \ldots, x_N\} = \bigcup_{\alpha\in A} \{x_{\alpha_3}, x_{\alpha_4}, x_{\alpha_5}\}$. Define $\Pi' = \{\pi_\alpha \mid \alpha \in A\} \cup \{\pi_\beta \mid \beta \in B\}$, where π_α and π_β are the equations:

$$(\pi_\alpha): \quad x_0 \cdot x^\alpha = 0 \qquad (\pi_\beta): \quad x_{\beta_1} \cdot x_{\beta_2} = 0$$

A solution for (Σ', Π') is a truth assignment for X' such that all the above equations hold. A *distinguished solution* for (Σ', Π') is a solution which assigns value 1 to the distinguished variable x_0, and value 0 to all the auxiliary variables in $X' \setminus X$. Thus, every solution of (Σ, Π) extends to a distinguished solution of (Σ', Π'), and every distinguished solution of (Σ', Π') restricts to a solution of (Σ, Π). Before we construct $A(\Sigma, \Pi)$, let us state some properties of the set of solutions of (Σ', Π') showing the degrees of freedom introduced by the auxiliary variables.

Fact 3 *For each positive integer $i \leq N$, exists a family F_i of solutions $f : X' \to \mathbf{2}$ for (Σ', Π') such that $\{i\} = \{j \mid 0 \leq j \leq N \ \wedge \ (\forall f \in F_i) \cdot (f(x_j) = 1)\}$.*

Proof indication: For $0 < i < n$, F_i is the set of solutions $f : X \to \mathbf{2}$ of (Σ', Π') such that $f(x_0) = 0$ and $f(x_i) = 1$. ∎

Fact 4 *For each $\alpha \in A$, exists a family F_α of solutions $f : X' \to \mathbf{2}$ for (Σ', Π') such that $(\forall f \in F_\alpha) \cdot (f(x^\alpha) = 1)$ and $\{j \mid 0 \leq j \leq N \ \wedge \ (\forall f \in F_\alpha) \cdot (f(x_j) = 1)\} = \emptyset$.*

Proof indication: F_α is the set of solutions $f : X \to \mathbf{2}$ of (Σ', Π') such that $f(x_0) = 0$ and $f(x^\alpha) = 1$. ∎

We now proceed to the definition of the automaton $A(\Sigma, \Pi)$.

Definition 5 *An automaton with* entry states *is a transition system (S, E, T) together with a set $S_0 \subseteq S$ of entry states. Given an i-indexed family of automata (S^i, E^i, T^i, S_0^i), their* union $\cup_i(S^i, E^i, T^i, S_0^i)$ *is the automaton $(\cup_i S^i, \cup_i E^i, \cup_i T^i, \cup_i S_0^i)$, and their* lifted union *is the labelled transition system (S, E, T, s_0) with initial state s_0, set of states $S = \{s_0\} \cup (\cup_i S^i)$, set of events $E = (\cup_i E^i) \cup (\cup_i S_0^i)$, and set of transitions $T = (\cup_i T^i) \cup (\cup_i \{s_0 \xrightarrow{s} s \mid s \in S_0^i\})$.*

Notation 6 *For k ranging over $\{0, \ldots, N\}$, let $A(k) = \{\alpha \in A \mid \ \exists i \in \{0, \ldots, 5\} \cdot k = \alpha_i\}$, $B(k) = \{\beta \in B \mid \exists j \in \{1, 2\} \cdot k = \beta_j\}$, and $\Gamma(k) = A(k) \cup B(k)$.*

Definition 7 *Let $A(\Sigma, \Pi)$ and $A'(\Sigma', \Pi')$ be respectively the lifted union and the union of the families of automata S^α, T^β, $UV_k = \cup_{\gamma \in \Gamma(k)} UV_k^\gamma$, $W^\alpha = W_L^\alpha \cup W_R^\alpha$ indexed by $\alpha \in A$, $\beta \in B$, and $k \in \{0, \ldots, N\}$, where S^α, T^β, UV_k^γ, W_L^α, and W_R^α are the transition systems displayed in figures Fig. 1 to Fig. 4, with respective initial states s_0^α, t_0^β, u_k^γ, w_0^α, and w_3^α. $TS(\Sigma, \Pi)$ and $TS'(\Sigma', \Pi')$ denote the underlying transition systems of respectively $A(\Sigma, \Pi)$ and $A'(\Sigma', \Pi')$.*

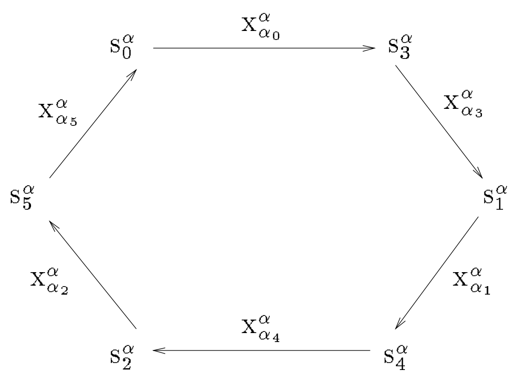

Figure 1: S^α

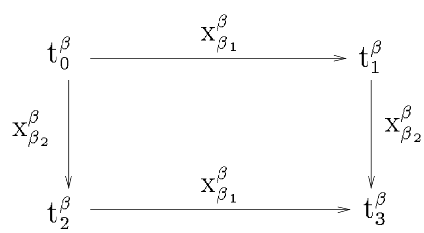

Figure 2: T^β

Fact 8 *Let ρ be a floating region of $TS(\Sigma, \Pi)$, then $\forall \gamma \in \Gamma(k)$ $\rho(\mathrm{x}_k^\gamma) = \rho(\mathrm{x}_k^*)$, and the mapping $f : X' \to \mathbf{2}$ given by $f(x_k) = \rho(\mathrm{x}_k^*)$ for $k \in [0, N]$, and $f(x^\alpha) = \rho(\mathrm{x}_*^\alpha)$ for $\alpha \in A$ is a solution of the system of equations (Σ', Π').*

Corollary 9 (Σ, Π) *is satisfiable if $A(\Sigma, \Pi)$ is an elementary transition system.*

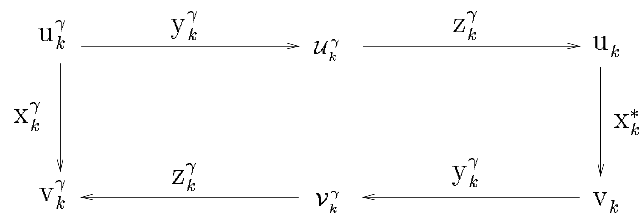

Figure 3: UV_k^γ

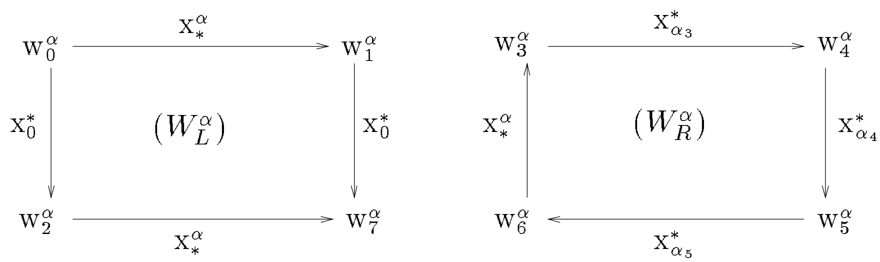

Figure 4: W^α

Proof: If $A(\Sigma,\Pi)$ is an elementary transition system, there exists at least one region $R\equiv(\sigma_R,\eta_R)$ such that $\sigma_R(\mathrm{w}_0^\alpha)\neq\sigma_R(\mathrm{w}_2^\alpha)$ and thus $|\eta_R|\,(\mathrm{x}_0^*)=1$. Then the floating region $\rho=|\eta_R|$ induces a solution f of (Σ',Π') such that $f(x_0)=1$ and which therefore restricts to a solution of (Σ,Π). ∎

Observe that $A(\Sigma,\Pi)$ is a pre-elementary transition system, with size polynomial in the size of (Σ,Π). It is an elementary transition system if and only if all instances of the separation problems SSP and ESSP can be solved in $TS(\Sigma,\Pi)$. In order to prove that CBR reduces polynomially to the synthesis problem, it remains to show that $A(\Sigma,\Pi)$ is elementary if (Σ,Π) is satisfiable. Thus, one should prove that every instance of SSP or ESSP can be solved in $TS(\Sigma,\Pi)$ or equivalently in $TS'(\Sigma',\Pi')$ if (Σ,Π) has a solution or yet equivalently if (Σ',Π') has a distinguished solution.

4 Satisfiability of systems of clauses reduces to the synthesis problem

We show in this section that every instance of SSP or ESSP can be solved in $TS'(\Sigma', \Pi')$ provided there exists a distinguished solution for (Σ', Π'). CBR reduces therefore to the synthesis problem for elementary net systems, and this problem is NP-complete like announced in the title of the paper.

As a starting point, let us observe again that every region $R \equiv (\sigma_R, \eta_R)$ in $TS'(\Sigma', \Pi')$ determines a solution $f : X' \to \mathbf{2}$ for (Σ', Π'), given by $f(x_k) = |\eta_R|\,(\mathrm{x}_k^*)$ $(= |\eta_R|\,(\mathrm{x}_k^\gamma))$ and $f(x^\alpha) = |\eta_R|\,(\mathrm{x}_*^\alpha)$. In order to carry on, we need a precise statement for the converse relationship, enabling us to construct sets of regions from arbitrary (i.e. possibly not distinguished) solutions of (Σ', Π').

Notation 10 *In order to simplify the notation, let* $TS' = TS'(\Sigma', \Pi')$. *Given an arbitrary solution* f *for* (Σ', Π'), *let* $\mathbf{R}(f)$ *be the set of regions* $R \equiv (\sigma, \eta)$ *in* TS' *such that* $f(x_k) = |\eta|\,(\mathrm{x}_k^*)$ *for all* $k \in \{0, \ldots, N\}$ *and* $f(x^\alpha) = |\eta|\,(\mathrm{x}_*^\alpha)$ *for all* $\alpha \in A$. *Finally let* A_* *be a disjoint copy of* A *and let* $\cdot_*$ *be a bijective mapping from* A *onto* A_*.

Definition 11 (Type of a region) *The* type *of region* $R \equiv (\sigma, \eta) \in \mathbf{R}(f)$ *is the unique map* $\tau : A \cup B \cup [0, n-1] \cup A_* \to \{0, 1\}$ *such that the following hold for all* $\alpha \in A$, $\beta \in B$, $k \in [0, n-1]$, *and* $\alpha_* \in A_*$:

$$\tau(\alpha) = \sigma(\mathrm{s}_0^\alpha) \qquad \tau(\beta) = \sigma(\mathrm{t}_0^\beta) \qquad \tau(k) = 0 \Leftrightarrow \eta(\mathrm{x}_k^*) = |\eta(\mathrm{x}_k^*)| \qquad \tau(\alpha_*) = \sigma(\mathrm{w}_3^\alpha)$$

Definition 12 (Dependent type of a region) *The* dependent type *of region* $R \equiv (\sigma, \eta) \in \mathbf{R}(f)$ *is the induced restriction of the map* η *on the set* $YZ = \cup\,\{\,\{\mathrm{y}_k^\gamma, \mathrm{z}_k^\gamma\} \mid k \in [0, \ldots, N] \,\wedge\, \gamma \in \Gamma(k)\}$.

Proposition 13 *Let* f *be a fixed solution for* (Σ', Π'). *A region* $R \equiv (\sigma, \eta) \in \mathbf{R}(f)$ *is totally determined from its type, its dependent type, its value* $\sigma(\mathrm{w}_0^\alpha)$ *at the initial state of each component* W_L^α, *and its value* $\sigma(\mathrm{u}_k^\gamma)$ *at one initial state in each component* UV_k. *Conversely, every map* $\tau : A \cup B \cup A_* \cup [0, n-1] \to \{0, 1\}$ *types a non empty set of regions* $(\sigma, \eta) \in \mathbf{R}(f)$ *and determines uniquely the common restriction* η_τ *of their* η*-component on* $E \setminus YZ$ *(where* E *is the set of events of* TS' *and* YZ *is the set introduced in the above definition). Moreover,* $|\eta_\tau(\mathrm{x}_k^*)| = |\eta_\tau(\mathrm{x}_k^\gamma)|$ *for every* $k \in [0, \ldots, N]$ *and* $\gamma \in \Gamma(k)$, *and a map* $\delta : YZ \to \{-1, 0, 1\}$ *is the dependent type of a region of type* τ *if and only if the following are satisfied for all* k *and* γ:

1. $\eta_\tau(\mathrm{x}_k^*) = \eta_\tau(\mathrm{x}_k^\gamma) = 0 \quad \Rightarrow \quad \delta(\mathrm{y}_k^\gamma) + \delta(\mathrm{z}_k^\gamma) = 0$
2. $\eta_\tau(\mathrm{x}_k^*) = \eta_\tau(\mathrm{x}_k^\gamma) \neq 0 \quad \Rightarrow \quad \delta(\mathrm{y}_k^\gamma) = \delta(\mathrm{z}_k^\gamma) = 0$
3. $\eta_\tau(\mathrm{x}_k^*) = -\eta_\tau(\mathrm{x}_k^\gamma) \neq 0 \quad \Rightarrow \quad [\delta(\mathrm{y}_k^\gamma) = \eta_\tau(\mathrm{x}_k^\gamma) \wedge \delta(\mathrm{z}_k^\gamma) = 0]$
   $\vee[\delta(\mathrm{y}_k^\gamma) = 0 \wedge \delta(\mathrm{z}_k^\gamma) = \eta_\tau(\mathrm{x}_k^\gamma)]$

Proof: In order to establish the last claim made in the proposition, it suffices to verify that all and only the maps from $\{\mathrm{x}_k^*,\ \mathrm{x}_k^\gamma,\ \mathrm{y}_k^\gamma,\ \mathrm{z}_k^\gamma\}$ to $\{-1,\ 0, 1\}$ that coincide with abstract regions in UV_k^γ appear as entries in the following table.

x_k^*	x_k^γ	y_k^γ	z_k^γ
0	0	0	0
0	0	1	-1
0	0	-1	1
1	1	0	0
-1	-1	0	0
1	-1	-1	0
-1	1	1	0
1	-1	0	-1
-1	1	0	1

Once this verification has been done, one may delete temporarily all components UV_k and focus on regions (σ, η) in $TS' \setminus \bigcup_k UV_k$ subject to the constraints $\rho(\mathrm{x}_k^\gamma) = \rho(\mathrm{x}_k^*)$ $(= f(x_k))$ imposed by the omitted components (where $\rho = |\eta|$ is the induced floating region). One may also delete temporarily all components W_L^α, imposing constraints $\rho(\mathrm{x}_0^*) \cdot \rho(\mathrm{x}_*^\alpha) = 0$ automatically satisfied by assumption on f as soon as $f(x^\alpha) = \rho(\mathrm{x}_*^\alpha)$. We are left with a transition system $TS'' = TS' \setminus ((\bigcup_k UV_k) \cup (\bigcup_\alpha W_l^\alpha))$ with the unique occurrence property, meaning that each event has at most one occurrence.

The events which occur in TS'' may have the form x_k^α, x_k^β, or x_l^* where $\alpha \in A$, $0 \leq k \leq N$, $\beta \in B$, and $n < l < N$. In particular, all events x_k^* with $k < n$ have disappear (since $n \leq \alpha_j$ for $j \leq 3$). By the unique occurrence property and because f is a solution of (Σ', Π'), every subset of $\{\mathrm{s}_0^\alpha |\ \alpha \in A\} \cup \{\mathrm{t}_0^\beta |\ \beta \in B\} \cup \{\mathrm{w}_0^\alpha |\ \alpha \in A\}$ determines a unique region (σ, η) in TS'', such that $\rho(\mathrm{x}_k^\alpha) = f(x_k)$, $\rho(\mathrm{x}_k^\beta) = f(x_k)$, $\rho(\mathrm{x}_l^*) = f(x_l)$, and $\rho(\mathrm{x}_*^\alpha) = f(x^\alpha)$, for $\alpha \in A$, $0 \leq k \leq N$, $\beta \in B$, and $n < l < N$. The choice for σ is encoded bijectively by the map $\tau : A \cup B \cup A^* \rightarrow \{0, 1\}$ given by $\tau(\alpha) = \sigma(\mathrm{s}_0^\alpha)$, $\tau(\beta) = \sigma(\mathrm{t}_0^\beta)$, and $\tau(\alpha_*) = \sigma(\mathrm{w}_*^\alpha)$.

Observe by the way that $\eta(\mathrm{x}_*^\alpha)$ has been fixed by the above process. In order to extend η into an abstract region of TS', it remains to fix $\eta(\mathrm{x}_k^*)$ for all $k < n$, and subsequently to fix $\eta(\mathrm{y}_k^\gamma)$ and $\eta(\mathrm{z}_k^\gamma)$ for all $0 \leq k \leq N$, and $\gamma \in \Gamma(k)$. By the first part of this proof, every map $\tau : [0, n-1] \rightarrow \{0, 1\}$ given jointly with a map $\delta : YZ \rightarrow \{-1, 0, 1\}$, compatible with the extented map $\tau : A \cup B \cup A^* \cup [0, n-1] \rightarrow \{0, 1\}$, defines uniquely an abstract region of type τ and dependent type δ in TS'.

In order to get a fully defined region (σ, η) in TS', it remains to extend σ on all components UV_k and W_L^α. For each component $G = UV_k$ or $G = W_L^\alpha$, two situations can occur. In case when $\eta(e) \neq 0$ for some event e occurring in G, the map η determines uniquely the map σ at all states in G since G is connected. In case when $\eta(e) = 0$ for all event e occuring in G, the map σ has a constant value on G which may be chosen freely in the set $\{0, 1\}$. The value of σ on G is then entirely fixed from the data of $\sigma(\mathrm{u}_k)$ if $G = UV_k$ or $\sigma(\mathrm{w}_0^\alpha)$ if $G = W_L^\alpha$. ∎

Armed with this proposition, and assuming a distinguished solution f_0 for (Σ', Π'), we now start to prove that all instances of ESSP can be solved in TS'. Once this result has been established, one verifies easily that all instances of SSP can be solved: most pairs of distinct states (s_1, s_2) in TS' are split by some event e, enabled at s_1 and disabled at s_2 (or the converse), and SSP is then automatically solved at (s_1, s_2) when ESSP is solved at (s_2, e). The pairs of states which remain to be checked for separation are all pairs of sink states, plus the pairs $(\mathrm{u}_0, \mathrm{w}_1^\alpha)$, $(\mathrm{w}_2^\alpha, \mathrm{w}_6^\alpha)$, $(\mathrm{u}_{\alpha_j}, \mathrm{w}_j^\alpha)$, and $(\mathcal{U}_k^\gamma, \mathcal{V}_k^\gamma)$, where $\alpha \in A, j \in \{3, 4, 5\}, k \in [0, N]$ and $\gamma \in \Gamma(k)$. If we except pairs $(\mathcal{U}_k^\gamma, \mathcal{V}_k^\gamma)$, all these pairs are assembled from states in two different connected components of TS', and their separation makes no problem since the set of states of a connected component is always a region. For the remaining pairs $(\mathcal{U}_k^\gamma, \mathcal{V}_k^\gamma)$, separation follows from Prop. 13 applied to any solution f of (Σ', Π') such that $f(x_k) = 1$ (where $\tau(\gamma)$ is chosen so that $\eta_\tau(\mathrm{x}_k^*) = \eta_\tau(\mathrm{x}_k^\gamma)$).

In order to divide the proof into meaningful pieces, let us introduce one more definition.

Definition 14 *For any pair (G_1, G_2) of connected components of TS', let $G_1 \triangleleft G_2$ if exists in TS' a region which inhibits e at s for every event e occurring in G_1 and for every state s in G_2 such that e is disabled at s.*

In the appendix we prove that $G_1 \triangleleft G_2$ for every pair of connected components (G_1, G_2), where possibly $G_1 = G_2$, thus establishing the announced result, namely

Theorem 15 *The synthesis problem for elementary net systems is NP-complete.*

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Appendix

In this appendix we prove that $G_1 \lhd G_2$ for every pair of connected components (G_1, G_2) of TS'. In several occasions we shall use the following fact about regions which was observed in [Ber93] and can easily be verified by the reader.

Fact 16 *The union of* disjoint *regions is a region. If R_1 and R_2 are regions with $R_1 \subset R_2$, then $R_2 \setminus R_1$ is a region.*

Lemma 17 $G \lhd G'$ *for* $G \in \{S^\alpha, T^\beta, W_L^\alpha, W_R^\alpha\}$ *and* $G' \in \{S^{\alpha'}, T^{\beta'}, W_L^{\alpha'}, W_R^{\alpha'}\}$, *where* $\alpha \neq \alpha'$ *and* $\beta \neq \beta'$.

Proof: An event e occurring in G may have form x_k^α or x_k^β for $k \leq N$, or x_*^α, or x_0^*, or x_k^* for $k \geq n$ (viz. $k = \alpha_j$ and $j \geq 3$). According to the case, let $x' \in X'$ be the variable x_k, or x^α, or x_0, or x_k then $f(x') = 1$ for some solution f of (Σ', Π'). From Prop. 13 applied to f, one may construct a region $R \equiv (\sigma, \eta) \in \mathbf{R}(f)$ such that $\eta(e) = -1$ with an adequate choice for $\tau(\alpha)$, $\tau(\beta)$, $\tau(\alpha_*)$, or $\tau(0)$ according to the form of e. If $\eta(e') = 0$ for every e' occurring in G', then the region $R' = (R \setminus G')$ inhibits e at every state s' in G'. Let us examine the converse case. If $G' \in \{S^{\alpha'}, T^{\beta'}, W_R^{\alpha'}\}$, let τ' be defined like τ except at α', β', or α'_* according to the form of G'. From Prop. 13, there exists a region $R' \equiv (\sigma', \eta') \in \mathbf{R}(f)$ with type τ'. Now the maps σ and σ' take complementary values at every state in G', hence either R or R' inhibits e at each state s' in G'. Finally let $G' = W_L^{\alpha'}$. If $e = x_0$, R inhibits e at w_2^α and w_7^α. In the converse case, let τ' be defined like τ except at 0 if $\eta(\mathrm{x}_0^*) \neq 0$ or at α'_* if $\eta(\mathrm{x}_*^{\alpha'}) \neq 0$. From Prop. 13, there exists a region $R' \equiv (\sigma', \eta') \in \mathbf{R}(f)$ with type τ'. The maps σ and σ' take complementary values on G', and the conclusion follows as above. ∎

Fact 18 $S^\alpha \lhd W_L^\alpha$, $\quad S^\alpha \lhd W_R^\alpha$, $\quad W_L^\alpha \lhd S^\alpha$, *and* $\quad W_R^\alpha \lhd S^\alpha$.

Lemma 19 $W_L^\alpha \lhd W_L^\alpha$, $\quad W_L^\alpha \lhd W_R^\alpha$, $\quad W_R^\alpha \lhd W_L^\alpha$, *and* $\quad W_R^\alpha \lhd W_R^\alpha$.

Proof: An event e occurring in W^α may have form x_0^*, or x_k^* for $k \geq n$, or x_*^α. Let us examine separately the three cases.

If $e = \mathrm{x}_0^*$, one applies Prop. 13 to the distinguished solution f_0 of (Σ', Π'), providing for a suitable choice of $\tau(0)$ a region $R \equiv (\sigma, \eta) \in \mathbf{R}(f_0)$ such that $\eta(\mathrm{x}_0^*) = -1$, and $\eta(e') = 0$ for every e' occurring in W_R^α. Now by Fact 16, the region $R \setminus W_R^\alpha$ inhibits x_0^* wherever it is not enabled in W^α.

If $e = \mathrm{x}_k^*$ and $k = \alpha_j$ $(j \geq 3)$, consider the solution f of (Σ', Π') defined by $f(x_h) = 1$ for $h \in \{\alpha_j, \alpha_{3+(j-1)mod 3}\}$ and $f(x') = 0$ for all the other variables. From Prop. 13 applied to f with a suitable choice for $\tau(\alpha_*)$, one constructs a region $R \equiv (\sigma, \eta) \in \mathbf{R}(f)$ such that $\eta(e) = -1$. We proceed by case analysis according to the value of j. If $j = 4$ or $j = 5$, the region $R \setminus W_L^\alpha$ inhibits e at every state s where it is not enabled in W^α. If $j = 3$, the same holds except for state w_6^α. In order to deal with this exception, we construct another solution f' for (Σ', Π'), such that $f'(x^\alpha) = f'(x_{\alpha_3}) = 1$. For that purpose, choose $i \in \{0, 1, 2\}$ such that $\alpha_i \neq 0$ and let $h = \alpha_i$. Then set $f'(x_h) = 1$, $f'(x^\gamma) = 1$ and $f'(x_{\gamma_3}) = 1$ iff $h \in \{\gamma_0, \gamma_1, \gamma_2\}$, and $f'(x') = 0$ for all the other variables. From Prop. 13 applied to f' for a suitable choice for $\tau(\alpha_*)$, one obtains a region $R' \equiv (\sigma', \eta') \in \mathbf{R}(f')$ such that $\eta(e) = -1$, and R' inhibits e at state w_6^α.

If $e = \mathrm{x}_*^\alpha$, we construct in a similar way a solution f' for (Σ', Π') such that $f'(x^\alpha) = f'(x_{\alpha_5}) = 1$ and $f'(x_{\alpha_3}) = f'(x_{\alpha_4}) = 0$. We obtain therefrom a region $R' \equiv (\sigma', \eta') \in \mathbf{R}(f')$ such that $\eta'(e) = -1$, and R' inhibits e wherever it is not enabled in W^α. ∎

Fact 20 *Let f be a solution for (Σ', Π'), then for each $\alpha \in A$, $i \in \{0, 1, 2\}$ and $j \in \{1, 2\}$, the functionf' defined by $f'(x_{\alpha_k}) = 1 + f(x_{\alpha_k})$ if $k \in \{3 + i\,,\, 3 + (i + j) \bmod 3\}$, and $f'(x) = f(x)$ for all the other variables in X', is also a solution for (Σ', Π').*

Lemma 21 $S^\alpha \lhd S^\alpha$.

Proof: Let $e = \mathrm{x}_k^\alpha$ and $k = \alpha_i$. We examine separately the cases $i \leq 2$ and $i \geq 3$.

If $i \leq 2$, let f be a solution of (Σ', Π') such that $f(x_k) = 1$. From Fact 5, exists another solution f' such that $f'(x_{\alpha_j}) = 1 + f(x_{\alpha_j})$ for $j \in \{3 + i, 3 + (i + 2) mod 3\}$, and $f'(x') = f(x')$ for all the other variables in X'. From Prop. 13, applied to f and f' with a suitable choice for $\tau(\alpha)$, one may construct respective regions $R \equiv (\sigma, \eta) \in \mathbf{R}(f)$ and $R' \equiv (\sigma', \eta') \in \mathbf{R}(f')$ such that $\eta(e) = \eta'(e) = -1$, whence either R or R' inhibits e in S^α at every state $s \neq \mathrm{s}_i^\alpha$.

If $i \geq 3$, we set $j = i - 3$ and $h = \alpha_j$, and we proceed separately for cases $h \neq 0$ and $h = 0$.

If $h \neq 0$, consider the solution f of (Σ', Π') defined by $f(x_h) = 1$, $f(x^\gamma) = 1$ for $\gamma \in A$ iff $h = \gamma_l$ for some l (≤ 2), $f(x_{\gamma_{l+3}}) = 1$ for such γ and l, and $f(x') = 0$ for all the other variables in X'. From Prop. 13 applied to f with a suitable choice for $\tau(\alpha)$, one obtains a region $R \equiv (\sigma, \eta) \in \mathbf{R}(f)$ such that $\eta(e) = -1$, and R inhibits e in S^α at every state $s \neq \mathrm{s}_i^\alpha$.

If $h = 0$, consider the distinguished solution f_0 of (Σ', Π'). Let l be the (unique) integer in $\{0, 1, 2\}$ such that $l \neq j$ and $f_0(x_{\alpha_l}) = 1$. One derives from f_0 another solution f for (Σ', Π') by setting $f(x_{\alpha_i}) = f(x_{\alpha_{l+3}}) = 1$ and $f(x') = f_0(x')$ for all the other variables in X'. From Prop. 13 applied to f with a suitable choice for $\tau(\alpha)$, one obtains a region $R \equiv (\sigma, \eta) \in \mathbf{R}(f)$ such that $\eta(e) = -1$, and R inhibits e in S^α at all states $s \neq \mathrm{s}_i^\alpha$ except at state s_{l+3}^α. In order to cope with this exception, consider the solution f' of (Σ', Π') defined by $f'(x_{\alpha_i}) = f'(x_{\alpha_{l+3}}) = 1$ and $f'(x') = 0$ for all the other variables in X'. From Prop. 13 applied to f' with a suitable choice for $\tau(\alpha)$, one obtains a region $R' \equiv (\sigma', \eta') \in \mathbf{R}(f')$ such that $\eta'(e) = -1$, and R' inhibits e at state s_{l+3}^α. ∎

Fact 22 $T^\beta \lhd T^\beta$.

Lemma 23 *$G \lhd UV_k$ for $G \in \{S^\alpha, T^\beta, W_L^\alpha, W_R^\alpha\}$.*

Proof: Let e be an event occurring in G. Let $x' = x_h$ if $e = \mathrm{x}_h^\alpha$ or $e = \mathrm{x}_h^\beta$ or $e = \mathrm{x}_h^*$, and let $x' = x^\alpha$ if $e = \mathrm{x}_*^\alpha$. We proceed separately with cases $x' \notin \{x_0, x_k\}$, $x' = x_k$, and $x' = x_0$.

Suppose $x' \notin \{x_0, x_k\}$. From facts 3 and 4, there exists in that case a solution f for (Σ', Π') such that $f(x') = 1$ and $f(x_k) = 0$. From Prop. 13, applied to f with a suitable choice for $\tau(\alpha)$ (if $G = S^\alpha$) or $\tau(\beta)$ (if $G = T^\beta$) or $\tau(\alpha_*)$ (if $G = W_L^\alpha$ or $G = W_R^\alpha$), one may construct a region $R \equiv (\sigma, \eta) \in \mathbf{R}(f)$ such that $\eta(e) = -1$ and $\eta(\mathrm{y}_k^\gamma) = \eta(\mathrm{z}_k^\gamma) = 0$ for all $\gamma \in \Gamma(k)$. Now by Fact 16, $R \setminus UV_k$ is a region which inhibits e at all states in UV_k.

Suppose $x' = x_k$ then either $k < n$ or $k = \alpha'_j$ for some $\alpha' \in A$ and $j \geq 3$. In either case, let f be a solution for (Σ', Π') such that $f(x') = 1$. If $e = \mathrm{x}_k^*$, one obtains from Prop. 13, applied to f with the suitable choice for $\tau(k)$ (if $k < n$ i.e. if $k = 0$) or $\tau(\alpha'_*)$ and all possible choices for $\{\tau(\gamma) | \gamma \in \Gamma(k)\}$, a family of regions $R_i \equiv (\sigma_i, \eta_i) \in \mathbf{R}(f)$ such that $\eta_i(e) = -1$ and every map $\eta : \{\mathrm{x}_k^\gamma | \gamma \in \Gamma(k)\} \rightarrow \{-1, 1\}$ coincides with the restriction of some abstract region η_i. If $e = \mathrm{x}_k^\delta$ for some $\delta \in \Gamma(k)$, one obtains from Prop. 13, applied to f with the suitable choice for $\tau(\delta)$ and with all possible choices for $\{\tau(\gamma) | \gamma \neq \delta \wedge \gamma \in \Gamma(k)\}$ and $\tau(k)$ (if $k < n$) or $\tau(\alpha'_*)$, a family of regions $R_i \equiv (\sigma_i, \eta_i) \in \mathbf{R}(f)$ such that $\eta_i(e) = -1$ and every map $\eta : \{\mathrm{x}_k^*\} \cup \{\mathrm{x}_k^\gamma | \gamma \neq \delta \wedge \gamma \in \Gamma(k)\} \rightarrow \{-1, 1\}$ coincides with the restriction of some abstract region η_i. In both situations, some region R_i in the resulting set inhibits e at each state u_k, v_k, u_k^γ, or v_k^γ in which it is not enabled. By varying now the dependent type of each region R_i according to the conditions stated in Prop. 13 (with η_i substituted for η_τ and δ), one obtains additional regions which inhibit e at all the remaining states $\mathcal{U}_k^\gamma$ and $\mathcal{V}_k^\gamma$.

Suppose finally $x' = x_0$ and $k \neq 0$. Consider the distinguished solution f_0 of (Σ', Π'). If $f_0(x_k) = 0$, which is always true for $k \geq n$, one obtains from Prop. 13, applied to f_0 with a suitable choice for $\tau(\alpha)$ (if $e = \mathrm{x}_0^\alpha$) or $\tau(0)$ (if $e = \mathrm{x}_0^*$), [1] a region $R \equiv (\sigma, \eta) \in \mathbf{R}(f_0)$ such that $\eta(e) = -1$, $\eta(\mathrm{x}_k^*) = 0$ and $\eta(\mathrm{x}_k^\gamma) = 0$ for all $\gamma \in \Gamma(k)$. From Prop. 13, the dependent type of R may be adjusted so that $\eta(\mathrm{y}_k^\gamma) = \eta(\mathrm{z}_k^\gamma) = 0$ for all $\gamma \in \Gamma(k)$. Now by Fact 16, $R \setminus UV_k$ is a region which inhibits e at all states in UV_k. If $f_0(x_k) = 1$ (thus $0 < k < n$), we proceed separately for the case $e = \mathrm{x}_0^\alpha \wedge \alpha \in \Gamma(k)$ and for the other cases.

Let $e = \mathrm{x}_0^*$, or $e = \mathrm{x}_0^\alpha$ and $\alpha \notin \Gamma(k)$. From Prop. 13, applied to f_0 with a suitable choice for $\tau(0)$ or $\tau(\beta)$ or $\tau(\alpha)$, and with all possible choices for $\{\tau(\gamma) | \gamma \in \Gamma(k)\}$, one obtains a family of regions $R_i \equiv (\sigma_i, \eta_i) \in \mathbf{R}(f_0)$ such that $\eta_i(e) = -1$ and every map $\eta : \{\mathrm{x}_k^\gamma | \gamma \in \Gamma(k)\} \to \{-1, 1\}$ coincides with the restriction of some abstract region η_i. The expected conclusion follows like in case $x' = x_k$.

Let finally $e = \mathrm{x}_0^\alpha$ and $\alpha \in \Gamma(k)$, whence $0 = \alpha_i$ and $k = \alpha_j$ for two distinct integers $i, j \in \{0, 1, 2\}$. Since f_0 is a distinguished solution of (Σ', Π'), $f_0(x_{\alpha_p}) = 0$ for all $p \in [0, 5] \setminus \{i, j\}$. From fact 5, exists another solution f for (Σ', Π') such that $f(x_{\alpha_p}) = 1$ for $p \in \{3 + j, 3 + (j + 2) mod 3\}$, and $f(x') = f_0(x')$ for all the other variables in X'. From Prop. 13, applied to f with a suitable choice for $\tau(\alpha)$, and with all possible choices for $\tau(k)$ and for $\{\tau(\gamma) | \gamma \neq \alpha \wedge \gamma \in \Gamma(k)\}$, one obtains a family of regions $R_i \equiv (\sigma_i, \eta_i) \in \mathbf{R}(f)$ such that $\eta_i(e) = -1$ and every map $\eta : \{\mathrm{x}_k^*\} \cup \{\mathrm{x}_k^\gamma | \gamma \neq \alpha \wedge \gamma \in \Gamma(k)\} \to \{-1, 1\}$ coincides with the restriction of some abstract region η_i. The expected conclusion follows again like in case $x' = x_k$. ∎

Fact 24 *For all k and $\gamma \in \Gamma(k)$, the sets $\{\mathcal{U}_k^\gamma, \mathcal{V}_k^\gamma\}$ and $UV_k \setminus \{\mathcal{U}_k^\gamma, \mathcal{V}_k^\gamma\}$ define respective regions $R \equiv (\sigma, \eta)$ and $R' \equiv (\sigma', \eta')$ in TS' such that $\eta(\mathrm{z}_k^\gamma) = -1$ and $\eta'(\mathrm{y}_k^\gamma) = -1$. Similarly, the set $\cup\{\{\mathcal{U}_k^\gamma, \mathcal{V}_k^\gamma\} \mid \gamma \in \Gamma(k)\}$ and its complement in UV_k are regions.*

Lemma 25 *$UV_k \lhd UV_l$ for $k \neq l$.*

Proof: In view of fact 24, it suffices to show that ESSP may be solved at all instances (e, s) in which s is a state in UV_l and $e = \mathrm{x}_k^*$ or $e = \mathrm{x}_k^\beta$ or $e = \mathrm{x}_k^\alpha$. Now in every case except when $e = \mathrm{x}_k^*$ and $0 < k < n$, ESSP may be solved at (e, s) as a consequence from $S^\alpha \lhd UV_l$, or $T^\beta \lhd UV_l$, or $W_L^\alpha \lhd UV_l$, or $W_R^\alpha \lhd UV_l$. So assume

[1] Remember that variable x_0 does not occur in equations Π thus there is no event of the form x_0^β for $\beta \in B$

$e = \mathrm{x}_k^*$ and $0 < k < n$. There exists a solution f for (Σ', Π') such that $f(x_k) = 1$ and $f(x_l) = 0$. From Prop. 13, applied to f with a suitable choice for $\tau(k)$, exists a region $R \equiv (\sigma, \eta) \in \mathbf{R}(f)$ such that $\eta(e) = -1$ and $\eta(e') = 0$ for every e' in UV_l. Then $R \setminus UV_l$ is a region that inhibits e everywhere in UV_l. ∎

Lemma 26 *$UV_k \lhd UV_k$.*

Proof: In view of the properties $S^\alpha \lhd UV_k$, $T^\beta \lhd UV_k$, $W_L^\alpha \lhd UV_k$, and $W_R^\alpha \lhd UV_k$, it suffices to solve ESSP at (e, s) for s in UV_k and $e = \mathrm{z}_k^\gamma$, or $e = \mathrm{y}_k^\gamma$, or $e = \mathrm{x}_k^*$ and $0 < k < n$. The first case is immediate (from fact 24). We examine separately the remaining cases.

Suppose $e = \mathrm{x}_k^*$ (with $0 < k < n$). Let then f be a solution for (Σ', Π') such that $f(x_k) = 1$. From Prop. 13, applied to f with the suitable choice for $\tau(k)$ and with all possible choices for $\{\tau(\gamma) |\, \gamma \in \Gamma(k)\}$, one obtains a family of regions $R_i \equiv (\sigma_i, \eta_i) \in \mathbf{R}(f)$ such that $\eta_i(e) = -1$ and every map $\eta : \{\mathrm{x}_k^\gamma |\, \gamma \in \Gamma(k)\} \to \{-1, 1\}$ coincides with the restriction of some abstract region η_i. As a consequence, some region in the family inhibits e at each state u_k, v_k, u_k^γ, or v_k^γ in which it is not enabled. By varying the dependent type of each region R_i according to the conditions stated in Prop. 13 (with η_i substituted for η_τ and δ), one obtains additional regions which inhibit e at all the remaining states $\mathcal{U}_k^\gamma$ and $\mathcal{V}_k^\gamma$.

Suppose now $e = \mathrm{y}_k^\gamma$, where either $k < n$ or $k = \alpha_j'$ $(j \geq 3)$. Since the set $\cup\{\{\mathcal{U}_k^\delta, \mathcal{V}_k^\delta\} \mid \delta \in \Gamma(k)\}$ and its complement in UV_k are regions, it suffices in fact to solve ESSP at (e, s) for $s = \mathrm{u}_k$ or v_k^γ, and for $s = \mathrm{u}_k^\delta$ or v_k^δ with $\delta \neq \gamma$. Let f be a solution for (Σ', Π') such that $f(x_k) = 1$. From Prop. 13, applied to f with a suitable choice for $\tau(k)$ (if $k < n$) or $\tau(\alpha_*')$ (if $k \geq n$) and for $\{\tau(\delta) |\, \delta \in \Gamma(k)\}$, one may construct a region $R \equiv (\sigma, \eta) \in \mathbf{R}(f)$ such that $\eta(\mathrm{x}_k^*) = 1$ and $\eta(\mathrm{x}_k^\delta) = -1$ for all $\delta \in \Gamma(k)$ (including γ). From Prop. 13, one may adjust the dependent type of R so that $\eta(\mathrm{y}_k^\gamma) = -1$, and then R inhibits e at u_k and at all states v_k^δ $(\delta \in \Gamma(k))$. The converse choice for $\{\tau(\delta) |\, \delta \neq \gamma \wedge \delta \in \Gamma(k)\}$ produces a region $R' \equiv (\sigma', \eta') \in \mathbf{R}(f)$ such that $\eta'(\mathrm{x}_k^*) = 1$, $\eta'(\mathrm{x}_k^\gamma) = -1$, and $\eta(\mathrm{x}_k^\delta) = 1$ for $\delta \neq \gamma$. This region R' inhibits e at all states u_k^δ $(\delta \in \Gamma(k)$ and $\delta \neq \gamma)$. ∎

Lemma 27 *$UV_k \lhd S^\alpha$ and $UV_k \lhd T^\beta$.*

Proof: In view of fact 24 and properties $S^\gamma \lhd S^\alpha$, $T^\gamma \lhd S^\alpha$, $W_L^\gamma \lhd S^\alpha$, $W_R^\gamma \lhd S^\alpha$, and $S^\gamma \lhd T^\beta$, $T^\gamma \lhd T^\beta$, $W_L^\gamma \lhd T^\beta$, $W_R^\gamma \lhd T^\beta$, it suffices to solve ESSP at (e, s) for s in S^α or

T^β and $e = \mathrm{x}_k^*$ with $0 < k < n$. Let f be a solution for (Σ', Π') such that $f(x_k) = 1$. From Prop. 13, applied to f with a suitable choice for $\tau(k)$, one may construct a region $R \equiv (\sigma, \eta)$ of type τ in $\mathbf{R}(f)$ such that $\eta(\mathrm{x}_k^*) = -1$. Let type τ' be defined like τ except at α (respectively at β). From Prop. 13, exists a region $R' \equiv (\sigma', \eta')$ of type τ' in TS'. Now the maps σ and σ' take complementary values at each state s in S^α (respectively T^β), hence at each state s, either R or R' inhibits x_k^*. ∎

Lemma 28 *$UV_k \lhd W_L^\alpha$ and $UV_k \lhd W_R^\alpha$.*

Proof: For the same reasons as in the preceding lemma, it suffices to solve ESSP at (e, s) for s in W_L^α or W_R^α and $e = \mathrm{x}_k^*$ with $0 < k < n$. Let f be a solution for (Σ', Π') such that $f(x_k) = 1$. From Prop. 13, applied to f with a suitable choice for $\tau(k)$, one may construct a region $R \equiv (\sigma, \eta)$ of type τ in $\mathbf{R}(f)$ such that $\eta(\mathrm{x}_k^*) = -1$. Let type τ' be defined like τ except at α_*. From Prop. 13, exists a region $R' \equiv (\sigma', \eta')$ of type τ' in TS'. Now the maps σ and σ' take complementary values at each state s in W_R^α, hence at each state s in W_R^α, either R or R' inhibits x_k^* at state s. This line of reasoning is also valid for W_L^α if $f(x^\alpha) = 1$ (τ' is like τ except at α_*) or $f(x^\alpha) = 0$ and $f(x_0) = 1$ (τ' is defined like τ except at 0). Consider finally the case where $f(x^\alpha) = f(x_0) = 0$. Then $R \setminus W_L^\alpha$ is a region which inhibits x_k^* at all states in W_L^α. ∎

The above series of lemmas show that $G_1 \lhd G_2$ for every pair of connected components (G_1, G_2) in TS' as required.

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